STRATIFICATIONS ASSOCIATED TO REDUCTIVE GROUP ACTIONS ON AFFINE SPACES

by VICTORIA HOSKINS[†]

(*Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*)

Abstract

For a complex reductive group G acting linearly on a complex affine space V with respect to a character ρ, we show two stratifications of V associated to this action (and a choice of invariant inner product on the Lie algebra of the maximal compact subgroup of G) coincide. The first is Hesselink's stratification by adapted 1-parameter subgroups and the second is the Morse theoretic stratification associated to the norm square of the moment map. We also give a proof of a version of the Kempf–Ness theorem, which states that the geometric invariant theory quotient is homeomorphic to the symplectic reduction (both taken with respect to ρ). Finally, for the space of representations of a quiver of fixed dimension, we show that the Morse theoretic stratification and Hesselink's stratification coincide with the stratification by Harder–Narasimhan types.

1. Introduction

When a complex reductive group G acts linearly on a complex projective variety $X \subset \mathbb{P}^n$, then Mumford's geometric invariant theory (GIT) [13] associates to this action a projective GIT quotient $X /\!/ G$ whose homogeneous coordinate ring is the G-invariant part of the homogeneous coordinate ring of X. The inclusion of the G-invariant subring induces a rational map $X \dashrightarrow X /\!/ G$ which restricts to a morphism on the open subset $X^{\mathrm{ss}} \subset X$ of semistable points. Topologically, the projective GIT quotient $X /\!/ G$ is X^{ss}/G modulo the equivalence relation that two orbits are equivalent if and only if their closures meet in X^{ss}. If X is smooth, then it is a symplectic manifold with symplectic form given by restricting the Kähler form on $\mathbb{P}^n$ and we can assume, without loss of generality, that the action of the maximal compact subgroup $K \subset G$ preserves this symplectic form. Then there is an associated moment map $\mu : X \to \mathfrak{K}^*$, where $\mathfrak{K}$ is the Lie algebra of K. The symplectic reduction of Marsden and Weinstein [11] and Meyer [12] is the quotient $\mu^{-1}(0)/K$. The Kempf–Ness theorem (see [8, 13, Section 8]) gives a homeomorphism between the symplectic reduction and GIT quotient

$$\mu^{-1}(0)/K \simeq X^{\mathrm{ss}} /\!/ G.$$

Kirwan [10] and Ness [14] show that two stratifications of the smooth projective variety X associated to this action (and a choice of K-invariant inner product on $\mathfrak{K}$) coincide. The first stratification is Hesselink's stratification by adapted 1-parameter subgroups (1-PSs) of G. We recall that the Hilbert–Mumford criterion is a numerical criterion used to determine the semistability of points in terms of 1-PSs (see [13, Section 2]). Kempf builds on these ideas and associates to any unstable point a

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[†]E-mail: victoria.hoskins@math.uzh.ch

conjugacy class for a parabolic subgroup of 'adapted' 1-PSs which are most responsible for the instability of that point [7] (where the norm associated to the given inner product is used to give a precise definition of what is meant by most responsible). Hesselink shows the unstable locus $X - X^{ss}$ can be stratified by conjugacy classes of 1-PSs [5]. We view this as a stratification of X with the open stratum given by X^{ss} (which we can think of as the stratum corresponding to the trivial 1-PS). The second stratification is a Morse theoretic stratification associated to the norm square of the moment map $\|\mu\|^2 : X \to \mathbb{R}$, where we use the norm associated to the given inner product. The strata are indexed by adjoint orbits $K \cdot \beta$ for $\beta \in \mathfrak{K}$ (or equivalently coadjoint orbits as the inner product allows us to identify $\mathfrak{K}^* \cong \mathfrak{K}$). For an adjoint orbit $K \cdot \beta$, we let $C_{K \cdot \beta}$ denote the set of subsets of critical points for $\|\mu\|^2$ on which μ takes a value in the coadjoint orbit corresponding to $K \cdot \beta$. The corresponding stratum $S_{K \cdot \beta}$ consists of all points whose negative gradient flow under $\|\mu\|^2$ converges to $C_{K \cdot \beta}$. For both stratifications, one can determine the index set for the stratification from the weights of the action of a maximal torus.

In this paper, we suppose that G is a complex reductive group acting linearly on a complex affine space V and ask whether the same results still hold. One immediate difference from the above set-up is that V is not compact and so we may have some issues with convergence properties; luckily this turns out not to be a problem as in the algebraic setting we are only interested in actions of 1-PSs $\lambda(t)$ on $v \in V$ for which the limit as $t \to 0$ exists and, in the symplectic setting, the convergence of the negative gradient flow of the norm square of the moment map has already been shown by Harada and Wilkin [4] and Sjamaar [16].

The affine GIT quotient for the action of G on V is the morphism $V \to V//G := \operatorname{Spec} \mathbb{C}[V]^G$ associated to the inclusion of the invariant subring $\mathbb{C}[V]^G \hookrightarrow \mathbb{C}[V]$. There is no notion of semistability yet and so the GIT quotient is topologically V/G modulo the equivalence relation that two orbits are equivalent if and only if their closures meet. Therefore, Hesselink's stratification of V is the trivial stratification. If we take a Hermitian inner product on V such that the maximal compact subgroup $K \subset G$ acts unitarily, then there is a natural moment map $\mu : V \to \mathfrak{K}^*$ for the action of K. In this case, there is only one index 0 for the Morse stratification associated to $\|\mu\|^2$ (for any choice of inner product) with the critical subset corresponding to $C_0 = \mu^{-1}(0)$ and so the Morse stratification is also trivial. Therefore, trivially the stratifications agree. The affine GIT quotient is homeomorphic to the symplectic reduction (this is a special case of Theorem 4.2). However, in the compact setting the proof is given by exhibiting a continuous bijection between the compact symplectic reduction and the separated GIT quotient which is then a homeomorphism. In the affine case, we no longer have compactness, and so we instead provide a continuous inverse going in the opposite direction by using the retraction $V \to \mu^{-1}(0)$ associated to the negative gradient flow for $\|\mu\|^2$.

More generally, we can use a character ρ of G to get a non-trivial notion of semistability by using ρ to lift the action of G on V to the trivial line bundle $L = V \times \mathbb{C}$. In this situation, there is a GIT quotient $V//_\rho G$ (with respect to ρ) which is a quotient of an open subset $V^{\rho\text{-ss}} \subset V$ of ρ-semistable points. This construction was used by King to construct moduli space of quiver representations [9]. More precisely, this GIT quotient is the quasi-projective variety

$$V//_\rho G = \operatorname{Proj} \bigoplus_{n \geq 0} \mathbb{C}[V]^G_{\rho^n},$$

where $\mathbb{C}[V]^G_{\rho^n} = \{f(g \cdot v) = \rho^n(g) f(v) \text{ for all } v, g\}$ is the ring of semi-invariants of weight ρ^n. Thus, a point v is ρ-semistable if and only if there is a semi-invariant f of weight ρ^n for $n > 0$ such that $f(v) \neq 0$. For infinite groups the origin is always ρ-unstable (for non-trivial characters ρ) and

so Hesselink's stratification is non-trivial. On the symplectic side, if ρ is a character of K, then we can use this to shift the moment map. We write μ^ρ for the shifted moment map and the symplectic reduction (with respect to ρ) is $(\mu^\rho)^{-1}(0)/K$. We give a proof of an affine version of the Kempf–Ness theorem which states that the GIT quotient is homeomorphic to the symplectic reduction (both taken with respect to ρ).

Next we compare Hesselink's stratification with the Morse theoretic stratification of V associated to a fixed K-invariant inner product on $\mathfrak{K}$. The main difference for Hesselink's stratification in the affine case is that, for an unstable point v, to determine which 1-PSs are adapted to v, we only consider those for which $\lim_{t \to 0} \lambda(t) \cdot v$ exists. On the Morse theory side, since the negative gradient flow of the norm square of the moment map converges, much of the picture remains the same as in the projective setting. In the projective setting, Kirwan [10, Section 6] gave a further description of the Morse strata $S_{K \cdot \beta}$ in terms of Morse strata for the functions $\mu_\beta : V \to \mathbb{R}$ given by $v \mapsto \mu(v) \cdot \beta$ and $\|\mu - \beta\|^2$. In the affine case, we also provide a similar description; however, our description differs slightly from that of Kirwan due to the fact that the negative gradient flow of μ_β on V does not always converge. We prove that the Morse theoretic stratification coincides with Hesselink's stratification (both taken with respect to ρ) for a fixed K-invariant inner product on $\mathfrak{K}$. Furthermore, we show that the index set for Hesselink's stratification and the Morse theoretic stratification can be determined combinatorially from the weights associated to the action of a maximal torus (cf. Section 2.4 and Section 3.3).

Finally, we apply this to the case in which V is the space of representations of a quiver of fixed dimension and G is a reductive group acting on V such that the orbits correspond to isomorphism classes of representations. It follows from above that the Morse stratification coincides with Hesselink's stratification, but we can also compare this to the stratification by Harder–Narasimhan types (where the notion of Harder–Narasimhan filtration depends on a choice of invariant inner product on the Lie algebra of the maximal compact subgroup of G). For a fixed inner product, we prove that all three stratifications coincide. For the case when the chosen inner product is the Killing form, the Harder–Narasimhan stratification has been described by Reineke [15] and Harada and Wilkin show that, for this inner product, the Harder–Narasimhan stratification and Morse theoretic stratification on V coincide with [4]. In his Ph.D. thesis, Tur [17] shows that Hesselink's stratification by adapted 1-PSs agrees with the Harder–Narasimhan stratification (for any choice of invariant inner product) and so it follows from this and the result we gave above that all three stratifications coincide. However, we provide a concise proof of this fact for completeness of the paper. Whilst this paper was being completed, we note that Zamora has also given a proof that the Kempf filtration (this is a natural filtration associated to an adapted 1-PS) is equal to the Harder–Narasimhan filtration for quiver representations [18].

The layout of this paper is as follows. In Section 2, we give some results on affine GIT. In particular, we give results for semistability of points with respect to a character ρ and also describe Hesselink's stratification by adapted 1-PSs. In Section 3, we describe the moment map, symplectic reduction and the Morse stratification associated to the norm square of the moment map for complex affine spaces. In Section 4, we show, for a fixed invariant inner product, that the Morse stratification agrees with Hesselink's stratification. In this section, we also give a proof of the affine Kempf–Ness theorem and prove an alternative description of the Morse strata. In Section 5, we apply the above to the space of representations of a quiver of fixed dimension. We define a notion of Harder–Narasimhan filtration that depends on the choice of the invariant inner product, and show that the stratification by Harder–Narasimhan types coincides with both the Morse stratification and Hesselink's stratification.

2. Affine GIT

In this section, we can work over an arbitrary algebraically closed field k of characteristic zero. Let G be a reductive group acting linearly on an affine space V over k. Let $k[V]$ denote the k-algebra of regular functions on V; then there is an induced action of G on $k[V]$ given by $g \cdot f(v) = f(g^{-1} \cdot v)$ for $f \in k[V]$ and $g \in G$. The inclusion of the invariant subalgebra $k[V]^G \hookrightarrow k[V]$ induces a morphism of affine varieties $V \to V /\!/ G$ which is known as the affine GIT quotient. In general, this is not the same as the topological quotient V/G as the affine GIT quotient identifies orbits whose closures meet. In particular, if one orbit is contained in the closure of every other orbit (as is the case when $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by scalar multiplication), then the affine GIT quotient $V /\!/ G$ is simply a point.

To avoid such collapsing for reductive actions on affine spaces, we can instead use a non-trivial character $\rho : G \to \mathbb{G}_m$ to linearize the action so that we obtain a better quotient of an open subset of V as follows. Let $L = V \times k$ denote the trivial line bundle on L; then we use ρ to lift the action of G on V to L so that

$$g \cdot (v, c) = (g \cdot v, \rho(g)c)$$

for $g \in G$ and $(v, c) \in L = V \times k$. We write L_ρ to denote the linearization consisting of the line bundle L and lift of the G-action given by ρ. We note that as linearizations $L_\rho^{\otimes n} = L_{\rho^n}$ for all $n \in \mathbb{Z}$ where $L_\rho^{\otimes -1} := L_\rho^{-1} = L_{\rho^{-1}}$ is the dual linearization to L_ρ. There is an induced action of G on $H^0(V, L_\rho^{\otimes n})$ given by

$$g \cdot \sigma(v) = \rho^n(g)\sigma(g^{-1}v)$$

for $\sigma \in H^0(V, L_\rho^{\otimes n})$ and $g \in G$. We note that the invariant sections

$$H^0(V, L_\rho^{\otimes n})^G \cong k[V]_{\rho^n}^G := \{f \in k[V] : f(g \cdot v) = \rho^n(g)f(v) \text{ for all } v \in V, g \in G\}$$

are equal to the semi-invariants on V of weight ρ^n. Consider the graded algebra

$$R := \bigoplus_{n \geq 0} H^0(V, L_\rho^{\otimes n})$$

and its invariant graded subalgebra $R^G = \bigoplus_n R_n^G$, where $R_n^G = k[V]_{\rho^n}^G$ is the algebra of semi-invariants on V of weight ρ^n. The inclusion $R^G \hookrightarrow R$ induces a rational map

$$V \dashrightarrow V /\!/_\rho G = \operatorname{Proj} R^G, \tag{1}$$

which is undefined on the null cone

$$N = \left\{ v \in V : f(v) = 0 \ \forall \ f \in \bigoplus_{n > 0} R_n^G \right\}. \tag{2}$$

Following Mumford [**13**, Definition 1.7] we have the following definition.

DEFINITION 2.1   Let $v \in V$; then

(i) v is ρ-semistable if there is an invariant section $\sigma \in H^0(V, L_\rho^{\otimes n})^G = k[V]_{\rho^n}^G$ for some $n > 0$ such that $\sigma(v) \neq 0$;

(ii) v is ρ-stable if $\dim G_v = 0$ and there is an invariant section $\sigma \in H^0(V, L_\rho^{\otimes n})^G = k[V]_{\rho^n}^G$ for some $n > 0$ such that $\sigma(v) \neq 0$ and the action of G on the open affine subset $V_\sigma := \{u \in V : \sigma(u) \neq 0\}$ is closed (that is, all G-orbits in V_σ are closed);

(iii) the points which are not ρ-semistable are called ρ-unstable.

The open subsets of ρ-stable and ρ-semistable will be denoted by $V^{\rho\text{-s}}$ and $V^{\rho\text{-ss}}$, respectively.

By definition, the semistable locus $V^{\rho\text{-ss}}$ is the complement of the null cone N and we call the morphism $V^{\rho\text{-ss}} \to V /\!/_\rho G$ the GIT quotient with respect to ρ. This approach of using a character to twist the trivial linearization on an affine space was used by King to construct moduli spaces of representations of finite-dimensional algebras [9]. We note that King modifies the notion of stability so that one can have a subgroup of dimension greater than zero contained in the stabilizer of each point and still have stable points. In fact, Mumford's original notion of stability [13] does not require points to have zero-dimensional stabilizers; the modern notion of stability (where one asks for zero-dimensional stabilizers) is what Mumford refers to as proper stability.

THEOREM 2.2 (Mumford) *The GIT quotient $\varphi : V^{\rho\text{-ss}} \to V /\!/_\rho G$ is a good quotient for the action of G on $V^{\rho\text{-ss}}$. Moreover, there is an open subset $V^{\rho\text{-s}}/G \subset V /\!/_\rho G$ whose preimage under φ is $V^{\rho\text{-s}}$ and the restriction $\varphi : V^{\rho\text{-s}} \to V^{\rho\text{-s}}/G$ is a geometric quotient (which in particular is an orbit space).*

REMARK 2.3 In general, the GIT quotient $V /\!/_\rho G = \operatorname{Proj} R^G$ with respect to ρ is only quasi-projective. It is projective over the affine GIT quotient $\operatorname{Spec} k[V]^G = \operatorname{Spec} R_0^G$ and so is projective if $k[V]^G = k$. We note that if ρ is the trivial character, then in this construction we recover the affine GIT quotient $V \to V /\!/ G$.

2.1. *Criteria for stability*

We note that, for finite groups, the notion of semistability is trivial for any character ρ and so, from now on, we may as well assume that our group is infinite. The following lemma gives a topological criterion for semistability. The proof follows in the same way as the original projective version (see [13, Proposition 2.2] and also [9, Lemma 2.2]). We let $L_\rho^{-1} = L_{\rho^{-1}}$ denote the dual linearization to L_ρ.

LEMMA 2.4 *Let $\tilde{v} = (v, a) \in L_\rho^{-1}$ be a point lying over $v \in V$ for which $a \neq 0$. Then*

(i) *v is ρ-semistable if and only if the orbit closure $\overline{G \cdot \tilde{v}}$ of $\tilde{v}$ in L_ρ^{-1} is disjoint from the zero section $V \times \{0\} \subset L_\rho^{-1} = V \times k$;*

(ii) *v is ρ-stable if and only if the orbit $G \cdot \tilde{v}$ of $\tilde{v}$ in L_ρ^{-1} is closed and $\dim G \cdot \tilde{v} = \dim G$.*

The topological criterion can be reformulated as a numerical criterion, known as the Hilbert–Mumford criterion, by using 1-PSs of G; that is, non-trivial homomorphisms $\lambda : \mathbb{G}_m \to G$. As every semi-invariant for the action of G is also a semi-invariant for the action of any subgroup of G, we see that if v is ρ-(semi)stable for the action of G, it must also be ρ-(semi)stable for the action of $\lambda(\mathbb{G}_m)$ for any 1-PS λ. We shall see that the notion of ρ-(semi)stability for a one-dimensional torus $\lambda(\mathbb{G}_m)$ can

easily be reformulated as a numerical condition. By the topological criterion for the one-dimensional torus $\lambda(\mathbb{G}_m)$:

(i) v is ρ-(semi)stable for the action of $\lambda(\mathbb{G}_m)$ if and only if $\lim_{t \to 0} \lambda(t) \cdot \tilde{v} \notin V \times \{0\}$ and $\lim_{t \to \infty} \lambda(t) \cdot \tilde{v} \notin V \times \{0\}$;

(ii) v is ρ-stable for the action of $\lambda(\mathbb{G}_m)$ if and only if neither limit exists.

Let (ρ, λ) denote the integer such that $\rho \circ \lambda(t) = t^{(\rho, \lambda)}$; then

$$\lambda(t) \cdot \tilde{v} = (\lambda(t) \cdot v, t^{-(\rho, \lambda)} a) \in L_\rho^{-1}.$$

The limit $\lim_{t \to 0} \lambda(t) \cdot \tilde{v}$ exists if and only if $\lim_{t \to 0} \lambda(t) \cdot v$ exists and $(\rho, \lambda) \leq 0$. In particular, if $\lim_{t \to 0} \lambda(t) \cdot v$ exists, then $\lim_{t \to 0} \lambda(t) \cdot \tilde{v} \notin V \times \{0\}$ if and only if $(\rho, \lambda) \geq 0$ (if it is positive the limit does not exist and if it is zero, then the limit exists, but does not belong to the zero section).

For $v \in V$ and a 1-PS λ of G, we let $\mu^\rho(v, \lambda) := (\rho, \lambda)$. We note that the quantity $\mu^\rho(v, \lambda)$ is independent of the point v, but we use this notation in analogy with the projective case.

PROPOSITION 2.5 (Hilbert–Mumford criterion)   *Let $v \in V$; then*

(i) *v is ρ-semistable if and only if $\mu^\rho(v, \lambda) \geq 0$ for every 1-PS λ of G for which the limit $\lim_{t \to 0} \lambda(t) \cdot v$ exists;*

(ii) *v is ρ-stable if and only if $\mu^\rho(v, \lambda) > 0$ for every 1-PS λ of G for which the limit $\lim_{t \to 0} \lambda(t) \cdot v$ exists.*

The discussion preceding this proposition proves the 'only if' direction of the Hilbert–Mumford criterion and the converse follows by using the following well-known theorem from GIT (the first version of this goes back to Hilbert; for a proof see [7, Theorem 1.4] or [13, Section 2]):

THEOREM 2.6   *Let G be a reductive group acting linearly on an affine space V. If $v \in V$ and $w \in \overline{G \cdot v}$, then there is a 1-PS λ of G such that $\lim_{t \to 0} \lambda(t) \cdot v$ exists and is equal to w.*

If we fix a maximal torus $T \subset G$, then the action of T on V gives a weight decomposition

$$V = \bigoplus_{\chi \in \chi^*(T)} V_\chi, \tag{3}$$

where $\chi^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ and $V_\chi := \{v \in V : t \cdot v = \chi(t)v \,\forall\, t \in T\}$. We refer to the finite set of characters χ for which $V_\chi \neq 0$ as the T-weights for the action. For any $v \in V$, we write $v = \sum v_\chi$ with respect to this decomposition and define $\mathrm{wt}_T(v) = \{\chi : v_\chi \neq 0\}$. We let $\chi_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ denote the set of cocharacters. If $T \cong (\mathbb{G}_m)^n$ is an n-dimensional torus, then there are natural identifications

$$\chi_*(T) \cong \mathbb{Z}^n \cong \chi^*(T),$$

$$\lambda(t) = (t^{m_1}, \ldots, t^{m_n}) \mapsto (m_1, \ldots, m_n) \mapsto \chi(t_1, \ldots, t_n) = \Pi t_i^{m_i}$$

and the natural pairing between characters and cocharacters corresponds to the dot product on $\mathbb{Z}^n$. We define the cone of allowable 1-PSs for v to be

$$C_v := \bigcap_{\chi \in \mathrm{wt}_T(v)} H_\chi \subset \mathbb{R}^n, \tag{4}$$

where $H_\chi := \{\lambda \in \chi_*(T)_\mathbb{R} \cong \mathbb{R}^n : (\chi, \lambda) \geq 0\}$. Then, by construction of this cone, a 1-PS λ belongs to C_v if and only if

$$\lim_{t \to 0} \lambda(t) \cdot v = \sum_\chi \lim_{t \to 0} t^{(\chi, \lambda)} v_\chi$$

exists. Then the Hilbert–Mumford criterion can be restated as:

(i) v is ρ-semistable for the action of T if and only if $C_v \subset H_\rho$;
(ii) v is ρ-stable for the action of T if and only if $C_v - \{0\} \subset H_\rho^o$, where

$$H_\rho^o := \{\lambda \in \chi_*(T)_\mathbb{R} \cong \mathbb{R}^n : (\rho, \lambda) > 0\}.$$

Moreover, as every 1-PS of G is conjugate to a 1-PS of T, we can use the above criteria to give criteria for ρ-(semi)stability with respect to G.

PROPOSITION 2.7 *Let $v \in V$; then*

(i) *v is ρ-semistable if and only if, for all $g \in G$, we have $C_{gv} \subset H_\rho$;*
(ii) *v is ρ-stable if and only if, for all $g \in G$, we have $C_{gv} - \{0\} \subset H_\rho^o$.*

REMARK 2.8 If $C_{gv} = \{0\}$ for all $g \in G$, then $G \cdot v$ is ρ-stable for any character ρ. We shall refer to such points as strongly stable points. We note that if $v = 0$, then $C_0 := \mathbb{R}^n$ and so 0 is unstable for any non-trivial character ρ. We also note that if $\rho = 0$ is the trivial character, then $H_\rho = \mathbb{R}^n$ and $H_\rho^o = \emptyset$, thus all points are semistable and the strongly stable points are the only stable points.

In the projective setting, the set of torus weights can also be used to study (semi)stability of points (for example, see [2, Theorem 9.3]); although a weight polytope is used (which is equal to the convex hull of some torus weights) rather than a cone of allowable 1-PSs.

2.2. *Instability*

To study the null cone N of unstable points, Kempf gave a notion for a 1-PS to be adapted to an unstable point which makes use of a normalized Hilbert–Mumford function $\mu^\rho(v, \lambda)/\|\lambda\|$ where $\| - \|$ is a fixed norm on the set $\chi_*(G)/G$ of conjugacy classes of 1-PSs. As the set of orbits for the action of G on G (by conjugation) is equal to the set of orbits for the action of the Weyl group W on a maximal torus T of G, to fix such a norm is equivalent to fixing a maximal torus T of G and a norm on $\chi_*(T)/W$. Under the natural identification $\chi_*(T) \cong \mathbb{Z}^n$, we can take the standard Euclidean norm on $\mathbb{R}^n$ and average over the action of the (finite) Weyl group W to produce a norm that is invariant under the action of W.

ASSUMPTION 2.9 We assume that $\|\lambda\|^2 \in \mathbb{Z}$ for all 1-PSs λ.

EXAMPLE 2.10 If $G = GL(n)$ and T is the maximal torus of diagonal matrices, then the dot product on $\mathbb{Z}^n \cong \chi_*(T)$ is invariant under the Weyl group $W = S_n$, which acts on T by permuting the diagonal entries.

REMARK 2.11 Often we shall work over the complex numbers, in which case a complex reductive group G is equal to the complexification of its maximal compact subgroup $K \subset G$. As any 1-PS of

G is conjugate to a 1-PS which sends the maximal compact subgroup $U(1) \subset \mathbb{C}^*$ to $K \subset G$, we can just consider 1-PSs of this form. There is a natural isomorphism $\chi_*(K)_{\mathbb{R}} = \text{Hom}(U(1), K) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathfrak{K} := \text{Lie } K$ given by

$$\lambda \mapsto d\lambda(2\pi i) := d_1\lambda(2\pi i) = \frac{d}{dt}\lambda(\exp(2\pi i t))|_{t=0} \in \mathfrak{K}$$

whose inverse is given by sending $\alpha \in \mathfrak{K}$ to the real 1-PS $\exp(\mathbb{R}\alpha) \subset K$ (although we note that the map $\lambda : S^1 \to K$ given by $\lambda(\exp(2\pi i t)) = \exp(t\alpha)$ is not necessarily a group homomorphism). We refer to points $\alpha \in \mathfrak{K}$ which lie in the image of $\chi_*(K)$ under this isomorphism as integral weights (in this case, the corresponding λ is a group homomorphism). A G-invariant norm on the set of 1-PSs of G is then equivalent to a K-invariant norm on $\mathfrak{K}$ and Assumption 2.9 is equivalent to requiring that $\|\alpha\|^2 \in \mathbb{Z}$ for all integral weights α. For example, any positive scalar multiple of the Killing form on $\mathfrak{K}$ is a K-invariant inner product and a suitably scaled version of the Killing form would satisfy Assumption 2.9.

DEFINITION 2.12   Let v be ρ-unstable for the action of G and define

$$M_G^\rho(v) = \inf \frac{\mu^\rho(v, \lambda)}{\|\lambda\|},$$

where the infimum is taken over all 1-PSs λ of G for which $\lim_{t \to 0} \lambda(t) \cdot v$ exists. We say that a subgroup $H \subset G$ is optimal for v if $M_G^\rho(v) = M_H^\rho(v)$. We shall often write $M^\rho(v)$ rather than $M_G^\rho(v)$ when it is clear that we are considering the action of a fixed group G. A 1-PS λ is said to be ρ-adapted to v if it achieves this minimum value; that is,

$$M_G^\rho(v) = \frac{\mu^\rho(v, \lambda)}{\|\lambda\|}.$$

We let $\wedge^\rho(v)$ denote the set of indivisible 1-PSs that are ρ-adapted to v.

For torus actions, we can combinatorially determine $\wedge^\rho(v)$ from the cone C_v defined at (4):

LEMMA 2.13   *Let v be an unstable point for the action of a torus $T \cong \mathbb{G}_m^n$ on a vector space V linearized by a non-trivial character ρ; then $\wedge^\rho(v)$ consists of a unique indivisible 1-PS.*

Proof. As v is ρ-unstable, the cone of allowable 1-PSs C_v defined at (4) is not contained in the half-space H_ρ. By definition

$$M^\rho(v) = \inf \frac{(\rho, \lambda)}{\|\lambda\|},$$

where the infimum is taken over 1-PSs $\lambda \in C_v \cap \mathbb{Z}^n - \{0\}$. As $(\rho, \lambda) = \|\lambda\| \|\rho\| \cos\theta_{\lambda,\rho}$ where $\theta_{\lambda,\rho}$ is the angle between the vectors λ and ρ, this is equivalent to taking the supremum over $\lambda \in C_v \cap \mathbb{Z}^n - \{0\}$ of the angle between ρ and λ. Hence, there is a unique ray R contained in the cone C_v along which the normalized Hilbert–Mumford function is minimized. Since R is the ray consisting of vectors whose angle to ρ is maximal over all vectors contained in the cone C_v, either R is the ray spanned by $-\rho$ or R is a ray contained in the boundary of the cone (that is; it is contained in

$\chi^\perp := \{\alpha \in \mathbb{R}^n : (\alpha, \rho) = 0\}$ for some $\chi \in \mathrm{wt}_T(v)$). In either case, R is the span of an integral point in $\mathbb{R}^n \cong \chi_*(T)_\mathbb{R}$ and so contains a unique point which corresponds to an indivisible 1-PS. $\qquad\square$

As any two maximal tori of a reductive group G are conjugate and $\mu^\rho(v, \lambda) = \mu^\rho(g \cdot v, g\lambda g^{-1})$, we see that one can calculate the possible values of M^ρ by fixing a maximal torus and calculating

$$\inf_{\lambda \neq 0 \in C} \frac{(p, \lambda)}{\|\lambda\|}$$

for all cones C which are constructed using some subset of the T-weights as at (4). In particular, we see that, for any reductive group, $M^\rho : V \to \mathbb{R}$ takes only finitely many values and $\wedge^\rho(v) \neq \emptyset$ for any ρ-unstable point.

DEFINITION 2.14   For any 1-PS λ of G we define a parabolic subgroup of G:

$$P(\lambda) := \left\{ g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t^{-1}) \text{ exists in } G \right\}.$$

One can easily modify the results of Kempf [7] on the sets $\wedge(v)$ to our setting of a reductive group action G on an affine space V linearized by a character ρ:

THEOREM 2.15 (Kempf)   *Let G be a reductive group acting on an affine V and suppose that ρ is a character that we use to linearize this action. If $v \in V$ is ρ-unstable, then*

(i) $\wedge^\rho(g \cdot v) = g \wedge^\rho(v) g^{-1}$ *for all $g \in G$;*
(ii) *there is a parabolic subgroup $P(v, \rho)$ such that $P(v, \rho) = P(\lambda)$ for all $\lambda \in \wedge^\rho(v)$;*
(iii) *all elements of $\wedge^\rho(v)$ are conjugate to each other by elements of $P(v, \rho)$. Moreover, the stabilizer subgroup G_v is contained in $P(v, \rho)$;*
(iv) *let $T \subset P(v, \rho)$ be a maximal torus of G; then there is a unique 1-PS of T which belongs to $\wedge^\rho(v)$;*
(v) *if $\lambda \in \wedge^\rho(v)$ and $w = \lim_{t \to 0} \lambda(t) \cdot v$, then also $\lambda \in \wedge^\rho(w)$.*

2.3. *The stratification of Hesselink*

When a reductive group G acts linearly on a projective space $\mathbb{P}^n$, there is a stratification of the unstable locus $\mathbb{P}^n - (\mathbb{P}^n)^{ss} = \sqcup S_{[\lambda], d}$ associated to this action and any norm as above which is indexed by pairs $([\lambda], d)$, where $[\lambda]$ denotes the conjugacy class of a 1-PS of G and d is a positive integer (see Hesselink [5] and also Kirwan [10, Section 12]). In this projective case, the strata are defined as

$$S_{[\lambda], d} := \{x : M(x) = -d \text{ and } \wedge(v) \cap [\lambda] \neq \emptyset\}$$

where $M(x) := \inf \mu(x, \lambda')/\|\lambda'\|$. We can easily modify these ideas to our setting of a reductive group G acting on an affine space V with respect to a character ρ to obtain a stratification of the null cone $N = V - V^{\rho\text{-ss}}$. We fix a norm on the set of conjugacy classes of 1-PSs of G which we assume

comes from a conjugation invariant inner product on the set of 1-PSs of G. In the affine setting, the index d is redundant as it is determined by λ and ρ; that is,

$$d = -M^\rho(v) = -\frac{\mu^\rho(v, \lambda)}{\|\lambda\|} = -\frac{(\rho, \lambda)}{\|\lambda\|} > 0$$

for $\lambda \in \wedge^\rho(v)$. We define

$$S_{[\lambda]} := \{v \in N : \wedge^\rho(v) \cap [\lambda] \neq \emptyset\};$$

then these subsets stratify the unstable locus $N := V - V^{\rho\text{-ss}}$ into G-invariant subsets (for details on what we mean by a stratification see Theorem 2.16). The strata $S_{[\lambda]}$ may also be described in terms of the 'blades'

$$S_\lambda = \{v \in N : \lambda \in \wedge^\rho(v)\} \subset S_{[\lambda]} \tag{5}$$

and the limit sets

$$Z_\lambda = \{v \in N : \lambda \in \wedge^\rho(v) \cap G_v\} \subset S_\lambda, \tag{6}$$

where G_v denotes the stabilizer subgroup of v and λ is a representative of the conjugacy class $[\lambda]$. It follows from Theorem 2.15 that $S_{[\lambda]} = G S_\lambda$. We let $p_\lambda : S_\lambda \to Z_\lambda$ be the retraction given by sending a point v to $\lim_{t \to 0} \lambda(t) \cdot v$. There is a strict partial ordering $<$ on the indices given by $[\lambda] < [\lambda']$ if

$$\frac{(\rho, \lambda)}{\|\lambda\|} > \frac{(\rho, \lambda')}{\|\lambda'\|};$$

that is, $M^\rho(v) > M^\rho(v')$ for $v \in S_{[\lambda]}$ and $v' \in S_{[\lambda']}$.

THEOREM 2.16   *Given a reductive group G acting on an affine space V with respect to a character ρ, there is a decomposition of the null cone $N := V - V^{\rho\text{-ss}}$*

$$N = \bigsqcup_{[\lambda]} S_{[\lambda]}$$

into finitely many disjoint G-invariant locally closed subvarieties $S_{[\lambda]}$ of V. Moreover, the strict partial ordering describes the boundary of a given stratum:

$$\partial S_{[\lambda]} \cap S_{[\lambda']} \neq \emptyset$$

only if $[\lambda] < [\lambda']$. (We refer to such a decomposition with such an ordering $<$ as a stratification).

Proof. By Theorem 2.15, the strata are disjoint and G-invariant. Let λ be a 1-PS of G which indexes a Hesselink stratum $S_{[\lambda]}$; then we write $V = \bigoplus_r V_r$, where $V_r = \{v \in V : \lambda(t) \cdot v = t^r v\}$. We let $V^\lambda = V_0$ denote the fixed point locus for the action of λ and $V_+^\lambda = \bigoplus_{r \geq 0} V_r$ denote the locus consisting of points $v \in V$ for which $\lim_{t \to 0} \lambda(t) \cdot v$ exists. Then both V^λ and V_+^λ are closed subsets of V. There is a projection $p_\lambda : V_+^\lambda \to V^\lambda$ given by sending a point to its limit under $\lambda(t)$ as $t \to 0$. Clearly, we have a containment of the blade $S_\lambda \subset V_+^\lambda$ and its limit set $Z_\lambda \subset V^\lambda$. In fact, Z_λ is an open subset of V^λ and $S_\lambda = p_\lambda^{-1}(Z_\lambda)$ by Proposition 2.18 and Lemma 2.19. It follows from this that $S_{[\lambda]} = G S_\lambda$ is a locally closed subvariety of V.

To prove the claim about the boundary, we note that, as $S_{[\lambda]}$ is contained in the closed set GV_+^λ, so is its closure. If $v \in \partial S_{[\lambda]} \cap S_{[\lambda']}$, then $v' = g \cdot v \in V_+^\lambda$. As $\lim_{t \to 0} \lambda(t) \cdot v'$ exists,

$$M^\rho(v) = M^\rho(v') = \frac{(\rho, \lambda')}{\|\lambda'\|} < \frac{(\rho, \lambda)}{\|\lambda\|};$$

that is, $[\lambda] < [\lambda']$. □

REMARK 2.17   From this stratification, we get a stratification

$$V = \bigsqcup S_{[\lambda]}, \tag{7}$$

where the minimal stratum $S_{[0]} := V^{\rho\text{-ss}}$ is open and is indexed by the trivial 1-PS. We shall refer to this stratification of V as Hesselink's stratification.

If λ is a non-trivial indivisible 1-PS which indexes a Hesselink stratum, we let G_λ denote the subgroup of G consisting of elements $g \in G$, which commute with $\lambda(t)$ for all t; then this group acts on the fixed point locus

$$V^\lambda := \{v \in V : \lambda(\mathbb{G}_m) \subset G_v\}$$

and we can use a character ρ_λ of G_λ to linearize the action. We use λ^* to denote the character of G_λ which is dual to the 1-PS λ of G_λ under the given inner product and (using additive notation for the group of characters) define

$$\rho_\lambda := \|\lambda\|^2 \rho - (\rho, \lambda)\lambda^*. \tag{8}$$

We note that $\|\lambda\|^2$ and $-(\rho, \lambda)$ are both positive integers and so ρ_λ is a character of G_λ. Our aim is to compare the semistable set $(V^\lambda)^{\rho_\lambda\text{-ss}}$ for this action of G_λ with the limit set $Z_\lambda \subset V^\lambda$ consisting of points v fixed by λ and for which λ is ρ-adapted to v. For the projective version of the following proposition, see Kirwan [10, Remark 12.21] and Ness [14, Theorem 9.4].

PROPOSITION 2.18   *Let λ be a non-trivial 1-PS that indexes a Hesselink stratum. Then the limit set Z_λ is equal to the semistable subset for the action of G_λ on V^λ with respect to the linearization given by ρ_λ.*

Proof. Let $v \in V^\lambda$; then first we claim that $G_\lambda \subset G$ is optimal for v in the sense of Definition 2.12. By Theorem 2.15(ii), if $\lambda' \in \wedge^\rho(v)$, then λ' is a 1-PS of $P(v, \rho)$ and, by (iii) of the same theorem, $G_v \subset P(v, \rho)$. Hence, there exists $p \in P(v, \rho)$ such that $p\lambda'p^{-1}$ and λ commute; that is, $p\lambda'p^{-1} \in \chi_*(G_\lambda) \cap \wedge^\rho(v)$, which proves the claim.

First, suppose that λ is not ρ-adapted to $v \in V^\lambda$ so that $v \notin Z_\lambda$. Then there is a 1-PS λ' for which $\lim_{t \to 0} \lambda'(t) \cdot v$ exists and such that

$$\frac{(\rho, \lambda')}{\|\lambda'\|} < \frac{(\rho, \lambda)}{\|\lambda\|} \tag{9}$$

and as G_λ is optimal for v, we can assume that λ' is a 1-PS of G_λ. By the definition of ρ_λ, we have

$$\mu^{\rho_\lambda}(v, \lambda') = (\rho_\lambda, \lambda') = \|\lambda\|^2(\rho, \lambda') - (\rho, \lambda)(\lambda, \lambda')$$

and using (9) this gives

$$\mu^{\rho_\lambda}(v, \lambda') < (\rho, \lambda)(\|\lambda\| \|\lambda'\| - (\lambda, \lambda')).$$

Then, since $(\lambda, \lambda') = \|\lambda\| \|\lambda'\| \cos\theta_{\lambda,\lambda'} \leq \|\lambda\| \|\lambda'\|$ and $(\rho, \lambda) < 0$, we have

$$\mu^{\rho_\lambda}(v, \lambda') < (\rho, \lambda)(\|\lambda\| \|\lambda'\| - (\lambda, \lambda')) \leq 0$$

which implies that v is not ρ_λ-semistable by Proposition 2.5.

Conversely, if v is unstable with respect to the action of G_λ linearized by ρ_λ, then there is a 1-PS λ' of G_λ such that

$$\mu^{\rho_\lambda}(v, \lambda') := (\rho_\lambda, \lambda') < 0.$$

Take a maximal torus T of G_λ which contains both λ and λ'; then in $\mathbb{R}^n \cong \chi^*(T)_\mathbb{R} \cong \chi_*(T)$ consider the vectors ρ, ρ_λ, λ and λ'. By definition of ρ_λ, this vector is contained in the interior of the cone spanned by ρ and λ, and is orthogonal to λ:

$$(\rho_\lambda, \lambda) = \|\lambda\|^2 (\rho, \lambda) - (\rho, \lambda)(\lambda, \lambda) = 0.$$

As $(\rho, \lambda) < 0$ and $(\rho_\lambda, \lambda') < 0$, we also know that the angles $\theta_{\rho,\lambda}$ and $\theta_{\rho_\lambda,\lambda'}$ are both at least $\pi/2$; therefore $\theta_{\rho,\lambda+\lambda'} > \theta_{\rho,\lambda} > \pi/2$ and so

$$\frac{\mu^\rho(v, \lambda + \lambda')}{\|\lambda + \lambda'\|} = \|\rho\| \cos(\theta_{\rho,\lambda+\lambda'}) < \|\rho\| \cos(\theta_{\rho,\lambda}) = \frac{\mu^\rho(v, \lambda)}{\|\lambda\|},$$

which shows that λ is not ρ-adapted to v. $\square$

LEMMA 2.19   *Let λ be a non-trivial 1-PS of G that indexes a Hesselink stratum $S_{[\lambda]}$; then*

$$S_\lambda = p_\lambda^{-1}(Z_\lambda),$$

where $p_\lambda : V_+^\lambda \to V^\lambda$ is the retraction given by sending v to $\lim_{t\to 0} \lambda(t) \cdot v$. Moreover, S_λ is a locally closed subvariety of V.

Proof. By Theorem 2.15, if $v \in S_\lambda$, then $p_\lambda(v) \in Z_\lambda$. Now suppose $v \in V_+^\lambda$ and $p_\lambda(v) \in Z_\lambda$. If $v \notin S_{[\lambda]}$, then v is ρ-unstable as $\mu^\rho(v, \lambda) = (\rho, \lambda) < 0$ and so we can assume $v \in S_{[\lambda']}$. Then $p_\lambda(v) \in \partial S_{[\lambda']} \cap S_{[\lambda]}$ and so $[\lambda'] < [\lambda]$. However, as both $\lim_{t\to 0} \lambda(t) \cdot v$ and $\lim_{t\to 0} \lambda'(t) \cdot v$ exist,

$$M^\rho(v) = \frac{(\rho, \lambda')}{\|\lambda'\|} < \frac{(\rho, \lambda)}{\|\lambda\|}, \tag{10}$$

which contradicts $[\lambda'] < [\lambda]$. Hence $v \in S_{[\lambda]} \cap V_+^\lambda = S_\lambda$. We note that Z_λ is open in V^λ as it is the semistable set for a reductive group action and since also $S_\lambda = p_\lambda^{-1}(Z_\lambda)$, we have that S_λ is locally closed. $\square$

2.4. *The indices for Hesselink's stratification*

If we fix a maximal torus T of G, then every conjugacy class $[\lambda]$ has a representative λ which is a 1-PS of T. It follows that to calculate the indices $[\lambda]$ occurring in Hesselink's stratification, we can just calculate the 1-PSs of T that are ρ-adapted to ρ-unstable points for the action of T. This can be done combinatorially as in Lemma 2.13, where we find the unique indivisible 1-PS which is adapted to an unstable point v by finding the ray in the cone C_v whose angle to ρ is maximal. As there are only finitely many T-weights, there are only finitely many cones C_v to consider (which can be determined combinatorially from subsets of the T-weights).

3. The moment map and symplectic reduction

Let K be a compact real Lie group acting smoothly on a complex vector space V. We fix a Hermitian inner product $H : V \times V \to \mathbb{C}$ on V and assume that K acts by unitary transformations of V. The imaginary part of H is a symplectic inner product on V which we can multiply by any non-zero real scalar. We let $\omega = (1/\pi) \operatorname{Im} H : V \times V \to \mathbb{R}$ denote our symplectic inner product on V; then

$$\omega(v, w) = \frac{1}{2\pi \mathrm{i}}(H(v, w) - \overline{H(v, w)}) = \frac{1}{2\pi \mathrm{i}}(H(v, w) - H(w, v)).$$

By identifying $TV \cong V \times V$, we see that (V, ω) is a symplectic manifold and ω is K-invariant. The infinitesimal action is a Lie algebra homomorphism $\mathfrak{K} \to \operatorname{Vect}(V)$ given by $\alpha \mapsto \alpha_V$ where $\alpha_v := (\alpha_V)_v$ is given by

$$\alpha_v = \frac{\mathrm{d}}{\mathrm{d}t} \exp(t\alpha) \cdot v \bigg|_{t=0} \in T_v V$$

for $\alpha \in \mathfrak{K} = \operatorname{Lie} K$.

DEFINITION 3.1 A moment map for the action of K on V is a smooth map $\mu : V \to \mathfrak{K}^*$, where $\mathfrak{K} = \operatorname{Lie} K$ such that

(1) μ is equivariant for the given action of K on V and the coadjoint action of K on $\mathfrak{K}^*$;
(2) for $v \in V$, $\alpha \in \mathfrak{K}$ and $\zeta \in V \cong T_v V$, we have

$$d_v \mu(\zeta) \cdot \alpha = \omega(\alpha_v, \zeta),$$

where $\cdot : \mathfrak{K}^* \times \mathfrak{K} \to \mathbb{R}$ denotes the natural pairing, $d_v \mu$ denotes the derivative of μ at v and $\alpha_v \in V \cong T_v V$ denotes the infinitesimal action of α on v.

LEMMA 3.2 *For the standard representation of* $\mathrm{U}(V)$ *on* V, *there is a natural moment map* $\mu : V \to \mathfrak{u}(V)^*$ *given by*

$$\mu(v) \cdot \alpha := \frac{1}{2}\omega(\alpha v, v) = \frac{1}{2\pi \mathrm{i}} H(\alpha v, v)$$

for $v \in V$ *and* $\alpha \in \mathfrak{u}(V)$. *Moreover, if* K *acts unitarily on* V, *then the composition of the above map with the projection* $\mathfrak{u}(V)^* \to \mathfrak{K}^*$ *associated to this action is a moment map for the action on* V.

Proof. We note that the second equality in the definition of μ above is due to the fact that

$$H(\alpha v, w) + H(v, \alpha w) = 0 \tag{11}$$

for all $v, w \in V$ and $\alpha \in \mathfrak{u}(V)$. The equivariance of μ follows immediately from the fact that H is invariant for the action of $U(V)$. The infinitesimal action of $\alpha \in \mathfrak{u}(V)$ at $v \in V$ is

$$\alpha_v := \frac{\mathrm{d}}{\mathrm{d}t} \exp(t\alpha) \cdot v = \alpha v \in V \cong T_v V.$$

For $v \in V$, $\alpha \in \mathfrak{u}(V)$ and $\zeta \in V \cong T_v V$, we have

$$d_v \mu(\zeta) \cdot \alpha := \frac{\mathrm{d}}{\mathrm{d}t} \mu(v + t\zeta) \cdot \alpha|_{t=0} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \omega(\alpha(v + t\zeta), v + t\zeta) \Big|_{t=0}$$

$$= \frac{1}{2} [\omega(\alpha v, \zeta) + \omega(\alpha\zeta, v)] = \frac{1}{4\pi \mathrm{i}} [H(\alpha v, \zeta) - H(\zeta, \alpha v) + H(\alpha\zeta, v) - H(v, \alpha\zeta)]$$

$$\overset{(11)}{=} \frac{1}{2\pi \mathrm{i}} [H(\alpha v, \zeta) - H(\zeta, \alpha v)] =: \omega(\alpha v, \zeta)$$

and hence μ is a moment map for the action of $U(V)$. If $\varphi : K \to U(V)$ denotes the representation corresponding to the action of K on V and we use $\varphi : \mathfrak{K} \to \mathfrak{u}(V)$ to also denote the associated Lie algebra homomorphism, then the natural moment map for the K-action on V is

$$\mu_K = \varphi^* \circ \mu_{U(V)} : V \to \mathfrak{u}(V)^* \to \mathfrak{K}^*,$$

where φ^* is dual to φ. Thus,

$$\mu_K(v) \cdot \alpha = \frac{1}{2\pi \mathrm{i}} H(\varphi(\alpha)v, v),$$

although we shall often, for simplicity, write α to mean $\varphi(\alpha)$ in such expressions. $\square$

REMARK 3.3   Given a character $\rho : K \to U(1)$, we can identify $U(1) \cong S^1$ and Lie $S^1 \cong 2\pi \mathrm{i}\mathbb{R}$. Let $\mathrm{d}\rho : \mathfrak{K} \to 2\pi \mathrm{i}\mathbb{R}$ denote the derivative of this character at the identity element of K. Then $(1/2\pi \mathrm{i}) \mathrm{d}\rho \in \mathfrak{K}^*$ is fixed by the coadjoint action of K on $\mathfrak{K}^*$ and so we can define a shifted moment map $\mu^\rho : V \to \mathfrak{K}^*$ by

$$\mu^\rho(v) \cdot \alpha = \frac{1}{2} \omega(\alpha v, v) - \frac{1}{2\pi \mathrm{i}} \mathrm{d}\rho \cdot \alpha = \frac{1}{2\pi \mathrm{i}} (H(\alpha v, v) - \mathrm{d}\rho \cdot \alpha).$$

We shall refer to this as the natural moment map for the action of K shifted by ρ.

In symplectic geometry, the symplectic reduction is used as a quotient:

DEFINITION 3.4   The symplectic reduction for the action of K on V (with respect to the character ρ) is given by $V^{\rho\text{-red}} = (\mu^\rho)^{-1}(0)/K$.

If 0 is a regular value of μ^ρ and the action of K on V is proper, then the symplectic reduction $V^{\rho\text{-red}}$ is a real manifold of dimension $\dim_{\mathbb{R}} V - 2 \dim_{\mathbb{R}} K$. Moreover, Marsden and Weinstein [11] and Meyer [12] show that there is a canonical induced symplectic form ω^{red} on $V^{\rho\text{-red}}$ such that $i^* \omega = \pi^* \omega^{\mathrm{red}}$, where $i : (\mu^\rho)^{-1}(0) \hookrightarrow V$ is the inclusion and $\pi : (\mu^\rho)^{-1}(0) \to V^{\rho\text{-red}}$ is the projection.

3.1. *The norm square of the moment map*

Given an inner product $(-, -)$ on the Lie algebra $\mathfrak{K}$ of K which is invariant under the adjoint action of K on $\mathfrak{K}$, we can identify $\mathfrak{K} \cong \mathfrak{K}^*$. We let $\| - \|$ denote the associated norm on $\mathfrak{K} \cong \mathfrak{K}^*$ given by $\|\alpha\|^2 = (\alpha, \alpha)$.

Suppose that we have an action of K on a complex affine space V (by unitary transformations) with natural moment map $\mu := \mu^\rho : V \to \mathfrak{K}^*$ shifted by a character ρ as above. We let $\mu^* : V \to \mathfrak{K}$ denote the map constructed from μ and the isomorphism $\mathfrak{K}^* \cong \mathfrak{K}$; thus,

$$\mu(v) \cdot \alpha = (\mu^*(v), \alpha).$$

We consider the norm square of the moment map

$$\|\mu\|^2 : \quad V \to \mathbb{R},$$
$$v \mapsto \|\mu(v)\|^2,$$

which is a real analytic function. The derivative of this function is a 1-form $d\|\mu\|^2$ on V which, at $v \in V$ and $\zeta \in V \cong T_v V$, is given by

$$d_v\|\mu\|^2(\zeta) := \frac{\mathrm{d}}{\mathrm{d}t}\|\mu(v + t\zeta)\|^2|_{t=0} = 2(d_v\mu(\zeta), \mu(v))$$

$$= 2d_v\mu(\zeta) \cdot \mu^*(v) = 2\omega(\mu^*(v)_v, \zeta). \tag{12}$$

The critical locus $\mathrm{crit}\|\mu\|^2 := \{v : d\|\mu\|_v^2 = 0\}$ is a closed subset of V.

LEMMA 3.5   *Let $v \in V$; then the following are equivalent*:

(1) *v is a critical point of $\|\mu\|^2 : V \to \mathbb{R}$;*
(2) *the infinitesimal action of $\mu^*(v)$ at v is zero*: $\mu^*(v)_v = 0$;
(3) *$\mu^*(v) \in \mathfrak{K}_v$, where $\mathfrak{K}_v$ is the Lie algebra of the stabilizer K_v of v.*

Clearly, $\mu^{-1}(0) = (\mu^*)^{-1}(0)$ is a subset of the critical locus $\mathrm{crit}\|\mu\|^2$; we refer to these critical points as minimal critical points. For $\beta \in \mathfrak{K}$, we can consider the function $\mu_\beta : V \to \mathbb{R}$ given by $\mu_\beta(v) = \mu(v) \cdot \beta$. As $d\mu_\beta = \omega(\beta_V, -)$, the critical locus of μ_β is the set of points $v \in V$ for which the infinitesimal action of β is zero; that is,

$$\mathrm{crit}\,\mu_\beta = \{v \in V : \beta_v = 0\}.$$

It follows from Lemma 3.5, that the intersection $\mathrm{crit}\,\mu_\beta \cap (\mu^*)^{-1}(\beta)$ is contained in the critical locus $\mathrm{crit}\|\mu\|^2$. Since $\|\mu\|^2 : V \to \mathbb{R}$ is K-invariant, the critical locus $\mathrm{crit}\|\mu\|^2$ is invariant under the action of K. Therefore, the disjoint sets

$$C_{K \cdot \beta} := K(\mathrm{crit}\,\mu_\beta \cap (\mu^*)^{-1}(\beta)) = \mathrm{crit}\|\mu\|^2 \cap (\mu^*)^{-1}(K \cdot \beta)$$

indexed by adjoint orbits $K \cdot \beta$ cover the critical locus of $\|\mu\|^2$ computed at (12) and we refer to these subsets $C_{K \cdot \beta}$ as the critical subsets.

We take a metric compatible with the complex and symplectic structure on V and this defines a duality between 1-forms and vector fields where $\omega(-, \psi)$ is dual to $i\psi$. The 1-form $d\|\mu\|^2$ is dual (under this metric) to a vector field on V, which we denote by $\text{grad}\|\mu\|^2$, and from the calculation of $d\|\mu\|^2$ at (12), we have at $v \in V$ that

$$\text{grad}\|\mu\|_v^2 = -2i\mu^*(v)_v. \tag{13}$$

DEFINITION 3.6   For $v \in V$, the negative gradient flow of $\|\mu\|^2$ at v is a path $\gamma_v(t)$ in V which is defined in some open neighbourhood of $0 \in \mathbb{R}_{\geq 0}$ and satisfies

$$\gamma_v(0) = v \quad \gamma_v'(t) = -\text{grad}\|\mu\|^2_{\gamma_v(t)}.$$

The negative gradient flow exists by the local existence of solutions to ordinary differential equations and it is well known that $\gamma_v(t)$ exists for all real t (for example, see [6]). If v is a critical point of $\|\mu\|^2$, then $\text{grad}\|\mu\|_v^2 = 0$ and so $\gamma_v(t) = v$ is the constant path. From the expression for $\text{grad}\|\mu\|_v^2$ given at (13), we see that the finite time negative gradient flow is contained in the orbit of v under the action of the complexified group $K_{\mathbb{C}}$ whose Lie algebra $\mathfrak{K}_{\mathbb{C}} \cong \mathfrak{K} \oplus i\mathfrak{K}$ is the complexification of $\mathfrak{K}$. Moreover, even though we are working in a (non-compact) affine space, the gradient $\gamma_v(t)$ converges to a critical value v_∞ of $\|\mu\|^2$ by the work of Harada and Wilkin [4] and Sjamaar [16].

3.2. *Morse-theoretic stratification*

We use the negative gradient flow of $\|\mu\|^2$ to produce a Morse-theoretic stratification of V as follows. The index set for the stratification is a finite number of adjoint orbits which index non-empty critical subsets:

$$\mathcal{B} := \{K \cdot \beta : \beta \in \mathfrak{K} \text{ and } C_{K \cdot \beta} \neq \emptyset\}. \tag{14}$$

For $K \cdot \beta \in \mathcal{B}$, we define the associated stratum

$$S_{K \cdot \beta} := \left\{ v \in V : \lim_{t \to \infty} \gamma_v(t) \in C_{K \cdot \beta} \right\}$$

to be the set of points whose negative gradient flow converges to $C_{K \cdot \beta}$. We refer to the stratification

$$V = \bigsqcup_{K \cdot \beta \in \mathcal{B}} S_{K \cdot \beta} \tag{15}$$

as a Morse stratification; the partial order on the indices is given by using the K-invariant norm $\| - \|$ on $\mathfrak{K}$ in the natural way. We say one stratum is lower or higher than another if its corresponding index is with respect to this norm. In particular, the lowest stratum is indexed by $0 \in \mathfrak{K}$ and $S_0 \subset V$ is open with limit set $C_0 = \mu^{-1}(0)$.

This ordering allows us to describe the closure of a Morse stratum as in [10, Lemma 10.7]. Our argument follows that of [10], but also uses the fact that the negative gradient flow of each point $v \in V$ under $\|\mu\|^2$ is contained in a compact subset of V (see [4, Lemma 3.3]).

LEMMA 3.7   *The boundary of a Morse stratum is contained in a union of strictly higher strata:*

$$\partial S_{K \cdot \beta} \subset \bigcup_{\beta' > \beta} S_{K \cdot \beta'}.$$

Proof. If $v \in S_{K \cdot \beta}$, then by definition its path of steepest descent under $\|\mu\|^2$ converges to a point in $C_{K \cdot \beta}$ and so $\|\mu(v)\|^2 \geq \|\mu(C_{K \cdot \beta})\|^2 = \|\beta\|^2$. If $v \in \bar{S}_{K \cdot \beta}$, then the closure of its path of steepest descent $\gamma_v(t)$ under $\|\mu\|^2$ is also contained in $\bar{S}_{K \cdot \beta}$; therefore, $v \in S_{K \cdot \beta'}$ for which $\|\beta'\|^2 \geq \|\beta\|^2$. In particular,

$$\bar{S}_{K \cdot \beta} \subset \bigcup_{\beta' \geq \beta} S_{K \cdot \beta'}.$$

We must now show that the boundary of a stratum $S_{K \cdot \beta}$ meets another stratum $S_{K \cdot \beta'}$ only if $\beta' > \beta$. Take open neighbourhoods U_β of the critical subsets $C_{K \cdot \beta}$ whose closures are disjoint; such neighbourhoods exist as the negative gradient flow paths are contained in compact subsets by Harada and Wilkin [4, Lemma 3.3]. As $\partial U_\beta \cap \bar{S}_{K \cdot \beta}$ is closed and does not contain $C_{K \cdot \beta}$, there is $\delta > 0$ such that $\|\mu(v)\|^2 \geq \|\beta\|^2 + \delta$ for all $v \in \partial U_\beta \cap \bar{S}_{K \cdot \beta}$.

Suppose $v \in \bar{S}_{K \cdot \beta} \cap S_{K \cdot \beta'}$; then the path of steepest descent $\gamma_v(t)$ of v under $\|\mu\|^2$ converges to $C_{K \cdot \beta'}$. In particular, there exists $t_0 \geq 0$ such that $\gamma_v(t_0) \in U_{\beta'}$ and $\|\mu(\gamma_v(t_0))\|^2 - \|\beta'\|^2 < \delta$. Then it follows that there is a neighbourhood W_v of v such that if $w \in W_v$, then

$$\|\mu(\gamma_w(t_0))\|^2 - \|\beta'\|^2 < \delta$$

and $\gamma_w(t_0) \in U_{\beta'}$. As $v \in \bar{S}_{K \cdot \beta}$, the neighbourhood W_v of v must have non-empty intersection with $S_{K \cdot \beta}$. Suppose $w \in W_v \cap S_{K \cdot \beta}$; then $\|\mu(\gamma_w(t_0))\|^2 - \|\beta'\|^2 < \delta$ and $\gamma_w(t_0) \in U_{\beta'}$. As $\gamma_w(t)$ converges to $C_{K \cdot \beta}$ and the closures of the neighbourhoods U_β are disjoint, there exists $t_1 > t_0$ such that $\gamma_w(t_1) \in \partial U_\beta$. Therefore,

$$\|\beta'\|^2 + \delta > \|\mu(\gamma_w(t_0))\|^2 \geq \|\mu(\gamma_w(t_1))\|^2 \geq \|\beta\|^2 + \delta$$

as $w \in W_v$; the norm square of the moment map $\|\mu\|^2$ decreases along $\gamma_w(t)$; and $\gamma_w(t_1) \in \partial U_\beta \cap S_{K \cdot \beta}$. □

Finally, we note that if we fix a maximal torus $T \subset K$ and a positive Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t} = \text{Lie } T$, then an adjoint orbit $K \cdot \beta$ meets $\mathfrak{t}_+$ in a single point β and so we can view the index set $\mathcal{B}$ as a finite set of elements in $\mathfrak{t}_+$.

3.3. *Description of the indices*

The indices $\beta \in \mathcal{B}$ can be computed from the weights of a maximal torus T of K acting on V similarly to the projective case given in [10, 14], although the role of weight polytopes is now played by shifted weight cones. A torus action gives an orthogonal decomposition $V = \bigoplus_\chi V_\chi$, where the indices χ are characters of T and $V_\chi = \{v \in V : t \cdot v = \chi(t)v \text{ for all } t \in T\}$. We use the natural identification $\chi^*(T)_{\mathbb{R}} \cong \mathfrak{t}^*$ given by

$$\chi \mapsto \frac{1}{2\pi i} \, d\chi$$

to identify (real) characters with elements of $\mathfrak{t}^* \cong \mathfrak{t}$ and will write χ to mean either a character or an element of $\mathfrak{t}^* \cong \mathfrak{t}$. We refer to the points in the lattice $\chi^*(T) \subset \mathfrak{t}^*$ as integral points. Let $T\text{-wt} := \{\chi : V_\chi \neq 0\}$ denote the set of T-weights for the action. The advantage of working with a maximal torus is that the image of a complex vector space under a moment map for a torus action (shifted by ρ) is a cone generated by the torus weights (shifted by $-\rho$) [1, 3].

If B is a (possibly empty) subset of the T-weights, then we associate to B a cone $C(B)$ in $\mathfrak{t}$ given by the positive span of these weights (where the cone associated to the empty set is the origin 0).

Let $C_{-\rho}(B) := C(B) - \rho$ denote the translation of this cone by $-\rho$; then we define $\beta(B)$ to be the closest point of $C_{-\rho}(B)$ to the origin. We claim that

$$\mathcal{B} = \{\beta(B) \in \mathfrak{t}_+ : B \subset T\text{-wt}\}.$$

The proof follows in exactly the same way as the projective case (for example, see [**10**, Section 3]) except that the weight polytopes are replaced by our shifted weight cones: to show $\beta(B)$ is an index, one can find $v \in V_B = \bigoplus_{\chi \in B} V_\chi$ such that $v \in S_{K \cdot \beta(B)}$ and, given $\beta \in \mathcal{B}$, one can find $v \in S_v$ such that $\mu^*(v) \in \mathfrak{t}$ and then $\beta = \beta(B)$, where $B = \{\chi : v_\chi \neq 0\}$ and $v = \sum v_\chi$.

From this description, we see that the indices β (which are the closest points to the origin of cones defined by integral weights χ shifted by an integral weight ρ) are rational weights; that is, $n\beta \in \chi^*(T) \cong \chi_*(T) \cong \mathbb{Z}^{\mathrm{rk}\,T}$ for some natural number n.

REMARK 3.8   If ρ is the trivial character, then any cone $C_0(B)$ associated to a subset B of the torus weights will contain 0 and so $\beta(B) = 0$ for all such subsets B. Thus, if $\rho = 0$, there is only one Morse stratum $S_0 = V$.

3.4. *An alternative description of the strata*

In this section, we given an alternative description of the Morse strata which we call the Kirwan stratification as it is inspired by a similar description in the projective setting given by Kirwan [**10**, Section 6]. We note that our notation differs slightly from that given in [**10**].

For $\beta \in \mathfrak{K}$, we let $\mu_\beta : V \to \mathbb{R}$ be given by $\mu_\beta(v) = \mu(v) \cdot \beta$ as above. Then $\mathrm{d}\mu_\beta = \omega(\beta_V, -)$ and

$$\mathrm{crit}\,\mu_\beta = \{v \in V : \beta \in \mathfrak{K}_v\} =: V^\beta.$$

We note that μ_β is constant on this critical locus: $\mu_\beta(V^\beta) = -\mathrm{d}\rho \cdot \beta/2\pi\mathrm{i}$ (in the projective setting this is not always the case and so one only considers certain connected components of this critical locus). The gradient vector field associated to μ_β is $\mathrm{grad}\,\mu_\beta = -\mathrm{i}\beta_V$ and so the negative gradient flow of μ_β at a point $v \in V$ is given by

$$\gamma_v^\beta(t) = \exp(\mathrm{i}t\beta) \cdot v.$$

We let $V_+^\beta := \{v \in V : \lim_{t \to \infty} \exp(\mathrm{i}t\beta) \cdot v \text{ exists}\}$ denote the subset of points for which the limit of the negative gradient flow exists; then there is a retraction $p_\beta : V_+^\beta \to V^\beta$ given by the negative gradient flow. In the projective setting, the negative gradient flow of any point under μ_β converges and the notation $p_\beta : Y_\beta \to Z_\beta$ is used.

Let K_β denote the stabilizer of β under the adjoint action of K and let $\mathfrak{K}_\beta$ denote the associated Lie algebra. Then V^β is invariant under the action of K_β and, for $v \in V^\beta$, we note that $\mu(v) \in \mathfrak{K}_\beta^*$ (for example, in [**14**] see the proof of Theorem 9.2). Hence $\mu : V^\beta \to \mathfrak{K}_\beta^*$ is a moment map for the action of K_β on V^β. As $\beta^* = (\beta, -) \in \mathfrak{K}_\beta^*$ is a central element, $\mu - \beta^*$ is also a moment map for the action of K_β on V^β. If we wish to emphasize that we are considering μ as a moment map for the action of K_β on V^β or K on V, we will write μ_{K_β} or μ_K, respectively.

The critical subset of $\|\mu\|^2$ associated to β (or more precisely the adjoint orbit of β) is

$$C_{K \cdot \beta} := K(\mathrm{crit}\,\mu_\beta \cap \mu^{-1}(\beta^*)) = \mathrm{crit}\|\mu\|^2 \cap \mu^{-1}(K \cdot \beta^*)$$

and we let $C_\beta := \mathrm{crit}\,\mu_\beta \cap \mu^{-1}(\beta^*) = V^\beta \cap \mu^{-1}(\beta^*) \subset C_{K \cdot \beta}$.

LEMMA 3.9   *For $\beta \in \mathfrak{K}$, the set C_β is equal to the set of minimal critical points for the norm square of the moment map $\mu - \beta^* : V_\beta \to \mathfrak{K}_\beta^*$; that is,*

$$C_\beta = (\mu - \beta^*)^{-1}(0).$$

Proof. We verify that $(\mu_{K_\beta} - \beta^*)^{-1}(0) = \mu_{K_\beta}^{-1}(\beta^*) = \mu_K^{-1}(\beta^*) \cap V_\beta = C_\beta$.   □

REMARK 3.10   If C_β is non-empty, then $\mu_\beta(C_\beta) = \|\beta\|^2$ but, as $\beta_v = 0$ for $v \in V^\beta$, we also have $\mu_\beta(V^\beta) = -\mathrm{d}\rho \cdot \beta/2\pi\mathrm{i}$. Therefore, the norm of β satisfies

$$\|\beta\|^2 = -\mathrm{d}\rho \cdot \beta/2\pi\mathrm{i}.$$

Then $\|\mu - \beta^*\|^2$ is used to obtain a description of the Morse stratum $S_{K \cdot \beta}$ for $\|\mu\|^2$ as in [10].

DEFINITION 3.11   Let $Z_\beta^{\min} \subset V^\beta$ be the minimal Morse stratum for the function $\|\mu - \beta^*\|^2$ on V^β (whose points flow to C_β by Lemma 3.9) and $Y_\beta^{\min} \subset V_+^\beta$ be the preimage of $Z_\beta^{\min} \subset V^\beta$ under $p_\beta : V_+^\beta \to V^\beta$.

As in [10], we can consider a parabolic subgroup P_β of the complexified group $G := K_\mathbb{C}$

$$P_\beta = \{g \in G : \lim_{t \to \infty} \exp(\mathrm{i}t\beta)g \exp(\mathrm{i}t\beta)^{-1} \text{ exists}\},$$

which leaves V_+^β and $Y_\beta^{\min}$ invariant. As $G = K P_\beta$, we have that $G V_+^\beta = K V_+^\beta$ and similarly $G Y_\beta^{\min} = K Y_\beta^{\min}$. In Theorem 4.17, we prove that $S_{K \cdot \beta} = G Y_\beta^{\min}$.

4. A comparison of the algebraic and symplectic descriptions

We recall that a group G is complex reductive if and only if it is the complexification of its maximal compact subgroup $K \subset G$. In this section, we suppose that we have a linear action of a complex reductive group $G = K_\mathbb{C}$ on a complex vector space V for which K acts unitarily (with respect to a fixed Hermitian inner product H on V). We linearize the action as in Section 2 by choosing a character $\rho : G \to \mathbb{C}^*$ and use this to construct a linearization L_ρ whose underlying line bundle is the trivial line bundle $L = V \times \mathbb{C}$. We assume that $\rho(K) \subset \mathrm{U}(1) \cong S^1$ so that we can consider the restriction of ρ to K as a compact character. Then the main results of this section are the following.

(i) The GIT quotient $V /\!/_\rho G$ with respect to ρ is homeomorphic to the symplectic reduction $V^{\rho\text{-red}} = (\mu^\rho)^{-1}(0)/K$ with respect to ρ (cf. Theorem 4.2).
(ii) If we fix a K-invariant inner product on $\mathfrak{K}$, then Hesselink's stratification of V by ρ-adapted 1-PSs agrees with the Morse stratification of V associated to $\|\mu^\rho\|^2$ (cf. Theorem 4.18).

4.1. *Affine Kempf–Ness theorem with respect to a character*

Before we can state the affine Kempf–Ness theorem, we need one additional definition from GIT. We recall that the action is lifted to $L^{-1} := L_\rho^{-1} = V \times \mathbb{C}$ by using the character $-\rho$.

DEFINITION 4.1   An orbit $G \cdot v$ is said to be ρ-polystable if $G \cdot \tilde{v}$ is closed in L^{-1} where $\tilde{v} = (v, a) \in L^{-1}$ and $a \neq 0$.

One can show that an orbit is ρ-polystable if and only if it is ρ-semistable and closed in $V^{\rho\text{-ss}}$. By Lemma 2.4, we have inclusions

$$V^{\rho\text{-s}} \subset V^{\rho\text{-ps}} \subset V^{\rho\text{-ss}},$$

where $V^{\rho\text{-ps}}$ denotes the locus of ρ-polystable points. From now on we use the terms stability, polystability and semistability to all mean with respect to the character ρ. As the closure of every semistable orbit in the semistable locus contains a unique closed orbit (which is polystable), the GIT quotient $V /\!/_\rho G$ is topologically the orbit space $V^{\rho\text{-ps}}/G$.

THEOREM 4.2 (Affine Kempf–Ness theorem)   *Let $G = K_{\mathbb{C}}$ be a complex reductive group acting linearly on a complex vector space V and suppose that K acts unitarily with respect to a fixed Hermitian inner product on V. Given a character ρ for this action, let $\mu := \mu^\rho : V \to \mathfrak{K}^*$ denote the moment map for this action (shifted by ρ). Then the following statements hold.*

  (i) *A G-orbit meets the preimage of 0 under μ if and only if it is polystable; that is, $G\mu^{-1}(0) = V^{\rho\text{-ps}}$.*
 (ii) *A polystable G-orbit meets $\mu^{-1}(0)$ in a single K-orbit.*
(iii) *A point $v \in V$ is semistable if and only if its G-orbit closure meets $\mu^{-1}(0)$.*
 (iv) *The lowest Morse stratum S_0 agrees with the GIT semistable locus $V^{\rho\text{-ss}}$.*
  (v) *The inclusion $\mu^{-1}(0) \subset V^{\rho\text{-ss}}$ induces a homeomorphism*

$$V^{\rho\text{-red}} := \mu^{-1}(0)/K \cong V /\!/_\rho G.$$

We note that King proves the first two parts of this theorem in [**9**, Section 6], although we shall also provide a proof here in part for completeness and also in part as it allows us to state some more general results that we shall need for the proof of Theorem 4.18.

To prove the above theorem, we introduce a function $p_v : \mathfrak{K} \to \mathbb{R}$ for each $v \in V$:

$$p_v(\alpha) = \frac{1}{4\pi} H(\exp(i\alpha)v, \exp(i\alpha)v) + \frac{1}{2\pi i} \, d\rho \cdot \alpha,$$

where we write $\exp(i\alpha)v$ to mean the action of $\exp(i\alpha) \in G$ on $v \in V$.

LEMMA 4.3   *An element $\alpha \in \mathfrak{K}$ is a critical point of p_v if and only if $\mu(\exp(i\alpha)v) = 0$. Moreover, every critical point is a minimum.*

Proof. The first- and second-order derivatives of p_v taken at α and evaluated at $\beta \in \mathfrak{K}$ are

$$\frac{d}{dt} p_v(\alpha + t\beta)|_{t=0} = \frac{1}{4\pi}[H(i\beta \exp(i\alpha)v, \exp(i\alpha)v) + H(\exp(i\alpha)v, i\beta \exp(i\alpha)v)] + \frac{1}{2\pi i} \, d\rho \cdot \beta$$

$$= \frac{i}{4\pi}[H(\beta \exp(i\alpha)v, \exp(i\alpha)v) - H(\exp(i\alpha)v, \beta \exp(i\alpha)v)] + \frac{1}{2\pi i} \, d\rho \cdot \beta$$

$$\stackrel{(11)}{=} \frac{i}{2\pi} H(\beta \exp(i\alpha)v, \exp(i\alpha)v) + \frac{1}{2\pi i} \, d\rho \cdot \beta = -\mu(\exp(i\alpha)v) \cdot \beta$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} p_v(\alpha + t\beta)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{i}{2\pi} H(\beta \exp(i\alpha + t\beta)v, \exp(i\alpha)v) + \frac{1}{2\pi i} \mathrm{d}\rho \cdot \beta \right)|_{t=0}$$

$$= \frac{-1}{2\pi} [H(\beta^2 \exp(i\alpha)v, \exp(i\alpha)v) - H(\beta \exp(i\alpha)v, \beta \exp(i\alpha)v)]$$

$$\overset{(11)}{=} \frac{1}{\pi} H(\beta \exp(i\alpha)v, \beta \exp(i\alpha)v) \geq 0.$$

The lemma then follows from these calculations. □

PROPOSITION 4.4 *Let $v \in V$; then p_v has a minimum if and only if v is polystable.*

Proof. If α is a critical point of p_v, then $u = \exp(i\alpha)v \in \mu^{-1}(0)$ by Lemma 4.3. Given a 1-PS λ of G, we can consider the limit

$$\lim_{t \to 0} \lambda(t) \cdot (u, 1) \tag{16}$$

in $L^{-1} = V \times \mathbb{C}$. If the limit exists, then we can assume (by conjugating λ by g and replacing u by gu if necessary) that $\lambda(S^1) \subset K$. By Lemma 4.5, if we set

$$\beta := \mathrm{d}\lambda(2\pi i) = \frac{\mathrm{d}}{\mathrm{d}t}\lambda(\exp(2\pi i t))\Big|_{t=0} \in \mathfrak{K},$$

then $\mu(u) \cdot \beta \geq 0$ with equality if and only if λ fixes $(u, 1)$. As $\mu(u) = 0$, we conclude that λ fixes $(u, 1)$ and so it follows from Theorem 2.6 that $G \cdot (u, 1)$ is closed in L^{-1}; that is $G \cdot u = G \cdot v$ is polystable.

Conversely, if $G \cdot v$ is polystable, then $G \cdot (v, 1)$ is closed in L^{-1}. It follows from this that for $\alpha \in \mathfrak{K}$:

1. if $\exp(i\alpha) \in G_v$, then $(1/2\pi i) \mathrm{d}\rho \cdot \alpha = 0$;
2. if $\exp(i\alpha) \notin G_v$, then either $\lim_{t \to \infty} \exp(it\alpha) \cdot v$ does not exist or $(1/2\pi i) \mathrm{d}\rho \cdot \alpha > 0$.

Hence the convex function $p_v : \mathfrak{K} \to \mathbb{R}$ is bounded below and achieves a minimum. □

LEMMA 4.5 *Let $v \in V$ and $\lambda : \mathbb{C}^* \to G$ be a group homomorphism such that $\lambda(S^1) \subset K$ and let $V = \bigoplus_{r \in \mathbb{Z}} V_r$ be the weight decomposition associated to the action of $\lambda(\mathbb{C}^*)$ on V; then*

$$\mu(v) \cdot \alpha = \sum_{r \in \mathbb{Z}} r H(v_r, v_r) - (\rho, \lambda),$$

where $v = \sum_r v_r$ is written with respect to the above weight decomposition and

$$\alpha = \mathrm{d}\lambda(2\pi i) = \frac{\mathrm{d}}{\mathrm{d}t}\lambda(\exp(2\pi i t))\Big|_{t=0} \in \mathfrak{K}.$$

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Furthermore,

(i) if $\lim_{t \to 0} \lambda(t) \cdot v$ exists in V, then $\mu(v) \cdot \alpha \geq -(\rho, \lambda)$ *with equality if and only if* λ *fixes* v;

(ii) if $\lim_{t \to 0} \lambda(t) \cdot (v, 1)$ exists in $L^{-1} = V \times \mathbb{C}$, then $\mu(v) \cdot \alpha \geq 0$ *with equality if and only if* λ *fixes* $(v, 1)$.

Proof. By definition $V_r := \{v \in V : \lambda(s) \cdot v = t^r v \text{ for all } t \in \mathbb{C}^*\}$, and so the infinitesimal action of $\alpha \in \mathfrak{K}$ on $v_r \in V_r$ is $\alpha_{v_r} = \alpha v_r = 2\pi i r v_r$. As λ is a 1-PS of K, the corresponding weight decomposition $V = \oplus V_r$ is orthogonal with respect to the Hermitian inner product H and so $H(v_r, v_s) = 0$ for $s \neq r$. Therefore

$$\mu(v) \cdot \alpha = \frac{1}{2\pi i}(H(\alpha v, v) - d\rho \cdot \alpha) = \frac{1}{2\pi i}\left(\sum_{r,s} H(2\pi i r v_r, v_s) - d\rho \cdot \alpha\right)$$

$$= \sum_r r H(v_r, v_r) - (\rho, \lambda),$$

where $2\pi i(\rho, \lambda) = d\rho \cdot \alpha$ as $\alpha = d\lambda(2\pi i)$. As $\lim_{t \to 0} \lambda(t) \cdot v$ exists, this implies $v_r = 0$ for all $r < 0$. Hence $\sum_r r H(v_r, v_r) \geq 0$ with equality if and only if $v = v_0$, which proves part (i). If $\lim_{t \to 0} \lambda(t) \cdot (v, 1)$ exists, then $v_r = 0$ for all $r < 0$ and $(\rho, \lambda) \leq 0$. Hence part (ii) follows and we have equality if and only if $v = v_0$ and $(\rho, \lambda) = 0$; that is, $(v, 1)$ is fixed by λ. $\qquad\square$

We just need to state one additional lemma before we can prove the affine version of the Kempf–Ness theorem.

LEMMA 4.6   *Let* $v \in V$ *be a critical point of* $\|\mu\|^2$ *where* $\| - \|$ *is the norm associated to a* K-*invariant inner product on* $\mathfrak{K}$; *then* $G \cdot v$ *meets the critical locus in precisely the* K-*orbit of* v.

Proof. If $w = gv$ is also critical for $\|\mu\|^2$, then we want to show that $w \in K \cdot v$. Since $G = K \exp(i\mathfrak{K})$, it suffices to prove this when $g = \exp(i\alpha)$ for some $\alpha \in \mathfrak{K}$. In this case, both v and w belong to some critical subset $C_{K \cdot \beta}$ and so there exists $k \in K$ such that

$$\beta := \mu^*(v) = k \cdot \mu^*(w) = \mu^*(kw).$$

As $w \in K \cdot v$ if and only if $kw \in K \cdot v$, we can assume $\beta = \mu^*(v) = \mu^*(w)$. Then, following [**10**, Lemma 7.2], we consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = 2\pi \mu(\exp(it\alpha)v) \cdot \alpha$. Since

$$h(0) = 2\pi \mu(v) \cdot \alpha = 2\pi(\beta, \alpha) = 2\pi \mu(w) \cdot \alpha = h(1),$$

there exists $t \in (0, 1)$ such that $h'(t) = 0$. Then

$$0 = h'(t) = \frac{d}{dt} \frac{1}{i}(H(\alpha \exp(it\alpha)v, \exp(it\alpha)v) + d\rho \cdot \alpha)$$

$$= H(\alpha^2 \exp(it\alpha)v, \exp(it\alpha)v) - H(\alpha \exp(it\alpha)v, \alpha \exp(it\alpha)v)$$

$$\stackrel{(11)}{=} -H(\alpha \exp(it\alpha)v, \alpha \exp(it\alpha)v).$$

Hence $\alpha \exp(it\alpha)v = 0$ and so α is contained in $\mathfrak{K}_{\exp(it\alpha)v}$. Therefore, $\alpha \in \mathfrak{K}_v$ and $i\alpha \in \mathfrak{g}_v$, so that $w = \exp(i\alpha)v = v \in K \cdot v$, as required. $\qquad\square$

We now give the proof of Theorem 4.2 (Affine Kempf–Ness theorem):

Proof. (i) follows immediately from Lemma 4.3 and Proposition 4.4 and (ii) follows from Lemma 4.6. For (iii), if $G \cdot v$ is semistable, then there is a unique closed orbit $G \cdot w$ in $\overline{G \cdot v}$ which is polystable and so, by (i), $G \cdot w$ meets $\mu^{-1}(0)$. Conversely, if $w \in \mu^{-1}(0) \cap \overline{G \cdot v}$, then w is polystable by (i). It then follows from the openness of $V^{\rho\text{-ss}}$ in V that v is also semistable.

For (iv), if $v \in S_0$, then the limit of its negative gradient flow $\lim_{t\to\infty} \gamma_v(t)$ is contained in $C_0 \cap \overline{G \cdot v}$ and as $C_0 = \mu^{-1}(0)$ by (iii) we conclude that v is semistable. Conversely, if $G \cdot v$ is semistable, then there is a polystable orbit $G \cdot w \subset \overline{G \cdot v}$ and by (i) we have that $G \cdot w$ meets $\mu^{-1}(0)$. Therefore, $w \in S_0$ and as S_0 is open in V, also $v \in S_0$.

For (v), it is clear from parts (i) and (ii) that we have a bijection of sets

$$\mu^{-1}(0)/K \cong V^{\rho\text{-ps}}/G \cong V/\!\!/_\rho G.$$

The inclusion $\mu^{-1}(0) \subset V^{\rho\text{-ss}}$ induces a continuous bijection

$$\mu^{-1}(0)/K \to V/\!\!/_\rho G$$

and the inverse of this map is constructed by using the gradient flow $V^{\rho\text{-ss}} = S_0 \to C_0 = \mu^{-1}(0)$ associated to $\|\mu\|^2$ with respect to a K-invariant norm on $\mathfrak{K}$. $\square$

4.2. *A comparison of the Hesselink and Kirwan stratifications*

We now fix an inner product $(-, -)$ on $\mathfrak{K}$ which is invariant under the adjoint action of K on its Lie algebra $\mathfrak{K}$. We use $\|-\|$ to denote the associated K-invariant norm on $\mathfrak{K}$ and also the norm on the set of conjugacy classes of 1-PSs. We assume that $\|-\|^2$ takes integral values on integral weights in $\mathfrak{K}$ (that is, if $\lambda : S^1 \to K$ is a group homomorphism, then $\|d\lambda(2\pi i)\|^2 \in \mathbb{Z}$, as required by Assumption 2.9). Our main aim is to compare Hesselink's stratification of V by ρ-adapted 1-PSs of $G = K_{\mathbb{C}}$

$$V = \bigsqcup_{[\lambda]} S_{[\lambda]},$$

which is indexed by conjugacy classes of homomorphisms $\lambda : \mathbb{C}^* \to G = K_{\mathbb{C}}$, with the Morse stratification of V associated to $\|\mu\|^2$

$$V = \bigsqcup_{K \cdot \beta} S_{K \cdot \beta},$$

which is indexed by adjoint orbits in $\mathfrak{K}$. If ρ is the trivial character, then Hesselink's stratification of V and the Morse stratification both consist of a single stratum indexed by the trivial homomorphism $\lambda : \mathbb{C}^* \to G$ and $0 \in \mathfrak{K}$, respectively. Therefore, we assume that ρ is a non-trivial character. In this section, we show that the Hesselink strata agree with the Kirwan strata $GY_\beta^{\min}$ defined in Section 3.4; then the agreement of the Hesselink and Morse stratifications will follow from this result and the alternative description of the Morse strata as the Kirwan strata (see Theorem 4.17).

For $v \in V$ and a 1-PS λ of G such that $\lim_{t\to 0} \lambda(t) \cdot v$ exists, in Section 2 we defined the Hilbert–Mumford function

$$\mu(v, \lambda) := (\rho, \lambda),$$

where we have now omitted the superscript ρ as this character is fixed. We recall that $v \in V$ is strongly stable if there are no 1-PSs of G for which $\lim_{t \to 0} \lambda(t) \cdot v$ exists (cf. Remark 2.8). For every $v \in V$ that is not strongly stable, we define

$$M(v) := \inf_{\lambda} \frac{\mu(v, \lambda)}{\|\lambda\|},$$

where the infimum is taken over all 1-PSs λ of G for which $\lim_{t \to 0} \lambda(t) \cdot v$ exists. For strongly stable points v, we define $M(v) := 0$.

Given a 1-PS λ of G such that $\lambda(S^1) \subset K$, we let

$$\alpha = \alpha(\lambda) = d\lambda(2\pi i) := \left. \frac{d}{dt} \lambda(\exp(2\pi i t)) \right|_{t=0} \in \mathfrak{K}$$

denote the associated integral weight. Conversely, given $\alpha \in \mathfrak{K}$, we define an associated real 1-PS subgroup $\exp(\alpha \mathbb{R})$ of K. This correspondence defines an isomorphism between real 1-PSs of K and $\mathfrak{K}$. We recall that α is said to be integral if the map $\lambda : S^1 \to K$ defined by $\lambda(\exp(2\pi i \mathbb{R})) := \exp(\alpha \mathbb{R})$ is a group homomorphism. For any α, we can find a positive real number c such that $c\alpha$ is integral and then we shall refer to the 1-PS λ associated to $c\alpha$ as a 1-PS associated to α.

LEMMA 4.7   *Let λ be a 1-PS of G such that $\lambda(S^1) \subset K$ and let $\alpha \in \mathfrak{K}$ be the associated integral weight. Then, for $v \in V$, we have the following.*

  (i) *If $\lim_{t \to 0} \lambda(t) \cdot v$ exists, then $\mu_\alpha(v) \geq -(\rho, \lambda)$ with equality if and only if $\lambda(t) \subset G_v$. In this case*

$$\frac{\mu(v, \lambda)}{\|\lambda\|} \geq -\frac{\mu_\alpha(v)}{\|\alpha\|} \geq -\|\mu(v)\|.$$

  (ii) *Moreover, $M(v) \geq -\|\mu(v)\|$ with equality if and only if v is a critical point of μ_β where $\beta = \mu^*(v)$.*

Proof. The first statement of part (i) follows immediately from Lemma 4.5(i). The second statement of part (i) follows from the first and the fact that

$$\frac{\mu_\alpha(v)}{\|\alpha\|} = \left(\mu^*(v), \frac{\alpha}{\|\alpha\|} \right) \leq \|\mu(v)\|$$

with equality if and only if $\mu^*(v)$ is a positive scalar multiple of α.

For (ii), if v is strongly stable, then $M(v) = 0 \geq -\|\mu(v)\|$ with equality if and only if $\mu(v) = 0$. If $\mu(v) = 0$, then v is critical for $\mu_0 = 0$ where $0 = \mu^*(v)$. If $\beta = \mu^*(v) \neq 0$ and the strongly stable point v is critical for μ_β, then this would contradict Theorem 4.2(iv): $V^{ss} = S_0$.

If v is not strongly stable, then the inequality in (ii) follows from (i). If v is a critical point of μ_β where $\beta = \mu^*(v)$, then, as β is rational (cf. Section 3.3), we know that $n\beta$ is integral for some

positive integer n. The 1-PS λ associated to $n\beta$ fixes v (by definition of v being a critical point of μ_β). Hence

$$M(v) \le \frac{(\rho, \lambda)}{\|\lambda\|} = -\frac{\mu_{n\beta}(v)}{\|n\beta\|} = -\left(\beta, \frac{\beta}{\|\beta\|}\right) = -\|\beta\| = -\|\mu(v)\|$$

and so $M(v) = -\|\mu(v)\|$. Conversely, if $M(v) = -\|\mu(v)\|$, then by (i) there is a 1-PS λ (corresponding to $\alpha \in \mathfrak{K}$) that fixes v and

$$\mu_\alpha(v) = -(\rho, \lambda) \text{ and } \|\mu(v)\| = \frac{\mu_\alpha(v)}{\|\alpha\|} = \left(\mu^*(v), \frac{\alpha}{\|\alpha\|}\right).$$

Hence $\beta := \mu^*(v)$ is a positive scalar multiple of α and $\alpha_v = 0$ as λ fixes v. Therefore, $\beta_v = 0$ and so v is critical for μ_β where $\beta = \mu^*(v)$. $\qquad\square$

COROLLARY 4.8 *Let $v \in V$ and $\beta = \mu^*(v) \neq 0$. If $\beta_v = 0$, then v is unstable. Moreover, if λ is a 1-PS associated to β, then λ is adapted to v.*

Proof. Let $\lambda : S^1 \to K$ be the 1-PS associated to $c\beta$, where c is a positive number such that $c\beta$ is integral; we note that this is non-trivial as $\beta \neq 0$. The 1-PS λ fixes v and so, by Lemma 4.7(i), we have that

$$0 < c\|\beta\|^2 = \mu_{c\beta}(v) = -(\rho, \lambda) = -\mu(v, \lambda).$$

Hence v is unstable by Proposition 2.5. Moreover,

$$M(v) \le \frac{\mu(v, \lambda)}{\|\lambda\|} = -\frac{\mu_{c\beta}(v)}{\|c\beta\|} = -\|\beta\| = -\|\mu(v)\|$$

and so, by Lemma 4.7(ii), $M(v) = -\|\mu(v)\|$ and λ is adapted to v. $\qquad\square$

We now have a nice description of the non-minimal critical points of $\|\mu\|^2$ (for the projective version, see [**14**, Theorem 6.1]):

THEOREM 4.9 *Let $v \in V$ and $\beta = \mu^*(v)$. Then the following are equivalent*:

(i) *v is a non-minimal critical point of $\|\mu\|^2$;*
(ii) *$\beta \neq 0$ and $\beta_v = 0$;*
(iii) *$-M(v) = \|\beta\| > 0$.*

Proof. The equivalence of (i) and (ii) is given in Lemma 3.5 and the equivalence between (ii) and (iii) is given by Lemma 4.7(ii). $\qquad\square$

Let $\beta \neq 0$ be an element of $\mathfrak{K}$, such that the adjoint orbit $K \cdot \beta \in \mathcal{B}$ indexes a Morse stratum $S_{K \cdot \beta}$. As β is a rational weight, we let n be the smallest positive integer such that $n\beta$ is integral and we let $\lambda := \lambda_\beta$ be the indivisible 1-PS (considered as a 1-PS $\lambda : S^1 \to K$ or $\lambda : \mathbb{C}^* \to G$) associated to this integral weight.

In Section 3.4, we considered the action of the stabilizer subgroup K_β of β under the adjoint action of K on the critical locus $V^\beta := \operatorname{crit} \mu_\beta$, for which there is a moment map $\mu - \beta^* : V^\beta \to \mathfrak{K}_\beta^*$. We

defined $Z_\beta^{\min} \subset V^\beta$ to be the minimal Morse stratum for $\|\mu - \beta^*\|^2$ whose corresponding critical subset is $C_\beta := V^\beta \cap \mu^{-1}(\beta^*)$ by Lemma 3.9.

In Section 2.3, we considered the action of the subgroup G_λ consisting of elements of G that commute with λ on the fixed point locus V^λ and used a character ρ_λ of G_λ defined at (8) to linearize this action. The semistable set for this action (with respect to ρ_λ) is equal to the limit set Z_λ defined at (6) of the blade S_λ of the Hesselink stratum $S_{[\lambda]}$ associated to this 1-PS λ by Proposition 2.18.

For $\lambda = \lambda_\beta$, we note that $V^\lambda = V^\beta$ and G_λ is the complexification of K_β; thus, from now on we shall write $G_\beta := G_\lambda$ for $\lambda = \lambda_\beta$.

THEOREM 4.10   *Let $\beta \neq 0$ be a point in $\mathfrak{K}$ that indexes a non-empty Morse stratum and let $\lambda = \lambda_\beta$ be the associated 1-PS as above. Then $Z_\lambda = Z_\beta^{\min}$ and the set of closed G_β-orbits in Z_λ is equal to $G_\beta C_\beta$.*

Proof. We recall that λ is the 1-PS associated to the integral weight $n\beta$ for some $n > 0$ and so $n\|\beta\| = \|\lambda\|$. If $v \in C_\beta$, then by Corollary 4.8 the 1-PS λ is adapted to v and by Theorem 4.9 $-M(v) = \|\beta\|$. Hence,

$$-M(v) = -\frac{(\rho, \lambda)}{\|\lambda\|} = \|\beta\| = \frac{\|\lambda\|}{n}. \tag{17}$$

As moment maps for the action of K_β on V^β, we claim that $\mu - \beta^*$ is equal to the natural moment map for this action shifted by $\rho_\lambda/\|\lambda\|^2$ which we denote by μ'. For $v \in V^\beta$ and $\alpha \in \mathfrak{K}_\beta$,

$$(\mu - \beta^*)(v) \cdot \alpha := \mu(v) \cdot \alpha - \beta^* \cdot \alpha = \frac{1}{2\pi i}(H(\alpha v, v) - d\rho \cdot \alpha) - \beta^* \cdot \alpha$$

and

$$\mu'(v) \cdot \alpha := \frac{1}{2\pi i}\left(H(\alpha v, v) - \frac{1}{\|\lambda\|^2}d\rho_\lambda \cdot \alpha\right) = \frac{1}{2\pi i}\left(H(\alpha v, v) - d\rho \cdot \alpha + \frac{(\rho, \lambda)}{\|\lambda\|^2}d\lambda^* \cdot \alpha\right),$$

where $\lambda^* : K \to S^1$ is the character dual to $\lambda : S^1 \to K$ under our fixed inner product (we note that, as λ is the 1-PS associated to the rational weight $n\beta$, we have that $d\lambda^* = 2n\pi i\beta^*$). It follows from the relation given at (17) that

$$\frac{1}{2\pi i}\frac{(\rho, \lambda)}{\|\lambda\|^2}d\lambda^* = -\frac{1}{2n\pi i}d\lambda^* = -\beta^*,$$

which proves the claim and so $\mu' = \mu - \beta^*$ as moment maps for the K_β-action on V^β.

For any character ρ and $m > 0$ we have that the GIT (semi)stable sets for ρ and $m\rho$ agree:

$$V^{\rho\text{-(s)s}} = V^{m\rho\text{-(s)s}}$$

(the easiest way to see this is to use the description of (semi)stability coming from the Hilbert–Mumford criterion given in Proposition 2.5). We note that, although ρ_λ is an honest character of G_λ, the element $\rho_\lambda/\|\lambda\|^2$ is only a rational character. However, for the purposes of GIT, we can still use this rational character to linearize the action and from the observation above we note that

$$(V^\lambda)^{\rho_\lambda/\|\lambda\|^2\text{-ss}} = (V^\lambda)^{\rho_\lambda\text{-ss}} = Z_\lambda, \tag{18}$$

where the second equality comes from Proposition 2.18. Then, by part (iv) of the affine Kempf–Ness theorem applied to the moment map μ' for the action of K_β on V^β, we have that the GIT semistable set of (18) is the lowest Morse stratum $Z_\beta^{\min}$ for the norm square of the moment map $\mu' = \mu - \beta^* : V_\beta \to \mathfrak{K}_\beta^*$. The final statement is part (i) of the affine Kempf–Ness theorem. $\square$

COROLLARY 4.11 *Let $\beta \neq 0$ be a point in $\mathfrak{K}$ which indexes a non-empty Morse stratum and let $\lambda = \lambda_\beta$ be the associated 1-PS as above; then*

$$S_\lambda = Y_\beta^{\min}.$$

Proof. This follows immediately from the result above, given the fact that $p_\lambda : V_+^\lambda \to V^\lambda$ is equal to $p_\beta : V_+^\beta \to V^\beta$ and $S_\lambda = p_\lambda^{-1}(Z_\lambda)$ and $Y_\beta^{\min} = p_\beta^{-1}(Z_\beta^{\min})$. $\square$

We have seen how to associate to $\beta \in \mathfrak{K}$ (which indexes a Morse stratum $S_{K \cdot \beta}$) a 1-PS λ_β that indexes a Hesselink stratum $S_{[\lambda_\beta]}$ and Theorem 4.10 shows that the associated Hesselink stratum is non-empty. We now describe how to associate to a 1-PS λ indexing an unstable Hesselink stratum $S_{[\lambda]}$ a rational weight $\beta = \beta(\lambda)$ which indexes a Morse stratum. First, we can assume (by conjugating λ if necessary) that $\lambda(S^1) \subset K$ and, from above, we know that β should be a positive scalar multiple of $\mathrm{d}\lambda(2\pi\mathrm{i}) \in \mathfrak{K}$. If we write $\beta = c\,\mathrm{d}\lambda(2\pi\mathrm{i})$, we have

$$c^2\|\lambda\|^2 = \|\beta\|^2 = -\frac{1}{2\pi\mathrm{i}}\,\mathrm{d}\rho \cdot \beta = -c(\rho, \lambda) \tag{19}$$

by Remark 3.10 and so

$$\beta := -\frac{(\rho, \lambda)}{\|\lambda\|^2}\,\mathrm{d}\lambda(2\pi\mathrm{i}) \in \mathfrak{K}$$

is the desired element which satisfies (19). Moreover, it follows that the associated Morse stratum $S_{K \cdot \beta}$ is non-empty by Theorem 4.10.

Let $G = K_{\mathbb{C}}$ be a complex reductive group acting linearly on a complex vector space V for which K acts unitarily with respect to a fixed Hermitian inner product on V. We recall that we use a character ρ to linearize the action and also to shift the natural moment map associated to this action.

THEOREM 4.12 *Let β be the rational weight associated to the 1-PS λ_β; then*

$$S_{[\lambda_\beta]} = GY_\beta^{\min}.$$

Hence the Hesselink and Kirwan stratifications agree.

Proof. We have seen above how to associate to β a 1-PS λ_β and conversely how to associate to a 1-PS λ a rational weight $\beta(\lambda)$. The theorem follows from Corollary 4.11 and the fact that $S_{[\lambda_\beta]} = GS_{\lambda_\beta}$. $\square$

4.3. *The proof of the alternate description of the Morse strata*

In this section, we prove the alternative description of the Morse strata $S_{K \cdot \beta}$, that is, that they coincide with the Kirwan strata $GY_\beta^{\min}$ defined in Section 3.4. Our proof follows the same lines of that of the projective setting of [**10**], where Kirwan proved that there is a neighbourhood U_β of each critical

subset $C_{K \cdot \beta}$ on which the Morse strata $S_{K \cdot \beta}$ coincided with a 'minimizing submanifold' Σ_β which is an open submanifold of the Kirwan stratum $GY_\beta^{\min}$. For our purposes, it suffices to show the existence of a neighbourhood U_β of $C_{K \cdot \beta}$ in V such that

$$U_\beta \cap S_{K \cdot \beta} \subset GY_\beta^{\min}$$

(see Proposition 4.16), from which the equivalence of the Morse and Kirwan stratifications will easily follow. To find these neighbourhoods, we use manifolds Σ_β analogous to the minimizing submanifolds considered by Kirwan.

We start with a useful proposition to describe the Kirwan strata. We fix a positive Weyl chamber $\mathfrak{t}_+$ and let β denote the unique point of the adjoint orbit $K \cdot \beta$ which meets $\mathfrak{t}_+$.

PROPOSITION 4.13   *For $v \in GV_+^\beta$, we have $\|\mu(v)\| \geq \|\beta\|$. Moreover, if $v \in GY_\beta^{\min}$, then β is the closest point to the origin of $\mu^*(\overline{G \cdot v}) \cap \mathfrak{t}_+$.*

Proof.   For the first statement, we can assume $v \in V_+^\beta$ as $\|\mu\|^2$ is K-invariant and $GV_+^\beta = KV_+^\beta$. It follows from Lemma 4.5 that

$$\mu(v) \cdot \beta \geq -\frac{1}{2\pi i} \, d\rho \cdot \beta,$$

where the right-hand side is equal to $\|\beta\|^2$ by Remark 3.10. Therefore, $\|\mu(v)\| \geq \|\beta\|$. For the second statement, we can assume $v \in Y_\beta^{\min}$ and, as $\|\mu(v)\| \geq \|\beta\|$, it suffices to show that $\beta \in \mu^*(\overline{G \cdot v})$. Since $p_\beta(v) \in Z_\beta^{\min}$ is contained in the orbit closure $\overline{G \cdot v}$, it suffices to show that $\beta \in \mu^*(\overline{G \cdot v})$ for $v \in Z_\beta^{\min}$. As $Z_\beta^{\min}$ is the minimal Morse stratum for $\|\mu - \beta^*\|^2$ with critical subset C_β and the flow is also contained in G-orbit closures, it suffices to show that $\beta \in \mu^*(\overline{G \cdot v})$ for points $v \in C_\beta$. Since μ^* takes the value β on C_β, this completes the proof. $\square$

The above proposition is an affine version of [**10**, Corollaries 6.11 and 6.12], which gives a description of the Kirwan strata $GY_\beta^{\min}$ as the set of points $v \in V$ for which β is the closest point to the origin of $\mu^*(\overline{G \cdot v}) \cap \mathfrak{t}_+$.

The existence of the manifolds Σ_β mentioned above (which are minimizing manifolds as considered by Kirwan [**10**, Definition 10.1]) follows in exactly the same way as that of [**10**, Section 4]. We give the outline of the proof of the existence of Σ_β as well as certain properties of these manifolds in Proposition 4.14. First, we need one additional piece of notation: for $v \in V$, we let $H_v := H_v(\|\mu\|^2)$ denote the Hessian which is a symmetric bilinear form on $T_v V$, defined as the matrix of second-order derivatives of $\|\mu\|^2$. We view H_v as a self-adjoint linear endomorphism of $T_v V$ via the Kähler metric on V.

PROPOSITION 4.14   *There is a locally closed manifold Σ_β of V such that*:

   (i) *Σ_β is a K-invariant open neighbourhood of $C_{K \cdot \beta}$ in GV_+^β;*
   (ii) *on Σ_β, the norm square of the moment map $\|\mu\|^2$ takes its minimum value on $C_{K \cdot \beta}$;*
   (iii) *for $v \in C_{K \cdot \beta}$, the Hessian H_v is positive semidefinite when restricted to $T_v \Sigma_\beta$ and negative definite on its orthogonal complement.*

Proof. Let $\sigma : K \times V \to V$ denote the action and consider the induced map

$$\tilde{\sigma} : K \times_{K_\beta} V_+^\beta \to V$$

whose image is $K V_+^\beta$ where $K_\beta \subset K$ is the stabilizer subgroup of β under the adjoint action. Let $v \in C_\beta$ and let $[1, v] \in K \times_{K_\beta} V_+^\beta$ denote the class of $(1, v) \in K \times V_+^\beta$. For $v \in C_\beta$, we claim that $d_{[1,v]}\tilde{\sigma}$ is injective. We have a short exact sequence

$$0 \longrightarrow \mathfrak{K}_\beta \longrightarrow \mathfrak{K} \times T_v V_+^\beta \longrightarrow T_{[1,v]}(K \times_{K_\beta} V_+^\beta) \longrightarrow 0,$$

where $\mathfrak{K}_\beta := \text{Lie } K_\beta$. Following the argument of [**10**, Lemma 4.10], we see that

$$\mathfrak{K}_\beta = \{A \in \mathfrak{K} : A_v \in T_v V_+^\beta\}.$$

Then, for $(A, \xi) \in \mathfrak{K} \times T_v V_+^\beta$, we have $d_{[1,v]}\tilde{\sigma}([A, \xi]) = 0$ if and only if $0 = d_{(1,v)}\sigma(A, \xi) := A_v + \xi$. In this case, as $A_v = -\xi \in T_v V_+^\beta$, we see that $[A, \xi] = 0 \in T_{[1,v]}(K \times_{K_\beta} V_+^\beta)$. Hence, this proves our claim and, moreover, $d\tilde{\sigma}$ is injective in a neighbourhood U_v of $[1, v]$. Then $W_v := \tilde{\sigma}(U_v)$ is a neighbourhood of v in $K V_+^\beta$ (cf. [**10**, Corollary 4.11]) and so, by the inverse function theorem, $K V_+^\beta$ is smooth in the neighbourhood W_v of v for each $v \in C_{K \cdot \beta}$. Hence $K V_+^\beta$ is smooth in the K-invariant neighbourhood $\Sigma_\beta := K(\cup_{v \in C_\beta} W_v)$ of $C_{K \cdot \beta} = K C_\beta$, which proves (i).

By definition, the set V_+^β consists of the points of V whose negative gradient flow under μ_β converges to V^β; therefore $\mu_\beta(v) := \mu(v) \cdot \beta \geq \|\beta\|^2 = \mu_\beta(V^\beta)$. Hence, $\|\mu(v)\|^2 \geq \|\beta\|^2$ for $v \in V_+^\beta$ (and also for $v \in K V_+^\beta = G V_+^\beta$ by K-invariance of $\|\mu\|^2$) with equality if and only if $v \in C_{K \cdot \beta}$. As $\Sigma_\beta \subset G V_+^\beta$, this proves (ii) and that H_v is positive semidefinite on $T_v \Sigma_\beta$.

The restriction of the symplectic form ω to $T_v G V_+^\beta$ is non-degenerate for $v \in C_\beta$ (and hence also, by K-invariance of ω, for $v \in C_{K \cdot \beta}$) by applying the argument of [**10**, Lemma 4.13]. Thus, we can write $T_v V = T_v \Sigma_\beta \oplus (T_v \Sigma)^\perp$ for $v \in C_\beta$. As V_+^β is the Morse stratum associated to the critical set V^β for the function μ_β, the Hessian $H_v(\mu_\beta)$ restricted to $(T_v V_+^\beta)^\perp$ is negative definite. Then, for $v \in C_\beta$, exactly as in the proof of [**10**, Proposition 4.15], one can check that on $(T_v \Sigma_\beta)^\perp$, the Hessian $H_v(\|\mu\|^2)$ is a scalar multiple of $H_v(\mu_\beta)$ which, after making use of the action of K, completes the proof of (iii). $\qquad\square$

Analogously to the above, Kirwan goes on to show in [**10**, Section 6] that $G Y_\beta^{\min}$ is smooth and diffeomorphic to $G \times_{P_\beta} Y_\beta^{\min}$ and, moreover, that we can take a minimizing manifold Σ_β as an open subset of $G Y_\beta^{\min}$. However, the description of Σ_β given in Proposition 4.14 is sufficient for our purposes. The next lemma is obtained by modifying [**10**, Corollary 10.15] to our setting.

LEMMA 4.15 *There is a neighbourhood U_β of $C_{K \cdot \beta}$ in V such that*

$$U_\beta \cap S_{K \cdot \beta} \subset \Sigma_\beta.$$

Proof. The properties of Σ_β listed above and the calculations of [**10**] given at 10.13 and 10.14 show there is a neighbourhood W_β of $C_{K \cdot \beta}$ and a constant $C > 1$ such that, whenever $\gamma_v(t) \in W_\beta$ for $t \in [0, 1]$, we have

$$\text{dist}(\gamma_v(1), \Sigma_\beta) \geq C \, \text{dist}(\gamma_v(0), \Sigma_\beta). \tag{20}$$

On the Morse stratum $S_{K \cdot \beta}$ and its closure, the function $\|\mu\|^2$ takes its minimum value precisely on the critical set $C_{K \cdot \beta}$. Hence, for $v \in \partial W_\beta \cap \overline{S_{K \cdot \beta}}$, we have that $\|\mu(v)\|^2 \geq \|\beta\|^2 + \delta$ for $\delta > 0$. Let

$$U_\beta := W_\beta \cap \{v : \|\mu(v)\|^2 < \|\beta\|^2 + \delta\};$$

then we claim that if $v \in U_\beta \cap S_{K \cdot \beta}$, then $\gamma_v(t) \in W_\beta$ for all $t \geq 0$. To prove the claim, suppose $\gamma_v(t_0) \in \partial W_\beta$; then

$$\|\mu(v)\|^2 \geq \|\mu(\gamma_v(t_0)\|^2 \geq \|\beta\|^2 + \delta,$$

where the second inequality is due to the fact that $\gamma_v(t_0) \in \partial W_\beta \cap S_{K \cdot \beta}$. This cannot be the case as this inequality contradicts $v \in U_\beta$. Hence, if $v \in U_\beta \cap S_{K \cdot \beta}$, then $\gamma_v(t) \in W_\beta$ for all $t \geq 0$. In particular, we can recursively apply the inequality (20) to conclude, for $v \in U_\beta \cap S_{K \cdot \beta}$, that

$$\text{dist}(\gamma_v(n), \Sigma_\beta) \geq C^n \, \text{dist}(\gamma_v(0), \Sigma_\beta)$$

for all $n \in \mathbb{N}$. However, as $\gamma_v(t)$ converges to $C_{K \cdot \beta} \subset \Sigma_\beta$, we see that we must have $\text{dist}(\gamma_v(0), \Sigma_\beta) = 0$; that is, $v = \gamma_v(0) \in \Sigma_\beta$. $\square$

PROPOSITION 4.16   *There is a neighbourhood U_β of $C_{K \cdot \beta}$ in V such that*

$$U_\beta \cap S_{K \cdot \beta} \subset GY_\beta^{\min}.$$

Proof. We let U_β be the neighbourhood of $C_{K \cdot \beta}$ given by Lemma 4.15; then

$$U_\beta \cap S_{K \cdot \beta} \subset \Sigma_\beta \subset GV_+^\beta.$$

We will prove that the intersection $U_\beta \cap S_{K \cdot \beta}$ is contained in the open subset $GY_\beta^{\min} \subset GV_+^\beta$, by showing that $S_{K \cdot \beta} \cap GV_+^\beta \subset GY_\beta^{\min}$.

First, we claim that

$$S_{K \cdot \beta} \subset GY_\beta^{\min} \cup \left(\bigcup_{\beta' < \beta} GY_{\beta'}^{\min} \right).$$

To prove the claim, we suppose that there exists $v \in S_{K \cdot \beta}$ such that $v \in GY_{\beta'}^{\min}$ for some $\beta' \neq \beta$. Let v_∞ denote the limit point of the negative gradient flow of v under $\|\mu\|^2$; then, by definition of the Morse strata, we have that $v_\infty \in C_{K \cdot \beta}$. As $C_\beta \subset Z_\beta^{\min}$ by Theorem 4.10, we have an inclusion $C_{K \cdot \beta} = KC_\beta \subset GY_\beta^{\min}$. Furthermore, the negative gradient flow under $\|\mu\|^2$ is contained in G-orbits so that v_∞ lies in the orbit closure of v and

$$v_\infty \in GY_\beta^{\min} \cap \overline{GY_{\beta'}^{\min}}.$$

By Theorem 4.12, the Kirwan strata $GY_\beta^{\min}$ equal the Hesselink strata $S_{[\lambda_\beta]}$ and the closures of the Hesselink strata are described by Theorem 2.16; in particular, we have that $\beta' < \beta$, which completes the proof of the claim.

It follows immediately from the claim that

$$S_{K \cdot \beta} \cap GV_+^{\beta} \subset GY_{\beta}^{\min} \cup \left(\bigcup_{\beta' < \beta} GY_{\beta'}^{\min} \cap GV_+^{\beta} \right).$$

However, if $v \in GY_{\beta'}^{\min} \cap GV_+^{\beta}$ and $\beta' < \beta$, then, by the second statement of Proposition 4.13, there exists $v' \in \overline{G \cdot v}$ such that $\mu^*(v') = \beta'$ and so $\|\mu(v')\| = \|\beta'\| < \|\beta\|$, which contradicts the first statement of Proposition 4.13 as $v' \in \overline{G \cdot v} \subset GV_+^{\beta}$. Hence $S_{K \cdot \beta} \cap GV_+^{\beta} \subset GY_{\beta}^{\min}$, which completes the proof of the proposition. □

THEOREM 4.17 *The Morse strata coincide with the Kirwan strata*:

$$S_{K \cdot \beta} = GY_{\beta}^{\min}.$$

Proof. As we can write V as a disjoint union of the Morse strata $S_{K \cdot \beta}$ and also as a disjoint union of the Kirwan strata $GY_{\beta}^{\min}$, it suffices to give an inclusion of each Morse stratum $S_{K \cdot \beta}$ in the corresponding Kirwan stratum $GY_{\beta}^{\min}$. For $v \in S_{K \cdot \beta}$, we let $\gamma_v(t)$ denote the path of steepest descent of v under $\|\mu\|^2$; by definition of the Morse stratification $\gamma_v(t)$ converges to a point of the critical subset $C_{K \cdot \beta}$. Therefore, the path $\gamma_v(t)$ eventually meets the neighbourhood U_{β} of $C_{K \cdot \beta}$ described in Proposition 4.16, and so there is some $t_0 \geq 0$ such that $\gamma_v(t_0) \in U_{\beta}$. As the path $\gamma_v(t)$ is contained in the Morse stratum $S_{K \cdot \beta}$, we see that

$$\gamma_v(t_0) \in U_{\beta} \cap S_{K \cdot \beta} \subset GY_{\beta}^{\min}$$

by Proposition 4.16. Since the finite time negative gradient flow of v lies in the G-orbit of v and the Kirwan strata are G-invariant, we conclude that $v \in GY_{\beta}^{\min}$. Hence $S_{K \cdot \beta} \subset GY_{\beta}^{\min}$, which completes the proof that the Morse strata and Kirwan strata coincide. □

In particular, we obtain two descriptions of the Morse stratum $S_{K \cdot \beta} = GY_{\beta}^{\min}$: either as the set of points whose path of steepest descent converges to a point of $C_{K \cdot \beta}$ under $\|\mu\|^2$ or as the set of points $v \in V$ for which β is the closest point to the origin of $\mu^*(\overline{G \cdot v}) \cap \mathfrak{t}_+$.

4.4. *Main theorem*

Let $G = K_{\mathbb{C}}$ be a complex reductive group acting linearly on a complex vector space V such that K acts unitarily with respect to a fixed Hermitian inner product on V. We pick a K-invariant inner product on the Lie algebra $\mathfrak{K}$ and choose a character ρ which is used both to linearize the action and to shift the natural moment map associated to this action. Associated to the linearized action, we obtain an algebraic stratification of the complex affine space V due to ideas of Hesselink, which we described in Section 2.3. The open stratum is the GIT semistable set (with respect to ρ) and the higher strata correspond to finitely many different instability types that depend on our chosen inner product and are indexed by conjugacy classes $[\lambda]$ of adapted 1-PSs. We write the stratification of Hesselink as

$$V = \bigsqcup_{[\lambda]} S_{[\lambda]}.$$

Using the ρ-shifted moment map of $\mu = \mu^{\rho} : V \to \mathfrak{K}^*$ with the norm associated to our chosen inner product on $\mathfrak{K}$, we can consider the norm square of this moment map $\|\mu\|^2 : X \to \mathbb{R}$ as in

Section 3.1. By taking the flow of points under the negative gradient flow associated to $\|\mu\|^2$, we obtain a Morse-type stratification of V (see Section 3.2 for details)

$$V = \bigsqcup_{K \cdot \beta} S_{K \cdot \beta}$$

indexed by coadjoint orbits $K \cdot \beta$ for which the critical locus of $\|\mu\|^2$ meets $(\mu^*)^{-1}(K \cdot \beta)$.

THEOREM 4.18   *For any character ρ of G and any K-invariant inner product on the Lie algebra $\mathfrak{K}$, the associated Morse stratification and Hesselink stratification coincide*; *that is*,

$$S_{K \cdot \beta} = S_{[\lambda_\beta]},$$

where λ_β is the 1-PS associated to β (or β is the rational weight associated to the 1-PS λ_β).

Proof. The proof follows immediately from Theorems 4.12 and 4.17. $\square$

5. Stratifications of spaces of quiver representations

As mentioned above, there is a construction due to King of the moduli space of 'semistable' representations of a quiver Q with fixed invariants d as a GIT quotient of a reductive group G acting on an affine space $\text{Rep}(Q, d)$ with respect to a character ρ [9]. The notion of (semi)stability is determined by a stability parameter θ which is also used to construct the character ρ.

Over the complex numbers, we know that the Morse stratification agrees with Hesselink's stratification for the action of $G = K_\mathbb{C}$ on the complex vector space $\text{Rep}(Q, d)$ linearized by ρ for any invariant inner product on $\mathfrak{K}$. In this section, we describe another stratification of $\text{Rep}(Q, d)$ by Harder–Narasimhan types τ, where the Harder–Narasimhan filtration of a quiver representation is defined by using both the stability parameter θ and the chosen inner product on $\mathfrak{K}$ (cf. Definition 5.2). We shall see that a Harder–Narasimhan stratum S_τ has a description as $S_\tau = GY_\tau^{ss}$ and there is a retraction $p_\tau : Y_\tau^{ss} \to Z_\tau^{ss}$, where Z_τ^{ss} can be explicitly described. Moreover, we shall see that the Harder–Narasimhan stratification, Morse stratification and Hesselink stratification all agree.

We briefly summarize here the known results on Harder–Narasimhan stratifications for quivers. Reineke describes a Harder–Narasimhan stratification for quivers and obtains formulae for the Betti numbers of the associated moduli spaces [15]. However, Reineke's notion of Harder–Narasimhan filtration does not depend on a choice of invariant inner product on $\mathfrak{K}$ (the definition given by Reineke corresponds to the Harder–Narasimhan filtration associated to the Killing form in our definition below). Harada and Wilkin [4] show for the Killing form that the Morse stratification agrees with the Harder–Narasimhan stratification. The group G is a product of general linear groups, and so one can construct families of invariant inner products on $\mathfrak{K}$ by using a weighted sum of the Killing forms on the unitary Lie algebras that make up $\mathfrak{K}$. This idea is used by Tur [17] in his thesis, where he shows that the Harder–Narasimhan stratification and Hesselink's stratification coincide for any inner product associated to a collection of weights. It follows from this result and Theorem 4.18 above that all three stratifications coincide; however, we provide a short proof of this result below for completeness.

5.1. *GIT construction of quiver moduli*

A ($\mathbb{C}$-)representation of a quiver $Q = (V, A, h, t)$ is a tuple $W = (W_v, \phi_e)$ consisting of a complex vector space W_v for each vertex v and a linear map $\phi_a : W_{t(a)} \to W_{h(a)}$ for each arrow a. A morphism of quiver representations $f : W \to W'$ is given by linear maps $f_v : W_v \to W'_v$ for each vertex v such that $f_{h(a)} \circ \phi_a = \phi'_a \circ f_{t(a)}$ for every arrow a. One forms the obvious notions of isomorphism, subrepresentation and quotient representation. The dimension of a quiver representation is dim $W = (\dim W_v) \in \mathbb{N}^V$.

For a dimension vector $d = (d_v) \in \mathbb{N}^V$, the complex affine space

$$\mathrm{Rep}(Q, d) = \bigoplus_{a \in A} \mathrm{Hom}(\mathbb{C}^{d_{t(a)}}, \mathbb{C}^{d_{h(a)}})$$

parametrizes representations W of Q of dimension d with a choice of isomorphism $W_v \cong \mathbb{C}^{d_v}$. The group

$$G(Q, d) = \prod_{v \in V} \mathrm{GL}(d_v)$$

acts naturally on $\mathrm{Rep}(Q, d)$ by conjugation: $(g \cdot \varphi)_a = g_{h(a)} \varphi_a g_{t(a)}^{-1}$ for $g = (g_v) \in G(Q, d)$ and $\varphi = (\varphi_a) \in \mathrm{Rep}(Q, d)$. We note that there is a copy of the multiplicative group $\mathbb{C}^*$ embedded in $G(Q, d)$ as $t \mapsto (t I_{d_v})$ which acts trivially on $\mathrm{Rep}(Q, d)$ and so if we want to have stable points, we must consider the action of $G(Q, d)/\mathbb{C}^*$ or modify the definition of stability so that stable orbits can have positive-dimensional stabilizers (the second approach is taken by King [9]). However, for the purposes of studying stratifications associated to this action, the presence of this copy of $\mathbb{C}^*$ does not matter and so we work with the action of $G = G(Q, d)$.

The action is linearized by choosing a character ρ of G. Following King [9], we let $\theta = (\theta_v) \in \mathbb{Z}^V$ denote a tuple of integers, such that $\sum_v \theta_v d_v = 0$, and associate to θ the character $\rho = \rho_\theta : G \to \mathbb{C}^*$ given by

$$\rho(g_v) = \prod_v \det(g_v)^{\theta_v}.$$

King uses the Hilbert–Mumford criterion to reinterpret the notion of ρ-semistability for points in $\mathrm{Rep}(Q, d)$ as a condition for the corresponding quiver representation.

DEFINITION 5.1   A representation W of Q of dimension d is θ-semistable if, for all proper subrepresentations $W' \subset W$, we have $\theta(W') := \sum_v \theta_v \dim W'_v \geq 0$.

The moduli space of θ-semistable quiver representations is constructed by King as the quotient of G acting on $\mathrm{Rep}(Q, d)$ with respect to the character ρ defined by θ.

5.2. *Harder–Narasimhan filtrations for quivers*

As $G = K_\mathbb{C} = G(Q, d)$ is a product of general linear groups $\mathrm{GL}(d_v)$, the Lie algebra $\mathfrak{K}$ is a sum of unitary Lie algebras $\mathfrak{u}(\mathbb{C}^{d_v})$. As every invariant inner product on $\mathfrak{u}(\mathbb{C}^{d_v})$ is a positive scalar multiple of the Killing form κ_v, the invariant inner products $(-, -)$ on $\mathfrak{K}$ are weighted sums $\sum \alpha_v \kappa_v$ for positive

α_v. Moreover, we shall assume that α_v are integral so that the norm square of an integral element in $\mathfrak{K}$ is integral (cf. Assumption 2.9). For $\alpha = (\alpha_v) \in \mathbb{N}_+^V$, we let $(-,-)_\alpha$ denote the associated inner product.

DEFINITION 5.2   For a representation W of Q (of any dimension), we say that W is θ-semistable if, for all proper subrepresentations,

$$\frac{\theta(W')}{\alpha(W')} \geq \frac{\theta(W)}{\alpha(W)},$$

where $\alpha(W) := \sum \alpha_v \dim W_v$. A Harder–Narasimhan filtration of W (with respect to α and θ) is a filtration $0 = W_{(0)} \subset W_{(1)} \subset \cdots \subset W_{(s)} = W$ by subrepresentations, such that the quotient representations $W_i := W_{(i)}/W_{(i-1)}$ are θ-semistable and

$$\frac{\theta(W_1)}{\alpha(W_1)} < \frac{\theta(W_2)}{\alpha(W_2)} < \cdots < \frac{\theta(W_s)}{\alpha(W_s)}.$$

The Harder–Narasimhan type of W (with respect to α and θ) is $\tau(W) := (\dim W_1, \ldots, \dim W_s)$.

The standard techniques are used to show the existence and uniqueness of the Harder–Narasimhan filtration. The trivial Harder–Narasimhan type for representations of dimension d is $\tau_0 = (d)$ and the representations with this type are θ-semistable. We have a decomposition

$$\mathrm{Rep}(Q, d) = \bigsqcup_\tau S_\tau,$$

where S_τ indexes the subset of representations with Harder–Narasimhan type τ.

5.3. *All three stratifications coincide*

Let $\tau = (d_1, \ldots, d_s)$ be a Harder–Narasimhan type (with respect to α and θ) for a representation of Q of dimension d. We write $W_v := \mathbb{C}^{d_v} \cong W_{1,v} \oplus \cdots \oplus W_{s,v}$, where $W_{i,v} = \mathbb{C}^{(d_i)_v}$; then every $(\varphi_a) \in \mathrm{Rep}(Q, d)$ can be written as

$$\varphi_a = \begin{pmatrix} \varphi_a^{11} & \cdots & \varphi_a^{1s} \\ \vdots & \ddots & \vdots \\ \varphi_a^{s1} & \cdots & \varphi_a^{ss} \end{pmatrix},$$

where $\varphi_a^{ij} : W_{j,t(a)} \to W_{i,h(a)}$. We let

$$Z_\tau := \{(\varphi_a) \in \mathrm{Rep}(Q, d) : \varphi_a^{ij} = 0 \text{ for } i \neq j\} \cong \bigoplus_{i=1}^s \mathrm{Rep}(Q, d_i)$$

and

$$Y_\tau := \{(\varphi_a) \in \mathrm{Rep}(Q, d) : \varphi_a^{ij} = 0 \text{ for } i > j\};$$

thus we have a projection $p_\tau : Y_\tau \to Z_\tau$. Let $\mathrm{Rep}(Q, d_i)^{\theta\text{-ss}} \subset \mathrm{Rep}(Q, d_i)$ denote the open subset of θ-semistable representations of dimension d_i. Then we define $Z_\tau^{\mathrm{ss}} \subset Z_\tau$ to be the image

of $\bigoplus_i \text{Rep}(Q, d_i)^{\theta\text{-ss}}$ under the above isomorphism and $Y_\tau^{\text{ss}} := p_\tau^{-1}(Z_\tau^{\text{ss}})$. By construction of Y_τ^{ss}, every point in Y_τ^{ss} has a Harder–Narasimhan filtration of type τ.

DEFINITION 5.3   We define $\beta = \beta(\tau) \in \mathfrak{K}$ by

$$\beta_v = 2\pi \mathrm{i}\, \text{diag}(\beta_1, \ldots, \beta_1, \beta_2, \ldots, \beta_2, \ldots, \beta_s, \ldots, \beta_s) \in \mathfrak{u}(\mathbb{C}^{d_v}),$$

where $\beta_i := -\theta(d_i)/\alpha(d_i)$ appears $(d_i)_v$ times in β_v.

We let λ_β denote the 1-PS associated to β, so $d\lambda_\beta(2\pi \mathrm{i}) = n\beta$ for some $n > 0$ and we let $S_{[\lambda_\beta]}$ denote the Hesselink strata for $(-, -)_\alpha$.

PROPOSITION 5.4   *Let $\beta = \beta(\tau)$; then*

(i) $V^{\lambda_\beta} = Z_\tau$;
(ii) $V_+^{\lambda_\beta} = Y_\tau$;
(iii) $Z_{\lambda_\beta} = Z_\tau^{\text{ss}}$;
(iv) $S_{\lambda_\beta} = Y_\tau^{\text{ss}}$.

Proof. The first two statements follow immediately from the definition of β, where we note that $\beta_1 > \cdots > \beta_s$ as τ is a Harder–Narasimhan type, and so the 'slopes' θ/α are increasing. As (iv) follows from (iii), it suffices to prove (iii). Let $\varphi \in Z_\tau^{\text{ss}}$; then, by Proposition 2.18, we know that $\varphi \in Z_{\lambda_\beta}$ if, for every 1-PS, $\lambda' = (\lambda'_{i,v})$ of $G_{\lambda_\beta} = \Pi_{i,v}\text{GL}(W_{i,v})$ for which $\lim_{t\to 0} \lambda'(t) \cdot \varphi$ exists, we have

$$\mu^{\rho_{\lambda_\beta}}(\varphi, \lambda') = \|\lambda_\beta\|_\alpha^2(\rho, \lambda') - (\rho, \lambda_\beta)(\lambda_\beta, \lambda')_\alpha = \|\lambda_\beta\|_\alpha^2[(\rho, \lambda') + (\beta, d\lambda'(2\pi \mathrm{i}))_\alpha] \geq 0, \quad (21)$$

where the second equality follows as β is scaled so that $\|\beta\|_\alpha^2 = -(1/2\pi \mathrm{i})\, d\rho \cdot \beta = -(1/n)(\rho, \lambda_\beta)$.

We simultaneously diagonalize the action of each $\lambda'_{i,v}$ on $W_{i,v}$, so we have weights $\gamma_1 > \cdots > \gamma_r$ and decompositions $W_{i,v} = W_{i,v}^1 \oplus \cdots \oplus W_{i,v}^r$ such that $\lambda_{i,v}(t)$ acts on $W_{i,v}^j$ by t^{γ_j}. We note that $\lim_{t\to 0} \lambda'(t) \cdot \varphi$ exists if and only if $W_i^{(j)} = (W_{i,v}^1 \oplus \cdots \oplus W_{i,v}^j, \varphi_a^{ii}|)$ is a subrepresentation of $W_i = (\mathbb{C}^{(d_i)_v}.\varphi_a^{ii})$ for all i and j. Then (21) is equivalent to

$$\sum_{i,j} \gamma_j(\theta(W_i^j) + \beta_i\alpha(W_i^j)) = \sum_{i,j} \gamma_j\left(\theta(W_i^j) - \frac{\theta(d_i)}{\alpha(d_i)}\alpha(W_i^j)\right) \geq 0. \quad (22)$$

As $\varphi \in Z_\tau^{\text{ss}}$, the representations $W_i = (W_{i,v}.\varphi_a^{ii})$ of dimension d_i are θ-semistable. Moreover, as $W_i^{(j)}$ are subrepresentations of W_i, semistability for all i implies that (22) holds.

Conversely, if $\varphi \in Z_\tau - Z_\tau^{\text{ss}}$, then there must be a destabilizing subrepresentation W_i^1 of W_i for some $1 \leq i \leq s$. We can pick an orthogonal complement $W_{i,v}^2$ to $W_{i,v}^1 \subset W_{i,v}$ and define a 1-PS λ' of G_{λ_β} by

$$\lambda_{i,v}(t) = \begin{pmatrix} tI_{W_{i,v}^1} & 0 \\ 0 & I_{W_{i,v}^2} \end{pmatrix} \quad \text{and} \quad \lambda_{k,v}(t) = I_{W_{k,v}} \text{ for } k \neq i.$$

Then, as W_i^1 destabilizes W_i, we have

$$\frac{\mu^{\rho_{\lambda_\beta}}(\varphi, \lambda')}{\|\lambda_\beta\|_\alpha^2} = (\rho, \lambda') + (\beta, d\lambda'(2\pi i))_\alpha = \theta(W_i^1) + \beta_i \alpha(W_i^1) < 0,$$

which proves $\varphi \notin Z_{\lambda_\beta}$ by Proposition 2.18.　　　　　　　　　　□

THEOREM 5.5　*The Harder–Narasimhan stratification of the space of representations of a quiver of fixed dimension agrees with both the Morse and Hesselink stratification for $(-,-)_\alpha$. Moreover, the Harder–Narasimhan strata have the form $S_\tau = GY_\tau^{ss}$.*

Proof. Let τ be a Harder–Narasimhan type as above; then clearly $Y_\tau^{ss} \subset S_\tau$ and, as the Harder–Narasimhan strata are G-invariant, we have $GY_\tau^{ss} \subset S_\tau$. However, by Proposition 5.4, $GY_\tau^{ss} = GS_{\lambda_\beta} = S_{[\lambda_\beta]}$ where $\beta = \beta(\tau)$. Since the Harder–Narasimhan strata and the Hesselink strata both form a stratification of $\mathrm{Rep}(Q, d)$ and every Hesselink stratum is contained in a Harder–Narasimhan stratum, these stratifications must coincide and $S_\tau = GY_\tau^{ss}$. It follows from Theorem 4.18 that all three stratifications coincide.　　　　　　　　　　□

Acknowledgements

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