# OPTIMAL REINSURANCE REVISITED – POINT OF VIEW OF CEDENT AND REINSURER

BY

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## ABSTRACT

It is known that the partial stop-loss contract is an optimal reinsurance form under the VaR risk measure. Assuming that market premiums are set according to the expected value principle with varying loading factors, the optimal reinsurance parameters of this contract are obtained under three alternative single and joint party reinsurance criteria: (i) strong minimum of the total retained loss VaR measure; (ii) weak minimum of the total retained loss VaR measure and maximum of the reinsurer's expected profit; (iii) weak minimum of the total retained loss VaR measure and minimum of the total variance risk measure. New conditions for financing in the mean simultaneously the cedent's and the reinsurer's required VaR economic capital are revealed for situations of pure risk transfer (classical reinsurance) or risk and profit transfer (design of internal reinsurance or reinsurance captive owned by the captive of a corporate firm).

## KEYWORDS

Optimal reinsurance, reinsurance captive, risk and profit transfer, partial stop-loss, economic capital, VaR, expected value principle, loading factor, mean financing property

## 1. INTRODUCTION

Reinsurance is an important risk transfer instrument that leads to a more effective risk management through reduction of required economic capital. Optimal reinsurance is a widely discussed and complex actuarial topic that finds a great variety of different answers. There are two kinds of optimization problems:

P1) Find the *optimal reinsurance form* under given criteria for a given set of ceded and/or retained loss functions.

P2) For a given optimal reinsurance form, that is a solution to problem P1, determine the *optimal reinsurance parameters* (e.g. optimal retention, expected profit, reinsurance price, etc.) under given criteria.

Concerning problem P1 early results have shown that the stop-loss contract is an optimal reinsurance form under the *variance risk measure* by minimizing the variance of a portfolio's retained loss for a fixed reinsurance premium (e.g. Borch (1960), Kahn (1961), Arrow (1963/74), Ohlin (1969), etc.), which yield optimality under the cedent's point of view. By minimizing the variance of a portfolio's ceded loss for a fixed reinsurance premium, Vajda (1962) has identified the quota-share contract as optimal reinsurance form, which yield optimality under the reinsurer's point of view. Obviously, there is a conflict of interests between the parties involved in a reinsurance program. As pointed out by Borch (1969) "an arrangement which is very attractive to one party may be quite unacceptable to the other". Since publication of Borch's seminal work (see Borch (1990) for collected papers), optimal solutions for both the cedent and the reinsurer have scarcely been discussed, although some papers devoted to joint optimality criteria have been published (e.g. Ignatov et al. (2004), Kaishev and Dimitrova (2006), Dimitrova and Kaishev (2010)).

In recent years various solutions to problem P1 and P2 under the value-at-risk measure (VaR) and the conditional value-at-risk measure (CVaR) have been obtained (e.g. Cai and Tan (2007), Cai et al. (2008), Bernard and Tian (2009), Cheung (2010), Chi and Tan (2010)). In the present paper, we show that within this framework it is possible to determine solutions that are optimal from the cedent's and reinsurer's point of views. Alternatively, we obtain optimal solutions under the *total variance risk measure* (sum of the ceded and retained variance of the loss) considered earlier by the author (in particular Hürlimann (1994a/b, 1996, 1999)).

The paper is organized as follows. Section 2 recalls the necessary and sufficient conditions for the existence of the optimal reinsurance form under the VaR risk measure as first identified by Cai et al.(2008). The Sections 3 and 4 are devoted to the optimal design of the partial stop-loss contract with ceded loss function $f(x) = a(x-d)_+$, $a \in (0,1]$, $d > 0$, which is an optimal reinsurance form under the VaR risk measure. We assume throughout that the market premium and the reinsurance premium are set according to the expected value principle with varying loading factors. Proposition 3.1 determines the optimal reinsurance parameters by minimizing the VaR measure of the total retained loss for an arbitrary confidence level (strong minimum). Proposition 3.2 proposes an alternative solution by minimizing the VaR measure of the total retained loss for a sufficiently high confidence level (weak minimum) and by maximizing the reinsurer's expected profit. A third optimal solution is found in Proposition 4.1 under a weak minimum of the total retained loss VaR measure and a minimum of the total variance risk measure. Interesting and useful conditions for financing in the mean simultaneously the cedent's and the reinsurer's required economic capital are derived for situations of pure risk transfer (classical reinsurance) or risk and profit transfer (design of internal reinsurance or reinsurance captive owned by the captive of a corporate firm). Section 5 illustrates the obtained results for a lognormal and a gamma approximate distribution of the loss. Section 6 summarizes and concludes.

## 2. Optimal reinsurance forms under the VaR risk measure

A reinsurance contract determines the rules according to which premium payments and unearned premium reserves, as well as claim payments, case reserves and IBNR reserves are split between the ceding and the reinsurance companies. In a simplified approach let $S$ be the (aggregate) loss of an insurance portfolio, which is supposed to be insured against a market premium $P$. We assume that $S$ is a non-negative integrable random variable with cumulative distribution $F_S(x) = P(S \le x)$, survival distribution $\bar{F}_S(x) = 1 - F_S(x)$, and positive mean $\mu = E[S] > 0$. We assume that $F_S(x)$ is strictly increasing and continuous on $(0,\infty)$, with a possible jump at 0. The quantile function of $S$ is defined and denoted by $Q_S(u) = \inf\{x : F_S(x) \ge u\}$. The stop-loss transform of $S$ is denoted by $\pi_S(x) = E[(S-x)_+]$. We use that $\pi_S(x) = \bar{F}_S(x) \cdot m_S(x)$, where $m_S(x)$ denotes the mean excess function. Let $S_r$, $S_c$ be the loss random variables representing the reinsurer's loss and the cedent's loss in the presence of reinsurance such that $S = S_c + S_r$ holds, and denote the expected values of these losses by $\mu_r = E[S_r]$, $\mu_c = E[S_c]$. We assume that the market premium and the reinsurance premium $P_r$ are set according to the expected value principle such that

$$P = (1 + \theta)\mu, \; P_r = (1 + \theta_r)\mu_r, \tag{2.1}$$

where $\theta$ is the market loading factor without reinsurance, and $\theta_r$ is the loading factor of the reinsurer. Note that this assumption is in agreement with actuarial practice, where usually either the reinsurance premium or the reinsurance loading factor is predetermined. Then, the cedent's retained premium and the corresponding (implicit) loading factor $\theta_c$ are determined by $P_c = P - P_r = (1 + \theta_c)\mu_c$. Given an insured loss $(\theta, S)$ and the reinsurance loading factor $\theta_r$, the actuarial problem of optimal reinsurance is to determine, under given criteria, the optimal reinsurance premium $P_r = (1 + \theta_r)\mu_r$, or equivalently, either the optimal retained premium $P_c = P - P_r$ or the cedent's loading factor $\theta_c = \theta - (\theta_r - \theta)\mu_r / (\mu - \mu_r)$.

The VaR and CVaR of a random variable $X$ at the confidence level $\alpha \in (0,1)$ are defined as $VaR_\alpha[X] = Q_X(\alpha)$ and $CVaR_\alpha[X] = E[X \mid X > VaR_\alpha[X]]$ respectively. Let $f(x)$ be a real, increasing and convex function on $(0,\infty)$, which satisfies $0 \le f(x) \le x$, called *ceded loss function*, such that $S_r = f(S)$ represents the ceded loss. The set of all possible ceded loss functions is denoted as $C$. The reinsurance premium corresponding to $f \in C$ is denoted by $P_r^f(S) = (1 + \theta_r)E[f(S)]$. The total retained loss, which is the sum of the cedent's loss and the reinsurance premium, is denoted by $T_c^f(S) = S - f(S) + P_r^f(S)$. The optimal reinsurance problems of type P1 studied by Cai et al. (2008) and Cheung (2010) are stated as follows:

$$VaR_\alpha\left[T_c^{f^*}(S)\right] = \min_{f \in C} VaR_\alpha\left[T_c^f(S)\right], CVaR_\alpha\left[T_c^{f^*}(S)\right] = \min_{f \in C} CVaR_\alpha\left[T_c^f(S)\right]. \tag{2.2}$$

It is convenient to introduce the following notation and functions

$$d^* = d(\theta_r),\ d(x) = Q_S(x/(1+x)),\ g(x) = x + (1+\theta_r)\pi_S(x). \tag{2.3}$$

One observes that $g(0) = (1+\theta_r)\mu$. The necessary and sufficient conditions for the existence of the optimal reinsurance form under the VaR risk measure have been identified by Cai et al. (2008). It is possible to rewrite the objective function under the VaR measure as $H(f) := Var_\alpha\left[T_c^f(S)\right] = Q_S(\alpha) - f(Q_S(\alpha)) + (1+\theta_r)E\left[f(S)\right]$ (e.g. Cheung (2010), equation (3)).

**Theorem 2.1.** For a given confidence level $\alpha \in (F_S(0), 1)$ the following statements hold true:

CASE 1: If $\theta_r > F_S(0)/\bar{F}_S(0)$ and $Q_S(\alpha) > g(d^*)$, then $\min_{f\in C} H(f) = g(d^*)$, and the stop-loss ceded loss function $f^*(x) = (x-d^*)_+$ is optimal.

CASE 2: If $\theta_r > F_S(0)/\bar{F}_S(0)$ and $Q_S(\alpha) = g(d^*)$, then $\min_{f\in C} H(f) = g(d^*)$, and the partial stop-loss ceded loss function $f^*(x) = a(x-d^*)_+$, $a \in [0,1]$ is optimal.

CASE 3: If $\theta_r \le F_S(0)/\bar{F}_S(0)$ and $Q_S(\alpha) > g(0)$, then $\min_{f\in C} H(f) = g(0)$, and the full reinsurance ceded loss function $f^*(x) = x$ is optimal.

CASE 4: If $\theta_r \le F_S(0)/\bar{F}_S(0)$ and $Q_S(\alpha) > g(0)$, then $\min_{f\in C} H(f) = g(0)$, and the quota-share ceded loss function $f^*(x) = ax$, $a \in [0,1]$ is optimal.

CASE 5: For all other cases one has $\min_{f\in C} H(f) = Q_S(\alpha)$, and full retention with ceded loss function $f^*(x) = 0$ is optimal.

**Proof.** Consult Cheung (2010), Theorem 1. □

Theorem 2.1 identifies the *partial stop-loss* contract with ceded loss function $f(x) = a\cdot(x-d)_+$, as an optimal reinsurance form. At this stage, some remarks on related but alternative reinsurance arrangements, which are optimal forms under different risk measures, are in order. The *limited stop-loss* contract $f(x) = \min\{(x-d)_+, b\}$, which is most popular in practice, has been identified as optimal reinsurance form in Cummins and Mahul (2004). In fact, Kaluska and Okolewski (2008) have derived optimality under the criteria of maximizing either the expected utility or the stability of the cedent for a fixed reinsurance premium under the maximal possible claims premium calculation principle. The same contract has been shown optimal under more general symmetric and even asymmetric risk measures in Gajek and Zagrodny (2004a). Similarly, the *truncated stop-loss* contract with ceded loss function $f(x) = 1\{x \le c\}\cdot(x-d)_+$, $c > d \ge 0$, has been identified as optimal reinsurance form in Gajek and Zagrodny (2004b), Kaluszka (2005), Kaluska and Okolewski (2008), and Bernard and Tian (2009). All these quite recent results are most interesting from a general risk management perspective. But, in view of the current implementation of regulatory solvency systems, the VaR and CVaR risk measures that are relevant to *standard solvency models* (e.g. Solvency II and Swiss Solvency Test (SST)) have first priority. In this respect, a unified analysis under the VaR and CVaR risk measures of the above three stop-loss related reinsurance forms over different classes of ceded loss functions with increasing degrees of generality has been undertaken in Chi and Tan (2010). Finally, note that for the CVaR risk measure, a result similar to Theorem 2.1 has also been obtained in Cai et al. (2008) and Cheung (2010). Given some optimal reinsurance form, it is possible to tackle problem P2 for it. In the present paper, we restrict ourselves to the fixed set of increasing and convex ceded loss functions, for which the partial stop-loss contract is relevant, and restrict ourselves to the VaR risk measure. Similar investigations for the related limited and truncated stop-loss contracts, as well as a study under the CVaR risk measure, are not undertaken here.

## 3. Joint party optimality with the maximum reinsurer's expected profit

Consider the partial stop-loss contract with ceded loss function $f(x) = a \cdot (x-d)_+$, $a \in [0,1]$, $d > 0$, and reinsurance premium $P_r = P_r(a,d) = (1+\theta_r) a\pi_S(d)$. The cedent's loss is described by $S_c(a,d) = S - a \cdot (S-d)_+$ and the total retained loss by $T_c(a,d) = S_c(a,d) + P_r(a,d)$. The optimization problem P2 will be formulated for different criteria that are all related to a minimization of the VaR measure of the total retained loss. The first approach requires a minimum for an arbitrary confidence level (called *strong minimum*)

$$\text{(P2.1)} \qquad VaR_\alpha\left[T_c(a^*,d^*)\right] = \min_{a\in[0,1],\, d>0} VaR_\alpha\left[T_c(a,d)\right]$$

By abuse of notation $(a^*, d^*)$ denotes throughout the optimal parameters of any (P2.1) related optimization problem. If the confidence level is sufficiently large the Lemma 3.1 shows that the optimal priority $d^*$ of the diverse problems is always of the form $d(\theta_r)$ and takes different values for different reinsurance loading factors (optimal or not). The chosen unified notation is somewhat confusing, but this will be ruled out through discussion whenever felt necessary.

**Lemma 3.1.** Assuming $Q_S(\alpha) > d^* > 0$ the optimal parameters of problem (P2.1) satisfy the following properties. One has $d^* = d(\theta_r)$, or equivalently $\theta_r = F_S(d^*) / \bar{F}_S(d^*) > F_S(0) / \bar{F}_S(0)$, and either (i) $a^* = 1$ if $Q_S(\alpha) > g(d^*)$ or (ii) $a^* \in [0,1]$ if $Q_S(\alpha) = g(d^*)$.

**Proof.** With Cheung (2010), equation (3.3) (see also Section 2), and if $Q_S(\alpha) > d$, one has

$$h(a,d) = VaR_\alpha\left[T_c(a,d)\right] = Q_S(\alpha) - a \cdot \left[Q_S(\alpha) - d\right] + (1+\theta_r)a\pi_S(d)$$
$$= a \cdot \left[d + (1+\theta_r)\,\pi_S(d)\right] + (1-a)\cdot Q_S(\alpha).$$

The minimum depends upon the partial derivatives (use that $\frac{d}{dx}\pi_S(x) = -\bar{F}_S(x)$)

$$h_a = d + (1+\theta_r)\,\pi_S(d) - Q_S(\alpha), \quad h_d = a \cdot \left\{1 - (1+\theta_r)\,\bar{F}_S(d)\right\}.$$

The necessary first order condition $h_d = 0$ implies that $1 + \theta_r = 1/\bar{F}_S(d^*)$, or equivalently $\theta_r = F_S(d^*)/\bar{F}_S(d^*)$. By definition of the function $d(x)$ in (2.3) this is equivalent to $d^* = d(\theta_r)$. Now, for $d = d^* = d(\theta_r)$, one has by definition of $g(x)$ in (2.3) that

$$h_a = d^* + m_S(d^*) - Q_S(\alpha) = g(d^*) - Q_S(\alpha),$$

and similarly

$$h(a,d^*) = a \cdot g(d^*) + (1-a)\cdot Q_S(\alpha).$$

If $h_a < 0$ then $h(a,d^*)$ is strictly decreasing, hence $a^* = 1$, and if $h_a = 0$ any $a^* \in [0,1]$ is optimal. In both cases one has $VaR_\alpha\left[T_c(a^*,d^*)\right] = h(a^*,d^*) = g(d^*)$ as in Theorem 2.1. □

Similarly to the above, the optimal cedent's loading factor is denoted by $\theta_c^*$. The next result describes the complete solution of problem P2 with respect to criterion (P2.1). The obtained result is a reformulation of the Cases 1 and 2 of Theorem 2.1.

**Proposition 3.1.** Given an insured loss $(\theta, S)$ and a reinsurance loading factor satisfying $F_S(0)/\bar{F}_S(0) < \theta_r \neq \theta$, assume that $0 \le \theta_c^* \le \theta$, as well as $Q_S(\alpha) > \mu/\bar{F}_S(0)$. Then, the optimal ceded partial stop-loss function $f^*(x) = a^* \cdot (x - d^*)_+$, $a^* \in [0,1]$, $d^* > 0$, under criterion (P2.1) is completely described by the following conditions. The optimal priority is given by $d^* = d(\theta_r) \in (0, d^*_{\max}]$, where $d^*_{\max}$ is the unique solution of the implicit equation

(C1.1) $$d^*_{\max} + m_S(d^*_{\max}) = Q_S(\alpha).$$

The loading factors are necessarily given by and must satisfy the inequalities

(C1.2)

$$0 \le \theta_c^* = \frac{\theta\mu - \theta_r\pi_S(d^*)}{\mu - \pi_S(d^*)} \le \theta < \theta_r = \frac{\bar{F}_S(d^*)}{F_S(d^*)}, \quad 0 < d^* < d^*_{\max},$$

$$\max\left\{\frac{\theta\mu - \theta_r\pi_S(d^*_{\max})}{\mu - \pi_S(d^*_{\max})}, 0\right\} \le \theta_c^* = \frac{\theta\mu - \theta_r a^*\pi_S(d^*)}{\mu - a^*\pi_S(d^*)} \le \theta < \theta_r = \frac{\bar{F}_S(d^*)}{F_S(d^*)},\ d^* = d^*_{\max},$$

and the optimal partial stop-loss factor $a^* \in [0,1]$ is determined as follows:

$$a^* = \begin{cases} 1, & 0 < d^* < d^*_{\max}, \\ \dfrac{\theta - \theta_c^*}{\theta_r - \theta_c^*} \cdot \dfrac{\mu}{\pi_S(d^*_{\max})} \in [0,1], & d^* = d^*_{\max}. \end{cases} \tag{C1.3}$$

**Proof.** By Lemma 3.1 and its proof one has $d^* = d(\theta_r)$ and $g(d^*) = d^* + m_S(d^*)$. One has $d^* \in (0, d^*_{\max}]$ because $g(x)$ is strictly increasing. Since the function $h(x) = x + m_S(x) - Q_S(\alpha)$ is strictly increasing on $(0, Q_S(\alpha))$ with $h(Q_S(\alpha)) > 0$ and $\bar{F}_S(0) \cdot h(0) = \mu - \bar{F}_S(0) \cdot Q_S(\alpha) < 0$ by assumption, the solution to (C1.1) is unique. On the other hand, the decomposition $P = P_r + P_c$ is equivalent to the equality (here for the optimal solution)

$$(\theta - \theta_c^*)\mu = (\theta_r - \theta_c^*)\, a^* \pi_S(d^*). \tag{3.1}$$

Assume for the moment that $\theta_c^* \leq \theta < \theta_r$, which will be shown below. If $0 < d^* < d^*_{\max}$ one must have $a^* = 1$ by the Case 1 of Theorem 2.1, which shows the first formula in (C1.3). The first formula in (C1.2) follows from (3.1). If $d^* = d^*_{\max}$ the second formula in (C1.3) follows also from (3.1). Since $a^* \leq 1$ the equation (3.1) implies the inequality $\theta\mu - \theta_r \pi_S(d^*) \leq \theta_c^*(\mu - \pi_S(d^*))$, which implies the first inequality for $\theta_c^*$ in the second formula in (C1.2). Moreover, using the assumption $\theta_c^* \leq \theta$, one obtains from the same inequality that $\theta \leq \theta_r$ and more stringently $\theta < \theta_r$ by the assumption $\theta_r \neq \theta$, hence $\theta_c^* \leq \theta < \theta_r$. □

**Remarks 3.1.**

(i) The condition $\theta_c^* \leq \theta < \theta_r$ means that the reinsurer covers stop-loss reinsurance at a higher loading factor than the cedent. In fact, if $\theta_c^* = \theta$ either the optimal priority is the unique solution $d^* \in (0, d^*_{\max})$ of $\theta\mu = \theta_r \pi_S(d^*)$ (if it exists) or one must have $a^* = 0$ (no reinsurance) in case $d^* = d^*_{\max}$. Excluding the first situation as a singular exception, reinsurance occurs only if the strict inequality $\theta_c^* < \theta < \theta_r$ holds. In fact, the inequality $\theta_c^* < \theta_r$ has been derived in the Appendix of Hürlimann (2010) by means of "ordering of risks" considerations. The assumption $\theta_r \neq \theta$ is not restrictive. Indeed, if $\theta_r = \theta \neq \theta_c^*$ the equality (3.1) is only possible for the choice $a^* = 1$, $d^* = 0$ of full reinsurance (excluded by assumption), and the remaining possibility $\theta_r = \theta = \theta_c^*$ contradicts the inequality $\theta_c^* < \theta_r$.

(ii) The pure stop-loss case $a^* = 1$ is somewhat ambiguous. While the cedent's loading factor is uniquely determined as a function of the optimal priority through $d^* = Q_S(\theta_r / (1 + \theta_r))$, the reinsurance loading factor varies in the range $\theta_r \in (F_S(0)/\bar{F}_S(0), F_S(d^*_{\max})/\bar{F}_S(d^*_{\max})]$. However, this fact may be of practical value, because the reinsurance loading factor can usually be chosen by the reinsurer (see also the comment (i) of the Remarks 3.2). Extending the optimality criterion taking into account the reinsurer's point of view will resolve this ambiguity and yield well-defined optimal reinsurance loading factors that are denoted $\theta_r^*$ (by abuse of notation).

A drawback of the obtained optimal design is the fact that it only provides optimality from the cedent's point of view. To overcome this possibly unrealistic limitation we consider alternative optimization criteria, which take into account both the cedent's and reinsurer's point of views. Our considerations will include a solvency perspective. We take into account the required VaR economic capitals of both parties and compare them to their underwriting expected profits. The retained VaR economic capital for a given confidence level $\alpha$ depends only upon the centered retained loss and it is defined by $EC_\alpha^{VaR}[S_c] := VaR_\alpha[S_c - \mu_c]$. One has the relationship

$$EC_\alpha^{VaR}[S_c] = VaR_\alpha[T_c] - P_r - \mu_c = VaR_\alpha[T_c] - EG_r - \mu, \tag{3.2}$$

where

$$EG_r = EG_r(a,d) = P_r - \mu_r = \theta_r a\pi_S(d) \tag{3.3}$$

denotes the reinsurer's expected profit. Since the VaR measure is additive for comonotonic random variables, and $S_c, S_r$ are comonotonic, the ceded VaR economic capital at the confidence level $\alpha$ is defined and determined by

$$EC_\alpha^{VaR}[S_r] := VaR_\alpha[S_r - \mu_r] = EC_\alpha^{VaR}[S] - EC_\alpha^{VaR}[S_c], \tag{3.4}$$

where $EC_\alpha^{VaR}[S] = VaR_\alpha[S - \mu]$ represents the VaR economic capital without reinsurance. For later use we introduce also the retained expected profit defined and denoted by

$$EG_c = EG_c(a,d) = P_c - \mu_c = \theta_c(\mu - a\pi_S(d)). \tag{3.5}$$

Before proceeding with alternatives let us mention the following interesting and useful properties. We use the notations $S_r^* = S_r(a^*, d^*)$, $S_c^* = S_c(a^*, d^*)$, $EG_r^* = EG_r(a^*, d^*)$, $EG_c^* = EG_c(a^*, d^*)$.

**Corollary 3.1.** Under the assumptions of Proposition 3.1 the reinsurer's expected profit of the optimal design, as a function of the ceded VaR economic capital, is given by

$$EG_r^* = EC_\alpha^{VaR}\left[S_r^*\right] + d^* + m_S(d^*) - Q_S(\alpha). \tag{3.6}$$

Moreover, in the special case $P = Q_S(\alpha)$ (percentile premium calculation principle), the retained expected profit, as a function of the retained VaR economic capital, is given by

$$EG_c^* = EC_\alpha^{VaR}\left[S_c^*\right] + Q_S(\alpha) - d^* - m_S(d^*). \tag{3.7}$$

**Proof.** By the proof of Lemma 3.1, one has for the optimal design $VaR_\alpha\left[T_c^*\right] = g(d^*)$. Substituting this and (3.2) into (3.4) one gets

$$EC_\alpha^{VaR}\left[S_r^*\right] = EC_\alpha^{VaR}[S] - VaR_\alpha\left[T_c^*\right] + EG_r^* + \mu = Q_S(\alpha) - g(d^*) + EG_r^*.$$

Since $g(d^*) = d^* + m_S(d^*)$ one obtains (3.6). In the special case one has

$$\begin{aligned} EC_\alpha^{VaR}\left[S_c^*\right] &= EC_\alpha^{VaR}[S] - EC_\alpha^{VaR}\left[S_r^*\right] \\ &= Q_S(\alpha) - \mu - EG_r^* + g(d^*) - Q_S(\alpha) = EG_c^* + g(d^*) - Q_S(\alpha), \end{aligned}$$

where the last equality follows from the relationship $Q_S(\alpha) - \mu = P - \mu = EG_c^* + EG_r^*$. □

The interpretation of Corollary 3.1 is quite instructive from an economic point of view. For this purpose we need a concept that describes the fact that a given deterministic quantity is financed in the mean by another stochastic quantity.

**Definition 3.1.** A deterministic liability $L$ is *weakly (strongly) mean financed* by a stochastic profit $G$ if the mean inequality $E[G] \geq L$ (mean equality $E[G] = L$) holds.

In the Case 2 of Theorem 2.1, i.e. $d^* = d_{\max}^*$, Corollary 3.1 simplifies by equation (C1.1) to the relationships $EG_r^* = EC_\alpha^{VaR}\left[S_r^*\right]$ and $EG_c^* = EC_\alpha^{VaR}\left[S_c^*\right]$. From an economic management perspective the first one means that the reinsurer's required VaR economic capital is strongly mean financed by the reinsurer's profit $G_r = P_r - S_r$. Even more, in the mentioned special case, the cedent's required VaR economic capital is also strongly mean financed, namely by the cedent's profit $G_c = P_c - S_c$. In practice, these situations might be unrealistic. First, the special case implies that the market premium $P = Q_S(\alpha)$, which is set according to the percentile premium calculation principle, might not be competitive. Second, a reinsurer's might not enter into such a zero-sum profit strategy and opt for higher expected profit at the cost of higher required economic capital.

Fortunately, the relationship (3.2) suggests at least two further optimization problems. First, one sees that minimizing $VaR_\alpha[T_c]$ and maximizing $EG_r$ renders actually $EC_\alpha^{VaR}[S_c]$ minimum. This can be formulated as single party optimization problem (point of view of cedent):

(P2.1') $$EC_{\alpha}^{VaR}\left[S_c(a^*, d^*)\right] = \min_{a\in[0,1], d\geq 0} EC_{\alpha}^{VaR}\left[S_c(a,d)\right]$$

The optimal solution of this problem coincides with full reinsurance.

**Corollary 3.2.** Assume $Q_S(\alpha) > \mu$. Given an insured loss $(\theta, S)$ and the reinsurance loading factor $\theta_r$, the optimal ceded partial stop-loss function $f^*(x) = a^* \cdot (x-d^*)_+$, $a^* \in [0,1]$, $d^* \geq 0$, under criterion (P2.1') is full reinsurance, i.e. $a^* = 1$, $d^* = 0$.

**Proof.** With Cheung (2010), equation (3.3), (see also Section 2), and (3.3), rewrite (3.2) as

$$EC_{\alpha}^{VaR}[S_c] = VaR_{\alpha}[T_c] - EG_r - \mu = Q_S(\alpha) - a(Q_S(\alpha) - d)_+ + (1+\theta_r)a\pi_S(d)$$
$$-EG_r - \mu = Q_S(\alpha) - a(Q_S(\alpha) - d)_+ + a\pi_S(d) - \mu.$$

Since this expression does not contain the variable $\theta_r$, one sees that problem (P2.1') is formally identical to the minimization problem $\min_{a\in[0,1], d\geq 0}\{VaR_{\alpha}[T_c(a,d)] - \mu\}$ for a vanishing reinsurance loading factor $\theta_r = 0$. Since $Q_S(\alpha) > g(0) = \mu$, the result follows from Case 3 of Theorem 2.1. □

A second meaningful two party optimization problem (point of view of cedent and reinsurer) is to minimize for a sufficiently high confidence level the VaR measure of the total retained loss and maximize the reinsurer's expected profit:

(P2.2) $VaR_{\alpha}\left[T_c(a^*, d^*)\right] = \min_{a\in(0,1], d>0} EG_r(a,d)$ for a sufficiently high confidence level $\alpha$,

$$EG_r(a^*, d^*) = \max_{a\in(0,1], d>0} EG_r(a,d)$$

To distinguish it from the strong minimum required by criterion (P2.1) the less stringent minimum in (P2.2) is called *weak minimum*. The slightly restricted optimization problem with a fixed partial stop-loss factor $a \in (0,1]$ is considered in Proposition 3.2 while Corollary 3.3 handles the unrestricted problem. As already stated in the comment (ii) of the Remarks 3.1 the obtained optimal reinsurance loading factors are denoted $\theta_r^*$.

**Proposition 3.2.** Given an insured loss $(\theta, S)$, assume that $0 \leq \theta_c^* \leq \theta$, $\theta_r^* \neq \theta$, and $Q_S(\alpha) > d^* > 0$. Assume that the density function $f_S(x) = F_S'(x) > 0$ and its derivative $f_S'(x)$ exist for all $x > 0$. Then, the optimal ceded partial stop-loss function $f(x) = a \cdot (x-d^*)_+$, $d^* > 0$, for fixed $a \in (0,1]$, under criterion (P2.2) is completely described as follows. A local maximum for the reinsurer's expected profit exists at the optimal priority $d^* = d(\theta_r^*) > 0$, if and only if the implicit equation

(C2.1) $$f_S(d^*) \cdot m_S(d^*) = F_S(d^*) \cdot \bar{F}_S(d^*)$$

has a solution and one has

(C2.2) $$f_S(d^*) \cdot (2 - 3 \cdot F_S(d^*)) \cdot \bar{F}_S(d^*) - f_S{}'(d) \cdot \pi_S(d) > 0.$$

The optimal reinsurance loading factor is necessarily given by

(C2.3) $$\theta_r^* = \frac{F_S(d^*)}{\bar{F}_S(d^*)} > \frac{F_S(0)}{\bar{F}_S(0)}.$$

The optimal cedent's loading factor is determined by and satisfies the inequality

(C2.4) $$0 \le \theta_c^* = \frac{\theta\mu - \theta_r^* a \pi_S(d^*)}{\mu - a\pi_S(d^*)} \le \theta < \theta_r^*.$$

**Proof.** The necessary condition (C2.3) of optimality, that is $\theta_r = F_S(d)/\bar{F}_S(d)$, has been shown in Lemma 3.1. It follows that $EG_r(a,d) = \theta_r a \pi_S(d) = a \cdot F_S(d) \cdot m_S(d)$. For ease of notation, set $h(d) = f_S(d) \cdot \pi_S(d) - F_S(d) \cdot \bar{F}_S(d)^2$. Since $m_S(x) = \pi_S(x) / \bar{F}_S(x)$, a calculation shows that $\frac{\partial EG_r(a,d)}{\partial d} = a \cdot \frac{h(d)}{\bar{F}_S(d)^2}$. A necessary condition for the existence of a local maximum is herewith (C2.1). Further, using that $h(d^*) = 0$ at the optimal priority, one obtains $\left.\frac{\partial^2 EG_r}{\partial d^2}\right|_{d=d^*} = a \cdot \frac{h'(d^*)}{\bar{F}_S(d^*)^2}$. A second order sufficient condition for a local maximum is herewith $h'(d^*) = f_S{}'(d) \cdot \pi_S(d^*) - f_S(d^*) \cdot (2 - 3F_S(d^*)) \cdot \bar{F}_S(d^*) < 0$, which yields condition (C2.2). The expression for the cedent's loading factor in (C2.4) follows from the equality (3.1) and the inequalities for the loading factors are shown as in the proof of Proposition 3.1. □

For a variable partial stop-loss factor $a \in (0,1]$ the optimal design is determined as follows.

**Corollary 3.3.** Under the assumptions of Proposition 3.2 and variable stop-loss factor $a \in (0,1]$ the optimal design under criterion (P2.2) has a ceded stop-loss function $f(x) = (x - d^*)_+$, $d^* > 0$, and is determined as follows. If it exists, the optimal priority $d^* = d(\theta_r^*) > 0$ is solution of the implicit equation (C2.1) and (C2.2) must hold. Moreover, the optimal reinsurance loading factor is given by (C2.3) and the remaining loading factors satisfy the conditions

(C2.4') $$\frac{\pi_S(d^*)}{\mu} \le \theta < \theta_r^* = \frac{F_S(d^*)}{\bar{F}_S(d^*)}, \quad \theta_c^* = \frac{\theta\mu - \theta_r^* \pi_S(d^*)}{\mu - \pi_S(d^*)}.$$

**Proof.** By the proof of Lemma 3.1 one has $VaR_\alpha[T_c(a,d^*] = a \cdot g(d^*) + (1-a) \cdot Q_S(\alpha)$. This is minimum, and $EG_r(a,d^*) = a \cdot F_S(d^*) \cdot m_S(d^*)$ is maximum, exactly when $a = 1$. The conditions (C2.4') follow from (C2.4) through rearrangement by setting $a = 1$. □

**Remarks 3.2.**

(i) In Proposition 3.2 and Corollary 3.3 the optimal reinsurance loading factor is by (C2.3) a predetermined function of the optimal priority. If $d^*$ is very large, then $\theta_r^*$ will be very large, which will be unrealistic in practice. In contrast to this, in Proposition 3.1 the reinsurance loading factor can be fixed by the reinsurer, which is in agreement with practice. But then, unless $g(d^*) = Q_S(\alpha)$ with $d^* = d^*_{\max}$, one has $g(d^*) < Q_S(\alpha)$ and only the pure stop-loss contract can be optimal by the Case 1 of Theorem 2.1. Based on reinsurance market data, it might be interesting to analyze whether Proposition 3.2 and Corollary 3.3 can generate unrealistic practical situations. As alternatives, it is still possible to consider similar optimization problems for the (optimal) limited and truncated stop-loss contracts mentioned at the end of Section 2.

(ii) Corollary 3.3 proposes a method to identify the unknown optimal priority in Case 1 of Theorem 2.1. The optimal priority in (C2.1) can be reinterpreted as fixed point of the quantile function

$$d^* = Q_S(h_S^0(d^*) \cdot m_S(d^*)) = Q_S\left(\frac{h_S^0(d^*)}{h_S^1(d^*)}\right), \tag{3.8}$$

where

$$h_S^0(x) = \frac{f_S(x)}{\bar{F}_S(x)} = -\tfrac{d}{dx}\ln\{\bar{F}_S(x)\},\ h_S^1(x) = \frac{\bar{F}_S(x)}{\pi_S(x)} = -\tfrac{d}{dx}\ln\{\pi_S(x)\}, \tag{3.9}$$

denote respectively the hazard rate function (also called failure rate, degree zero stop-loss rate), and the degree one stop-loss rate, (see Hürlimann (2000), Section 2).

(iii) Existence and uniqueness of the local maximum defined by (C2.1)-(C2.2) remain to be discussed. The numerical optimal priorities obtained in Section 5 are all uniquely determined. In general, it is felt that the fixed point equation (3.8) might be helpful for resolving the open existence question using appropriate conditions for such equations. Uniqueness depends upon the number of sign changes of the function $h(d)$ in the proof of Proposition 3.2. Both questions have so far not been analyzed and go beyond the scope of the present investigation.

## 4. Joint party optimality with the minimum of the total variance risk measure

From now on we will assume that the loss $S$ has a finite variance $\sigma_S^2 = Var[S]$ and we define and denote the stop-loss transform of degree two by $\pi_{2,S}(x) = E\left[(S-x)_+^2\right]$. The stop-loss transform variance is defined and denoted by $\sigma_S^2(x) = \pi_{2,S}(x) - \pi_S(x)^2$.

Instead of maximizing the reinsurer's expected profit besides minimizing in a weak sense the total retained loss VaR, one might consider stabilization of the total variance of the retained and ceded loss. It is generally recognized that an appropriate reinsurance program should minimize unexpected fluctuations and produce value through more stable insurance results (e.g. Venter (2001)). Suppose unexpected fluctuations are measured using the variance, and that the unexpected total fluctuations associated to the splitting of a risk in several components is measured using the *total variance risk measure* defined by the sum of the component variances. In the reinsurance situation this risk measure is defined and denoted by

$$R_S[S_c, S_r] = Var[S_c] + Var[S_r] = \sigma_S^2 - 2 \cdot Cov[S_c, S_r] \tag{4.1}$$

An alternative to (P2.2) is therefore the following optimization criterion (weak minimum of the cedent's VaR measure of the total retained loss and minimum of the total variance risk measure):

(P2.3) $VaR_\alpha\left[T_c(a^*, d^*)\right] = \min_{a \in (0,1], d > 0} VaR_\alpha\left[T_c(a,d)\right]$ for all sufficiently high confidence levels $\alpha$,

$$R_S\left[S_c(a^*, d^*),\ S_r(a^*, d^*)\right] = \min_{a \in (0,1], d > 0} R_S\left[S_c(a,d),\ S_r(a,d)\right]$$

We consider optimization under the restricted criterion (P2.3) for a fixed partial stop-loss factor $a \in (0,1]$. The "conjugate" or dual stop-loss transform defined by $\bar{\pi}_S(x) = x - \mu + \pi_S(x)$ and the dual mean excess function $\bar{m}_S(x)$ defined by $\bar{\pi}_S(x) = F_S(x) \cdot \bar{m}_S(x)$ are used.

**Proposition 4.1.** Given an insured loss $(\theta, S)$ with finite variance $\sigma_S^2$, assume that $0 \le \theta_c^* \le \theta$, $\theta_r^* \neq \theta$, and $Q_S(\alpha) > d^* > 0$. Assume that the density function $f_S(x) = F_S{}'(x) > 0$ exists for all $x > 0$. Then, the optimal ceded partial stop-loss function $f(x) = a \cdot (x - d^*)_+$, $d^* > 0$, for fixed $a \in (0,1]$, under criterion (P2.3) is completely described as follows. A local minimum for the total variance of the retained and ceded loss exists at the optimal priority $d^* = d(\theta_r^*) > 0$, if and only if, $a \in (\frac{1}{2}, 1]$, the implicit equation

(C3.1) $$(2a-1) \cdot m_S(d^*) - \bar{m}_S(d^*) = 0,,$$

has a solution and one has

(C3.2) $$2a \cdot F_S(d^*) \cdot \bar{F}_S(d^*) - f_S(d^*) \cdot \left(d^* - \mu + 2a \cdot \pi_S(d^*)\right) > 0.$$

The optimal reinsurance loading factor is necessarily given by

(C3.3) $$\theta_r^* = \frac{F_S(d^*)}{\bar{F}_S(d^*)} > \frac{F_S(0)}{\bar{F}_S(0)}.$$

The optimal cedent's loading factor is determined by and satisfies the inequality

(C3.4) $$0 \le \theta_c^* = \frac{\theta\mu - \theta_r^* a\pi_S(d^*)}{\mu - a\pi_S(d^*)} \le \theta < \theta_r^*.$$

**Proof.** The necessary condition (C3.3) has been shown in Lemma 3.1. According to (4.1) the total variance risk measure is minimal exactly when the covariance between the retained and ceded loss is maximum. To obtain an expression for this quantity, use that $S_c = S - a \cdot (S-d)_+$, $S_r = a \cdot (S-d)_+$ to get

$$\begin{aligned} Cov\left[S_c, S_r\right] &= a \cdot \left\{Cov\left[S, (S-d)_+\right] - a \cdot Var\left[(S-d)_+\right]\right\} \\ &= a \cdot \left\{E\left[S \cdot (S-d)_+\right] - \mu\pi_S(d) - a\pi_{2,S}(d) + a\pi_S^2(d)\right\}. \end{aligned}$$

But, one has $E\left[S \cdot (S-d)_+\right] = E\left[(S-d)_+^2 + d \cdot (S-d)_+\right] = \pi_{2,S}(d) + d\pi_S(d)$. Inserted in the preceding curly bracket one obtains

$$\begin{aligned} Cov\left[S_c, S_r\right] &= a \cdot \left\{(1-a) \cdot \pi_{2,S}(d) + (d - \mu + \pi_S(d)) \cdot \pi_S(d) - (1-a)\,\pi_S^2(d)\right\} \\ &= a \cdot h(d), \end{aligned}$$

with $h(d) = (1-a) \cdot \sigma_S^2(d) + \pi_S(d) \cdot \bar{\pi}_S(d)$. A necessary condition for a local maximum is $h'(d^*) = 0$. Using that $\pi_S(d)' = -\bar{F}_S(d)$, $\bar{\pi}_S(d)' = F_S(d)$, $\sigma_S^2(d)' = -2F_S(d)\,\pi_S(d)$ one obtains

$$\begin{aligned} h'(d) &= a \cdot \left\{-2(1-a)F_S(d)\pi_S(d) - \bar{F}_S(d)\bar{\pi}_S(d) + F_S(d)\pi_S(d)\right\} \\ &= a \cdot F_S(d)\bar{F}_S(d) \cdot \left\{(2a-1)m_S(d) - \bar{m}_S(d)\right\}, \end{aligned}$$

which implies the condition (C3.1). One notes that the condition $a \in (\frac{1}{2}, 1]$ is necessary for a local minimum. Indeed, if $a \le \frac{1}{2}$ then $h'(d) = 0$ can only be fulfilled if $\bar{\pi}_S(d) \le 0$. Since $\bar{\pi}_S(d)$ is strictly increasing and $\bar{\pi}_S(0) = 0$ one has $\bar{\pi}_S(d) < 0$ for all $d > 0$ and thus $h'(d) = 0$ is impossible for $a \le \frac{1}{2}$. A second order sufficient condition for a local maximum is $h''(d^*) < 0$. Since $h'(d) = a \cdot \{(2a-1)F_S(d)\pi_S(d) - \bar{F}_S(d)\bar{\pi}_S(d)\}$ one gets further

$$h''(d) = a \cdot \left\{(2a-1)(f_S(d)\pi_S(d) - F_S(d)\bar{F}_S(d)) + f_S(d)\bar{\pi}_S(d) - F_S(d)\bar{F}_S(d)\right\}$$
$$= a \cdot \left\{f_S(d)((2a-1)\pi_S(d) + \bar{\pi}_S(d)) - 2aF_S(d)\bar{F}_S(d)\right\},$$

which implies the condition (C3.2). The condition (C3.4) is shown as in the preceding proofs. □

Concerning the relationship between the cedent's and reinsurer's VaR economic capitals and the corresponding expected profits, a result similar to Corollary 3.1 is obtained.

**Corollary 4.1.** Under the assumptions of Proposition 4.1 and $a \in (\frac{1}{2}, 1]$, the reinsurer's expected profit of the optimal design, as a function of the retained VaR economic capital, is given by

$$EG_r^* = EC_\alpha^{VaR}\left[S_c^*\right] + (1-a) \cdot \left\{d^* + m_S(d^*) - Q_S(\alpha)\right\}. \qquad (4.1)$$

Moreover, in the special case $P = Q_S(\alpha)$ (percentile premium calculation principle), the retained expected profit, as a function of the ceded VaR economic capital, is given by

$$EG_c^* = EC_\alpha^{VaR}\left[S_r^*\right] + (1-a) \cdot \left\{Q_S(\alpha) - d^* - m_S(d^*)\right\}. \qquad (4.2)$$

**Proof.** The retained VaR economic capital equals by (3.2) (using (C3.3))

$$EC_\alpha^{VaR}\left[S_c^*\right] = VaR_\alpha\left[T_c^*\right] - EG_r^* - \mu = a \cdot g(d^*) + (1-a) \cdot Q_S(\alpha) - \theta_r^* a\pi_S(d^*) - \mu =$$
$$a \cdot \left(d^* + m_S(d^*)\right) + (1-a) \cdot Q_S(\alpha) - a \cdot F_S(d^*)m_S(d^*) - \mu$$
$$= d^* - \mu + \bar{F}_S(d^*)m_S(d^*) + (1-a) \cdot \left\{F_S(d^*)m_S(d^*) + Q_S(\alpha) - d^* + m_S(d^*)\right\}$$
$$= \bar{\pi}_S(d^*) + (1-a) \cdot \left\{F_S(d^*)m_S(d^*) + Q_S(\alpha) - d^* + m_S(d^*)\right\}.$$

On the hand, using that $a \cdot m_S(d^*) = \bar{m}_S(d^*) + (1-a) \cdot m_S(d^*)$ by (C3.1) (and (C3.3)), the reinsurer's expected profit in (3.3) can be rewritten as

$$EG_r^* = \theta_r^* a\pi_S(d^*) = a \cdot F_S(d^*) \cdot m_S(d^*) = \bar{\pi}_S(d^*) + (1-a) \cdot F_S(d^*)m_S(d^*).$$

Comparing both quantities one obtains (4.1). In the special case one gets from (3.4) that

$$EC_\alpha^{VaR}\left[S_r^*\right] = EC_\alpha^{VaR}[S] - EC_\alpha^{VaR}\left[S_c^*\right] = Q_S(\alpha) - \mu - EG_r^*$$
$$+(1-a) \cdot \left\{d^* + m_S(d^*) - Q_S(\alpha)\right\} = EG_c^* + (1-a) \cdot \left\{d^* + m_S(d^*) - Q_S(\alpha)\right\},$$

where the last equality follows from the relationship $Q_S(\alpha) - \mu = P - \mu = EG_c^* + EG_r^*$. □

**Remarks 4.1.**

(i) The conditions (C3.1) and (C3.2) for the pure stop-loss contract $a = 1$ have been further discussed and analyzed in Hürlimann (1999). The results in Corollary 4.1 are new. They provide an alternative to Corollary 3.3 for identifying the unknown optimal priority in Case 1 of Theorem 2.1 (see point (iii) of the Remarks 3.2). Moreover, they suggest further application in risk management:

(ii) Consider "internal reinsurance" in a (re)insurance company or a "reinsurance captive" owned by a corporate business firm that acts as insurer either directly or via another direct insurance captive. The property $EC_\alpha^{VaR}[S_c^*] = EG_r^*$ means that the cedent's required VaR economic capital is strongly mean financed by the reinsurer's profit. What does this mean in practice? Let split the insurance loss of a firm between a direct insurance captive (acting as cedent) and a reinsurance captive in this optimal way. Since a reinsurance captive does not need to generate profit from their reinsurance business, its profit can be transferred to the direct insurance captive that covers herewith in the mean its required VaR economic capital. As a consequence only the VaR economic capital of the reinsurance captive must be reserved by the firm. Though the reinsurer's required VaR economic capital, given by $EC_\alpha^{VaR}[S_r^*] = Q_S(\alpha) - \mu - \bar{\pi}_S(d^*)$, may be quite important, it is clearly less than the required VaR economic capital $EC_\alpha^{VaR}[S] = Q_S(\alpha) - \mu$ for the original loss faced without a reinsurance arrangement. Though this implies a release of required VaR economic capital this is not a surprise per se because the premium for the original loss will include and finance in the mean the reinsurer's profit. Therefore, from an economic point of view the release of economical capital is just transferred to the original premium without a priori any economic benefit.

(iii) In continuation of the discussion in (ii), in the special case of Corollary 4.1, the reinsurer's VaR economic capital also coincides with the retained expected profit. In this situation an overall strongly mean financing strategy for an internal reinsurance or for a two party captive structure (direct insurance captive and reinsurance captive) can be designed by exchanging the profits between cedent and reinsurer. Setting the market premium equal to the percentile premium $P = Q_S(\alpha)$ fulfills automatically the VaR economic capital requirements of both the cedent and the reinsurer. In this optimal way, the required VaR economic capital of a firm is not only mean financed but also mean self-financed by the cedent's and reinsurer's profits. Again, this is not a surprise because the VaR economic capital of the original loss is already strongly mean financed by the original profit in virtue of the equality $E[G] = E[P - S] = Q_S(\alpha) - \mu = EC_\alpha^{VaR}[S]$, and a splitting of the original loss does not make any difference in the overall.

The created value or relevance of the reinsurance splitting strategy is beforehand risk theoretical (also in the situation (ii) above). Indeed, under the optimal strategy, the total insurance loss fluctuations are minimized in terms of the total variance risk measure. By definition (4.1) these fluctuations are clearly less than the original insurance loss fluctuations as measured by the variance.

The next result states useful sufficient conditions for the existence of a solution to (C3.1).

**Proposition 4.2.** Assume the distribution $F_S(x)$ of the loss is strictly increasing and continuous on $(0,\infty)$ and has finite mean $\mu$ and variance $\sigma_S^2$. Two cases are distinguished.

**Case 1:** $a = 1$

If $F_S(\mu) \geq \frac{1}{2}$ then there exists $d \geq \mu$ such that $g(d) = F_S(d)\pi_S(d) - \bar{F}_S(d)\bar{\pi}_S(d) = 0$.

**Case 2:** $a \in (\frac{1}{2}, 1]$

If the squared coefficient of variation satisfies the inequality $k_S^2 = \left(\frac{\sigma_S}{\mu}\right)^2 < C(a,\mu) = \left(\frac{1-2(1-a)F_S(\mu)}{1-2aF_S(\mu)}\right)^2 - 1$ then there exists $0 < d \leq \mu$ such that $g(d) = (2a-1)F_S(d)\pi_S(d) - \bar{F}_S(d)\bar{\pi}_S(d) = 0$.

**Proof.** Case 1 is Theorem 4 in Hürlimann (1999). Its elementary probabilistic proof is based on the inequality of Bowers (1969)

$$\pi_S(d) \leq \tfrac{1}{2}\left(\sqrt{\sigma_S^2 + (d-\mu)^2} - (d-\mu)\right),$$

which is now used in a similar way to show Case 2. First, one observes that if $a > \frac{1}{2}$ there exists $d_1 > \mu$ such that $F_S(d_1) < 1/2a < 1$, hence (remember that $\bar{\pi}_S(x) = x - \mu + \pi_S(x)$)

$$g(d_1) = -\left[1 - 2aF_S(d_1)\right]\pi_S(d_1) - (d_1 - \mu)\bar{F}_S(d) < 0.$$

By continuity of the function $g(d)$ it suffices to show the existence of $0 < d_2 \leq \mu < d_1$ such that $g(d_2) \geq 0$. Let $0 < d \leq \mu$ and note that $F_S(d) \leq F_S(\mu) < F_S(d_1) < 1/2a$. It follows that

$$g(d) = (\mu - d)\bar{F}_S(d) - \left[1 - 2aF_S(d)\right]\pi_S(d) \geq (\mu - d)\bar{F}_S(\mu) - \left[1 - 2aF_S(\mu)\right]\pi_S(d).$$

By the above inequality of Bowers a sufficient condition for $g(d) \geq 0$ is

$\frac{1}{2}\left(\sqrt{\sigma_S^2+(d-\mu)^2}-(d-\mu)\right) \le \left(\frac{1-F_S(\mu)}{1-2aF_S(\mu)}\right)(\mu-d)$, or equivalently

$\sqrt{\sigma_S^2+(d-\mu)^2} \le \left(\frac{1-2(1-a)F_S(\mu)}{1-2aF_S(\mu)}\right)(\mu-d)$.

By definition of $C(a,\mu)$, which is non-negative for all $a \in (\frac{1}{2},1]$, the preceding inequality is further equivalent to $d \le \mu - \sigma_S/\sqrt{C(a,\mu)}$. With the made assumption, any $d_2 \in \left(0, \mu-\sigma_S/\sqrt{C(a,\mu)}\right]$ satisfies the inequality $g(d_2) \ge 0$. Case 2 is shown. □

To complete the results, we show that it is not possible to minimize the total variance risk measure through simultaneous variation of the partial stop-loss factor and the priority (a dual version of Theorem 2 in Hürlimann (1999)).

**Proposition 4.3.** Given is a loss $S$ with finite variance $\sigma_S^2$. The bi-dimensional optimization problem $Cov\left[S_c(a^*,d^*),S_r(a^*,d^*)\right] = \min_{a\in(0,1],d>0} Cov\left[S_c(a,d),S_r(a,d)\right]$ for the partial stop-loss function $f(x) = a\cdot(x-d)_+$, $a\in(0,1]$, $d>0$ has no solution unless $a = 1$.

**Proof.** The covariance between the retained and ceded loss reads $Cov[S_c,S_r] = h(a,d)$ with $h(a,d) = a\cdot\{(1-a)\cdot\sigma_S^2(d)+\pi_S(d)\cdot\bar{\pi}_S(d)\}$ (see the proof of Proposition 4.1). A calculation shows that the first order conditions $h_a = h_d = 0$ for a maximum over the domain $a\in(0,1]$, $d>0$, are equivalent to the equations (the expression for $h_d$ is found in the proof of Proposition 4.1)

$$(2a-1)\cdot\sigma_S^2(d) = \pi_S(d)\cdot\bar{\pi}_S(d),\ (2a-1)\cdot F_S(d)\cdot\pi_S(d) = \bar{F}_S(d)\cdot\bar{\pi}_S(d). \quad (4.3)$$

Solve both equations for the unknown parameter $a$ and equate the obtained expressions to get the condition

$$\sigma_S^2(d) = \frac{F_S(d)}{\bar{F}_S(d)}\cdot\pi_S(d)^2. \quad (4.4)$$

Applying the generalized inequality of Kremer (1990) (see Hürlimann (1994a, 1997a/b)) one has

$$\sigma_S^2(d) \ge \frac{F_S(d)}{\bar{F}_S(d)}\cdot\pi_S(d)^2. \quad (4.5)$$

Since there is in general strict inequality only the equality case must be considered. In this situation we argue as in Case 2 of the proof of Theorem 2 in Hürlimann (1999). □

## 5. Case study

Two frequently encountered distributions used to approximate the aggregate loss of an insurance portfolio are the lognormal and gamma distributions. We note that it is very natural to choose a log-normal distribution as it is compliant with the Solvency II non-life SCR standard formula. The gamma distribution can be justified as limiting distribution of a compound Poisson distribution with gamma distributed claim size (e.g. Hürlimann(2002), Theorem 2). For simplicity only these loss distributions are considered in the present case study.

### 5.1. Lognormal distribution of the loss

Suppose that the loss has a lognormal survival distribution $\bar{F}_S(x) = \bar{\Phi}((\ln x - \upsilon)/\tau)$, $\Phi(x)$ the standard normal distribution, whose parameters satisfy the relationships

$$\tau = \sqrt{\ln(1+k_S^2)}, \quad \upsilon = \ln(\mu) - \tfrac{1}{2}\tau^2, \quad k_S = \frac{\sigma_S}{\mu}. \tag{5.1}$$

Besides the distribution function, full optimization calculations depend upon the specification of the density, its first derivative, the quantile function and the stop-loss transform. These functions are determined by the following formulas:

$$f_S(x) = \frac{1}{\tau \cdot x}\,\varphi\!\left(\frac{\ln x - \upsilon}{\tau}\right), \varphi(x) = \Phi'(x),\ f_S'(x) = \frac{-1}{\tau \cdot x}\cdot\left(\frac{\ln x - \upsilon}{\tau} + \tau\right)\cdot f_S(x),$$
$$Q_S(u) = \exp\!\left(\Phi^{-1}(u)\cdot\tau - \frac{1}{2}\tau^2\right),\ \pi_S(x) = \mu\cdot\bar{F}_S(x\cdot\exp(-\tau^2)) - x\cdot\bar{F}_S(x). \tag{5.2}$$

### 5.2. Gamma distribution of the loss

Suppose that the loss has a gamma survival distribution $\bar{F}_S(x) = \bar{\Gamma}(\beta \cdot x; \gamma)$, $\Gamma(x;\gamma)$ the incomplete gamma function, whose parameters satisfy the relationships

$$\gamma = \frac{1}{k_S^2}, \quad \beta = \frac{1}{k_S^2\cdot\mu}, \quad k_S = \frac{\sigma_S}{\mu}. \tag{5.3}$$

Full optimization specification is based on the following formulas:

$$f_S(x) = \beta\cdot h(\beta\cdot x;\gamma),\ h(x;\gamma) = \frac{x^{\gamma-1}\cdot e^{-x}}{\Gamma(\gamma)},\ f_S'(x) = -\left(\beta - \frac{\gamma-1}{x}\right)\cdot f_S(x),$$
$$Q_S(u) = \beta^{-1}\cdot\Gamma^{-1}(x;\gamma),\ \pi_S(x) = \mu\cdot\bar{\Gamma}(\beta\cdot x;\ \gamma+1) - x\cdot\bar{F}_S(x). \tag{5.4}$$

### 5.3. Numerical implementation and illustration

To illustrate the risk management application of the obtained results we consider lognormal (lnN) and Gamma (G) distributed losses with a fixed mean $\mu = 100$ and varying coefficient of variation $k_S = 0.05 \cdot j$, $j = 1, 2, 3, 4, 5$. The market premium is set according to the percentile or the standard deviation premium calculation principle. We consider the optimal partial stop-loss contract under the three optimality criteria (P2.1), (P2.2) and (P2.3) for the four different cases:

Case 1: $P = Q_S(\alpha), \quad a = 1$

Case 2: $P = (1 + 3k_S) \cdot \mu, \quad a = 1$

Case 3: $P = Q_S(\alpha), \quad a = 0.75$

Case 4: $P = (1 + 3k_S) \cdot \mu, \quad a = 0.75$

To fix ideas the confidence level is set at $\alpha = 99.5\%$, which is regulatory compliant with Solvency II. In the ambiguous situation $a = 1$ of criterion (P2.1), which allows for varying reinsurance loading factors, only the extreme reinsurance loading factor $\theta_r = F_S(d^*_{\max}) / \bar{F}_S(d^*_{\max})$ is chosen in the numerical illustration. In the Cases 1 and 3 the market loading factor without reinsurance equals $\theta = (Q_S(\alpha) - \mu)/\mu$, and in the Cases 2 and 4 it is equal to $\theta = 3k_S$. Note that for the lognormal distribution with $k_S = 0.145$ the two market premiums approximately coincide, that is $Q_S(0.995) \cong (1 + 3k_S) \cdot \mu$. This corresponds to the distribution used to calibrate the Solvency II standard non-life SCR formula.

Numerical implementation is straightforward, especially if it is done with the aid of a computer algebra system (e.g. MATHCAD). For example, to find the optimal pair $(d^*, \theta_r^*)$ of problem (P2.2) for a distribution function $F_S(x)$ one proceeds as follows. Ensure first that the related probabilistic functions $f_S(x)$, $f_S'(x)$, $Q_S(\alpha)$, $\pi_S(x)$, $m_S(x)$ are available (formulas (5.2) and (5.4) in the illustration). Then, according to Proposition 3.2, one solves the implicit equation (C2.1) using a suitable software functionality (e.g. the "solver" in MATHCAD). If the condition (C2.2) for $d^*$ and the assumption $Q_S(\alpha) > d^*$ are satisfied a local maximum has been found. The optimal loading factors $(\theta_r^*, \theta_c^*)$ are obtained from the expressions in (C2.2) and (C.3).

### 5.4. Mean financing properties: risk transfer versus risk and profit transfer

Besides calculation of the economic capital and expected profit for the cedent and reinsurer, we compare the mean financing properties of the optimal solutions under the three optimality criteria. We do this for two kinds of risk sharing arrangements. The first one is classical reinsurance. This is a pure *risk transfer* (RT) under which the reinsurer takes over the ceded loss against a predetermined reinsurance premium. The second one is an alternative *risk and profit* *transfer* (RPT) under which the reinsurer takes over the ceded loss against a predetermined reinsurance premium and the profits of the cedent and reinsurer are exchanged (e.g. design of internal reinsurance or setting a reinsurance captive).

The interpretation of Corollary 3.1 has shown that the required VaR economic capitals for the optimal RT under criterion (P2.1) is strongly mean financed by the corresponding profits for both the cedent and reinsurer provided $a < 1$ (typically Case 3), for which one has necessarily $d^* = d^*_{\max}$. In Case 1, that is $a = 1$, this property holds only for the extreme reinsurance loading factor $\theta_r = F_S(d^*_{\max}) / \bar{F}_S(d^*_{\max})$, which is the choice made in our numerical case study. Nevertheless, for $\theta_r \in (F_S(0) / \bar{F}_S(0),\ F_S(d^*_{\max}) / \bar{F}_S(d^*_{\max}))$ an *overall strongly mean financing property* holds in the sense that the following equation holds (trivially fulfilled in case $P = Q_S(\alpha)$):

$$EC_\alpha^{VaR}[S] = EC_\alpha^{VaR}[S_c] + EC_\alpha^{VaR}[S_r] = EG_c + EG_r \tag{5.5}$$

For $a = 1$, Corollary 4.1 can be interpreted as a strongly mean financing property for the optimal RPT under criterion (P2.3) because the cedent and reinsurer exchange their profits. To measure the degree of mean finance of a RT and RPT, it is useful to consider the following indices.

**Definitions 5.1.** The *RT indices of mean finance* for the cedent and reinsurer are defined by

$$IMF_c^{RT} := \frac{EG_c - EC_\alpha^{VaR}[S_c]}{EC_\alpha^{VaR}[S]},\ IMF_r^{RT} := \frac{EG_r - EC_\alpha^{VaR}[S_r]}{EC_\alpha^{VaR}[S]}, \tag{5.6}$$

The *RPT indices of mean finance* for the cedent and reinsurer are defined by

$$IMF_c^{RPT} := \frac{EG_r - EC_\alpha^{VaR}[S_c]}{EC_\alpha^{VaR}[S]},\ IMF_r^{RPT} := \frac{EG_c - EC_\alpha^{VaR}[S_r]}{EC_\alpha^{VaR}[S]} \tag{5.7}$$

Clearly, a VaR economic capital is weakly (strongly) mean financed if and only if an index is non-negative (vanishes). According to (5.5) the required overall VaR economic capital is weakly (strongly) mean financed by the profits if the corresponding sums of indices in (5.6) respectively (5.7) are non-negative (vanish). The indices of mean finance are non-negative for both the cedent and reinsurer under the following conditions:

$$IMF_c^{RT} \geq 0 \wedge IMF_r^{RT} \geq 0 \quad \forall k_S, F_S(x) \Leftrightarrow (P2.1),\ P \geq Q_S(\alpha),\ a \in (0,1] \tag{5.8}$$

$$IMF_c^{RPT} \geq 0 \wedge IMF_r^{RPT} \geq 0 \quad \forall k_S, F_S(x) \Leftrightarrow (P2.3),\ P \geq Q_S(\alpha),\ a = 1 \tag{5.9}$$

Table 5.1 lists besides the optimal pairs $(d^*, \theta_r^*)$ the cedent's and reinsurer's VaR economic capitals and expected profits for the three optimality criteria under the four Cases. Table 5.2 displays the corresponding indices of mean finance. The abbreviation CV for coefficient of variation is used in the Tables and discussion.

The most important properties and dependencies can be read off from the Tables as follows:

(1) The optimal pairs $(d^*, \theta_r^*)$ depend on the loss distribution but not on the market premium, and depend on the partial stop-loss factor $a$ only under (P2.3). For criterion (P2.1) the values of $d^*, \theta_r^*$ are the highest possible ones (extreme reinsurance loading factor) but any other values satisfying $0 < d^* = d(\theta_r^*) \leq d^*_{\max}$ could have been chosen as already explained in the text.

(2) The optimal pairs $(d^*, \theta_r^*)$ for criterion (P2.1) are similar for the lognormal and gamma distributions. For the criteria (P2.2)-(P2.3) they differ more and more by increasing CV. In particular, $\theta_r^*$ increases much faster under a lognormal distribution than under a gamma distribution by increasing CV for the criterion (P2.2).

(3) The optimal priority is increasing with increasing CV except for criterion (P2.3), $a = 0.75$. In this situation it is decreasing with increasing CV.

(4) The optimal reinsurance loading factor is increasing with increasing CV for the criteria (P2.2)-(P2.3), but it is decreasing with increasing CV for criterion (P2.1).

(5) The properties of the optimal priority solving the equation (C3.1) for criterion (P2.3) are instructive. In case $a = 1$ the optimal priority is close to the mean but above it, in agreement with Case 1 of Proposition 4.2 and the observations already made in former papers (e.g. Hürlimann (1994b), Hürlimann (1999), Section 4). For $a \in (\frac{1}{2}, 1]$ the optimal priority is less than the mean, in agreement with Case 2 of Proposition 4.2. One can argue that a priority less than the mean is not meaningful, as done in Hürlimann(1999), hence only the pure stop-loss case $a = 1$ might be practical.

(6) For fixed $a$ the VaR economic capitals of the cedent and reinsurer as well as the reinsurer's expected profits do not depend upon the market premium but depend on the loss distribution. The expected profit of the cedent depends on $a$, the market premium and the loss distribution. Maximizing the reinsurer's expected profit under criterion (P2.2) requires a high increase of economic capital for a moderate increase in expected profit when compared to criterion (P2.1). However, the corresponding reinsurance premiums are much more competitive for criterion (P2.2) in this situation (much smaller reinsurance loading factors).

(7) The overall VaR economic capital is strongly mean financed by the profits in the Cases 1 and 3 under all criteria. Since $P = Q_S(\alpha)$ this is a trivial property. Accordingly, in the Cases 2 and 4, the same weak property only holds for those smaller CV's satisfying $P > Q_S(\alpha)$.

(8) The cedant's and reinsurer'sVaR economic capitals are strongly mean financed by their own profits (Cases 1 and 3 for the RT under criterion (P2.1)) respectively by their exchanged profits (Case 1 for the RPT under criterion (P2.3)). The same weak properties hold in the Cases 2 and 4 for the RT, respectively Case 2 for the RPT, provided the CV's are sufficiently small to satisfy $P > Q_S(\alpha)$.

TABLE 5.1

ECONOMIC CAPITAL AND EXPECTED PROFIT FOR OPTIMAL PAIRS $(d^*, \theta_r^*)$

CASE 1: Market premium = percentile premium to confidence level 99.5%, $a = 1$

| criterion: | | | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CV | model | P | $d^*$ | | | $\theta_r^*$ | | | EC cedent | | | EG cedent | | | EC reinsurer | | | EG reinsurer | | |
| 5% | lnN | 113.6 | 111.6 | 104.2 | 100.2 | 75.54 | 4.01 | 1.12 | 11.6 | 4.8 | 2.1 | 11.6 | 11.2 | 11.5 | 2.0 | 8.8 | 11.5 | 2.0 | 2.4 | 2.1 |
| | G | 113.3 | 111.5 | 104.0 | 100.1 | 75.75 | 3.72 | 1.08 | 11.5 | 4.6 | 2.1 | 11.5 | 11.0 | 11.3 | 1.9 | 8.8 | 11.3 | 1.9 | 2.3 | 2.1 |
| 10% | lnN | 128.7 | 124.2 | 109.7 | 100.9 | 74.97 | 5.11 | 1.25 | 24.2 | 10.7 | 4.4 | 24.2 | 23.6 | 24.2 | 4.4 | 18.0 | 24.2 | 4.4 | 5.1 | 4.4 |
| | G | 127.6 | 123.5 | 108.7 | 100.6 | 75.43 | 4.26 | 1.16 | 23.6 | 9.8 | 4.3 | 23.6 | 22.8 | 23.3 | 4.0 | 17.8 | 23.3 | 4.0 | 4.9 | 4.3 |
| 15% | lnN | 145.2 | 137.7 | 117.2 | 102.0 | 74.39 | 6.83 | 1.39 | 37.8 | 18.4 | 7.0 | 37.8 | 37.1 | 38.2 | 7.4 | 26.9 | 38.2 | 7.4 | 8.1 | 7.0 |
| | G | 142.8 | 136.2 | 114.2 | 101.3 | 75.13 | 4.91 | 1.25 | 36.3 | 15.8 | 6.7 | 36.3 | 35.3 | 36.2 | 6.5 | 27.1 | 36.2 | 6.5 | 7.6 | 6.7 |
| 20% | lnN | 163.3 | 152.1 | 127.7 | 103.5 | 73.82 | 9.97 | 1.55 | 52.2 | 28.9 | 9.9 | 52.2 | 51.7 | 53.4 | 11.1 | 34.5 | 53.4 | 11.1 | 11.6 | 9.9 |
| | G | 159.0 | 149.5 | 120.7 | 102.3 | 74.87 | 5.69 | 1.34 | 49.7 | 22.6 | 9.2 | 49.7 | 48.4 | 49.7 | 9.3 | 36.4 | 49.7 | 9.3 | 10.5 | 9.2 |
| 25% | lnN | 182.9 | 167.3 | 144.4 | 105.5 | 73.25 | 17.84 | 1.73 | 67.5 | 45.3 | 13.0 | 67.5 | 67.3 | 69.9 | 15.4 | 37.6 | 69.9 | 15.4 | 15.6 | 13.0 |
| | G | 176.0 | 163.4 | 128.3 | 103.7 | 74.63 | 6.62 | 1.44 | 63.6 | 30.3 | 12.0 | 63.6 | 62.3 | 64.1 | 12.4 | 45.7 | 64.1 | 12.4 | 13.7 | 12.0 |

CASE 2: Market premium = mean plus three standard deviations, $a = 1$

| criterion: | | | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CV | model | P | $d^*$ | | | $\theta_r^*$ | | | EC cedent | | | EG cedent | | | EC reinsurer | | | EG reinsurer | | |
| 5% | lnN | 115.0 | 111.6 | 104.2 | 100.2 | 75.54 | 4.01 | 1.12 | 11.6 | 4.8 | 2.1 | 13.0 | 12.6 | 12.9 | 2.0 | 8.8 | 11.5 | 2.0 | 2.4 | 2.1 |
| | G | 115.0 | 111.5 | 104.0 | 100.1 | 75.75 | 3.72 | 1.08 | 11.5 | 4.6 | 2.1 | 13.1 | 12.7 | 12.9 | 1.9 | 8.8 | 11.3 | 1.9 | 2.3 | 2.1 |
| 10% | lnN | 130.0 | 124.2 | 109.7 | 100.9 | 74.97 | 5.11 | 1.25 | 24.2 | 10.7 | 4.4 | 25.6 | 24.9 | 25.6 | 4.4 | 18.0 | 24.2 | 4.4 | 5.1 | 4.4 |
| | G | 130.0 | 123.5 | 108.7 | 100.6 | 75.43 | 4.26 | 1.16 | 23.6 | 9.8 | 4.3 | 26.0 | 25.1 | 25.7 | 4.0 | 17.8 | 23.3 | 4.0 | 4.9 | 4.3 |
| 15% | lnN | 145.0 | 137.7 | 117.2 | 102.0 | 74.39 | 6.83 | 1.39 | 37.8 | 18.4 | 7.0 | 37.6 | 36.9 | 38.0 | 7.4 | 26.9 | 38.2 | 7.4 | 8.1 | 7.0 |
| | G | 145.0 | 136.2 | 114.2 | 101.3 | 75.13 | 4.91 | 1.25 | 36.3 | 15.8 | 6.7 | 38.5 | 37.4 | 38.3 | 6.5 | 27.1 | 36.2 | 6.5 | 7.6 | 6.7 |
| 20% | lnN | 160.0 | 152.1 | 127.7 | 103.5 | 73.82 | 9.97 | 1.55 | 52.2 | 28.9 | 9.9 | 48.9 | 48.4 | 50.1 | 11.1 | 34.5 | 53.4 | 11.1 | 11.6 | 9.9 |
| | G | 160.0 | 149.5 | 120.7 | 102.3 | 74.87 | 5.69 | 1.34 | 49.7 | 22.6 | 9.2 | 50.7 | 49.5 | 50.8 | 9.3 | 36.4 | 49.7 | 9.3 | 10.5 | 9.2 |
| 25% | lnN | 175.0 | 167.3 | 144.4 | 105.5 | 73.25 | 17.84 | 1.73 | 67.5 | 45.3 | 13.0 | 59.6 | 59.4 | 62.0 | 15.4 | 37.6 | 69.9 | 15.4 | 15.6 | 13.0 |
| | G | 175.0 | 163.4 | 128.3 | 103.7 | 74.63 | 6.62 | 1.44 | 63.6 | 30.3 | 12.0 | 62.6 | 61.3 | 63.0 | 12.4 | 45.7 | 64.1 | 12.4 | 13.7 | 12.0 |

CASE 3: Market premium = percentile premium to confidence level 99.5%, $a = 0.75$

| criterion: | | | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CV | model | P | $d^*$ | | | $\theta_r^*$ | | | EC cedent | | | EG cedent | | | EC reinsurer | | | EG reinsurer | | |
| 5% | lnN | 113.6 | 111.6 | 104.2 | 96.5 | 75.54 | 4.01 | 0.32 | 12.1 | 7.0 | 3.9 | 12.1 | 11.8 | 12.6 | 1.5 | 6.6 | 9.7 | 1.5 | 1.8 | 1.0 |
| | G | 113.3 | 111.5 | 104.0 | 96.4 | 75.75 | 3.72 | 0.31 | 11.9 | 6.8 | 3.8 | 11.9 | 11.6 | 12.4 | 1.4 | 6.6 | 9.5 | 1.4 | 1.7 | 1.0 |
| 10% | lnN | 128.7 | 124.2 | 109.7 | 93.5 | 74.97 | 5.11 | 0.36 | 25.3 | 15.2 | 8.2 | 25.3 | 24.9 | 26.5 | 3.3 | 13.5 | 20.4 | 3.3 | 3.8 | 2.2 |
| | G | 127.6 | 123.5 | 108.7 | 93.1 | 75.43 | 4.26 | 0.34 | 24.6 | 14.3 | 7.9 | 24.6 | 22.8 | 25.6 | 3.0 | 13.4 | 19.7 | 3.0 | 3.6 | 2.1 |
| 15% | lnN | 145.2 | 137.7 | 117.2 | 91.1 | 74.39 | 6.83 | 0.41 | 39.6 | 25.1 | 13.0 | 39.6 | 39.1 | 41.8 | 5.6 | 20.1 | 32.2 | 5.6 | 6.1 | 3.4 |
| | G | 142.8 | 136.2 | 114.2 | 90.3 | 75.13 | 4.91 | 0.37 | 38.0 | 22.6 | 12.3 | 38.0 | 37.2 | 39.6 | 4.9 | 20.3 | 30.5 | 4.9 | 5.7 | 3.3 |
| 20% | lnN | 163.3 | 152.1 | 127.7 | 89.1 | 73.82 | 9.97 | 0.46 | 55.0 | 37.5 | 18.3 | 55.0 | 54.6 | 58.5 | 8.3 | 25.8 | 45.1 | 8.3 | 8.7 | 4.9 |
| | G | 159.0 | 149.5 | 120.7 | 87.8 | 74.87 | 5.69 | 0.40 | 52.0 | 31.7 | 17.0 | 52.0 | 51.1 | 54.4 | 7.0 | 27.3 | 42.0 | 7.0 | 7.9 | 4.5 |
| 25% | lnN | 182.9 | 167.3 | 144.4 | 87.6 | 73.25 | 17.84 | 0.51 | 71.3 | 54.7 | 23.9 | 71.3 | 71.2 | 76.5 | 11.6 | 28.2 | 59.0 | 11.6 | 11.7 | 6.4 |
| | G | 176.0 | 163.4 | 128.3 | 85.7 | 74.63 | 6.62 | 0.44 | 66.7 | 41.8 | 22.0 | 66.7 | 65.7 | 70.1 | 9.3 | 34.3 | 54.0 | 9.3 | 10.3 | 5.9 |

CASE 4: Market premium = mean plus three standard deviations, $a = 0.75$

| criterion: | | | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CV | model | P | $d^*$ | | | $\theta_r^*$ | | | EC cedent | | | EG cedent | | | EC reinsurer | | | EG reinsurer | | |
| 5% | lnN | 115.0 | 111.6 | 104.2 | 96.5 | 75.54 | 4.01 | 0.32 | 12.1 | 7.0 | 3.9 | 13.5 | 13.2 | 14.0 | 1.5 | 6.6 | 9.7 | 1.5 | 1.8 | 1.0 |
| | G | 115.0 | 111.5 | 104.0 | 96.4 | 75.75 | 3.72 | 0.31 | 11.9 | 6.8 | 3.8 | 13.6 | 13.3 | 14.0 | 1.4 | 6.6 | 9.5 | 1.4 | 1.7 | 1.0 |
| 10% | lnN | 130.0 | 124.2 | 109.7 | 93.5 | 74.97 | 5.11 | 0.36 | 25.3 | 15.2 | 8.2 | 26.7 | 26.2 | 27.8 | 3.3 | 13.5 | 20.4 | 3.3 | 3.8 | 2.2 |
| | G | 130.0 | 123.5 | 108.7 | 93.1 | 75.43 | 4.26 | 0.34 | 24.6 | 14.3 | 4.3 | 27.0 | 26.4 | 27.9 | 3.0 | 13.4 | 19.7 | 3.0 | 3.6 | 2.1 |
| 15% | lnN | 145.0 | 137.7 | 117.2 | 91.1 | 74.39 | 6.83 | 0.41 | 39.6 | 25.1 | 13.0 | 39.4 | 38.9 | 41.6 | 5.6 | 20.1 | 32.2 | 5.6 | 6.1 | 3.4 |
| | G | 145.0 | 136.2 | 114.2 | 90.3 | 75.13 | 4.91 | 0.37 | 38.0 | 22.6 | 12.3 | 40.1 | 39.3 | 41.7 | 4.9 | 20.3 | 30.5 | 4.9 | 5.7 | 3.3 |
| 20% | lnN | 160.0 | 152.1 | 127.7 | 89.1 | 73.82 | 9.97 | 0.46 | 55.0 | 37.5 | 18.3 | 51.7 | 51.3 | 55.1 | 8.3 | 25.8 | 45.1 | 8.3 | 8.7 | 4.9 |
| | G | 160.0 | 149.5 | 120.7 | 87.8 | 74.87 | 5.69 | 0.40 | 52.0 | 31.7 | 17.0 | 53.0 | 52.1 | 55.5 | 7.0 | 27.3 | 42.0 | 7.0 | 7.9 | 4.5 |
| 25% | lnN | 175.0 | 167.3 | 144.4 | 87.6 | 73.25 | 17.84 | 0.51 | 71.3 | 54.7 | 23.9 | 63.4 | 63.3 | 68.6 | 11.6 | 28.2 | 59.0 | 11.6 | 11.7 | 6.4 |
| | G | 175.0 | 163.4 | 128.3 | 85.7 | 74.63 | 6.62 | 0.44 | 66.7 | 41.8 | 22.0 | 65.7 | 64.7 | 69.1 | 9.3 | 34.3 | 54.0 | 9.3 | 10.3 | 5.9 |

TABLE 5.2

INDICES OF MEAN SELF-FINANCE FOR OPTIMAL RT AND RPT

CASE 1: Market premium = percentile premium to confidence level 99.5%, $a = 1$

| optimality criterion: | | | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CV | model | premium | optimal priority | | | IMF_RT cedent | | | IMF_RT reinsurer | | | IMF_RPT cedent | | | IMF_RPT reinsurer | | |
| 5% | lnN | 113.6 | 111.6 | 104.2 | 100.2 | 0.00 | 0.47 | 0.69 | 0.00 | -0.47 | -0.69 | -0.71 | -0.18 | 0.00 | 0.71 | 0.18 | 0.00 |
| | G | 113.3 | 111.5 | 104.0 | 100.1 | 0.00 | 0.48 | 0.69 | 0.00 | -0.48 | -0.69 | -0.72 | -0.17 | 0.00 | 0.72 | 0.17 | 0.00 |
| 10% | lnN | 128.7 | 124.2 | 109.7 | 100.9 | 0.00 | 0.45 | 0.69 | 0.00 | -0.45 | -0.69 | -0.69 | -0.20 | 0.00 | 0.69 | 0.20 | 0.00 |
| | G | 127.6 | 123.5 | 108.7 | 100.6 | 0.00 | 0.47 | 0.69 | 0.00 | -0.47 | -0.69 | -0.71 | -0.18 | 0.00 | 0.71 | 0.18 | 0.00 |
| 15% | lnN | 145.2 | 137.7 | 117.2 | 102.0 | 0.00 | 0.41 | 0.69 | 0.00 | -0.41 | -0.69 | -0.67 | -0.23 | 0.00 | 0.67 | 0.23 | 0.00 |
| | G | 142.8 | 136.2 | 114.2 | 101.3 | 0.00 | 0.45 | 0.69 | 0.00 | -0.45 | -0.69 | -0.70 | -0.19 | 0.00 | 0.70 | 0.19 | 0.00 |
| 20% | lnN | 163.3 | 152.1 | 127.7 | 103.5 | 0.00 | 0.36 | 0.69 | 0.00 | -0.36 | -0.69 | -0.65 | -0.27 | 0.00 | 0.65 | 0.27 | 0.00 |
| | G | 159.0 | 149.5 | 120.7 | 102.3 | 0.00 | 0.44 | 0.69 | 0.00 | -0.44 | -0.69 | -0.68 | -0.20 | 0.00 | 0.68 | 0.20 | 0.00 |
| 25% | lnN | 182.9 | 167.3 | 144.4 | 105.5 | 0.00 | 0.27 | 0.69 | 0.00 | -0.27 | -0.69 | -0.63 | -0.36 | 0.00 | 0.63 | 0.36 | 0.00 |
| | G | 176.0 | 163.4 | 128.3 | 103.7 | 0.00 | 0.42 | 0.68 | 0.00 | -0.42 | -0.68 | -0.67 | -0.22 | 0.00 | 0.67 | 0.22 | 0.00 |

CASE 2: Market premium = mean plus three standard deviations, $a = 1$

| optimality criterion: | | | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CV | model | premium | optimal priority | | | IMF_RT cedent | | | IMF_RT reinsurer | | | IMF_RPT cedent | | | IMF_RPT reinsurer | | |
| 5% | lnN | 115.0 | 111.6 | 104.2 | 100.2 | 0.10 | 0.58 | 0.79 | 0.00 | -0.47 | -0.69 | -0.71 | -0.18 | 0.00 | 0.82 | 0.28 | 0.10 |
| | G | 115.0 | 111.5 | 104.0 | 100.1 | 0.12 | 0.60 | 0.81 | 0.00 | -0.48 | -0.69 | -0.72 | -0.17 | 0.00 | 0.84 | 0.29 | 0.12 |
| 10% | lnN | 130.0 | 124.2 | 109.7 | 100.9 | 0.05 | 0.50 | 0.74 | 0.00 | -0.45 | -0.69 | -0.69 | -0.20 | 0.00 | 0.74 | 0.24 | 0.05 |
| | G | 130.0 | 123.5 | 108.7 | 100.6 | 0.09 | 0.55 | 0.77 | 0.00 | -0.47 | -0.69 | -0.71 | -0.18 | 0.00 | 0.79 | 0.27 | 0.09 |
| 15% | lnN | 145.0 | 137.7 | 117.2 | 102.0 | 0.00 | 0.41 | 0.68 | 0.00 | -0.41 | -0.69 | -0.67 | -0.23 | 0.00 | 0.67 | 0.22 | 0.00 |
| | G | 145.0 | 136.2 | 114.2 | 101.3 | 0.05 | 0.50 | 0.74 | 0.00 | -0.45 | -0.69 | -0.70 | -0.19 | 0.00 | 0.75 | 0.24 | 0.05 |
| 20% | lnN | 160.0 | 152.1 | 127.7 | 103.5 | -0.05 | 0.31 | 0.63 | 0.00 | -0.36 | -0.69 | -0.65 | -0.27 | 0.00 | 0.60 | 0.22 | -0.05 |
| | G | 160.0 | 149.5 | 120.7 | 102.3 | 0.02 | 0.46 | 0.70 | 0.00 | -0.44 | -0.69 | -0.68 | -0.20 | 0.00 | 0.70 | 0.22 | 0.02 |
| 25% | lnN | 175.0 | 167.3 | 144.4 | 105.5 | -0.10 | 0.17 | 0.59 | 0.00 | -0.27 | -0.69 | -0.63 | -0.36 | 0.00 | 0.53 | 0.26 | -0.10 |
| | G | 175.0 | 163.4 | 128.3 | 103.7 | -0.01 | 0.41 | 0.67 | 0.00 | -0.42 | -0.68 | -0.67 | -0.22 | 0.00 | 0.66 | 0.21 | -0.01 |

CASE 3: Market premium = percentile premium to confidence level 99.5%, $a = 0.75$

| optimality criterion: | | | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CV | model | premium | optimal priority | | | IMF_RT cedent | | | IMF_RT reinsurer | | | IMF_RPT cedent | | | IMF_RPT reinsurer | | |
| 5% | lnN | 115.0 | 111.6 | 104.2 | 96.5 | 0.10 | 0.46 | 0.74 | 0.00 | -0.36 | -0.64 | -0.78 | -0.38 | -0.21 | 0.89 | 0.49 | 0.32 |
| | G | 115.0 | 111.5 | 104.0 | 96.4 | 0.12 | 0.48 | 0.76 | 0.00 | -0.36 | -0.64 | -0.79 | -0.38 | -0.21 | 0.91 | 0.50 | 0.34 |
| 10% | lnN | 130.0 | 124.2 | 109.7 | 93.5 | 0.05 | 0.38 | 0.68 | 0.00 | -0.34 | -0.64 | -0.77 | -0.40 | -0.21 | 0.82 | 0.44 | 0.26 |
| | G | 130.0 | 123.5 | 108.7 | 93.1 | 0.09 | 0.44 | 0.85 | 0.00 | -0.35 | -0.64 | -0.78 | -0.38 | -0.08 | 0.87 | 0.47 | 0.30 |
| 15% | lnN | 145.0 | 137.7 | 117.2 | 91.1 | 0.00 | 0.31 | 0.63 | 0.00 | -0.31 | -0.64 | -0.75 | -0.42 | -0.21 | 0.75 | 0.41 | 0.21 |
| | G | 145.0 | 136.2 | 114.2 | 90.3 | 0.05 | 0.39 | 0.69 | 0.00 | -0.34 | -0.64 | -0.77 | -0.39 | -0.21 | 0.82 | 0.44 | 0.26 |
| 20% | lnN | 160.0 | 152.1 | 127.7 | 89.1 | -0.05 | 0.22 | 0.58 | 0.00 | -0.27 | -0.63 | -0.74 | -0.45 | -0.21 | 0.68 | 0.40 | 0.16 |
| | G | 160.0 | 149.5 | 120.7 | 87.8 | 0.02 | 0.35 | 0.65 | 0.00 | -0.33 | -0.63 | -0.76 | -0.40 | -0.21 | 0.78 | 0.42 | 0.23 |
| 25% | lnN | 175.0 | 167.3 | 144.4 | 87.6 | -0.10 | 0.10 | 0.54 | 0.00 | -0.20 | -0.63 | -0.72 | -0.52 | -0.21 | 0.63 | 0.42 | 0.12 |
| | G | 175.0 | 163.4 | 128.3 | 85.7 | -0.01 | 0.30 | 0.62 | 0.00 | -0.32 | -0.63 | -0.75 | -0.41 | -0.21 | 0.74 | 0.40 | 0.20 |

CASE 4: Market premium = mean plus three standard deviations, $a = 0.75$

| optimality criterion: | | | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) | (P2.1) | (P2.2) | (P2.3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| CV | model | premium | optimal priority | | | IMF_RT cedent | | | IMF_RT reinsurer | | | IMF_RPT cedent | | | IMF_RPT reinsurer | | |
| 5% | lnN | 115.0 | 111.6 | 104.2 | 96.5 | 0.10 | 0.46 | 0.74 | 0.00 | -0.36 | -0.64 | -0.78 | -0.38 | -0.21 | 0.89 | 0.49 | 0.32 |
| | G | 115.0 | 111.5 | 104.0 | 96.4 | 0.12 | 0.48 | 0.76 | 0.00 | -0.36 | -0.64 | -0.79 | -0.38 | -0.21 | 0.91 | 0.50 | 0.34 |
| 10% | lnN | 130.0 | 124.2 | 109.7 | 93.5 | 0.05 | 0.38 | 0.68 | 0.00 | -0.34 | -0.64 | -0.77 | -0.40 | -0.21 | 0.82 | 0.44 | 0.26 |
| | G | 130.0 | 123.5 | 108.7 | 93.1 | 0.09 | 0.44 | 0.85 | 0.00 | -0.35 | -0.64 | -0.78 | -0.38 | -0.08 | 0.87 | 0.47 | 0.30 |
| 15% | lnN | 145.0 | 137.7 | 117.2 | 91.1 | 0.00 | 0.31 | 0.63 | 0.00 | -0.31 | -0.64 | -0.75 | -0.42 | -0.21 | 0.75 | 0.41 | 0.21 |
| | G | 145.0 | 136.2 | 114.2 | 90.3 | 0.05 | 0.39 | 0.69 | 0.00 | -0.34 | -0.64 | -0.77 | -0.39 | -0.21 | 0.82 | 0.44 | 0.26 |
| 20% | lnN | 160.0 | 152.1 | 127.7 | 89.1 | -0.05 | 0.22 | 0.58 | 0.00 | -0.27 | -0.63 | -0.74 | -0.45 | -0.21 | 0.68 | 0.40 | 0.16 |
| | G | 160.0 | 149.5 | 120.7 | 87.8 | 0.02 | 0.35 | 0.65 | 0.00 | -0.33 | -0.63 | -0.76 | -0.40 | -0.21 | 0.78 | 0.42 | 0.23 |
| 25% | lnN | 175.0 | 167.3 | 144.4 | 87.6 | -0.10 | 0.10 | 0.54 | 0.00 | -0.20 | -0.63 | -0.72 | -0.52 | -0.21 | 0.63 | 0.42 | 0.12 |
| | G | 175.0 | 163.4 | 128.3 | 85.7 | -0.01 | 0.30 | 0.62 | 0.00 | -0.32 | -0.63 | -0.75 | -0.41 | -0.21 | 0.74 | 0.40 | 0.20 |

## 6. CONCLUSIONS AND OUTLOOK

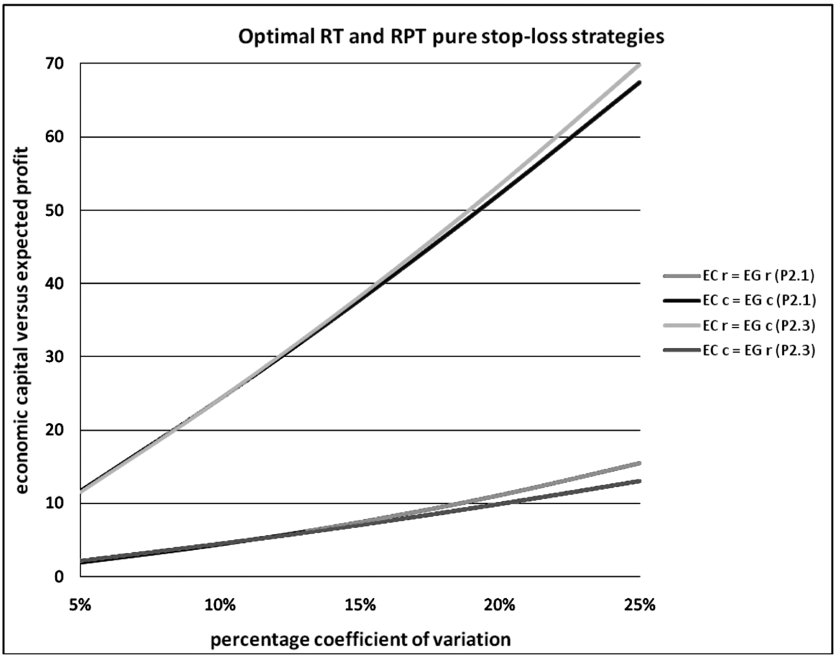
A summary of what has been accomplished by highlighting the main results and findings might be helpful. Our starting point has been the partial stop-loss contract. It has been identified as optimal reinsurance form under criterion (P2.1) by minimizing the value-at-risk measure of the total retained loss over the class of increasing and convex ceded loss functions (recent work by Cai et al. (2008), Cheung (2010) and Chi and Tan (2010)). Unless the maximum possible stop-loss priority is chosen, only the pure stop-loss contract will be optimal. This insight is a consequence of the reformulation in Proposition 3.1 of the Cases 1 and 2 of the original Theorem 2.1 (see (i) of the Remarks 3.2). But then, the reinsurance loading factor remains indeterminate and can vary over a finite range. This enables the reinsurer to take the opportunity to maximize his expected profit, which leads to the joint party optimality criterion (P2.2). Compared to (P2.1) the new criterion prescribes a sufficiently high confidence level. This restriction seems to be fulfilled in practical situations. Alternatively, stabilization of the total variance of the retained and ceded loss is considered by minimizing instead the total variance risk measure, which leads to the joint party optimality criterion (P2.3). As a first main result we obtain a complete description of the optimal reinsurance parameters under the single party criterion (P2.1) in Proposition 3.1. The Propositions 3.2 and 4.1 do the same for the criteria (P2.2) and (P2.3) by providing necessary and sufficient conditions for the local extremal solutions of the corresponding joint party optimization problems.

In view of the emerging importance of solvency systems, the relationship between the cedent's and reinsurer's VaR economic capitals and the expected profits is emphasized. Corollary 3.1 summarizes this for the criterion (P2.1) while Corollary 4.2 does it for the criterion (P2.3). The obtained results are interpreted in terms of mean financing properties, which turn out to be of some importance for the economics of pure risk transfer (classical reinsurance) or risk and profit transfer (design of internal reinsurance or reinsurance captive owned by the captive of a corporate firm). First of all, set the market premium equal to the percentile premium. Then, the following characterizations of the optimal risk transfer strategies can be formulated (consequences of the Corollaries 3.1 and 4.1):

(1) Consider an optimal pure risk transfer agreement under criterion (P2.1). The cedent's and reinsurer's required VaR economic capitals are both strongly mean financed by their profits, if and only if, the maximum optimal stop-loss priority is chosen. Moreover, in this situation, the cedent's VaR economic capital is minimum and the reinsurer's expected profit is maximum only for the pure stop-loss contract.

(2) Consider an optimal risk and profit transfer agreement under criterion (P2.3). The cedent's and reinsurer's required VaR economic capitals are both strongly mean financed by their exchanged profits, if and only if, the pure stop-loss contract is chosen. Moreover, in this situation, the cedent's VaR economic capital is minimum and the total variance of the retained and ceded loss is minimum.

To illustrate, assume a percentile market premium and that the characterizing property in (1) and (2) holds. It is remarkable to observe additionally that the optimal cedent's and reinsurer's expected profits under the criteria (P2.1) and (P2.3) are very close, especially for the lower coefficients of variation. The Graph 6.1 illustrates this finding for the lognormal insurance loss distribution. The big difference lies in the optimal priorities and reinsurance loading factors,




GRAPH 6.1: Comparison of the pure stop-loss optimal risk transfer strategies

which turn out to be much larger under criterion (P2.1). In particular, higher retained VaR economic capitals are required under (P2.1).

Finally, on cannot conclude without giving a brief outlook for possible work. Besides the CVaR risk measure, also called tail conditional expectation (CTE) risk measure, other important risk measures can be considered. For example, in the class of tail preserving risk measures, one finds the right-tail risk measure by Wang (1998) (see also Hürlimann (2004)) and the lookback risk measure (e.g. Hürlimann (1998/2003/2004)). Future research on this topic should include other optimal reinsurance forms like the limited and truncated stop-loss contracts (comments of Section 2) and the excess-of-loss reinsurance contract studied in Meng and Zhang (2010).

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