

An estimate for the measure theoretic entropy of geodesic flows

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Abstract. A new proof and a generalization of the Osseman–Sarnak estimate for the measure theoretic entropy of geodesic flows is presented.

In this note we discuss an estimate of Osseman and Sarnak [O–S] for the measure theoretic entropy of geodesic flows. Let M be a compact Riemannian manifold with nonpositive sectional curvature and denote by R the curvature tensor of M . Then for all $p \in M$ and all unit vectors $v \in T_p M$, $K(v) := R(\cdot, v)v$ is a nonpositive symmetric operator on $T_p M$. The estimate we will obtain is

$$h_\mu \geq \int_{SM} \operatorname{tr} \sqrt{-K(v)} \, d\mu(v), \quad (1)$$

where h_μ is the measure theoretic entropy of the geodesic flow of M , SM is the unit tangent bundle of M and μ denotes the normalized Liouville measure on SM . Moreover,

$$h_\mu = \int_{SM} \operatorname{tr} \sqrt{-K(v)} \, d\mu(v) \quad \text{iff } M \text{ is locally symmetric.} \quad (2)$$

The results, (1) and (2), were obtained by Osseman and Sarnak in the case that the sectional curvature of M is strictly negative. We refer the reader to [O–S] for a formulation and comparison of the estimates which preceded (1) and respective references.

The measure theoretic entropy of a measure preserving flow is an asymptotic quantity associated with the flow. In the case under consideration Pesin [P2] showed that h_μ is given by the average of the mean curvatures of the horospheres. The significance of (1) lies in the fact that it estimates the asymptotic quantity h_μ by an average of local quantities associated to M .

Our proof of (1) and (2) is a simplification of the proof of Osseman and Sarnak and it works under the weaker assumption of nonpositive sectional curvature. Note that this is the weakest assumption under which (1) and (2) make sense. An important

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feature of our approach is that we estimate the exponential rate of growth of certain volumes along individual but typical geodesics. In the course of the discussion we will also obtain formulas of Pesin [P2] and Freire and Mañe [F-M] for h_μ .

1. *Multiplicative ergodic theorem and entropy formula*

In this section we will state the Oseledets Multiplicative Ergodic Theorem [Os] in the form in which we will use it. Let $\{g^t\}$ be a smooth flow without fixed points on a compact manifold X preserving a smooth probability measure μ on X . We are interested in exponential growth rates associated with dg^t .

Let $u(x) = (d/dt)g^t x|_{t=0}$ be the velocity vector of the flow. Then $dg^t u(x) = u(g^t x)$ and therefore it is natural to consider, instead of dg^t , the quotient linear operators

$$L'_x : Q_x \rightarrow Q_{g^t x},$$

where Q_x is the quotient of $T_x X$ by the 1-dimensional subspace spanned by $u(x)$. These operators contain all the information about the exponential growth and decay under dg^t which we need.

Assume $\|\cdot\|$ is a Riemannian norm on the bundle $Q \rightarrow X$. Then for almost all $x \in X$, the limit

$$\lim_{t \rightarrow \pm\infty} \frac{1}{|t|} \ln \|L'_x w\| := \chi^\pm(x, w),$$

exists for all $x \in Q_x \setminus \{0\}$ and it is called a Lyapunov exponent. Lyapunov exponents are independent of the choice of $\|\cdot\|$. We define $\chi^\pm(x, 0) := -\infty$.

For almost every $x \in X$, we obtain subspaces

$$E^u(x) = \{w \in Q_x \mid \chi^-(x, w) < 0\},$$

$$E^s(x) = \{w \in Q_x \mid \chi^+(x, w) < 0\},$$

$$E^0(x) = \{w \in Q_x \mid \chi^-(x, w) \leq 0 \text{ and } \chi^+(x, w) \leq 0\}.$$

By the Oseledets Theorem for almost every $x \in X$ these subspaces are linearly independent and they span Q_x . Furthermore for any subspace E such that $E^u(x) \subset E \subset E^u(x) \oplus E^0(x)$ the exponential rates of volume growth in E , that is the limits

$$\lim_{t \rightarrow \pm\infty} \frac{1}{|t|} \ln |\det(L'_x|_E)| = \pm\chi(x) \tag{3}$$

exist and they differ by the sign alone in the future and in the past (cf. [F-M]). Also $\chi(x)$ does not depend on the choice of E as above and $\chi(x) > 0$ unless $E^u(x) = \{0\}$. In fact $\chi(x)$ is equal to the sum of the positive Lyapunov exponents taken with multiplicities but this will not be used in the sequel. By the Pesin Entropy Formula [P1] the metric entropy h_μ of the flow $\{g^t\}$ with respect to the measure μ is equal to the average exponential rate of volume growth, that is

$$h_\mu = \int_X \chi(x) d\mu(x). \tag{4}$$

2. *The geodesic flow*

Let M be a compact Riemannian manifold of dimension n and of nonpositive sectional curvature. By SM we denote the unit tangent bundle of M and by γ_v ,

$v \in SM$, the geodesic with initial velocity v . The geodesic flow $g^t : SM \rightarrow SM$, $t \in \mathbb{R}$, is defined by $g^t v = \dot{\gamma}_v(t)$. The flow g^t preserves the normalized Liouville measure μ on SM .

Let $v \in SM$ and $p \in M$ be the footpoint of v . Then $T_v SM$ is naturally isomorphic to $T_p M \oplus V(v)$, where $V(v) \subset T_p M$ is the orthogonal complement of v in $T_p M$. With this identification the differential dg^t of g^t at v is given by

$$dg^t(x, y) = (J(t), J'(t)),$$

where J is the Jacobi field along γ_v with the initial condition $J(0) = x$, $J'(0) = y$. Since $dg^t(V(v) \oplus V(v)) = V(g^t v) \oplus V(g^t v)$, the restriction of dg^t to $V(v) \oplus V(v)$ is a canonical representation of the quotient L^t discussed in § 1. This corresponds to considering Jacobi fields perpendicular to γ_v . Recall that Jacobi fields satisfy the Jacobi equation, namely

$$J'' + K(g^t v)J = 0, \tag{5}$$

where $K(v) : V(v) \rightarrow V(v)$ is defined by $K(v)x = R(x, v)v$ and R denotes the curvature tensor. The operator $K(v)$ is symmetric and since the sectional curvature is nonpositive it is also nonpositive. Note that this implies that $\|J(t)\|^2$ is a convex function of t . Indeed

$$\langle J, J \rangle'' = 2\langle J'', J \rangle + 2\langle J', J' \rangle = -2\langle KJ, J \rangle + 2\langle J', J' \rangle \geq 0.$$

It is useful to introduce the symplectic form ω on $V(v) \oplus V(v)$ defined by

$$\omega((x_1, y_1), (x_2, y_2)) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle.$$

By the force of (5) this form is preserved under dg^t .

We denote by $H(v)$ the linear subspace of $V(v) \oplus V(v)$ defined by the condition that $(x, y) \in H(v)$ if the Jacobi field J along γ_v determined by $J(0) = x$, $J'(0) = y$ is nondecreasing in norm for all t . By convexity we have

$$(x, y) \in H(v) \text{ iff } \|J(t)\| \text{ remains bounded for } t \leq 0. \tag{6}$$

It is well known that $H(v)$ is the graph of the second fundamental form $U(v) : V(v) \rightarrow V(v)$ of the horosphere determined by $-v$. The operator $U(v)$ is symmetric nonpositive and since $dg^t H(v) = H(g^t v)$ it satisfies by the force of (5) the Riccati equation

$$U' + U^2 + K(g^t v) = 0, \tag{7}$$

where U' denotes the covariant derivative of $U(g^t v)$ along γ_v . Note that $H(v)$ is a Lagrangian subspace of $V(v) \oplus V(v)$ since $U(v)$ is symmetric.

For notational simplicity we will sometimes suppress the dependence on v .

PROPOSITION 2.1. $E^u \subset H \subset E^u \oplus E^0$.

Proof. We will first establish the general fact that E^u and $E^u \oplus E^0$ are skeworthogonal complements of each other. Indeed, if $w_1 \in E^u$ and $w_2 \in E^u \oplus E^0$, then since $\|\omega\| = 1$ we have

$$\lim_{t \rightarrow -\infty} \frac{1}{|t|} \ln |\omega(dg^t w_1, dg^t w_2)| < 0.$$

The symplectic form ω is preserved by dg^t , so that $\omega(dg^t w_1, dg^t w_2)$ is constant in t , and we conclude that $\omega(w_1, w_2) = 0$. A skeworthogonal complement of a subspace

has the complementary dimension, hence

$$\dim E^u + \dim E^u \oplus E^0 \leq 2(n - 1).$$

Similarly

$$\dim E^s + \dim E^s \oplus E^0 \leq 2(n - 1).$$

But because $\dim E^s + \dim E^u + \dim E^0 = 2(n - 1)$ we conclude that there must be equalities and hence $E^u \oplus E^0$ is the skeworthogonal complement of E^s .

$E^u \subset H$ by (6). It follows that the skeworthogonal complement of E^u , which is $E^s \oplus E^0$, must contain the skeworthogonal complement of H , which is H itself because H is a Lagrangian subspace. \square

In the case of negative sectional curvatures $E^u(v) = H(v)$. When the sectional curvature is allowed to vanish $E^u(v)$ may be much smaller than $H(v)$. But $H(v)$ is a more geometric object than $E^u(v)$ is. In particular $H(v)$ is well defined for every $v \in SM$ and $E^u(v)$ is defined only almost everywhere.

By (3) and (4) and the above Proposition we can express the metric entropy of $\{g^t\}$ by the exponential rates of volume growth in $H(g^t v)$. However, to be able to treat the case of nonpositive sectional curvature as opposed to negative sectional curvature we have to work with a subspace of $H(v)$. Let $V_0(v)$ be the orthogonal complement of $\ker U(v)$ in $V(v)$.

LEMMA 2.2. (i) $x \in \ker U(v)$ iff the Jacobi field determined by $J(0) = x, J'(0) = 0$ satisfies $J'(t) = 0$ for all $t \leq 0$.

(ii) $K(v) = 0$ on $\ker U(v)$ and $U(v)$ and $K(v)$ preserve $V_0(v)$.

Proof. Since $\|J(t)\|^2$ is convex and nondecreasing and $\langle J'(0), J(0) \rangle = 0$ we conclude that $\|J(t)\| = \|J(0)\|$ for $t \leq 0$. Thus we get for $t \leq 0$

$$0 = \frac{1}{2} \langle J, J \rangle'' = \langle J', J' \rangle = -\langle KJ, J \rangle + \langle J', J' \rangle.$$

Since $K \leq 0$ we get $J' = 0$ and $KJ = 0$ for $t \leq 0$, and (i) follows immediately. Furthermore we see that $\ker U(v) \subset \ker K(v)$. Since $K(v)$ is symmetric we conclude that $K(v)$ preserves $V_0(v)$. This proves the Lemma. \square

It follows from the above Lemma that $\dim V_0(g^t v)$ is nondecreasing in t and that $dg^t(V_0(v) \oplus V_0(v)) \subset V_0(g^t v) \oplus V_0(g^t v)$ for $t \geq 0$.

LEMMA 2.3. $\dim V_0(g^t v) \equiv \text{constant}$ in t for v from a g^t -invariant subset $\Omega \subset SM$ of full measure.

Proof. Any measurable function, which is nondecreasing along the orbits of a measurable flow preserving a probability measure, is equal on a set of full measure to a function constant on the orbits. \square

Lemma 2.2 implies that $V_0(g^t v)$ is the parallel translate of $V_0(v)$ along γ_v for all $t \in \mathbb{R}$ and $v \in \Omega$.

Let us denote by $U_0(v)$ and $K_0(v)$ the restrictions of $U(v)$ and $K(v)$ to $V_0(v)$. We define

$$H_0(v) := H(v) \cap (V_0(v) \oplus V_0(v)) = \text{graph } U_0(v).$$

By Lemmas 2.2 and 2.3 $dg^t H_0(v) = H_0(g^t v)$ for $v \in \Omega$.

For $v \in \Omega$ a Jacobi field J along γ_v splits into $J = J_1 + J_2$, where $J_1(t) \in \ker U(g'v)$ and $J_2(t) \in V_0(g'v)$ are Jacobi fields since $K(g'v) \equiv 0$ on $\ker U(g'v)$ and $\ker U(g'v)$ is parallel along γ_v . We easily conclude that $E^u(v) \subset H_0(v)$ for $v \in \Omega$.

Note that by construction U_0 is an invertible operator.

3. Volume growth and entropy estimate

It is useful to consider different volume elements in $H_0(v)$. Since $H_0(v) = \text{graph } U_0(v)$, we may identify $H_0(v)$ with $V_0(v)$ by the projection onto the first factor and thus write a scalar product in $H_0(v)$ as $\langle A(v) \cdot, \cdot \rangle$, where $A(v)$ is a positive operator on $V_0(v)$. For example the standard scalar product in $H_0(v)$ is given by $A(v) = I + U_0^2$. Given scalar products in $H_0(g'v)$ let

$$a(t) = |\det dg'|_{H_0(v)} \tag{8}$$

be the Jacobian of dg' restricted to $H_0(v)$ with respect to the scalar products. A classical computation yields

$$r(t) := \frac{d}{dt} \ln a(t) = \frac{1}{2} \text{tr } A' A^{-1} + \text{tr } U_0. \tag{9}$$

Note that $a(t)$ is defined along a particular geodesic but r is actually a function on SM (defined almost everywhere).

PROPOSITION 3.1.

$$h_\mu = \int_{SM} \text{tr} (U_0 - K_0 U_0) (I + U_0^2)^{-1} d\mu.$$

Proof. Applying (9) to the standard scalar product we get by the Riccati equation (7)

$$A' = U_0' U_0 + U_0 U_0' = -2U_0^3 - K_0 U_0 - U_0 K_0$$

and

$$r(t) = \text{tr} (U_0 - K_0 U_0) (I + U_0^2)^{-1}.$$

Furthermore we have

$$\chi(v) = \lim_{T \rightarrow +\infty} \frac{1}{T} \ln a(T) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T r(t) dt$$

and since r is an integrable function on SM we get by the Birkhoff Ergodic Theorem

$$h_\mu = \int_{SM} \chi(v) d\mu(v) = \int_{SM} r d\mu. \tag{□}$$

For the purpose of proving the above Proposition we could have worked as well with H , U and K . Anyway, by our construction in § 2, replacing U_0 and K_0 by U and K will not change the trace so that we obtain immediately

$$h_\mu = \int_{SM} \text{tr} (U - KU) (I + U^2)^{-1} d\mu.$$

This is one of Pesin's formulas for the metric entropy of the geodesic flow, [P2]. It holds under the assumption that the metric has no conjugate points, if we define U appropriately and our proof works in that case also, once we establish that $E^u \subset \text{graph } U \subset E^u \oplus E^0$.

Now we will consider three other scalar products, namely

1. $A = I,$
2. $A = U_0^2,$
3. $A = U_0.$

The scalar product 1(2) is the standard scalar product in V_0 transported to H_0 by the projection onto the first (second) factor. All three scalar products are smaller than the standard scalar product. The norm of a nonzero vector in H_0 is increasing under dg' for the scalar products 1 and 3 (but not necessarily so for the scalar product 2).

Let a_i and $r_i, i = 1, 2, 3,$ be defined as in (8) and (9) for the respective scalar products 1, 2, 3 and let c_i be the ratio of the standard volume element in H_0 and the volume element determined by the i th scalar product. By straightforward computations, using (9) and the Riccati equation (7) we get

$$\begin{aligned} r_1 &= \text{tr } U_0, \\ r_2 &= \text{tr } (-K_0 U_0^{-1}), \\ r_3 &= \frac{1}{2} \text{tr } (U_0 - K_0 U_0^{-1}). \end{aligned}$$

We have

$$0 < a_i(t) < a(t)c_i(0) \quad \text{and} \quad r_i(t) > 0 \tag{10}$$

for all $t \in \mathbb{R}, i = 1, 2, 3.$ The first claim follows from the fact that the volume element defined by the i th scalar product is smaller than the standard volume element. The second claim follows from $\text{tr } (-K_0 U_0^{-1}) = \text{tr } (\sqrt{-K_0} U_0^{-1} \sqrt{-K_0}) > 0.$ It follows from (10) and (3) that

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T r_i(t) dt = \limsup_{T \rightarrow +\infty} \frac{1}{T} \ln a_i(T) \leq \lim_{T \rightarrow +\infty} \frac{1}{T} \ln a(T) = \chi(v),$$

and

$$\liminf_{T \rightarrow -\infty} \frac{1}{|T|} \int_T^0 r_i(t) dt = -\limsup_{T \rightarrow -\infty} \frac{1}{|T|} \ln a_i(T) \geq -\lim_{T \rightarrow -\infty} \frac{1}{|T|} \ln a(T) = \chi(v).$$

Putting the inequalities together we obtain

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T r_i(t) dt \leq \chi(v) \leq \liminf_{T \rightarrow -\infty} \frac{1}{|T|} \int_T^0 r_i(t) dt. \tag{11}$$

Under the assumption that r_i is integrable as a function on SM we obtain from (11) by the Birkhoff Ergodic Theorem and the equality of time averages in the future and in the past that

$$\int_{SM} r_i d\mu = \int_{SM} \chi(v) d\mu(v) = h_\mu.$$

Integrability of r_1 follows from the fact that the principal curvatures of horospheres are bounded, but we do not know how to establish the integrability of r_2 (and r_3) by geometric considerations. We will obtain the integrability of $r_i, i = 1, 2, 3,$ from (11) by a bit of Ergodic Theory.

LEMMA 3.2 *Let $\{g^t\}$ be a measure preserving flow on a probability space $(X, \mu),$ and*

let $f : X \rightarrow \mathbb{R}$ be a measurable nonnegative function. If for almost every $x \in X$

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(g^t x) dt \leq k(x),$$

where $k : X \rightarrow \mathbb{R}$ is a measurable function, then

$$\int_X f(x) d\mu(x) \leq \int_X k(x) d\mu(x).$$

Proof. Let

$$f_N(x) = \begin{cases} f(x) & \text{if } f(x) \leq N, \\ N & \text{if } f(x) \geq N. \end{cases}$$

We have $0 \leq f_N \leq f$ and $f_N \in L^1(X, \mu)$. By the Birkhoff Ergodic Theorem, for almost all x

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f_N(g^t x) dt = f_N^*(x)$$

and

$$\int_X f_N^*(x) d\mu(x) = \int_X f_N(x) d\mu(x).$$

By assumption $f_N^*(x) \leq k(x)$ and so

$$\int_X f_N(x) d\mu(x) \leq \int_X k(x) d\mu(x).$$

Hence

$$\int_X f(x) d\mu(x) \leq \int_X k(x) d\mu(x). \quad \square$$

Applying the above Lemma we get

PROPOSITION 3.3

$$h_\mu = \int_{SM} \text{tr } U_0 d\mu(v) = \int_{SM} \text{tr } (-K_0 U_0^{-1}) d\mu(v).$$

The first formula was obtained by Pesin [P2] in the case of a metric without conjugate points, see also [F–M]. One formula would follow immediately from the other if we would have the integrability of $\ln \det U_0$. Indeed $(\ln \det U_0)' = \text{tr } U_0' U_0^{-1} = \text{tr } (U_0 - K_0 U_0^{-1})$. We do not know though how to establish the integrability of $\ln \det U_0$ by a geometric argument (*a priori* $\det U_0$ may be arbitrarily small). In the case of a metric without conjugate points the formula $h_\mu = \int_{SM} \text{tr } U d\mu$ can also be obtained by our method once Proposition 2.1 is established. In this case $r_1 = \text{tr } U$ may be negative but it is known to be bounded and hence integrable.

In estimating the entropy from below we will use

$$h_\mu = \int_{SM} r_3 d\mu$$

and the following general fact.

LEMMA 3.5. For all linear symmetric operators U, M, N , on a Euclidean space, if $U > 0, M \geq 0$ and $N \geq 0$, then

$$\text{tr}(MU + NU^{-1}) \geq 2 \text{tr} \sqrt{M} \sqrt{N}$$

and the equality holds if and only if $\sqrt{M}U = \sqrt{N}$.

Proof. Let $S = \sqrt{M}\sqrt{U} - \sqrt{N}\sqrt{U^{-1}}$. We have $\text{tr} SS^* \geq 0$ and $\text{tr} SS^* = 0$ if and only if $S = 0$. At the same time

$$\begin{aligned} \text{tr} SS^* &= \text{tr}(\sqrt{M}\sqrt{U} - \sqrt{N}\sqrt{U^{-1}})(\sqrt{U}\sqrt{M} - \sqrt{U^{-1}}\sqrt{N}) \\ &= \text{tr} \sqrt{M}U\sqrt{M} + \text{tr} \sqrt{N}U^{-1}\sqrt{N} - \text{tr} \sqrt{M}\sqrt{N} - \text{tr} \sqrt{N}\sqrt{M}. \end{aligned}$$

Hence

$$\text{tr}(MU + NU^{-1}) \geq 2 \text{tr} \sqrt{M}\sqrt{N},$$

and the equality holds if and only if $\sqrt{M}\sqrt{U} = \sqrt{N}\sqrt{U^{-1}}$. □

THEOREM 3.6.

$$h_\mu \geq \int_{SM} \text{tr} \sqrt{-K(v)} d\mu(v),$$

and the equality holds if and only if the manifold M is locally symmetric.

Proof. By Lemma 3.5

$$h_\mu = \int_{SM} \frac{1}{2} \text{tr}(U_0 - K_0 U_0^{-1}) d\mu \geq \int_{SM} \text{tr} \sqrt{-K_0} d\mu = \int_{SM} \text{tr} \sqrt{-K} d\mu.$$

The equality holds if and only if $U_0(v) = \sqrt{-K_0(v)}$ for almost all $v \in SM$. Then for almost all $v \in SM, U(v)^2 = -K(v)$. By continuity we have the last equality for all $v \in SM$. The Riccati equation (7) implies that $U' = 0$ on SM . Hence U^2 and therefore also K are parallel along geodesics. But if K is parallel along geodesics the geodesic reflection in any point of M is a (local) symmetry. Thus M is locally symmetric. □

Remark 3.7. In the proof we estimate the exponential rate of volume growth pointwise. With our choice of volume element this rate has by Lemma 3.5 a global minimum in an open set of Lagrangian subspaces graph $U, U > 0$. So we could obtain our estimate without the construction of invertible U_0 , but then we would be unable to establish when the equality holds. In [W] such a method was formulated in the case of discrete time symplectic systems and applied to estimating the measure theoretic entropy of the gas of hard spheres.

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