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Discrete spectrum of perturbed Dirac systems with real and periodic coefficients

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Synopsis

This paper deals with the number of eigenvalues which appear in the gaps of the spectrum of a Dirac system with real and periodic coefficients when the coefficients are perturbed. The main results provide an upper bound and a condition under which exactly one eigenvalue appears in a given gap.

Introduction

Let us consider the differential expression

$$\sigma y(x) = -y''(x) + q(x)y(x),$$

where y is a complex valued function defined on $\mathbb{R}$ and $q\colon \mathbb{R} \to \mathbb{R}$ is periodic with period $a > 0$ and locally absolutely integrable.

The maximal operator S generated by σ on $\mathbb{R}$ **[7]** is self-adjoint and is called the *Hill's operator*. Its spectrum $\sigma(S) \subset \mathbb{R}$ is purely continuous, bounded from below but unbounded from above, and it is a locally finite union of closed intervals of positive length. In the following, we shall suppose that $\sigma(S)$ has an infinity of gaps; this is so, for example, if $q \in L^2_{\text{loc}}(\mathbb{R})$ and is not analytic **[5]**.

Rofe-Beketov **[3, 4]** studied the perturbed Hill's operator $\tilde{S}$, which is the maximal operator generated on $\mathbb{R}$ by the differential expression

$$\tilde{\sigma} y(x) = -y''(x) + \{q(x) + \Delta q(x)\}y(x),$$

where $\Delta q\colon \mathbb{R} \to \mathbb{R}$ is such that $|\Delta q(x)|\,(1 + |x|)$ is integrable on $\mathbb{R}$. This is a self-adjoint operator with the same essential spectrum $\sigma_e(\tilde{S})$ as S. He proved that there is only a finite number of eigenvalues of $\tilde{S}$ in each gap an at most two eigenvalues in each sufficiently remote gap; moreover, there is exactly one eigenvalue in each sufficiently remote gap if the following additional condition is satisfied: $\int \Delta q(x)\, dx \neq 0$.

The purpose of this paper is to prove analogous results for Dirac systems. Let τ be the differential expression

$$\tau u(x) = R(x)^{-1}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u'(x) + P(x)u(x)\right\},$$

where u is a $\mathbb{C}^2$-valued function defined on $\mathbb{R}$; P and R are symmetric 2×2 matrices, with locally absolutely integrable real entries which are periodic with period $a > 0$; R is positive definite almost everywhere; and let $L^2(]c, d[, R)$ be the Hilbert space defined by

$$L^2(]c, d[, R) = \left\{u:]c, d[\to \mathbb{C}^2 \mid \int_c^d (R(x)u(x), u(x))\, dx < \infty\right\}$$

and

$$\langle u, v\rangle = \int_c^d (R(x)u(x), v(x))\, dx,$$

where $(.\,,.)$ denotes the usual scalar product in $\mathbb{C}^2$ and $-\infty \leqq c < d \leqq +\infty$ (we do not distinguish between two functions which are equal almost everywhere). The maximal operator T generated by τ and defined in $L^2(\mathbb{R}, R)$ is self-adjoint, its spectrum $\sigma(T)$ is purely continuous, unbounded from above and below, and it is a locally finite union of closed intervals of positive length [**7**, theorem 12.5]. In the following, we shall suppose that $\sigma(T)$ has at least one gap; for example, if

$$a = 2, \quad R \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P|_{]0,1[} \equiv \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \quad P|_{]1,2[} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

there is an infinity of gaps [**7**, chap. 17.G].

We shall examine the perturbed operator $\tilde{T}$, which is the maximal operator generated on $\mathbb{R}$ by the differential expression

$$\tilde{\tau}u(x) = R(x)^{-1}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u'(x) + \{P(x) + \Delta P(x)\}u(x)\right\},$$

where $\Delta P(x)$ is a symmetric 2×2 matrix, with absolutely integrable real entries, and whose support satisfies $\operatorname{supp}(\Delta P) \subset [A, B](-\infty < A \leqq B < +\infty)$. The operator $\tilde{T}$ is self-adjoint and $\sigma_e(\tilde{T}) = \sigma_e(T)$; this can be proved by the method of decomposition [**1**].

Let $]\mu, \nu[$ be a gap of $\sigma_e(\tilde{T})$, let $r_2(x)$, $p_2(x)$, $\delta p_2(x)$ be the largest eigenvalues of respectively $R(x)$, $P(x)$, $\Delta P(x)$, and let $r_1(x)$, $p_1(x)$, $\delta p_1(x)$ be the corresponding lowest eigenvalues. We shall show that if for $N \in \mathbb{N}$,

$$\int_A^B \{(|\mu| + |\nu|)(r_2(x) - r_1(x)) + 2(p_2(x) - p_1(x)) + (\delta p_2(x) - \delta p_1(x))\}\, dx \leqq N\pi,$$

then there are at most $N + 1$ eigenvalues of $\tilde{T}$ in $]\mu, \nu[$, and if

$$\int_A^B \{\max(|\mu|, |\nu|)(r_2(x) - r_1(x)) + (p_2(x) - p_1(x)) + |\delta p_2(x)|\}\, dx \leqq \pi/2,$$

$$\delta p_1(x) = \delta p_2(x) \quad \text{almost everywhere on } \mathbb{R},$$

$$\delta p_2 \text{ is not equal almost everywhere to the null function,}$$

$$\delta p_2(x) \geqq 0 \quad \text{almost everywhere or} \quad \delta p_2(x) \leqq 0 \quad \text{almost everywhere on } \mathbb{R},$$

then there is exactly one eigenvalue in $]\mu, \nu[$.

But first we shall present an important tool, which is an adaptation of the oscillation theory for Dirac systems developed by Weidmann [**7**, chap. 16].

1. Oscillation theory for Dirac systems defined on $\mathbb{R}$ and which are in the limit point case at $-\infty$ and $+\infty$

Let τ be the differential expression

$$\tau u(x)=R(x)^{-1}\left\{2\begin{pmatrix}0 & q(x)\\ -q(x) & 0\end{pmatrix}u'(x)+\begin{pmatrix}0 & q'(x)\\ -q'(x) & 0\end{pmatrix}u(x)+P(x)u(x)\right\},$$

where u is a $\mathbb{C}^2$-valued function defined on $\mathbb{R}$; P and R are symmetric 2×2 matrices, with locally absolutely integrable real entries; R is positive definite almost everywhere; q is a locally absolutely continuous real valued function and for all values of $x\in\mathbb{R}$: $q(x)>0$. If u is a non-trivial real solution of $\tau u=\lambda u$, we introduce the transformation

$$u(x)=\rho(x)\begin{pmatrix}\cos\theta(x)\\ \sin\theta(x)\end{pmatrix},\quad \rho(x)>0,$$

where ρ and θ are continuous and θ is defined up to $2k\pi$. If θ is completely defined (for instance, if we know its value at a given x_0), we shall call it *a determination of the angular part* of u.

It is easy to check that

$$\theta'(x)=\left(G(x)\begin{pmatrix}\cos\theta(x)\\ \sin\theta(x)\end{pmatrix},\begin{pmatrix}\cos\theta(x)\\ \sin\theta(x)\end{pmatrix}\right),$$

with

$$G(x)=\frac{1}{2q(x)}(\lambda R(x)-P(x)).$$

We write $\theta(\lambda,\alpha,c,x)(\lambda,\alpha,c$ in $\mathbb{R})$ for the angular part which

(i) corresponds to the solution $v(x)=(v_1(x),v_2(x))$ of $\tau u=\lambda u$, satisfying $v_1(c)=\cos(\alpha)$ and $v_2(c)=\sin(\alpha)$;

(ii) is such that $\theta(\lambda,\alpha,c,c)=\alpha$.

PROPOSITION 1.1. *If the maximal operator T generated by τ on $\mathbb{R}$ is self-adjoint and $-\infty<\mu<\lambda<+\infty$, then*

$$\dim R(E_T(\lambda-)-E_T(\mu))\leqq\liminf_{n\to\infty}\left[\frac{\theta(\lambda,\alpha_n,c_n,d_n)-\theta(\mu,\alpha_n,c_n,d_n)}{\pi}\right],$$

where $(\alpha_n)\subset\mathbb{R}$, $c_n\to-\infty$, $d_n\to+\infty$, $E_T(.)$ is the right continuous spectral resolution of T and $[x]=\max\{k\in\mathbb{Z}\mid k\leqq x\}$.

Proof. Let $\delta>0$ be such that $\mu+\delta$ and $\lambda-\delta$ are not eigenvalues of T and $\mu+\delta<\lambda-\delta$, let β_n be defined by $\beta_n=\theta(\mu+\delta,\alpha_n,c_n,d_n)$ and let us define the self-adjoint operators B_n, O_n and T_n as follows:

$$B_n:\begin{cases}D(B_n)\subset L^2(]c_n,d_n[,R)\to L^2(]c_n,d_n[,R),\\ B_nu=\tau u,\end{cases}$$

with

$$D(B_n)=\{u\in L^2(]c_n,d_n[,R)\mid u \text{ is loc. abs. cont., } \tau u\in L^2(]c_n,d_n[,R) \text{ and}$$
$$\sin(\alpha_n)u_1(c_n)-\cos(\alpha_n)u_2(c_n)=\sin(\beta_n)u_1(d_n)-\cos(\beta_n)u_2(d_n)=0\},$$

O_n is the null operator on $L^2(]-\infty,c_n[,R)\oplus L^2(]d_n,+\infty[,R)$ and $T_n=B_n\oplus O_n$.

The sequence (T_n) converges to T in the sense of the strong resolvent convergence and therefore

$$\dim R(E_T(\lambda-\delta)-E_T(\mu+\delta)) \leqq \liminf_{n\to\infty} \dim R(E_{T_n}(\lambda-\delta)-E_{T_n}(\mu+\delta))$$
$$= \liminf_{n\to\infty} \dim R(E_{B_n}(\lambda-\delta)-E_{B_n}(\mu+\delta))$$

(see [**6**, theorems 9.16.(i) and 9.19]). We have

$$\begin{aligned}\dim R(E_{B_n}(\lambda-\delta)-E_{B_n}(\mu+\delta))\\ &= \operatorname{card}(\{t\in\,]\mu+\delta,\lambda-\delta]\mid \theta(t,\alpha_n,c_n,d_n)=\beta_n \bmod \pi\}),\\ &= \left[\frac{\theta(\lambda-\delta,\alpha_n,c_n,d_n)-\theta(\mu+\delta,\alpha_n,c_n,d_n)}{\pi}\right]\\ &\leqq \left[\frac{\theta(\lambda,\alpha_n,c_n,d_n)-\theta(\mu,\alpha_n,c_n,d_n)}{\pi}\right],\end{aligned}$$

because $\theta(t,\alpha_n,c_n,d_n)$ is increasing in t [**7**, theorem 16.1]. Therefore,

$$\dim R(E_T(\lambda-)-E_T(\mu)) \leqq \liminf_{\delta\to 0} \dim R(E_T(\lambda-\delta)-E_T(\mu+\delta))$$
$$\leqq \liminf_{n\to\infty}\left[\frac{\theta(\lambda,\alpha_n,c_n,d_n)-\theta(\mu,\alpha_n,c_n,d_n)}{\pi}\right]. \qquad \square$$

Proposition 1.2. *If T is any self-adjoint operator generated by τ on $\mathbb{R}$ and $-\infty<\mu<\lambda<+\infty$, then*

$$\dim R(E_T(\lambda)-E_T(\mu-)) \geqq \sup_{c<d,\,\alpha\in\mathbb{R}}\left[\frac{\theta(\lambda,\alpha,c,d)-\theta(\mu,\alpha,c,d)}{\pi}\right]-1.$$

Proof. Let us choose c, d, α in $\mathbb{R}$, and let us suppose that $c<d$ and

$$n := \left[\frac{\theta(\lambda,\alpha,c,d)-\theta(\mu,\alpha,c,d)}{\pi}\right] \geqq 2.$$

We introduce the operator

$$B:\begin{cases} D(B)\subset L^2(]c,d[,R)\to L^2(]c,d[,R),\\ Bu=\tau u,\end{cases}$$

with

$$/D(B)=\{u\in L^2(]c,d[,R)\mid u \text{ is loc. abs. cont., } \tau u\in L^2(]c,d[,R) \text{ and}$$
$$\sin(\alpha)u_1(c)-\cos(\alpha)u_2(c)=\sin(\beta)u_1(d)-\cos(\beta)u_2(d)=0\},$$

where $\beta=\theta(\mu,\alpha,c,d)$. Since $\dim R(E_B(\lambda)-E_B(\mu-))=\operatorname{card}(\{t\in[\mu,\lambda]\mid \theta(t,\alpha,c,d)=\beta \bmod \pi\})=n+1$, there exists a subspace $M\subset R(E_B(\lambda)-E_B(\mu-))$ such that (i) $\dim(M)\geqq n-1$, (ii) for all values of u in M: $u(c)=u(d)=(0,0)$ and $\|\{B-(\lambda+\mu)/2\}u\|\leq\{(\lambda-\mu)/2\}\|u\|$. We can consider M as a subspace of $D(T)$ and so we have $\dim R(E_T(\lambda)-E_T(\mu-))\geqq n-1$ (if we had $\dim R(E_T(\lambda)-E_T(\mu-))<n-1$, then there would exist $f\in M$ such that $f\neq 0$, $f\perp R(E_T(\lambda)-E_T(\mu-))$ and $\|\{T-(\lambda+\mu)/2\}f\|>\{(\lambda-\mu)/2\}\|f\|$). $\square$

PROPOSITION 1.3. *Let u and $\tilde{u}$ be two non-trivial real solutions of $\tau u = \lambda u$ (λ in $\mathbb{R}$) and let θ and $\tilde{\theta}$ be any two determinations of the corresponding angular parts. If there exists x_0 in $\mathbb{R}$ and k in $\mathbb{Z}$ such that $\tilde{\theta}(x_0) - \theta(x_0) \in [k\pi, (k+1)\pi[$, then for all values of $x \in \mathbb{R}$: $\tilde{\theta}(x) - \theta(x) \in [k\pi, (k+1)\pi[$.*

Proof. We have

$$\begin{aligned}\sigma'(x) = \tilde{\theta}'(x) - \theta'(x) &= g_{11}(x)\cos^2\tilde{\theta}(x) + g_{22}(x)\sin^2\tilde{\theta}(x) \\ &\quad + 2g_{12}(x)\sin\tilde{\theta}(x)\cos\tilde{\theta}(x) \\ &\quad - g_{11}(x)\cos^2\theta(x) - g_{22}(x)\sin^2\theta(x) - 2g_{12}(x)\sin\theta(x)\cos\theta(x) \\ &= \{g_{22}(x) - g_{11}(x)\}\{\sin^2\tilde{\theta}(x) - \sin^2\theta(x)\} + g_{12}(x)\{\sin 2\tilde{\theta}(x) - \sin 2\theta(x)\} \\ &= \{g_{22}(x) - g_{11}(x)\}\sin\{\tilde{\theta}(x) + \theta(x)\}\sin\sigma(x) \\ &\quad + 2g_{12}(x)\cos\{\tilde{\theta}(x) + \theta(x)\}\sin\sigma(x),\end{aligned}$$

where $\sigma(x) = \tilde{\theta}(x) - \theta(x)$. This differential equation verifies the local existence and uniqueness theorem. Since $\sigma \equiv k\pi$ and $\sigma \equiv (k+1)\pi$ are solutions and $\sigma(x_0) \in [k\pi, (k+1)\pi[$, we have $\sigma(x) \in [k\pi, (k+1)\pi[$ for all x in $\mathbb{R}$. □

2. Dirac systems with periodic and real coefficients

We suppose that $q \equiv \frac{1}{2}$ and that P and R have the period $a > 0$; τ becomes

$$\tau u(x) = R(x)^{-1}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u'(x) + P(x)u(x)\right\}.$$

For λ in $\mathbb{R}$, let us introduce the fundamental system of solutions of $\tau u(x) = \lambda u(x)$: $\varphi(x, \lambda)$, $\psi(x, \lambda)$, satisfying

$$\varphi(0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have

$$W[\varphi, \psi](x, \lambda) := \begin{vmatrix} \varphi_1(x, \lambda) & \psi_1(x, \lambda) \\ \varphi_2(x, \lambda) & \psi_2(x, \lambda) \end{vmatrix} \equiv 1.$$

We introduce the discriminant, which is the real valued function defined by $D(\lambda) = \varphi_1(a, \lambda) + \psi_2(a, \lambda) (\lambda \in \mathbb{R})$.

For $\lambda \in \mathbb{R}$ such that $|D(\lambda)| \geqq 2$, let $\rho_1(\lambda)$ and $\rho_2(\lambda)$ in $\mathbb{R}$ be the two roots of $\rho^2 - D(\lambda)\rho + 1$ with $|\rho_1(\lambda)| \leqq 1 \leqq |\rho_2(\lambda)|$, and let $k(\lambda)$ in $\mathbb{C}$ satisfy $\exp\{ak(\lambda)\} = \rho_2(\lambda)$. For $i = 1, 2$, there exists a real solution $e_i(x, \lambda)$ of $\tau u(x) = \lambda u(x)$ such that

$$e_i(x + a, \lambda) = \rho_i(\lambda)e_i(x, \lambda),$$

and if we define z_1 and z_2 by

$$e_1(x, \lambda) = \exp\{-k(\lambda)x\}z_1(x, \lambda) \quad \text{and} \quad e_2(x, \lambda) = \exp\{k(\lambda)x\}z_2(x, \lambda),$$

then $z_i(x + a, \lambda) = z_i(x, \lambda)$ for all x in $\mathbb{R}$ and $i = 1, 2$.

If $\rho_1(\lambda) \neq \rho_2(\lambda)$, then $e_1(x, \lambda)$ and $e_2(x, \lambda)$ can be chosen linearly independent; the same is possible if $\rho_1(\lambda) = \rho_2(\lambda) (= \pm 1)$ and $\varphi_2(a, \lambda) = \psi_1(a, \lambda) = 0$. If

$\rho_1(\lambda)=\rho_2(\lambda)(=\pm 1)$ and $|\varphi_2(a,\lambda)|+|\psi_1(a,\lambda)|>0$, then we can choose $e_i(x,\lambda)$ such that $e_1(x,\lambda)=e_2(x,\lambda)\neq(0,0)$ (for all values of $x\in\mathbb{R}$), and there exists a solution $y(x)$ of $\tau u=\lambda u$, linearly independent of $e_1(x,\lambda)$, such that $\limsup_{x\to+\infty}|y(x)|=\limsup_{x\to-\infty}|y(x)|=+\infty$.

We can take for example

$$\begin{aligned} e_i(x,\lambda)&=\psi_1(a,\lambda)\varphi(x,\lambda)+\{\rho_i(\lambda)-\varphi_1(a,\lambda)\}\psi(x,\lambda), \text{ or}\\ e_i(x,\lambda)&=\{\rho_i(\lambda)-\psi_2(a,\lambda)\}\varphi(x,\lambda)+\varphi_2(a,\lambda)\psi(x,\lambda), \end{aligned} \tag{2.1}$$

but these functions can be null ($i=1,2$). Note that they are analytic in λ on $\{\lambda\in\mathbb{R}\mid |D(\lambda)|>2\}$ for every $x\in\mathbb{R}$. The reader is referred to **[2]** and **[7]** for more information.

The maximal operator defined on $\mathbb{R}$ by τ is self-adjoint and its spectrum is equal to $\{\lambda\in\mathbb{R}\mid |D(\lambda)|\leqq 2\}$. Let $]\mu,\nu[$ be a gap of its spectrum and let e_μ and e_ν be two non-trivial, real and periodic solutions of, respectively, $\tau u=\mu u$ and $\tau u=\nu u$. We shall denote any two determinations of the corresponding angular parts by θ_μ and θ_ν. Let $c<d$ be in $\mathbb{R}$ such that $d-c$ is in a $\mathbb{Z}$.

PROPOSITION 2.1. (1) $\theta_\nu(d)-\theta_\mu(d)=\theta_\nu(c)-\theta_\mu(c)$;

(2) if θ is the determination of the angular part of a non-trivial real solution of $\tau u=\nu u$ such that $\theta(c)=\theta_\mu(c)$, then $\theta(d)-\theta_\mu(d)\in]0,\pi[$.

Proof. (1) Since $\theta_\mu(x)$ and $\theta_\nu(x)$ are the angular parts of periodic or semi-periodic functions of period or semi-period a, there exists k in $\mathbb{Z}$ such that $\{\theta_\nu(d)-\theta_\mu(d)\}-\{\theta_\nu(c)-\theta_\mu(c)\}=k\pi$.

Case (i). Let us suppose that $k>0$. For $n\in\mathbb{N}$, set $c_n=c-(d-c)n$ and $d_n=d+(d-c)n$. Then $\{\theta_\nu(d_n)-\theta_\mu(d_n)\}-\{\theta_\nu(c_n)-\theta_\mu(c_n)\}=(2n+1)k\pi$. Let θ_n be the determination of the angular part of a non-trivial real solution of $\tau u=\nu u$ such that $\theta_n(c_n)=\theta_\mu(c_n)$. We have

$$\begin{aligned}\theta_n(d_n)-\theta_\mu(d_n)=(\{\theta_n(d_n)-\theta_\nu(d_n)\}-\{\theta_n(c_n)-\theta_\nu(c_n)\})\\ +(\{\theta_\nu(d_n)-\theta_\mu(d_n)\}-\{\theta_\nu(c_n)-\theta_\mu(c_n)\}).\end{aligned}$$

By Proposition 1.3, the first term belongs to $]-\pi,\pi[$ and therefore

$$\theta_n(d_n)-\theta_\mu(d_n)\in](2n+1)k\pi-\pi,\quad (2n+1)k\pi+\pi[.$$

By Proposition 1.2, there is an infinity of eigenvalues of T in $[\mu,\nu]$. This assertion being false, we have proved that $k\leqq 0$.

Case (ii). Let us suppose that $k<0$. Then $\theta_n(d_n)-\theta_\mu(d_n)<0$ for $n\in\mathbb{N}$. Hence we have a contradiction with [**7**, theorem 16.1].

(2) There exists k in $\mathbb{Z}$ such that $\theta(c)-\theta_\nu(c)=\theta_\mu(c)-\theta_\nu(c)=\theta_\mu(d)-\theta_\nu(d)\in[k\pi,(k+1)\pi[$. By Proposition 1.3, $\theta(d)-\theta_\nu(d)\in[k\pi,(k+1)\pi[$. Thus

$$\theta(d)-\theta_\mu(d)=\{\theta(d)-\theta_\nu(d)\}-\{\theta_\mu(d)-\theta_\nu(d)\}\in]-\pi,\pi[.$$

By [**7**, theorem 16.1], $\theta(d)>\theta_\mu(d)$ and therefore $\theta(d)-\theta_\mu(d)\in]0,\pi[$.

3. The main results

Let τ and $\tilde{\tau}$ be as in the Introduction and let $]\mu, \nu[$ be a gap of $\sigma_e(\tilde{T})$ (we suppose that there is at least one gap).

PROPOSITION 3.1. *If for $N \in \mathbb{N}$,*

$$\int_A^B \{(|\mu| + |\nu|)(r_2(x) - r_1(x)) + 2(p_2(x) - p_1(x)) + (\delta p_2(x) - \delta p_1(x))\}\, dx \leqq N\pi,$$

then there are at most $(N+1)$ eigenvalues of $\tilde{T}$ in $]\mu, \nu[$.

Proof. Let e_μ be a non-trivial, real and periodic solution of $\tau u = \mu u$, and let θ_μ be any determination of its angular part. For $n \in \mathbb{N}$, we introduce the following notation: θ is the determination of the angular part of a real non-trivial solution of $\tau u = \nu u$ such that $\theta(-na) = \theta_\mu(-na)$; $\tilde{\theta}_\mu$ is the determination of the angular part of a real non-trivial solution of $\tilde{\tau} u = \mu u$ such that $\tilde{\theta}_\mu(-na) = \theta_\mu(-na)$; $\tilde{\theta}$ is the determination of the angular part of a real non-trivial solution of $\tilde{\tau} u = \nu u$ such that $\tilde{\theta}(-na) = \theta_\mu(-na)$.

We have

$$\theta_\mu'(x) = \left((\mu R(x) - P(x)) \begin{pmatrix} \cos \theta_\mu(x) \\ \sin \theta_\mu(x) \end{pmatrix}, \begin{pmatrix} \cos \theta_\mu(x) \\ \sin \theta_\mu(x) \end{pmatrix} \right),$$

$$\tilde{\theta}_\mu'(x) = \left((\mu R(x) - P(x) - \Delta P(x)) \begin{pmatrix} \cos \tilde{\theta}_\mu(x) \\ \sin \tilde{\theta}_\mu(x) \end{pmatrix}, \begin{pmatrix} \cos \tilde{\theta}_\mu(x) \\ \sin \tilde{\theta}_\mu(x) \end{pmatrix} \right),$$

$$\theta'(x) = \left((\nu R(x) - P(x)) \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \end{pmatrix}, \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \end{pmatrix} \right),$$

$$\tilde{\theta}'(x) = \left((\nu R(x) - P(x) - \Delta P(x)) \begin{pmatrix} \cos \tilde{\theta}(x) \\ \sin \tilde{\theta}(x) \end{pmatrix}, \begin{pmatrix} \cos \tilde{\theta}(x) \\ \sin \tilde{\theta}(x) \end{pmatrix} \right),$$

and thus

$$\begin{aligned} &|\{\tilde{\theta}'(x) - \theta'(x)\} - \{\tilde{\theta}_\mu'(x) - \theta_\mu'(x)\}| \\ &\qquad \leqq (|\mu| + |\nu|)(r_2(x) - r_1(x)) + 2(p_2(x) - p_1(x)) + (\delta p_2(x) - \delta p_1(x)). \end{aligned}$$

Let us suppose that n is such that $[A, B] \subset]-na, na[$ and let k be in $\mathbb{Z}$ such that $\tilde{\theta}_\mu(B) - \theta_\mu(B) \in [k\pi, (k+1)\pi[$. We have

$$\begin{aligned} &|\{\tilde{\theta}(B) - \theta(B)\} - \{\tilde{\theta}_\mu(B) - \theta_\mu(B)\}| \\ &\qquad \leqq \int_A^B |\{\tilde{\theta}'(x) - \theta'(x)\} - \{\tilde{\theta}_\mu'(x) - \theta_\mu'(x)\}|\, dx \leqq N\pi \end{aligned}$$

and therefore $\tilde{\theta}(B) - \theta(B) \in [(k-N)\pi, \ (k+N+1)\pi[$. By Proposition 1.3, $\tilde{\theta}_\mu(na) - \theta_\mu(na) \in [k\pi, (k+1)\pi[$ and $\tilde{\theta}(na) - \theta(na) \in [(k-N)\pi, \ (k+N+1)\pi[$. Using $\theta(na) - \theta_\mu(na) \in]0, \pi[$ (Proposition 2.1 (2)), we get

$$\begin{aligned} \tilde{\theta}(na) - \tilde{\theta}_\mu(na) &= \{\tilde{\theta}(na) - \theta(na)\} - \{\tilde{\theta}_\mu(na) - \theta_\mu(na)\} \\ &\quad + \{\theta(na) - \theta_\mu(na)\} < (k + N + 1 - k + 1)\pi = (N+2)\pi. \end{aligned}$$

Letting n tend to $+\infty$, the result now follows from Proposition 1.1 and the fact that the eigenvalues are of multiplicty one (for $\lambda \in]\mu, \nu[$, a solution in $L^2(\mathbb{R}, R)$ of $\tilde{\tau}u = \lambda u$ is a multiple of $e_1(., \lambda)$ on $[B, \infty[$ and a multiple of $e_2(., \lambda)$ on $]-\infty, A]$).

PROPOSITION 3.2. *If*

$$\int_A^B \{\max(|\mu|, |\nu|)(r_2(x) - r_1(x)) + (p_2(x) - p_1(x)) + |\delta p_2(x)|\}\, dx \leqq \pi/2,$$

$$\delta p_1(x) = \delta p_2(x) \quad \textit{almost everywhere on } \mathbb{R},$$

δp_2 is not equal almost everywhere to the null function,

$$\delta p_2(x) \geqq 0 \quad \textit{almost everywhere or} \quad \delta p_2(x) \leqq 0 \quad \textit{almost everywhere on } \mathbb{R},$$

then there is exactly one eigenvalue of $\tilde{T}$ in $]\mu, \nu[$.

Proof. We shall adapt a method of Rofe-Beketov **[4]**. Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be as in Section 2 and let $\tilde{\varphi}(x, \lambda)$ and $\tilde{\psi}(x, \lambda)$ be two solutions of $\tilde{\tau}u(x) = \lambda u(x)$ satisfying

$$\tilde{\varphi}(0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{\psi}(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us introduce the two regular matrices

$$L(x, \lambda) = (\varphi(x, \lambda)\ \psi(x, \lambda)) \quad \text{and} \quad \tilde{L}(x, \lambda) = (\tilde{\varphi}(x, \lambda)\ \tilde{\psi}(x, \lambda)),$$

let v and w be two real solutions of $\tau u = \lambda u$ ($\lambda \in \mathbb{R}$ is fixed) and let $\tilde{v}$ and $\tilde{w}$ be two real solutions of $\tilde{\tau}u = \lambda u$, such that v and $\tilde{v}$ are equal on $[B, +\infty[$, and w and $\tilde{w}$ are equal on $]-\infty, A]$.

Using the method of variation of constants, we get

$$\tilde{w}(x) = w(x) + L(x, \lambda)\int_A^x \left\{L^{-1}(t, \lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta P(t)\tilde{w}(t)\right\} dt,$$

$$v(x) = \tilde{v}(x) - \tilde{L}(x, \lambda)\int_B^x \left\{\tilde{L}^{-1}(t, \lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta P(t)v(t)\right\} dt,$$

and using

$$v^t(x)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} L(x, \lambda) = v^t(t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} L(t, \lambda)$$

and

$$\tilde{w}^t(x)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{L}(x, \lambda) = \tilde{w}^t(t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{L}(t, \lambda),$$

we obtain

$$W[\tilde{v}, \tilde{w}] = W[v, w] + v^t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(\tilde{w} - w) - \tilde{w}^t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(\tilde{v} - v)$$

$$= W[v, w] - \int_A^B (\Delta P(t)v(t), \tilde{w}(t))\, dt.$$

Now let us consider the case $v = e_1(x, \lambda)$ and $w = e_2(x, \lambda)$, where $e_1(x, \lambda)$ and $e_2(x, \lambda)$ are defined by one of the formulae (2.1) and $\lambda \in [\mu, v]$. We shall use $\dot{u}$ for the derivation in λ, u' for the derivation in x, and $E_i(x, \lambda)$ for $\tilde{e}_i(x, \lambda)$. Since $d/dx(E_{12}\dot{E}_{21} - E_{11}\dot{E}_{22}) = d/dx(E_{22}\dot{E}_{11} - E_{21}\dot{E}_{12}) = -(RE_1, E_2)$, we have

$$\frac{d}{d\lambda} W[E_1, E_2](\Lambda) = \{(E_{22}\dot{E}_{11} - E_{21}\dot{E}_{12}) - (E_{12}\dot{E}_{21} - E_{11}\dot{E}_{22})\}(0, \Lambda)$$

$$= \int_{-\infty}^{+\infty} (R(t)E_1(t, \Lambda), E_2(t, \Lambda))\, dt \neq 0$$

if for all values of $t \in \mathbb{R}$: $E_1(t, \Lambda) = E_2(t, \Lambda) \neq (0, 0)$, i.e. if $\Lambda \in]\mu, v[$ is an eigenvalue of $\tilde{T}$ such that $W(e_1, e_2)(\Lambda) \neq 0$.
Set $e_i(x, \lambda) = \psi_1(a, \lambda)\varphi(x, \lambda) + \{\rho_i(\lambda) - \varphi_1(a, \lambda)\}\psi(x, \lambda)$. We have

$$W[e_1, e_2](\lambda) = \psi_1(a, \lambda)\{\rho_2(\lambda) - \rho_1(\lambda)\} \neq 0$$

if $\lambda \in]\mu, v[$ and $\psi_1(a, \lambda) \neq 0$.

As for Sturm-Liouville operators with Dirichlet and Neumann boundary conditions [**7**, chap. 13], the spectrum of the operator generated by τ on $]0, a[$ with boundary conditions $u_2(0) = u_2(a) = 0$ (respectively $u_1(0) = u_1(a) = 0$) is equal to $\{\lambda \mid \varphi_2(a, \lambda) = 0\}$ (respectively $\{\lambda \mid \psi_1(a, \lambda) = 0\}$). We can also prove that in each maximal interval included in $\{\lambda \mid |D(\lambda)| \geqq 2\}$, $\varphi_2(a, \lambda)$ and $\psi_1(a, \lambda)$ have exactly one zero. In particular, there exists an unique $\kappa \in [\mu, v]$ such that $\psi_1(a, \kappa) = 0$.

Case (i). κ is not an eigenvalue of $\tilde{T}$ and $\kappa \in]\mu, v[$. If $\lambda \in \{\mu, v\}$, we have

$$W[E_1, E_2](\lambda) = -\int_A^B (\Delta P(t)e_2(t, \lambda), E_2(t, \lambda))\, dt.$$

The hypothesis

$$\int_A^B \{\max(|\mu|, |v|)(r_2(x) - r_1(x)) + (p_2(x) - p_1(x)) + |\delta p_2(x)|\}\, dx \leqq \pi/2$$

implies that there are at most two eigenvalues in $]\mu, v[$ (Proposition 3.1) and, with the fact that e_2 and E_2 are not trivial, that the cosine of the angle between e_2 and E_2 is not negative on $[A, B]$. Indeed,

$$\theta'(x) = \left((\lambda R(x) - P(x))\begin{pmatrix}\cos\theta(x)\\ \sin\theta(x)\end{pmatrix}, \begin{pmatrix}\cos\theta(x)\\ \sin\theta(x)\end{pmatrix}\right)$$

and

$$\tilde{\theta}'(x) = \left((\lambda R(x) - P(x) - \Delta P(x))\begin{pmatrix}\cos\tilde{\theta}(x)\\ \sin\tilde{\theta}(x)\end{pmatrix}, \begin{pmatrix}\cos\tilde{\theta}(x)\\ \sin\tilde{\theta}(x)\end{pmatrix}\right),$$

thus

$$|\tilde{\theta}'(x) - \theta'(x)| \leqq \max(|\mu|, |v|)(r_2(x) - r_1(x)) + (p_2(x) - p_1(x)) + |\delta p_2(x)|$$

and, for all x in $[A, B]$,

$$|\tilde{\theta}(x) - \theta(x)|$$
$$\leqq \int_A^x \{\max(|\mu|, |v|)(r_2(s) - r_1(s)) + (p_2(s) - p_1(s)) + |\delta p_2(s)|\} ds \leqq \pi/2,$$

where θ and $\tilde{\theta}$ are any two determinations of the angular parts of e_2 and E_2 respectively, such that $\theta(A)=\tilde{\theta}(A)$, and $\lambda \in \{\mu, \nu\}$. Since $\delta p_1 = \delta p_2$ has a constant sign and is not equal almost everywhere to the null function,

$$W[E_1, E_2](\mu) = -\int_A^B (\Delta P(t)e_2(t, \mu), E_2(t, \mu))\, dt$$

and

$$W[E_1, E_2](\nu) = -\int_A^B (\Delta P(t)e_2(t, \nu), E_2(t, \nu))\, dt$$

are not null and have the opposite sign of δp_2. Moreover, the function $W[E_1, E_2](\lambda)$ crosses the λ-axis at $\lambda=\kappa$ and at every eigenvalue. Therefore, there is exactly one eigenvalue in $]\mu, \nu[$.

In order to prove that $W[E_1, E_2](\lambda)$ crosses the λ-axis at $\lambda=\kappa$, we introduce

$$f_b(x, \lambda) = \psi_1(a, \lambda)\varphi(x, \lambda) + \{\rho_b(\lambda) - \varphi_1(a, \lambda)\}\psi(x, \lambda)$$

and

$$f_c(x, \lambda) = \{\rho_c(\lambda) - \psi_2(a, \lambda)\}\varphi(x, \lambda) + \varphi_2(a, \lambda)\psi(x, \lambda).$$

We suppose that $\{b, c\} = \{1, 2\}$ and $\rho_c(\kappa) \neq \psi_2(a, \kappa)$. Let $F_i(x, \lambda)$ $(i=1,2)$ be the corresponding perturbed functions such that $F_1(., \lambda)$ and $f_1(., \lambda)$ are equal on $[B, \infty[$, and $F_2(., \lambda)$ and $f_2(., \lambda)$ are equal on $]-\infty, A]$. It follows that

$$W[f_b, f_c](\lambda) = \{\rho_c(\lambda) - \psi_2(a, \lambda)\}\{\rho_c(\lambda) - \rho_b(\lambda)\},$$

$$W[e_1, e_2](\lambda) = \frac{\psi_1(a, \lambda)}{\rho_c(\lambda) - \psi_2(a, \lambda)} W[f_1, f_2](\lambda),$$

and

$$W[E_1, E_2](\lambda) = \frac{\psi_1(a, \lambda)}{\rho_c(\lambda) - \psi_2(a, \lambda)} W[F_1, F_2](\lambda).$$

Near κ, $W[f_b, f_c](\lambda)$ and $W[F_1, F_2](\lambda)$ are not null, and $(\partial/\partial\lambda)\psi_1(a, \kappa) \neq 0$ (see below); therefore the function $W[E_1, E_2](\lambda)$ crosses the λ-axis at $\lambda=\kappa$.

Case (ii). κ is an eigenvalue. Then $W[F_1, F_2](\lambda)$ and $\psi_1(a, \lambda)$ cross the λ-axis at $\lambda=\kappa$ and thus $W[E_1, E_2](\lambda)$ is zero at $\lambda=\kappa$ without crossing the λ-axis. The result follows in the same way as in case (i).

Case (iii). $\kappa \in \{\mu, \nu\}$. Let us introduce

$$g_i(x, \lambda) = \{\rho_i(\lambda) - \psi_2(a, \lambda)\}\varphi(x, \lambda) + \varphi_2(a, \lambda)\psi(x, \lambda) \quad (i=1, 2),$$

and let $G_i(x, \lambda)$ be the corresponding perturbed functions such that $G_1(., \lambda)$ and $g_1(., \lambda)$ are equal on $[B, \infty[$, and $G_2(., \lambda)$ and $g_2(., \lambda)$ are equal on $]-\infty, A]$. Since

$$W[g_1, g_2](\lambda) = \varphi_2(a, \lambda)\{\rho_1(\lambda) - \rho_2(\lambda)\},$$

it follows that

$$W[E_1, E_2](\lambda) = -\frac{\psi_1(a, \lambda)}{\varphi_2(a, \lambda)} W[G_1, G_2](\lambda).$$

Moreover $\varphi_2(a, \kappa) \neq 0$ and $g_1(., \kappa) = g_2(., \kappa)$ is not trivial. Hence $W[G_1, G_2](\kappa) \neq 0$ and $W[G_1, G_2](\kappa)$ has the opposite sign of δp_2. Since

$$\frac{\partial}{\partial \lambda} \psi_1(a, \lambda) = \int_0^a \{-(R(t)\psi(t, \lambda), \psi(t, \lambda))\varphi_1(a, \lambda) + (R(t)\psi(t, \lambda), \varphi(t, \lambda))\psi_1(a, \lambda)\}\, dt$$

and

$$\frac{\partial}{\partial \lambda} \varphi_2(a, \lambda) = \int_0^a \{(R(t)\varphi(t, \lambda), \varphi(t, \lambda))\psi_2(a, \lambda) - (R(t)\psi(t, \lambda), \varphi(t, \lambda))\varphi_2(a, \lambda)\}\, dt$$

(see [**2**, lemma 2.1]), we have $(\partial/\partial\lambda)\varphi_2(a, \delta) \neq 0$, where δ is the unique zero of $\varphi_2(a, \lambda)$ in $[\mu, \nu]$, and $\operatorname{sgn}\{(\partial/\partial\lambda)\psi_1(a, \kappa)\} = -\operatorname{sgn}\{\varphi_1(a, \kappa)\} = -\operatorname{sgn}\{D(\kappa)\} = -\operatorname{sgn}\{D(\delta)\} = -\operatorname{sgn}\{\psi_2(a, \delta)\} = -\operatorname{sgn}\{(\partial/\partial\lambda)\varphi_2(a, \delta)\}$, and thus $-(\psi_1(a, \lambda)/\varphi_2(a, \lambda))$ is negative between κ and λ. The results follows as in case (i).

Remarks 3.3. If $r_1 = r_2$ and $\delta p_1 = \delta p_2$, then Proposition 3.2 provides sufficient conditions for the perturbed operator to have exactly one eigenvalue in each gap, and Proposition 3.1 provides a sufficient condition on $\operatorname{supp}(\Delta P)$ for the perturbed operator to have at most $N + 1$ eigenvalues in each gap ($N \in \mathbb{N}$).

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