Characteristic Classes in Deformation Quantization

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In deformation quantization, one can associate five characteristic functions to (stable) formality morphisms on cochains and chains and to "two-brane" formality morphisms. We show that these characteristic functions agree.

1 Introduction

Let $T_{\mathrm{poly}}(\mathbb{R}^n) = \Gamma(\mathbb{R}^n, \wedge^\bullet T\mathbb{R}^n)$ be the space of multivector fields on $\mathbb{R}^n$ and let $D_{\mathrm{poly}}(\mathbb{R}^n)$ be the space of multidifferential operators on $\mathbb{R}^n$. The central result of deformation quantization is M. Kontsevich's Formality Theorem [16], stating that there is a Lie_∞ quasi-isomorphism

$$\mathcal{U}^{\mathrm{Kontsevich}} : T_{\mathrm{poly}}(\mathbb{R}^n)[1] \to D_{\mathrm{poly}}(\mathbb{R}^n)[1].$$

Here, we understand $T_{\mathrm{poly}}(\mathbb{R}^n)[1]$ as a Lie algebra endowed with the Schouten–Nijenhuis bracket and $D_{\mathrm{poly}}(\mathbb{R}^n)[1]$ as a Lie algebra endowed with the Gerstenhaber bracket. The differential forms $\Omega_\bullet(\mathbb{R}^n)$ on $\mathbb{R}^n$, with nonpositive grading, form a Lie module over $T_{\mathrm{poly}}(\mathbb{R}^n)[1]$, and similarly the (topological) Hochschild chains $C_\bullet(\mathbb{R}^n) = C_\bullet(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n))$ form a module over the multidifferential operators $D_{\mathrm{poly}}(\mathbb{R}^n)$. For a

more detailed description of these objects and the actions we refer the reader to [23]. It was conjectured by Tsygan [23] and shown by Shoikhet [22] that there is a Lie_∞ quasi-isomorphism of modules

$$\mathcal{V}^{\text{Shoikhet}} : C_\bullet(\mathbb{R}^n) \to \Omega_\bullet(\mathbb{R}^n).$$

A globalized version of this statement was shown by Dolgushev [10]. Here, the Lie_∞ action of $T_{\text{poly}}(\mathbb{R}^n)[1]$ on $C_\bullet(\mathbb{R}^n)$ is obtained by pulling back the action of $D_{\text{poly}}(\mathbb{R}^n)[1]$ on $C_\bullet(\mathbb{R}^n)$ via $\mathcal{U}^{\text{Kontsevich}}$. In particular, the statement that $\mathcal{V}^{\text{Shoikhet}}$ is a Lie_∞ morphism of modules implicitly references $\mathcal{U}^{\text{Kontsevich}}$.

The formality morphisms $\mathcal{U}^{\text{Kontsevich}}$ and $\mathcal{V}^{\text{Shoikhet}}$ are given by explicit "sum-of-graphs" formulas:

$$\mathcal{U}_k^{\text{Kontsevich}} = \sum_\Gamma c_\Gamma^{\text{Kontsevich}} D_\Gamma, \tag{1}$$

$$\mathcal{V}_k^{\text{Shoikhet}} = \sum_{\tilde{\Gamma}} \tilde{c}_{\tilde{\Gamma}}^{\text{Shoikhet}} \tilde{D}_{\tilde{\Gamma}}. \tag{2}$$

Here $\mathcal{U}_k$ (respectively, $\mathcal{V}_k$) is the kth component of the Lie_∞ morphism $\mathcal{U}$ (respectively, of $\mathcal{V}_k$). The top sum runs over the set of isomorphism classes of *Kontsevich graphs* with k Type I vertices. For the definition of these graphs, we refer the reader to [16], an example can be found in Figure 1. Finally,

$$D_\Gamma : S^k T_{\text{poly}}(\mathbb{R}^n)[2] \to D_{\text{poly}}(\mathbb{R}^n)[2]$$

is an operator naturally associated to a Kontsevich graph Γ. It implicitly depends on the dimension n of the underlying space $\mathbb{R}^n$. The coefficients $c_\Gamma^{\text{Kontsevich}}$ are numbers. Similarly, in (2) the sum ranges over all isomorphism classes of *Shoikhet graphs* with k Type I vertices (see [22] for the definition and Figure 1 for an example). The coefficients $\tilde{c}_{\tilde{\Gamma}}^{\text{Shoikhet}}$ are again numbers and

$$\tilde{D}_{\tilde{\Gamma}} : S^k T_{\text{poly}}[2](\mathbb{R}^n) \otimes C_\bullet(\mathbb{R}^n) \to \Omega_\bullet(\mathbb{R}^n)$$

are morphisms naturally associated to Shoikhet graphs, cf. [22].

In [11], formality morphisms given by sum-of-graphs formulas as above were called *stable*.
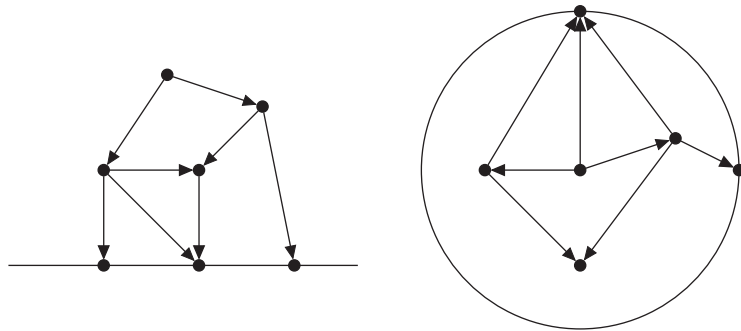
Fig. 1.   An example of a Kontsevich graph (left) and a Shoikhet graph (right).

Definition 1 (Following [11])**.**   A *stable formality morphism on cochains* is a collection of numbers $\{c_\Gamma\}_\Gamma$, one for each Kontsevich graph, such that the formulas

$$\mathcal{U}_k = \sum_\Gamma c_\Gamma D_\Gamma \tag{3}$$

define a Lie_∞ quasi-isomorphism of Lie algebras

$$\mathcal{U} : T_{\text{poly}}(\mathbb{R}^n)[1] \to D_{\text{poly}}(\mathbb{R}^n)[1]$$

for all $n(= \dim \mathbb{R}^n)$, and such that $\mathcal{U}_1$ is the Hochschild–Kostant–Rosenberg morphism.

A *stable formality morphism on cochains and chains* is a stable formality morphism on cochains together with a collection of numbers $\{\tilde{c}_{\tilde{\Gamma}}\}_{\tilde{\Gamma}}$, one for each Shoikhet graph, such that the formulas

$$\mathcal{V}_k = \sum_{\tilde{\Gamma}} \tilde{c}_{\tilde{\Gamma}} \tilde{D}_{\tilde{\Gamma}} \tag{4}$$

define a Lie_∞ quasi-isomorphism of modules

$$\mathcal{V} : C_\bullet(\mathbb{R}^n) \to \Omega_\bullet(\mathbb{R}^n)$$

for all n, and such that $\mathcal{V}_0$ is the Connes–Hochschild–Kostant–Rosenberg morphism.   □

Example 1.   In particular, to every Drinfeld associator Φ one may associate a stable formality morphism of cochains as follows:

    (1)   To the Alekseev–Torossian Drinfeld associator Φ_{AT}, see [2, 20], we associate the Kontsevich stable formality morphism $\mathcal{U}^{\text{Kontsevich}}$.

(2) Let Φ be any Drinfeld associator. The Grothendieck–Teichmüller group acts freely transitively on the set of Drinfeld associators. Hence there is a unique element g of the Grothendieck–Teichmüller group that maps Φ_{AT} to Φ. Using the pro-unipotence of the Grothendieck–Teichmüller group we may write

$$g = \exp(\psi)$$

For a unique ψ in the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$. This element ψ may be associated a graph cohomology class in M. Kontsevich's graph complex GC_2 (see [26, Section 3]), which is represented, say, by some degree 0 cocycle $\gamma \in \mathsf{GC}_2$. Now GC_2 naturally acts on the set of stable formality morphisms of cochains (see [11, 26, 27]). We define the stable formality morphism associated to Φ as

$$\exp(\gamma)\mathcal{U}^{\mathrm{Kontsevich}}.$$

Note that this is well defined only up to homotopy, since one had to make a choice in picking one representative of the graph cohomology class canonically associated to ψ.

In a similar way one may also obtain a stable formality morphism of cochains and chains as discussed in [28]. □

Remark 1. Definition 1 differs slightly from the one given in [11, Definition 5.1] by Dolgushev. There, a stable formality morphism was defined as an operad map from a colored operad governing open closed homotopy algebras to a colored operad KGra, satisfying some conditions. Elements of KGra are essentially linear combinations of Kontsevich graphs. We leave it to the reader to check that both definitions agree. □

Remark 2. Note that all formality morphism constructed as in Example 1 can be globalized, that is, they satisfy suitable properties (P1)–(P5) stated by Kontsevich [16]. □

1.1 A remark on signs and prefactors

The explicit definition of M. Kontsevich's formality morphism, correct with signs and prefactors, and the definition of the symbols D_Γ is quite lengthy to state. In fact, a separate paper [4] has been written just about the signs and prefactors. It involves conventional choices at various places in the construction. We want to avoid flooding this

paper with pages of definitions to fix the signs. To still obtain well-defined numbers c_Γ we adopt the following conventions:

(1) For each isomorphism class of Kontsevich (respectively, Shoikhet) graphs we fix once and for all a representative graph, together with an ordering of the edges. Below, when we introduce certain such graphs, we will indicate the ordering of the edges by writing numbers next to the edges.

(2) Our conventions regarding D_Γ are assumed to be chosen such that the formulas (1) are correct, for c_Γ given by Kontsevich's integral

$$c_\Gamma^{\text{Kontsevich}} = \int \prod_{(i,j)} \frac{1}{2\pi} d\arg\left(\frac{z_i - z_j}{\bar{z}_i - z_j}\right)$$

where the product is over all edges, in the order that was specified once and for all for this isomorphism class of Kontsevich graphs. Similarly, we choose our conventions regarding $\tilde{D}_{\tilde{F}}$ such that (2) is correct for $\tilde{c}_{\tilde{F}}$ being the usual Shoikhet integral, without any additional prefactors.

A careful discussion of signs for the Kontsevich morphism, which is somewhat shorter than [4] (but still spans many pages) has been given by the author in [29].

1.2 Homotopies and homotopy invariant functions

Recall that an L_∞ structure on $\mathfrak{g}$ is a degree 1, square zero coderivation on $S^+\mathfrak{g}[1]$, the cofree cocommutative coalgebra (without counit) cogenerated by $\mathfrak{g}[1]$. An L_∞ morphism between L_∞ algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a map of coalgebras

$$f : S^+ g[1] \to S^+\mathfrak{h}[1]$$

compatible with the given coderivations. Let us say that two L_∞ morphism f, g from $\mathfrak{g}$ to $\mathfrak{h}$ are directly homotopic if there is an L_∞ morphism

$$F : \mathfrak{g} \to \mathfrak{h}[t, dt]$$

such that the restriction to $t = 0$ (respectively, $t = 1$) agrees with f (respectively, with g). Concretely, F may be written as

$$F = f_t + h_t \, dt$$

where f_t is a (polynomial) family of L_∞ morphisms interpolating between $f_0 = f$ and $f_1 = g$. We call the other component, h_t the *homotopy*.

We say that two L_∞ morphisms f, g are *homotopic*, if there is some (finite) tuple of L_∞ morphisms $(a_1, \ldots, a_k)$ such that f is directly homotopic to a_1, each a_j is directly homotopic to a_{j+1} and a_k is directly homotopic to g. Clearly being homotopic is an equivalence relation on the set of L_∞ morphisms from $\mathfrak{g}$ to $\mathfrak{h}$. A function from the set of L_∞ morphisms from $\mathfrak{g}$ to $\mathfrak{h}$ to some other set is *homotopy invariant* if it is constant on equivalence classes. For a more detailed discussion of homotopies between homotopy morphisms we refer the reader to [12].

The above notion of homotopy may be transferred to stable formality morphisms with minor changes [11, Section 5]. So let $\mathcal{U}, \mathcal{U}'$ be stable formality morphisms (say of cochains, the case for cochains and chains is analogous). We say that $\mathcal{U}, \mathcal{U}'$ are directly homotopic if there is a collection of polynomials $c_\Gamma(t, \mathrm{d}t) \in \mathbb{R}[t, \mathrm{d}t]$ such that:

(1) The formulas

$$\tilde{\mathcal{U}}_k := \sum_\Gamma c_\Gamma(t, \mathrm{d}t) D_\Gamma$$

define an L_∞ morphism $T_{\mathrm{poly}}(\mathbb{R}^n) \to D_{\mathrm{poly}}(\mathbb{R}^n)$ for each n.

(2) Restricting $\tilde{\mathcal{U}}$ to fixed t yields a family of stable formality morphisms interpolating between $\mathcal{U}$ (for $t = 0$) and $\mathcal{U}'$ (reached at $t = 1$).

As above one may split

$$\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_t + h_t \, \mathrm{d}t \tag{5}$$

where $\tilde{\mathcal{U}}_t$ is the restriction of $\tilde{\mathcal{U}}$ to fixed t and we call h_t the homotopy.
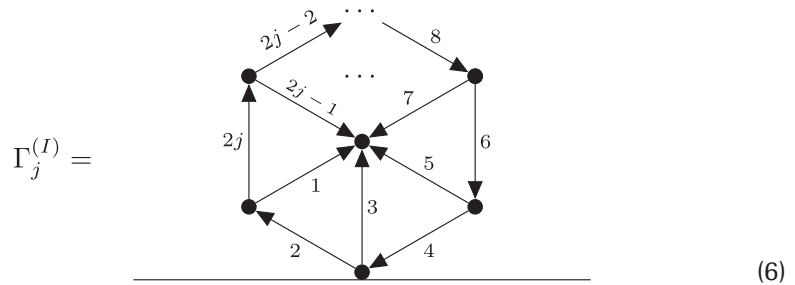
Again we define the equivalence relation of being homotopic as the transitive closure of the relation of being directly homotopic. For more details we refer the reader to [11].

A function on the set of stable formality morphisms is called homotopy invariant if it is constant on equivalence classes of the above equivalence relation. Of course, this is equivalent to saying that the function takes the same values on directly homotopic stable formality morphisms.
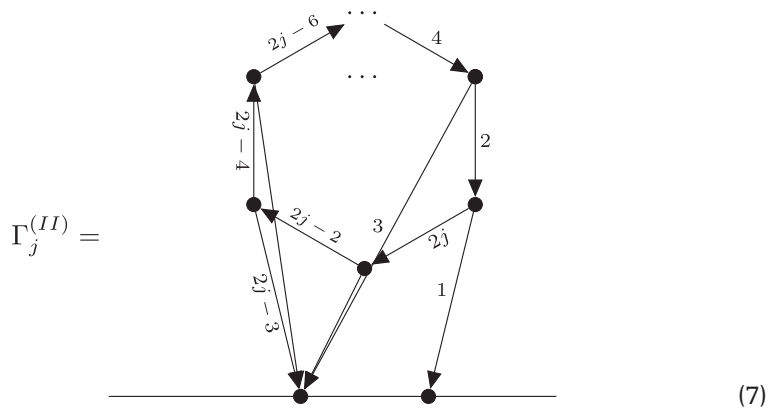
1.3 **Characteristic functions**

We will consider the following four characteristic functions:

- Let $\mathcal{U}$ be a stable formality morphism of cochains. We set $f^{\text{Duflo}}(x) = \sum_{j \geq 2} \lambda_j^{\text{Duflo}} x^j$ where $\lambda_j^{\text{Duflo}} = \frac{1}{j} c_{\Gamma_j^{(I)}} - \frac{1}{j} c_{\Gamma_j^{(II)}}$ and $c_{\Gamma_j^{(I)}}$ and $c_{\Gamma_j^{(II)}}$ are the coefficients of the graphs

$$\Gamma_j^{(I)} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6)$$

  in $\mathcal{U}_{j+1}$ and

$$\Gamma_j^{(II)} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7)$$

  in $\mathcal{U}_j$. The function f^{Duflo} appears in the proof of Duflo's Theorem through deformation quantization as in [16, Section 8] (more precisely, the series f^{Duflo} appears in the morphism I_{strange} of [16, Section 8.3.4]).

- Let $\mathcal{U}$ be a stable formality morphism of cochains. We set $f^{\text{curv}}(x) = \sum_{j \geq 2} \lambda_j^{\text{curv}} x^j$ where $\lambda_j^{\text{curv}} = \frac{1}{j} c_{\Gamma_j^{(III)}}$ and $c_{\Gamma_j^{(III)}}$ is the coefficient of the wheel graph

with spokes pointing outwards

$$\Gamma_j^{(III)} =$$

$$(8)$$

in $\mathcal{U}_{j+1}$. These graphs appear in [7, 24, 25], and in particular as a curvature term in the formality morphism with branes [5, 8].

- Let $(\mathcal{U}, \mathcal{V})$ be a stable formality morphism of cochains and chains. We set $f^{\text{chain}}(x) = \sum_{j\geq 2} \lambda_j^{\text{chain}} x^j$ where $\lambda_j^{\text{chain}} = \frac{1}{j}\tilde{c}_{\tilde{\Gamma}_j}$ and $\tilde{c}_{\tilde{\Gamma}_j}$ is the coefficient of the graph

$$(9)$$

in $\mathcal{V}_{j+1}$. These graphs determine the character map in deformation quantization, see [9].

- For any Drinfeld associator $\Phi(X, Y)$ one defines the formal function $f^{\text{assoc}}(x) = \sum_{j\geq 2} \lambda_j^{\text{assoc}} x^j$, where λ_j^{assoc} is the coefficient of $X^{j-1}Y$ in $\Phi(X, Y)$, divided by j. So

$$\Phi(X, Y) = 1 + \sum_{j\geq 2} j\lambda_j^{\text{assoc}} X^{j-1} Y + (\text{other terms})$$

The exponential of the function $-f^{\text{assoc}}$ has been called Duflo function in [3] and the Γ function in [13].

Example 2.   Several of these characteristic functions have been computed in the literature:

- Kontsevich computed [16] that for his stable formality morphism $\mathcal{U}^{\text{Kontsevich}}$

$$f^{\text{Duflo}} = -\sum_{k=1}^{\infty} \frac{1}{2k} \frac{B_{2k}}{2(2k)!} x^{2k} = -\frac{1}{2} \log \left(\frac{e^{x/2} - e^{-x/2}}{x} \right)$$

  where B_j is the jth Bernoulli number. In fact, it was shown by Shoikhet [21] that $c_{\Gamma_j^{(I)}} = 0$ in this case.
- For the Kontsevich stable formality morphism $\mathcal{U}^{\text{Kontsevich}}$ it has been computed in [24] that

$$f^{\text{curv}}(x) = -\sum_{k=1}^{\infty} \frac{1}{2k} \frac{B_{2k}}{2(2k)!} x^{2k} = -\frac{1}{2} \log \left(\frac{e^{x/2} - e^{-x/2}}{x} \right).$$

  Consider also the stable formality morphism of chains and cochains $(\mathcal{U}^{\text{Kontsevich}}, \mathcal{V}^{\text{Shoikhet}})$. In this case, the integral expressions defining $c_{\Gamma_j^{(III)}}$ and $\tilde{c}_{\tilde{\Gamma}_j}$ agree. This also shows that in this case

$$f^{\text{chain}}(x) = -\frac{1}{2} \log \left(\frac{e^{x/2} - e^{-x/2}}{x} \right).$$

- For the stable formality morphism obtained using the Kontsevich "$\frac{1}{2}$-propagator" (see [1]) it has been shown by Merkulov [18, Appendix A] that

$$f^{\text{curv}}(x) = \sum_{k=2}^{\infty} \frac{\zeta(k)}{k(2\pi i)^k} x^k = \log \left(\Gamma \left(1 - \frac{x}{2\pi i} \right) \right) - \frac{\gamma}{2\pi i} x$$

  where Γ, ζ, and γ are the Γ function, the Riemann ζ function and the Euler–Mascheroni constant as usual.
- It is known (see [17] or [3, Example 9.1]) that for the Knizhnik–Zamolodchikov associator

$$f^{\text{assoc}}(x) = \sum_{k=2}^{\infty} \frac{\zeta(k)}{k(2\pi i)^k} x^k = \log \left(\Gamma \left(1 - \frac{x}{2\pi i} \right) \right) - \frac{\gamma}{2\pi i} x.$$

- One can check that the even part of $f^{\text{assoc}}(x)$ must be the same for all Drinfeld associators. Since the Alekseev–Torossian associator [2, 20] is even, we obtain from the previous example that for the Alekseev–Torossian associator

$$f^{\text{assoc}}(x) = \sum_{k=1}^{\infty} \frac{k\zeta(2k)}{(2\pi i)^{2k}} x^{2k} = -\frac{1}{2} \log\left(\frac{e^{x/2} - e^{-x/2}}{x}\right). \qquad \Box$$

Lemma 1 (Homotopy invariance). Let $\mathcal{U}^1, \mathcal{U}^2$ be stable formality morphisms of cochains that are homotopic. Let f_1^{Duflo}, f_1^{curv} and f_2^{Duflo}, f_2^{curv} be the associated characteristic functions as defined above. Then $f_1^{\text{Duflo}} = f_2^{\text{Duflo}}$ and $f_1^{\text{curv}} = f_2^{\text{curv}}$.

Let furthermore $(\mathcal{U}^1, \mathcal{V}^1)$ and, $(\mathcal{U}^2, \mathcal{V}^2)$ be homotopic stable formality morphisms of cochains and chains and let f_1^{chain} and f_2^{chain} be the characteristic functions associated to $\mathcal{V}^1$ and $\mathcal{V}^2$ as above. Then $f_1^{\text{chain}} = f_2^{\text{chain}}$. $\qquad \Box$

Proof sketch. It is sufficient to consider only directly homotopic stable formality morphisms (see Section 1.2). Let us use the notation from Equation (5). The dt-components of the Lie_∞ relations for $\tilde{\mathcal{U}}$ say that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{U}}_t = \pm d_S h_t \pm d_H h_t \pm [\mathcal{U}_t, h_t] \qquad (10)$$

where d_S is a term containing the Schouten–Nijenhuis bracket, d_H is (induced from) the Hochschild differential and the bracket is (induced from) the Gerstenhaber bracket.

To see the invariance for f^{curv} one notes that (for large enough n) the right-hand side cannot contain any terms associated to graphs (8), as they could be produced by neither the differential d_S and d_H, nor by the Gerstenhaber bracket. Hence f^{curv} must be the same for each $\tilde{\mathcal{U}}_t$. For f^{chain} the argument is analogous.

The case of f^{Duflo} is more difficult, as the right-hand side of (10) may contain graphs of the forms (6) and (7). Concretely, both can be produced by terms corresponding to a unique graph in h_t, namely the following:

$$(11)$$
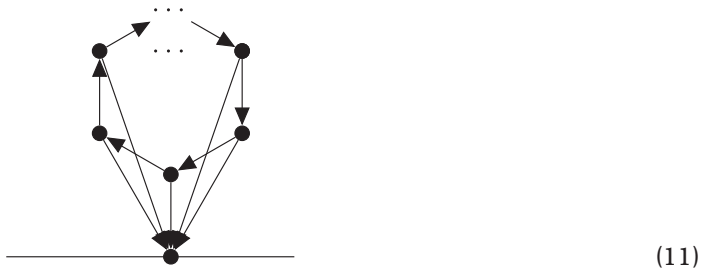
The term $d_H h_t$ (may) contain terms corresponding to the graph (7) and the term $[\mathcal{U}_t, h_t]$ (may) contain terms corresponding to the graph (6).

However, computing the signs and prefactors both contributions are equal and hence f^{Duflo} remains unchanged.

Note also that graphs of the form
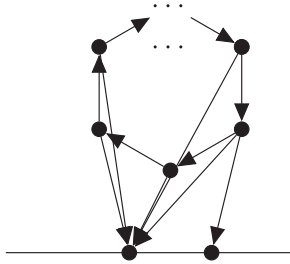
in h_t do not contribute since the two terms of the form (7) that can be produced through $d_S h_t$ occur with opposite signs and hence cancel. ∎

1.4   **Main result**

The main result of this paper is the following theorem.

Theorem 1 (Partially contained in [16, 26, Section 10])**.**

(1)   Let $\mathcal{U}$ be a stable formality morphism of cochains. Then

$$f^{\text{Duflo}} = f^{\text{curv}}.$$

(2)   If $\mathcal{U}$ is obtained from a Drinfeld associator Φ according to the procedure of Example 1, then furthermore

$$f^{\text{Duflo}} = f^{\text{curv}} = f^{\text{assoc}}.$$

(3)   Let $(\mathcal{U}, \mathcal{V})$ be an extension of $\mathcal{U}$ to a stable formality morphism of cochains and chains. Then
$$f^{\text{Duflo}} = f^{\text{curv}} = f^{\text{chain}}. \qquad \qquad \square$$

Theorem 1 can in fact almost be extracted from existing literature. The fact that $f^{\text{Duflo}} = f^{\text{curv}}$ is essentially contained in some form in [16], and the fact that $f^{\text{curv}} = f^{\text{assoc}}$ is contained (in an albeit sketchy way) in [26]. Nevertheless, we will give a self-contained proof in Section 3.

Remark 3. In fact, the even part of the characteristic functions above is the same for all stable formality morphisms and agrees with the function

$$-\frac{1}{2} \log \frac{e^{x/2} - e^{-x/2}}{x} = -\sum_{j \geq 1} \frac{B_{2j}}{4j(2j)!} x^{2j} \qquad \qquad \square$$

2 Action of the Graph Complex

M. Kontsevich's graph complex GC_2 is a complex formed by formal series of (isomorphism classes of) undirected, at least trivalent, connected graphs. The simplest nontrivial example of a graph giving rise to an element of GC_2 is the tetrahedron graph
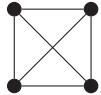
For more details, and the (lengthy) definition of GC_2 we refer the reader to [26, Section 3; 11, Section 6]. For us, the important fact is that there is a map of dg Lie algebras from GC_2 to the Chevalley complex of $T_{\mathrm{poly}}(\mathbb{R}^n)$ for each n. In particular, closed degree zero elements of GC_2 give rise to Lie_∞-derivations of $T_{\mathrm{poly}}(\mathbb{R}^n)[1]$. Denote the space of closed degree 0 elements by $\mathrm{GC}^0_{2,d} \subset GC_2$. It is a pro-nilpotent Lie algebra, and is the Lie algebra of a prounipotent group

$$\mathrm{Exp}\, GC^0_{2,d}$$

which may be realized as the group-like elements in the completed universal enveloping algebra of $\mathrm{GC}^0_{2,d}$. The action of $\mathrm{GC}^0_{2,d}$ on $T_{\mathrm{poly}}(\mathbb{R}^n)[1]$ by Lie_∞-derivations integrates to an action of $\mathrm{Exp}\, GC^0_{2,d}$ on $T_{\mathrm{poly}}(\mathbb{R}^n)[1]$ by Lie_∞-automorphisms. It is then not hard to check that precomposition yields an action of $\mathrm{Exp}\, GC^0_{2,d}$ on the set of stable formality morphisms (of cochains). It is clear that this action descends to an action of the homotopy classes of stable formality morphisms. Dolgushev showed the following theorem, which is important for us.

Theorem 2 ([11]). The induced action of $\mathrm{Exp}\, GC^0_{2,d}$ on the set of homotopy classes of stable formality morphisms is transitive. $\qquad \square$

3   **Proof of the Theorem 1**

First, let us reduce the statement to the cases involving only stable formality morphisms of cochains by showing that $f^{\text{curv}} = f^{\text{chain}}$. For this, consider the coefficient of the graph
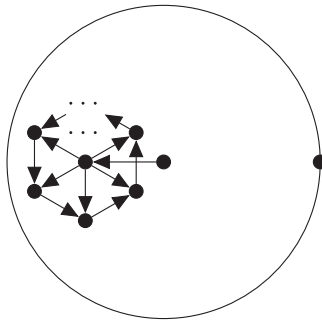

in the Lie_∞ relation for modules. Terms can be contributed by the graph $\Gamma_j^{(III)}$ (see (8)) and by the graph $\tilde{\Gamma}_j$ (see (9)) and by no other graphs. Checking the prefactors, It follows that the coefficients need to be equal, up to possibly an overall sign, which depends on conventions, but not on the particular stable formality morphism chosen. However, for the Kontsevich/Shoikhet morphism our conventions and Example 2 say that the sign is "+", hence it must be "+" for any stable formality morphism.

Next let us turn to the statement that $f^{\text{curv}} = f^{\text{Duflo}}$. By Lemma 1 and Theorem 2 it suffices to show the following two statements:

(1)   For one particular stable formality morphism $f^{\text{curv}} = f^{\text{Duflo}}$.
(2)   The action of degree zero cocycles in GC_2 leaves invariant the expression $f^{\text{Duflo}}(x) - f^{\text{curv}}(x)$.

We take for the particular formality morphism that constructed by M. Kontsevich, that is, $\mathcal{U}^{\text{Kontsevich}}$. In this case item (1) above is settled by Example 2.

Next consider the action of a degree zero cocycle $\Gamma \in \mathsf{GC}_2$. By the explicit description of the action it cannot change the coefficient of the graph $\Gamma_j^{(II)}$ (see (7)) in a stable formality morphism. Furthermore, it changes both the coefficients of the graphs $\Gamma_j^{(II)}$ and $\Gamma_j^{(III)}$ (see (6), (8)) by the coefficient of the wheel graph
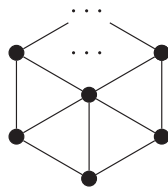
in Γ. In particular, the quantity $f^{\mathrm{Duflo}}(x) - f^{\mathrm{curv}}(x)$ is unchanged. Hence we have shown that $f^{\mathrm{curv}} = f^{\mathrm{Duflo}}$ for all stable formality morphisms.

To show the final assertion of Theorem 1 the proof is similar and has been given in [26]. We recall it here. It clearly suffices to show the following items:

(1) For the Kontsevich stable formality morphism and the Alekseev–Torossian Drinfeld associator, $f^{\mathrm{assoc}} = f^{\mathrm{curv}}$.

(2) The difference $f^{\mathrm{assoc}} - f^{\mathrm{curv}}$ is invariant under the action of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$, where to define its action on stable formality morphisms one uses the map from $\mathfrak{grt}_1$ to $H(\mathsf{GC}_2)$ as in Example 1.

Again, item (1) has been settled by Example 2. Furthermore, the cycle in graph homology s_n that picks out the coefficient of the wheel graph with n spokes (n odd) is shown in [26, Proposition 9.1] to correspond to the cochain of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$ that picks out the coefficient of

$$\mathrm{ad}_X^{n-1} Y$$

of elements in $\mathfrak{grt}_1$. The action of some $\mathfrak{grt}_1$ element on a Drinfeld associator changes the coefficient of $X^{n-1} Y$ of the associator by precisely this term. Hence Theorem 1 follows.

4 Application: Star Products on Duals of Lie Algebras

Let $\mathfrak{g}$ be any Lie algebra, $U\mathfrak{g}$ its universal enveloping algebra, and $S\mathfrak{g}$ the symmetric algebra. The Poincaré–Birkhoff–Witt isomorphism

$$\phi_{\mathrm{PBW}} : S\mathfrak{g} \to U\mathfrak{g}$$

endows $S\mathfrak{g}$ with an associative (but not necessarily commutative) product $\star_{\mathrm{PBW}}$ via pull-back, that is,

$$p \star_{\mathrm{PBW}} q := \phi_{\mathrm{PBW}}^{-1}(\phi_{\mathrm{PBW}}(p)\phi_{\mathrm{PBW}}(q))$$

for any $p, q \in S\mathfrak{g}$.

Furthermore, for any Lie algebra $\mathfrak{g}$ the dual space $\mathfrak{g}^*$ carries a canonical Poisson structure, the Kirillov–Kostant Poisson structure. A stable formality morphism provides us (in particular) with an associative product $\star$ on $S\mathfrak{g}$. This product in general depends on the stable formality morphism chosen. However, it is an elementary exercise to check that any such product is the pullback of $\star_{\mathrm{PBW}}$ via an automorphism of the vector space

$S\mathfrak{g}$ of the form

$$\Psi = \exp\left(\sum_{j\geq2} c_j \mathrm{tr}(\mathrm{ad}_\partial^j)\right) \tag{12}$$

for some constants c_j. Here

$$\mathrm{tr}(\mathrm{ad}_\partial^j) := f_{i_1}^{k_1 i_j} f_{i_2}^{k_2 i_1} \cdots f_{i_j}^{k_j i_{j-1}} \partial_{k_1} \cdots \partial_{k_j}$$

are differential operators where f_c^{ab} are the structure constants of the Lie algebra and summation over repeated indices is assumed. Note that the constants c_j are not characters of the stable formality morphism, that is, they may change upon changing the stable formality morphism to a gauge equivalent one. However, there is the following result.

Proposition 1. Given a stable formality morphism define the formal series $f(x) := -\sum_j \frac{(-1)^j}{j} c_j x^j$, where the c_j are as in (12). If the stable formality morphism is such that the weights $c_{\Gamma_j^{(I)}}$ of graphs $\Gamma_j^{(I)}$ (cf. (6)) vanish for all j, then f agrees with the characteristic function defined above, that is,

$$f = f^{\mathrm{Duflo}} = f^{\mathrm{curv}}. \qquad \square$$

Proof. Under the assumptions given $\lambda_j^{\mathrm{Duflo}} = -\frac{1}{j} c_{\Gamma_j^{(II)}}$. However, it is not hard to check that $\star_{PBW}$ does not contain terms corresponding to graphs $\Gamma_j^{(II)}$. They have to be produced via pullback with $\mathrm{tr}(\mathrm{ad}_\partial^j)$ and hence the respective coefficients need to agree, up to a combinatorial prefactor, independent of the stable formality morphism under consideration. Unwinding conventions left implicit in this paper the combinatorial prefactor could be computed. However, to settle the prefactors it is also sufficient to check that both characteristic functions agree for one stable formality morphism for which all of the coefficients of f are nonzero. It has been shown by Rossi [19] that for the Kontsevich formality morphism with $\frac{1}{2}$-propagator, $f = f^{\mathrm{curv}}$ (cf. also Example 2). Since in this case all $c_j \neq 0$, the combinatorial prefactors must all be $+1$. ∎

In the special case of the Kontsevich stable formality morphism, Ψ becomes the Duflo morphism, hence the name of f^{Duflo}. Special cases of the above proposition have been shown in [16, 14, Appendix F, 6, 19].

5   **The "Two Branes" Case**

The above results may be extended slightly to apply to the formality morphisms "with branes" introduced by Calaque et al. [5]. In particular, one may identify a characteristic function for "stable versions" (i.e., given by sum-of-graphs formulas) of such morphisms, which has been used implicitly in [6, 19]. This function turns out to agree with the characteristic functions discussed above.

Let us begin by reviewing the results of [5]. Consider the polynomial (or exterior) algebras $A = \mathbb{R}[X_1, \ldots, X_n]$ and $B = \mathbb{R}[\xi_1, \ldots, \xi_n]$ where the formal variables $X_1, \ldots, X_n$ live in degree 0, while the formal variables $\xi_1, \ldots, \xi_n$ live in degree 1. A and B are Koszul dual algebras. One may show this by showing that the Koszul complex

$$A \otimes B^*$$

has cohomology $\mathbb{R}$. Note also that $A \otimes B^*$ carries a natural A–B bimodule structure.

The first result of [5] is an explicit construction of an Assoc_∞ A–B bimodule structure on $K = \mathbb{R}$. It was shown in [15] that the bimodule K is in fact Assoc_∞ quasi-isomorphic to $A \otimes B^*$.

One may package A, B, and K into an A_∞ category $\mathsf{Cat}_\infty(A, B, K)$ (notation as in [5]) with objects A and B and the space of morphisms between A and B being K. The second result of [5] is the construction of a Lie_∞ morphism

$$T_{\mathrm{poly}}(\mathbb{R}^n)[1] \to C(\mathsf{Cat}_\infty(A, B, K))[1]$$

where the right-hand side is the Hochschild complex of $\mathsf{Cat}_\infty(A, B, K)$. This morphism contains the Kontsevich formality morphism $\mathcal{U}^{\mathrm{Kontsevich}}$ from above.

One may package both the Assoc_∞ bimodule structure and the Lie_∞ morphism into a "nonflat" Lie_∞ morphism, that is, a Lie_∞ morphism with nonvanishing zeroth term, which encodes the bimodule structure. This morphism is also given by a sum-of-graphs formula of the form

$$\mathcal{W}_k^{\mathrm{CFFR}} = \sum_{\bar{\Gamma}} \bar{c}_{\bar{\Gamma}}^{\mathrm{CFFR}} D_{\bar{\Gamma}}.$$

Here the graphs summed over are essentially Kontsevich graphs, possibly with one distinguished Type II vertex. For a more precise definition, we refer the reader to [5].

In analogy with Definition 1, we may define a stable formality morphism of Calaque, Felder, Ferrario, and Rossi (CFFR) type to be a collection of numbers $c_{\bar{\Gamma}}$ such

that

$$\mathcal{W}_k = \sum_{\bar{\Gamma}} c_{\bar{\Gamma}} D_{\bar{\Gamma}}.$$

defines a nonflat Lie_∞ morphism for all n, and such that (1) the restriction to Kontsevich-type graphs yields a stable formality morphism and (2) the two graphs below have coefficient 1.
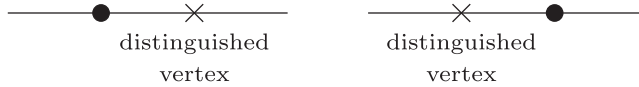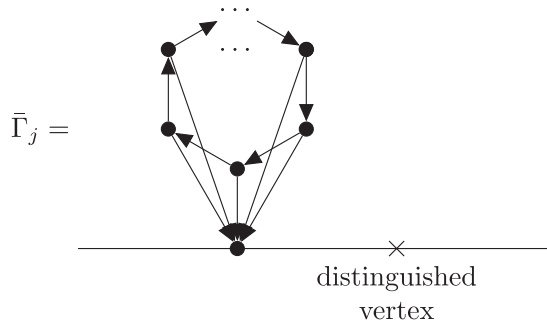

These graphs are the leading contribution to the bimodule structure.

Such stable formality morphisms possess a characteristic function

$$f^{\mathrm{brane}}(x) = \sum_{j \geq 2} \lambda_j^{\mathrm{brane}} x^j$$

where $\lambda_j^{\mathrm{brane}} = \frac{1}{j} c_{\Gamma_j^I} + \frac{1}{j} \bar{c}_{\bar{\Gamma}_j}$, with Γ_j^I as depicted in (6), and $\bar{\Gamma}_j$ as follows:

It may be verified that f^{brane} is indeed a characteristic function, that is, it does not change when changing the stable formality morphism of CFFR type to a homotopic one. Note that this is not true if one omits the term $c_{\Gamma_j^I}$ from the definition. The characteristic function f^{brane} is implicitly used in [6, 19, 25], where it is shown to agree with f^{curv} for two special stable formality morphisms of CFFR type. We have the following general result.

Proposition 2. $f^{\mathrm{brane}} = f^{\mathrm{curv}}$ for all stable formality morphisms of CFFR type. □

Proof sketch.   The statement is equivalent to saying that the coefficients of the terms associated to the graphs

distinguished
vertex
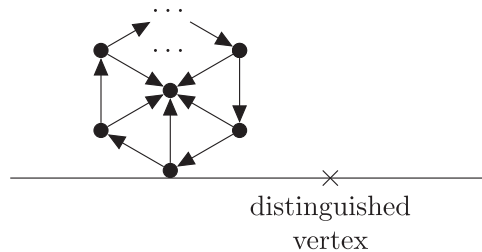
in the Lie_∞ relations vanish (for n big enough).   ∎

References

[1]   Alekseev, A., C. A. Rossi, C. Torossian, and T. Willwacher. "Logarithms and deformation quantization" (2014): preprint arXiv:1401.3200.

[2]   Alekseev, A. and C. Torossian. "Kontsevich deformation quantization and flat connections." *Communications in Mathematical Physics* 300, no. 1 (2010): 47–64.

[3]   Alekseev, A. and C. Torossian. "The Kashiwara–Vergne conjecture and Drinfeld's associators." *Annals of Mathematics* (2) 175, no. 2 (2012): 415–63.

[4]   Arnal, D., D. Manchon, and M. Masmoudi. "Choix des signes pour la formalite de M. Kontsevich." *Pacific Journal of Mathematics* 203, no. 1 (2002): 23–66.

[5]   Calaque, D., G. Felder, A. Ferrario, and C. A. Rossi. "Bimodules and branes in deformation quantization." *Compositio Mathematica* 147, no. 01 (2011): 105–60.

[6]   Calaque, D., G. Felder, and C. A. Rossi. "Deformation quantization with generators and relations." *Journal of Algebra* 337, no. 1 (2011): 1–12.

[7]   Calaque, D. and M. Van den Bergh. "Hochschild cohomology and Atiyah classes." *Advances in Mathematics* 224, no. 5 (2010): 1839–89.

[8]   Cattaneo, A. S. and G. Felder. "Relative formality theorem and quantisation of coisotropic submanifolds." *Advances in Mathematics* 208, no. 2 (2007): 521–48.

[9]   Cattaneo, A. S., G. Felder, and T. Willwacher. "The character map in deformation quantization." *Advances in Mathematics* 228, no. 4 (2011): 1966–89.

[10] Dolgushev, V. "A formality theorem for Hochschild chains." *Advances in Mathematics* 200, no. 1 (2006): 51–101.

[11] Dolgushev, V. "Stable formality quasi-isomorphisms for Hochschild Cochains I." (2011): preprint arXiv:1109.6031v2.

[12] Dotsenko, V. and N. Poncin. "A tale of three homotopies." (2012): preprint arXiv:1208.4695.

[13] Enriquez, B. "On the Drinfeld generators of $\mathfrak{grt}_1(\mathbf{k})$ and Γ-functions for associators." *Mathematical Research Letters* 13, no. 2–3 (2006): 231–43.

[14] Felder, G. and T. Willwacher. "On the (ir)rationality of Kontsevich weights." *International Mathematics Research Notices* 2010, no. 4 (2010): 701–16.

[15] Ferrario, A., C. Rossi, and T. Willwacher. "A Note on the Koszul complex in deformation quantization." *Letters in Mathematical Physics* 95, no. 1 (2011): 27–39.

[16] Kontsevich, M. "Deformation quantization of Poisson manifolds." *Letters in Mathematical Physics* 66, no. 3 (2003): 157–216.

[17] Le, T. T. Q. and J. Murakami. "Kontsevich's integral for the Kauffman polynomial." *Nagoya Mathematical Journal* 142 (1996): 39–65.

[18] Merkulov, S. "Exotic automorphisms of the Schouten algebra of polyvector fields." (2008): preprint arXiv:0809.2385v6.

[19] Rossi, C. A. " The explicit equivalence between the standard and the logarithmic star product for Lie algebras." (2012): preprint arXiv:1205.5512.

[20] Severa, P. and T. Willwacher. "Equivalence of formalities of the little discs operad." (2009): preprint arXiv:0905.1789.

[21] Shoikhet, B. "Vanishing of the Kontsevich integrals of the wheels." (2000): preprint arXiv:math/0007080.

[22] Shoikhet, B. "A proof of the Tsygan formality conjecture for chains." *Advances in Mathematics* 179, no. 1 (2003): 7–37.

[23] Tsygan, B. "Formality Conjectures for Chains." In *Differential Topology, Infinite-Dimensional Lie Algebras, and Applications*, American Mathematical Society Translations: Series 2, 194, 261–274. Providence, RI: American Mathematical Society, 1999.

[24] Van den Bergh, M. "The Kontsevich weight of a wheel with spokes pointing outward." *Algebras and Representation Theory* 12, no. 2–5 (2009): 443–79.

[25] Willwacher, T. "A counterexample to the quantizability of modules." *Letters in Mathematical Physics* 81, no. 3 (2007): 265–80.

[26] Willwacher, T. "M. Kontsevich's graph complex and the Grothendieck–Teichmüller Lie algebra." (2010): preprint arxiv:1009.1654v4 (to appear in *Inventiones Mathematicae*).

[27] Willwacher, T. "A Note on Br-infinity and KS-infinity formality." (2011): preprint arXiv:1109.3520.

[28] Willwacher, T. "The Grothendieck–Teichmüller group action on differential forms and formality morphism of chains." (2013): preprint arXiv:1308.6097.

[29] Willwacher, T. and D. Calaque. "Formality of cyclic cochains." *Advances in Mathematics* 231, no. 2 (2012): 624–50.