J. Aust. Math. Soc. **79** (2005), 183–212

CONJUGACY IN SINGULAR ARTIN MONOIDS

RUTH CORRAN

(Received 3 January 2003; revised 9 September 2004)

Communicated by D. Easdown

Abstract

We define a notion of conjugacy in singular Artin monoids, and solve the corresponding conjugacy problem for finite types. We show that this definition is appropriate to describe type (1) singular Markov moves on singular braids. Parabolic submonoids of singular Artin monoids are defined and, in finite type, are shown to be singular Artin monoids. Solutions to conjugacy-type problems of parabolic submonoids are described. Geometric objects defined by Fenn, Rolfsen and Zhu, called (j, k)-bands, are algebraically characterised, and a procedure is given which determines when a word represents a (j, k)-band.

2000 *Mathematics subject classification*: primary 20F36; secondary 20M05, 20F05.

1. Preliminaries

Singular Artin monoids are introduced in [9] as a generalisation of singular braid monoids, defined by presentations related to Coxeter matrices. The singular braid monoids are singular Artin monoids defined by type A Coxeter matrices. Artin groups (see [7] or [10]) are subgroups of singular Artin monoids, and Coxeter groups (see, for example, [20]) are quotients of Artin groups. The Artin groups of type A are the Artin braid groups, and the Coxeter groups of type A are the symmetric groups. Thus singular Artin monoids are generalisations of very natural objects.

The main result of this paper is a solution to the conjugacy problem in singular Artin monoids. In order to solve this problem, we need an appropriate definition of conjugacy. Perhaps the most natural choices are the following: to say that V and W are conjugate if

(1) there exists X such that $VX = XW$, or

(2) there exist Y and Z such that $V = YZ$ and $W = ZY$.

The first is reflexive and transitive, but not necessarily symmetric, while the second is reflexive and symmetric, but not necessarily transitive. We take the first of these to be the definition of conjugacy in this paper. This first section establishes and collates results which allow us to show that it is indeed a symmetric relation in the context of singular Artin monoids, and Sections 2 and 3 present the solution to the conjugacy problem in singular Artin monoids of finite type. The second notion of conjugacy defined above we call 'swap conjugacy', and we verify that the original definition is precisely its transitive closure.

Fenn, Rolfsen and Zhu [12] introduced the notion of (j, k)-bands, which are singular braids satisfying a certain geometric condition on the jth and $(j + 1)$th strings of the braid, and they find an equivalent algebraic condition. In the fourth section, we describe how to determine whether a braid is a (j, k)-band given a word in the generators which represents it.

In Section 5 we introduce parabolic submonoids of singular Artin monoids, which are generated by particular subsets of the generators. We show that parabolic submonoids of singular Artin monoids of finite type are themselves isomorphic to singular Artin monoids. This result mirrors that in Coxeter and Artin groups; although in both of these cases the result has been shown to hold for arbitrary type. The notion of a (J, K)-conjugator (a generalisation of a (j, k)-band), where J and K define subsets of the generators, is discussed, and a method for determining when a word is a (J, K)-conjugator is given. We show that the set of (J, K)-conjugators is not empty if and only if the parabolic submonoids defined by J and K are conjugate. We give a method for determining when two parabolic submonoids are conjugate.

The (j, k)-bands were first introduced in order to prove one case of a conjecture of Birman [5] about singular braid monoids embedding in the group algebra of the braid group. The last section of this paper discusses the singular braid monoid exclusively, particularly the problem of determining when two singular braids close to give equivalent singular links. Gemein [14] obtained an analogy for singular braids of Markov's theorem for braids, which describes the algebraic connections, called Markov moves, between braids which give equivalent links. We show how to determine when two singular braids are connected by one Markov move, and give some stronger results for positive braids connected by 'positive Markov moves'.

A solution has recently been obtained for Birman's conjecture by Paris [21]. The conjecture may be generalised to arbitrary Artin types, and a solution was subsequently obtained for the generalisation of this conjecture to FC-type (a distinct case to the finite type case mostly considered in this article) by Godelle and Paris [17]. However, the finite type case, other than the case originally solved by Paris, remains open. The result of [17] used results of Godelle's for Artin groups ([15, 16]) similar to some obtained here for singular Artin monoids. We hope that the results obtained here will contribute towards the resolution of the generalised Birman's conjecture for finite type, which remains an object of current study (see, for example, [1]).

Some of the techniques in this paper have been adapted from [7] (particularly the notion of an *a-chain*), in which solutions to the word, division and conjugacy problems for Artin groups were given. That paper, in turn, generalised notions of Garside [13] who originally solved those problems for the braid group.

We begin by defining *positive singular Artin monoids*. These turn out to be submonoids of the singular Artin monoids (defined in Section 3). Let I denote a finite set. A *Coxeter matrix* over I is a symmetric $I \times I$ matrix $M = (m_{ij})$ where $m_{ii} = 1$ for all $i \in I$, and $m_{ij} \in \{2, 3, 4, \dots, \infty\}$ for $i \neq j$. The *Coxeter graph* Γ_M associated with a Coxeter matrix M is the graph with vertices indexed by I, and where an edge labelled m_{ij} joins the vertices i and j precisely when $m_{ij} \geq 3$. The convention is to explicitly show this label only when $m_{ij} \geq 4$ — thus an unlabelled edge indicates that $m_{ij} = 3$. Figure 1 shows some Coxeter graphs.

Given a Coxeter matrix M, the *positive singular Artin monoid of type M* is the monoid generated by $S \cup T$ where $S = \{\sigma_i \mid i \in I\}$ and $T = \{\tau_i \mid i \in I\}$, subject to the relations $\mathscr{R}$ listed below

$$\begin{aligned}
\langle \sigma_i\sigma_j \rangle^{m_{ij}} &= \langle \sigma_j \sigma_i \rangle^{m_{ij}} && \text{whenever } 2 \leq m_{ij} < \infty, \\
\langle \sigma_i\sigma_j \rangle^{m_{ij}-1}\tau_k &= \tau_j \langle \sigma_i\sigma_j \rangle^{m_{ij}-1} && \text{whenever } 2 \leq m_{ij} < \infty, \\
& && \quad \text{and where } k = \begin{cases} i & \text{if } m_{ij} \text{ is odd;} \\ j & \text{if } m_{ij} \text{ is even,} \end{cases} \\
\tau_i\tau_j &= \tau_j\tau_i && \text{whenever } m_{ij} = 2, \quad \text{and} \\
\sigma_i\tau_i &= \tau_i\sigma_i && \text{for all } i \text{ in } I,
\end{aligned}$$

where $\langle ab \rangle^p$ denotes the alternating product $aba\cdots$ with p factors. Let $\mathscr{S}_M^+$ denote the positive singular Artin monoid of type M. If two words W and V represent the same element of $\mathscr{S}_M^+$, we say that W and V are *equivalent*, and write $W \sim V$. The symbol $\equiv$ is used to indicate when two words are the same letter for letter (in other words, equal in the free monoid on $S \cup T$, which is denoted $(S \cup T)^*$). Notice that since every relation is homogeneous — that is, both sides of the equation are words of the same length — whenever $W \sim V$, then the length of W and V must be the same. The length of a word W in the generators $S \cup T$ is denoted $\ell(W)$. Let $\mathscr{R}^\Sigma$ denote the set $\{(U, V), (V, U) \mid U = V \text{ is a relation from } \mathscr{R}\}$.

A word V is said to *divide* a word W if there exists a word X such that $W \sim VX$. A set of words Ω has a *common multiple* W if every element of Ω divides W. A *least common multiple* is a common multiple which divides all other common multiples. Notice that by homogeneity, the length of a divisor cannot exceed the length of its multiple. Thus only finite sets can have common multiples (as infinite sets contain words of arbitrary length). However, there are finite sets of words without common multiples (see the comments following Lemma 1.1, below, for an example).

Many properties of $\mathscr{S}_M^+$ were discussed in [9], where the word and division problems for such a monoid were solved, and a unique normal form was described. Furthermore, it was shown that $\mathscr{S}_M^+$ is both left and right cancellative, and that whenever a set Ω has a common multiple in $\mathscr{S}_M^+$, it has a least common multiple $L(\Omega)$ (which is unique in $\mathscr{S}_M^+$). The crucial result in proving the above properties is the following *reduction property* (so named after [7, Lemma 2.1], 'Reduction Lemma').

LEMMA 1.1 ([9, Lemma 15]). *For all $a, b \in S \cup T$, and for all words X and Y, the equation $aX \sim bY$ implies there exist words U, V and W in $(S \cup T)^*$ such that $X \sim UW$, $Y \sim VW$ and either (aU, bV) is in $\mathscr{R}^\Sigma$ or $aU \equiv bV$.*

Thus if $W_1 \sim W_2$, then there are words V, R_1 and R_2 such that the first letter of W_1 and R_1 coincide, the first letter of W_2 and R_2 coincide, $W_1 \sim R_1 V$ and $W_2 \sim R_2 V$, and either $(R_1, R_2) \in \mathscr{R}^\Sigma$ or $R_1 \equiv R_2$. This result is very useful: we can immediately apply it to the problem of the existence of common multiples of pairs of generators. For example, since there is no pair of the form $(\tau_i U, \tau_j V)$, where $m_{ij} > 2$, in $\mathscr{R}^\Sigma$, then the reduction property ensures that $\{\tau_i, \tau_j\}$ has no common multiple.

In this first section various results are obtained which are useful in the sequel. Notation and operators defined in full in [9] are more briefly described. The last part of this section deals specifically with the subset of positive singular Artin monoids of *finite type*, which turn out to be precisely the singular Artin monoids associated to finite disjoint unions of the Coxeter graphs in Figure 1. (The results preceding this are valid for positive singular Artin monoids of arbitrary type.)

LEMMA 1.2. *Suppose that $m_{ij} > 2$. Then there is no common multiple of τ_i and $\langle \sigma_j \sigma_i \rangle^{m_{ij}-3} \tau_p$, where p is i if m_{ij} is even and j otherwise.*

PROOF. If $m_{ij} = 3$, then, as remarked earlier, the reduction property precludes τ_i and τ_j having a common multiple. Suppose that $m_{ij} > 3$,

$$X \sim \tau_i W \sim \langle \sigma_j \sigma_i \rangle^{m_{ij}-3} \tau_p U,$$

and that X provides a minimal length counterexample to the lemma. By the reduction property, there is a word W_1 such that

$$W \sim \langle \sigma_j \sigma_i \rangle^{m_{ij}-1} W_1 \quad \text{and} \quad \langle \sigma_i \sigma_j \rangle^{m_{ij}-4} \tau_p U \sim \langle \sigma_i \sigma_j \rangle^{m_{ij}-2} \tau_p W_1.$$

Cancelling yields $\tau_p U \sim \sigma_p \sigma_q \tau_p W_1$. By further applications of the reduction property, there are words W_2, W_3 and W_4 such that

$$
\begin{array}{rcl}
U \sim \sigma_p W_2 & \text{and} & \sigma_q \tau_p W_1 \sim \tau_p W_2, \\
\tau_p W_1 \sim \langle \sigma_p \sigma_q \rangle^{m_{ij}-2} \tau_i W_3 & \text{and} & W_2 \sim \langle \sigma_q \sigma_p \rangle^{m_{ij}-1} W_3, \quad \text{and} \\
W_1 \sim \sigma_p W_4 & \text{and} & \langle \sigma_q \sigma_p \rangle^{m_{ij}-3} \tau_i W_3 \sim \tau_p W_4.
\end{array}
$$

But this last equivalence gives a common multiple of τ_p and $\langle \sigma_q \sigma_p \rangle^{m_{ij}-3} \tau_i$, and of length less than the length of X, contradicting the assumption of minimality. The lemma now follows by induction. □

Let a and b be letters from $S \cup T$. A nonempty word C is a *simple chain* with *source* a and *target* b if there are words U and V such that $(aU, CbV) \in \mathscr{R}^\Sigma$; we also say that C is a *simple a-chain* for short. A simple a-chain C is said to be *preserving* if $(aC, Cb) \in \mathscr{R}^\Sigma$. Inspection of the relations shows that any simple a-chain whose target is an element of T must be preserving, and that if C is a simple preserving a-chain to b, then a is in T precisely when b is in T.

A word C is called a *compound a-chain*, or just an a-chain, if $C \equiv C_1 \cdots C_k$ for simple chains $C_1, \dots, C_k$, where C_1 is an a-chain, and the source of C_{i+1} is the target of C_i for all $i \geq 1$. The *source* and *target* of C are defined to be the source of C_1 and the target of C_k respectively. An a-chain is said to be *preserving* if each of its component simple chains is preserving.

REMARK 1.3. It was shown in [9] that if C is an a-chain to b and CD is a common multiple of a and C, then CD is a common multiple of a and Cb; thus the target of C divides D. In particular, a does not divide C. For each a in $S \cup T$, a partial operator $K_a : (S \cup T)^* \to (S \cup T)^*$ was then defined, with the properties that

(1) $K_a(W)$ is defined whenever a and W have a common multiple;
(2) when it is defined, $K_a(W) \sim W$, and $K_a(W)$ begins with a if a divides W, or otherwise is an a-chain;
(3) $K_a(W)$ is calculable; and
(4) if a does not divide W, but a divides Wb for some generator b, then $K_a(W)$ is an a-chain with target b.

LEMMA 1.4. *A nonempty word W is equivalent to a preserving a-chain to b precisely when a does not divide W and $aW \sim Wb$. Moreover, any a-chain with target in T is preserving.*

PROOF. Suppose that W is equivalent to $C \equiv C_1 \cdots C_k$ where each C_i is a simple preserving a_{i-1} chain to a_i. Then for each i, $(a_{i-1}C_i, C_i a_i) \in \mathscr{R}^\Sigma$, so $a_{i-1}C_i \sim C_i a_i$. Hence $a_0 C \sim C a_k$, where $a_0 \equiv a$ is the source and $a_k \equiv b$ is the target of C. Thus $aW \sim Wb$, and by Remark 1.3, a does not divide W.

Now suppose that W is not divisible by a but $aW \sim Wb$. By the reduction property, there is some pair $(aC_1, U_1 d)$ in $\mathscr{R}^\Sigma$ and a word V such that $aW \sim aC_1 V$ and $Wb \sim U_1 d V$. Thus $C_1 V b \sim Wb \sim aW \sim aC_1 V \sim U_1 d V$. Suppose $C_1 \not\equiv U_1$. Inspection of the relations tells us that $aC_1 \equiv \langle \sigma_i \sigma_j \rangle^{m_{ij}-1} \tau_p$ and $U_1 d \equiv \tau_j \langle \sigma_i \sigma_j \rangle^{m_{ij}-1}$ for some i and j, where $p = i$ if m_{ij} is odd and j if m_{ij} is even. Substituting into the above, $\langle \sigma_j \sigma_i \rangle^{m_{ij}-2} \tau_p V b \sim \tau_j \langle \sigma_i \sigma_j \rangle^{m_{ij}-1} V$. By the reduction property, there is a word V' such that $\langle \sigma_i \sigma_j \rangle^{m_{ij}-3} \tau_p V b \sim \tau_j V'$, which contradicts Lemma 1.2. Hence $C_1 \equiv U_1$.

Thus C_1 must be preserving. Since $C_1 V b \sim C_1 d V$, left cancellativity gives $Vb \sim dV$, where V is shorter than W. Furthermore, V is not divisible by d since if it were, $V \sim dV''$ for some word V'', and $W \sim C_1 V \sim C_1 d V'' \sim aC_1 V''$, which contradicts that a does not divide W. So we can continue in this way replacing W with V, until $W \sim C_1 C_2 \cdots C_k$ where each C_i is preserving.

Finally, suppose that $C \equiv C_1 \cdots C_k$ is an a-chain to τ_j for some j. Let a_{i-1} be the source and a_i the target of C_i for each i. So C_k is a simple a_{k-1}-chain to τ_j. Thus it must be preserving, and have source $a_{k-1} \in T$ also. Continuing backwards in this way through C, we have that each simple component is preserving, and that each a_i is in T. □

THEOREM 1.5. *For any word W, any generators a and b and r any positive integer, we have $a^r W \sim W b^r$ if and only if $aW \sim Wb$.*

PROOF. The 'if' direction is evident; suppose henceforth that $a^r W \sim W b^r$. If a divides W, then cancellativity and an inductive hypothesis give the result quickly. We may suppose a does not divide W. We now use various parts of Remark 1.3: firstly, since Wb^r is a common multiple of a and W, then $K_a(W)$ is defined, and since a does not divide W, is an a-chain. Let t denote its target. Since $K_a(W) b^r \sim W b^r \sim a^r W$ is a common multiple of a and W, t divides b^r, implying $t \equiv b$. Thus $K_a(W)$ is an a-chain to b.

If $a \in T$, since the number of τ's is preserved by the relation $\sim$, then $a^r W \sim W b^r$ implies that $b \in T$ as well, so by Lemma 1.4, $K_a(W)$ is preserving, and $aW \sim Wb$.

From now on, we suppose that a, and hence b, are in S. Write $a = \sigma_i$. First we show that W is not divisible by τ_j for $m_{ij} > 2$. Let $e \geq 1$ and $w \equiv \tau_j \langle \sigma_i \sigma_j \rangle^{m_{ij}-1} (\langle \sigma_j \sigma_i \rangle^{m_{ij}})^{e-1}$; then $w \sim \sigma_i^e v$ where

$$v \equiv \begin{cases} (\langle \sigma_j \sigma_i \rangle^{m_{ij}-1})^{e-1} \langle \sigma_j \sigma_i \rangle^{m_{ij}-2} \tau_j & \text{if } m_{ij} \text{ even;} \\ (\langle \sigma_j \sigma_i \rangle^{m_{ij}-1} \langle \sigma_i \sigma_j \rangle^{m_{ij}-1})^{(e-1)/2} \langle \sigma_j \sigma_i \rangle^{m_{ij}-2} \tau_j & \text{if } m_{ij} \text{ and } e \text{ odd;} \\ (\langle \sigma_j \sigma_i \rangle^{m_{ij}-1} \langle \sigma_i \sigma_j \rangle^{m_{ij}-1})^{(e-2)/2} \langle \sigma_j \sigma_i \rangle^{m_{ij}-1} \langle \sigma_i \sigma_j \rangle^{m_{ij}-2} \tau_i & \text{otherwise.} \end{cases}$$

Further, v is a singleton $\sim$-equivalence class in each case. Since τ_j and σ_i^e have a common multiple in w, by [9, Corollary 13] they have a least common multiple, say

$L \equiv \sigma_i^e u$. Since L divides w, we have u divides v. Since the number of occurrences of letters from T is preserved under $\sim$, L must contain at least one such occurrence, so u cannot properly divide v; they must be equivalent. Thus w is the lcm of τ_j and σ_i^e. This lcm has length $m_{ij}e \geq 3e$. Now for any n, $a^{nr}W \sim Wb^{nr}$, so if W were divisible by τ_j, then $a^{nr}W$ would be a common multiple of σ_i^{nr} and τ_j, and hence would have length greater than $3nr$. Thus for every n, the length of W would have to be at least $2nr$, which is absurd. Thus W must not be divisible by τ_j for $m_{ij} > 2$.

Thus we have that W begins with x where either $x = \sigma_j \in S$, or $x = \tau_j$ where $i = j$ or $m_{ij} = 2$. From inspection of the relations, we see that the lcm of a and x must be $aC \sim Cd$ where C begins with x and $d \in S$; furthermore, $a^rC \sim Cd^r$ is the lcm of a^r and x. Thus a^rC divides $a^rW \sim Wb^r$, so $W \sim CV$ for some word V. Thus $a^rW \sim Cd^rV \sim CVb^r \sim Wb^r$, so $d^rV \sim Vb^r$. By an inductive hypothesis, $dV \sim Vb$, so $aW \sim aCV \sim CdV \sim CVb \sim Wb$, completing the proof. □

For any words $V_1, \dots, V_k$ over $S \cup T$ with a common multiple, a word $L(V_1, \dots, V_k)$ can be calculated which is a least common multiple of $V_1, \dots, V_k$ (see [9, Lemma 12]). Let $\Lambda(U)$ denote the set of letters from $S \cup T$ which divide the word U. Since U is a common multiple, $L(\Lambda(U))$ always exists, and divides U. If τ_i and τ_j are in $\Lambda(U)$, then $U \sim \tau_i U_1 \sim \tau_j U_2$ for some U_1 and U_2, so applying the reduction property, $m_{ij} = 2$. Thus the product of the elements of $\Lambda(U) \cap T$ is a common multiple of $\Lambda(U) \cap T$; and is, in fact, equivalent to $L(\Lambda(U) \cap T)$.

Lemma 1.6. *Let U be a nonempty word such that $\Lambda(U) \subset T$. Suppose that $VU \sim UW$ for some words V and W and $\Lambda(U) \cap \Lambda(V) = \emptyset$. Then $L(\Lambda(U))$ commutes with V and there is a word U' shorter than U such that $VU' \sim U'W$.*

Proof. Take any τ_j in $\Lambda(U)$. Since τ_j and V have a common multiple, Remark 1.3 says $K_{\tau_j}(V)$ is defined, and since τ_j is not in $\Lambda(V)$, $K_{\tau_j}(V)$ is a τ_j-chain. Let b be the target of $K_{\tau_j}(V)$. Again by Remark 1.3, since τ_j divides $K_{\tau_j}(V)U \sim VU$, b divides U. So $b \equiv \tau_k$ for some k, $K_{\tau_j}(V)$ is preserving, and $\tau_j V \sim V\tau_k$.

Thus for every τ_p in $\Lambda(U)$, there is a corresponding τ_q in $\Lambda(U)$ such that $\tau_p V \sim V\tau_q$. Moreover, if $\tau_r V \sim V\tau_q$ then $\tau_p V \sim \tau_r V$; right cancellation then gives $\tau_p \equiv \tau_r$. Thus V defines a permutation π_V on $\Lambda(U)$ by $\tau_p V \sim V\pi_V(\tau_p)$. By the comment preceding the statement of the lemma, $L(\Lambda(U))$ is the product of the elements of $\Lambda(U)$ in some order, and the letters all commute. Thus

$$L(\Lambda(U))V \equiv \tau_{i_1} \cdots \tau_{i_k} V \sim V\pi_V(\tau_{i_1}) \cdots \pi_V(\tau_{i_k}) \sim V\tau_{i_1} \cdots \tau_{i_k} \equiv VL(\Lambda(U)).$$

Now $U \sim L(\Lambda(U))U'$ for some word U' which is shorter than U. Thus

$$L(\Lambda(U))VU' \sim VL(\Lambda(U))U' \sim VU \sim UW \sim L(\Lambda(U))U'W.$$

Cancelling $L(\Lambda(U))$ from the left, $VU' \sim U'W$, and U' is shorter than U, as desired. $\square$

The operator $\mathtt{rev} : (S \cup T)^* \longmapsto (S \cup T)^*$ maps a word to its reverse:

$$\mathtt{rev}(a_1 a_2 \cdots a_{k-1} a_k) \equiv a_k a_{k-1} \cdots a_2 a_1.$$

LEMMA 1.7. *Let C be an a-chain to b. Then $b \in S$ if $a \in S$, and b does not divide $\mathtt{rev}(C)$ whenever $a \in S$ or $b \in T$.*

PROOF. By inspection of the relations, $b \in S$ if $a \in S$, and always, b does not divide $\mathtt{rev}(C)$ if C is simple. Suppose henceforth that $a \in S$ or $b \in T$, so either $\{a, b\} \subseteq S$, or $\{a, b\} \subseteq T$. We will show

(*) *if C is simple and W any word where b divides $\mathtt{rev}(WC)$, then a divides $\mathtt{rev}(W)$.*

The result then follows by induction on the number of simple components of C.

The proof of (*) falls into two cases: either (i) $\mathtt{rev}(C)$ is a simple b-chain to a, or (ii) $C \equiv \tau_j \langle \sigma_i \sigma_j \rangle^r$ for some i and j with $m_{ij} > 2$, and some $0 \leq r < m_{ij} - 1$. In case (i), we observe that $\mathtt{rev}(WC)$ is a common multiple of $\mathtt{rev}(C)$ and b, and so, by the comments beginning Remark 1.3, a (the target of the b-chain $\mathtt{rev}(C)$) divides $\mathtt{rev}(W)$.

In case (ii), C is a simple $a \equiv \sigma_i$-chain to b, where $b \equiv \sigma_j$ if r is odd and $b \equiv \sigma_i$ otherwise. However, $\mathtt{rev}(C)$ is not simple, but a compound b-chain to σ_j consisting of simple chains $\mathtt{rev}(\langle \sigma_i \sigma_j \rangle^r)$ and τ_j. Again using Remark 1.3, σ_j must divide $\mathtt{rev}(W)$, say $W \sim X\sigma_j$; hence

$$WC \sim X\sigma_j \tau_j \langle \sigma_i \sigma_j \rangle^r \sim X\tau_j \langle \sigma_j \sigma_i \rangle^{r+1}.$$

Now $\mathtt{rev}(\langle \sigma_j \sigma_i \rangle^{r+1})$ is a b-chain to σ_i, so σ_i must divide $\mathtt{rev}(X\tau_j) \equiv \tau_j\, \mathtt{rev}(X)$. By the reduction property, there exists a word V such that $\mathtt{rev}(X) \sim \langle \sigma_i \sigma_j \rangle^{m_{ij}-1} V$. Thus

$$\mathtt{rev}(W) \sim \sigma_j\, \mathtt{rev}(X) \sim \sigma_j \langle \sigma_i \sigma_j \rangle^{m_{ij}-1} V \equiv \langle \sigma_j \sigma_i \rangle^{m_{ij}} V \sim \langle \sigma_i \sigma_j \rangle^{m_{ij}} V,$$

so we have $\sigma_i \equiv a$ divides $\mathtt{rev}(W)$. $\square$

We thank the referee for the stronger version of this result as given above. He also pointed out that it is now as strong as it can be—there are a-chains C with target $b \in S$ for which b divides $\mathtt{rev}(C)$. An example is in type A_2, where $C \equiv \sigma_1\sigma_2\sigma_1\sigma_2$ is a compound τ_2-chain to σ_1, but $C \sim \sigma_1\sigma_1\sigma_2\sigma_1$, so $\mathtt{rev}(C)$ is divisible by its target, σ_1.

For the remainder of this section, we suppose that $\Delta \equiv L(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is defined, that is, the elements of S have a common multiple, and Δ is a least common multiple

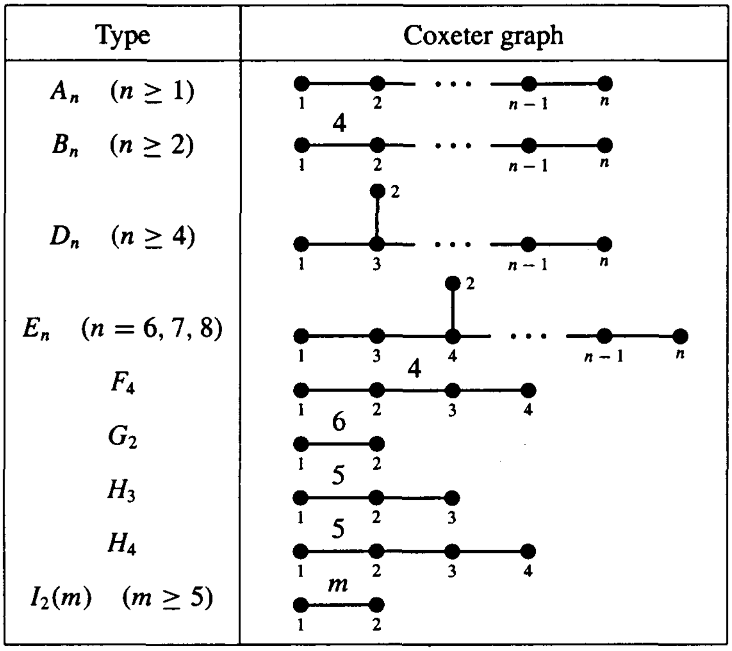

FIGURE 1. The irreducible Coxeter graphs of finite type. Unlabelled edges have value 3.

of S. Then we say that M is of *finite type*. It is known (see for example, [20, Chapter 2]) that the graphs for these types are precisely finite disjoint unions of those shown in Figure 1. The reader will notice that this list is closed under taking complete subgraphs. The element Δ is called the *fundamental element*, and has a number of properties which will be referred to shortly. First we need the definition of a square free word. A word W has a *quadratic factor* if there are words U and V over $S \cup T$ and a letter a such that $W \equiv UaaV$. A word is *square free* if all words equivalent to it have no quadratic factor. The following results can all be found in [7, Sections 5 and 8] and [9, Section 4].

REMARK 1.8. Properties of the fundamental element Δ. ([7, Sections 5, 8]; [9, Section 4].)

(1) Δ is an element of the positive Artin monoid (that is, a word over S).
(2) Δ is both a left and right least common multiple of S.
(3) $\texttt{rev}\,\Delta \sim \Delta$.
(4) A word over S is square free precisely when it is a divisor of Δ.
(5) Δ generates the centre of $\mathscr{S}^+_M$ for all finite Artin types except types A_n for $n \geq 2$, D_{2k+1}, E_6 and $I_2(2q+1)$, in which cases Δ^2 generates the centre.

We will denote by ζ either Δ or Δ^2, the generator of the centre of $\mathscr{S}^+_M$. Define

$\mathcal{QF}(S)$ to be the set of words over S for which there is a word $\bar{A}$ such that $A\bar{A} \sim \Delta$—in other words, $\mathcal{QF}(S)$ is all words (including the empty word) which divide the fundamental element. According to the previous result, we could define $\mathcal{QF}(S)$ equivalently to be the square free words over S (the letters $\mathcal{QF}$ stand for *quadrat frei*, or square free). There are only finitely many words whose length is at most $\ell(\Delta)$, so $\mathcal{QF}(S)$ is finite. In contrast, there are infinitely many square free words over $S \cup T$, as there are square free words of arbitrary length over $S \cup T$, for example $(\tau_i\tau_j)^k$ is square free for any k, provided $m_{ij} > 2$.

LEMMA 1.9. (1) *Suppose that $BxC \sim B'yC'$ where B, B', C, C' are words over S and $x, y \in T$. Then $BC \sim B'C'$.*
(2) *Suppose that A is in $\mathcal{QF}(S)$ and b is any letter in $S \cup T$. If b does not left divide A, then bA is square free. If b does not right divide A, then Ab is square free.*

PROOF. Part (1) follows by observing that the only relations involving an element t of T which may be applied here are of the form (tw, wu) or (wt, uw) for some $u \in T$ and word w over S. The result then follows by induction on the number of such applications required to transform BxC into $B'yC'$.

If $b \in S$, then the statement of part (2) follows from [7, Lemma 3.4] and an application of `rev`. If $b \in T$, then by part (1), bA and Ab are square free whenever A is in $\mathcal{QF}(S)$. □

LEMMA 1.10. *Let B and V be words over $S \cup T$. Suppose that A is the longest square free word over S which divides AB, and moreover, that A divides VAB. Then A divides VA.*

PROOF. By (3) and (4) of Remark 1.8, there is a word D over S such that $DA \sim \Delta$. So Δ divides $DVAB$. Hence every letter of S divides $DVAB$. In particular, $DVAB$ is a common multiple of σ_i and DVA for all σ_i in S, so $K_{\sigma_i}(DVA)$ is always defined.

Suppose σ_i does not divide DVA. Then $K_{\sigma_i}(DVA)$ is a σ_i-chain to b, for some $b \in S$. Further, b must divide B, since σ_i divides $DVAB$ (Remark 1.3). Moreover, Lemma 1.7 says b does not right divide DVA, so in particular b does not right divide A. Hence Ab is squarefree (by Lemma 1.9) and divides AB, contradicting the maximality of A.

Thus each σ_i divides DVA, so their least common multiple $\Delta \sim DA$ divides DVA. Cancelling D from the left, A must divide VA. □

2. Conjugacy in $\mathcal{S}_M^+$ when M is of finite type

Suppose that V and W are words over $S \cup T$. We say that V *is conjugate to* W *(relative to $\mathcal{R}$)*, denoted $V \asymp W$, if there exists a word X over $S \cup T$ such that

$VX \sim XW$. In this case we say $V \asymp W$ by X. It is not immediately obvious whether conjugacy is an equivalence relation. It is certainly reflexive, and if $V \asymp W$ by X and $W \asymp Z$ by Y then $V \asymp Z$ by XY, so conjugacy is transitive.

Throughout the whole of this section, M is assumed to be of finite type. We will see that this restriction is enough to ensure that conjugacy is also symmetric (although in the case when M is not of finite type, this is not known).

Let $\aleph = \mathcal{QF}(S) \cup T$. Say that V *is* $\aleph$-*conjugate to* W if $V \asymp W$ by some element of $\aleph$; that is, if there is a word A in $\aleph$ such that $VA \sim AW$. This is denoted $V \asymp_\aleph W$. If $V \asymp_\aleph W$ then $V \asymp W$. We will show that $\asymp$ is contained in the equivalence relation generated by $\asymp_\aleph$. (This will turn out to be $\asymp$.)

LEMMA 2.1. *Suppose that $V \asymp W$. Then there is a positive integer p and words $X_0, X_1, \dots, X_p$ with $X_0 \equiv V$, $X_p \equiv W$ and $X_{i-1} \asymp_\aleph X_i$ for $i = 1, \dots, p$.*

PROOF. Suppose $V \asymp W$ by U, so $VU \sim UW$. The argument is by induction on the length of U. If $\ell(U) = 0$ or 1, then $U \in \aleph$ so $p = 1$. Suppose now that $\ell(U) > 1$, so $\Lambda(U)$ must be nonempty.

First suppose that $\Lambda(U) \cap S \neq \emptyset$. Let A be the longest square free word over S which divides U. So $U \sim AB$ for some word B and A is not empty. Since A divides VAB, Lemma 1.10 says there is a word X such that $VA \sim AX$. But $A \in \aleph$ so $V \asymp_\aleph X$, and moreover $AXB \sim ABW$, so after cancelling, $XB \sim BW$. The result now follows by induction applied to B.

Now we suppose that $\Lambda(U) \subseteq T$. Suppose that $\Lambda(U) \cap \Lambda(V) \neq \emptyset$. Then there is a $\tau_i \in T$ such that $U \sim \tau_i U'$ and $V \sim \tau_i V'$. Let $X \equiv V'\tau_i$. Then $V\tau_i \sim \tau_i V'\tau_i \equiv \tau_i X$, so $V \asymp_\aleph X$. Moreover, $\tau_i X U' \sim \tau_i V'\tau_i U' \sim VU \sim UW \sim \tau_i U'W$. By cancellation, $XU' \sim U'W$, and the result follows by induction applied to U'.

The only case left is when $\Lambda(U) \subseteq T$ and $\Lambda(U) \cap \Lambda(V) = \emptyset$. These are precisely the conditions of Lemma 1.6, so there is a word U' shorter than U such that $VU' \sim U'W$, and so the result follows by induction applied to U'. □

LEMMA 2.2. *If $V \asymp_\aleph W$ then $W \asymp V$.*

PROOF. Suppose that A is an element of $\mathcal{QF}(S)$ such that $VA \sim AW$. Then there is a word D such that $AD \sim \Delta$, and $AD\Delta \sim \Delta^2$ is certainly central, so

$$AWD\Delta \sim VAD\Delta \sim V\Delta^2 \sim \Delta^2 V \sim AD\Delta V,$$

and after cancelling A we have $WD\Delta \sim D\Delta V$, and so $W \asymp V$.

Suppose alternatively that $V\tau_i \sim \tau_i W$ for some τ_i in T. If τ_i divides V then $V \sim \tau_i X$ for some word X, so $\tau_i W \sim \tau_i X \tau_i$, after which cancelling gives $W \sim X\tau_i$, yielding $WX \sim X\tau_i X \sim XV$, and so $W \asymp V$. If τ_i does not divide V, then $K_{\tau_i}(V)$

is a τ_i-chain to τ_i, by (4) of Remark 1.3. In this case, by Lemma 1.4, $K_{\tau_i}(V)$ is preserving, so $\tau_i W \sim V\tau_i \sim \tau_i V$; cancelling gives $W \sim V$, so $W \asymp V$. □

THEOREM 2.3. *Conjugacy is an equivalence relation.*

PROOF. As remarked earlier, conjugacy is reflexive and transitive. Suppose that $V \asymp W$. Then by Lemma 2.1 there is an integer p and words $X_0, X_1, \ldots, X_p$ such that $V \equiv X_0$, $X_p \equiv W$ and $X_{i-1} \asymp_{\aleph} X_i$ for $i = 1, \ldots, p$. By the previous lemma, for each $i = 1, \ldots, p$, $X_i \asymp X_{i-1}$. By transitivity, $X_p \asymp X_0$; so $W \asymp V$. Thus conjugacy is also symmetric, and hence an equivalence relation. □

Let Σ be a set of words over $S \cup T$. Then define

$$\varphi(\Sigma) = \{X \mid V \asymp_{\aleph} X \text{ for some } V \in \Sigma\}.$$

Observe that if V is conjugate to W, then there is an integer p and words $X_0, \ldots, X_p$ such that $V \equiv X_0 \asymp_{\aleph} X_1 \asymp_{\aleph} \cdots \asymp_{\aleph} X_p \equiv W$, so $W \in \varphi^p(\{V\})$. Furthermore, homogeneity forces $\ell(X) = \ell(V)$ whenever $V \asymp_{\aleph} X$, so

$$\varphi^k(\{V\}) \subseteq \{X \in (S \cup T)^* \mid \ell(X) = \ell(V)\},$$

which is finite—it has at most $(2n)^{\ell(V)}$ elements, where $|S| = |T| = n$. By reflexivity of $\asymp_{\aleph}$-conjugacy, $\varphi^k(\Sigma) \subseteq \varphi^{k+1}(\Sigma)$ for all numbers k and sets Σ. If $\varphi^k(\Sigma) = \varphi^{k+1}(\Sigma)$ then $\varphi^{k+r}(\Sigma) = \varphi^k(\Sigma)$ for all $r > 0$. So,

$$\varphi^{(2n)^{\ell(V)}}(\{V\}) = \varphi^{(2n)^{\ell(V)}+1}(\{V\}).$$

Let

$$\Phi(V) = \varphi^{(2n)^{\ell(V)}}(\{V\}).$$

Then $\Phi(V)$ is precisely the set of all words conjugate to V.

If $W \equiv bU$ for some letter b, then we write ${}^{b^{-1}}W \equiv U$. If W does not begin with b then ${}^{b^{-1}}W$ is not defined. We now define a partial operator $(\,/\,) : (S\cup T)^* \times (S\cup T)^* \to (S \cup T)^*$ which does the job of division. Suppose $V \equiv a_1a_2\cdots a_k$. If V divides W, then

$$(W/V) \equiv {}^{a_k^{-1}}K_{a_k}\big(\cdots {}^{a_2^{-1}}K_{a_2}({}^{a_1^{-1}}K_{a_1}(W))\cdots\big),$$

and (W/V) is undefined otherwise. (In fact, if V does not divide W then in trying to perform the calculation described will result in an 'undefined' answer at some stage.) Moreover, if V divides W then, by [9, Lemma 6], $W \sim V(W/V)$.

THEOREM 2.4. *The set $\Phi(V)$ of all words conjugate to V is calculable.*

PROOF. For any $k > 0$, $\varphi^{k+1}(\{V\}) = \bigcup_{X \in \varphi^k(\{V\})} \varphi(\{X\})$. Consider $X \in \varphi^k(\{V\})$. Then $\ell(X) = \ell(V)$, and

$$\begin{aligned}\varphi(\{X\}) &= \{Y \in (S \cup T)^{\ell(V)} \mid XA \sim AY \text{ for some word } A \text{ in } \aleph\} \\ &= \{Y \in (S \cup T)^{\ell(V)} \mid (AY/X) \not\equiv \infty \text{ and } ((AY/X)/A) \text{ is the} \\ &\qquad \text{empty word for some word } A \text{ in } \aleph\}.\end{aligned}$$

So to calculate $\varphi(\{X\})$, for each of the elements A of $\aleph$ and for each of the $(2n)^{\ell(V)}$ elements Y of $(S \cup T)^{\ell(V)}$, the two calculations (AY/X) and $((AY/X)/A)$ must be made. There are at most $|T| + (|S| + 1)^{\ell(\Delta)} = n + (n+1)^{\ell(\Delta)}$ words in $\aleph$, so this adds up to at most $2(2n)^{\ell(V)}(n + (n+1)^{\ell(\Delta)})$ calculations. Since there can be no more than $(2n)^{\ell(V)}$ elements of $\varphi^k(\{V\})$, at most $2(2n)^{\ell(V)}(n + (n+1)^{\ell(\Delta)})(2n)^{\ell(V)}$ calculations need to be done to calculate $\varphi^{k+1}(\{V\})$, once $\varphi^k(\{V\})$ is calculated. By definition $\Phi(V) = \varphi^{(2n)^{\ell(V)}}(\{V\})$, so at most

$$2(2n)^{\ell(V)}(n + (n+1)^{\ell(\Delta)})(2n)^{\ell(V)}(2n)^{\ell(V)} = 2^{3\ell(V)+1}n^{3\ell(V)}(n + (n+1)^{\ell(\Delta)})$$

calculations are required to determine $\Phi(V)$. □

Thus the conjugacy problem is solvable in positive singular Artin monoids. To determine if W is conjugate to V, calculate $\Phi(V)$ (which is finite) and see whether W is a member.

3. Conjugacy in singular Artin monoids of finite type

Suppose that M is an $I \times I$ Coxeter matrix as defined in the first section. Let S^{-1} denote the set $\{\sigma_i^{-1} \mid i \in I\}$ of formal inverses of S. Then the *singular Artin monoid of type M*, denoted $\mathcal{S}_M$, is the monoid generated by $S \cup S^{-1} \cup T$ subject to the relations $\mathcal{R}$ described in Section 1, and the free group relations on S:

$$\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 \qquad \text{for all } i \in I.$$

If two words V and W over $S \cup S^{-1} \cup T$ represent the same element in $\mathcal{S}_M$ then write $V \approx W$.

Throughout the rest of this section, M will be of finite type. In this case, the singular Artin monoid S_M is also said to be of finite type. The reader is reminded that this means the Coxeter graph of M is a finite disjoint union of graphs in the list in Figure 1. The singular Artin monoid of type A_n is commonly known as the *singular braid monoid on $n+1$ strings*, as defined in [4] and [5].

Theorem 20 of [9], known as the Embedding Theorem in the sequel, says that $\mathcal{S}_M^+$ embeds in $\mathcal{S}_M$. The proof of this theorem made much use of the fundamental element

Δ, and the central element ζ (which is either Δ or Δ^2, depending on the type, as per (5) of Remark 1.8). Suppose $V \equiv U_0\sigma_{i_1}^{-1}U_1\sigma_{i_2}^{-1}U_2 \cdots U_{k-1}\sigma_{i_k}^{-1}U_k$ where each U_i is a word over $S \cup T$. Then, as in [9, Section 5], define

$$\begin{aligned}\theta_1(V) &\equiv U_0\zeta_{i_1}U_1\zeta_{i_2}U_2 \cdots U_{k-1}\zeta_{i_k}U_k,\\ \theta_2(V) &= k,\end{aligned}$$

where $\zeta_i \equiv (\zeta/\sigma_i)$, which is defined because every letter of S divides ζ. Thus

$$\zeta^{-\theta_2(V)}\theta_1(V) \approx V.$$

Two words V and W over $S \cup S^{-1} \cup T$ are said to be *conjugate in $\mathscr{S}_M$* if there exists a word X over $S \cup S^{-1} \cup T$ such that $VX \approx XW$. The following result shows that conjugacy in $\mathscr{S}_M^+$ is just the restriction of conjugacy in $\mathscr{S}_M$. Thus it will make sense to use the notation $V \asymp W$ for conjugacy in $\mathscr{S}_M$ also.

LEMMA 3.1. *If V and W are conjugate in $\mathscr{S}_M$, then there is a word X over $S \cup T$ such that $VX \approx XW$.*

PROOF. Suppose that V, W, and Y are words over $S \cup S^{-1} \cup T$ such that $VY \approx YW$. Since $\zeta^{-\theta_2(Y)}\theta_1(Y) \approx Y$, multiplying through by $\zeta^{\theta_2(Y)}$ gives $V\theta_1(Y) \approx \theta_1(Y)W$, so in fact V and W are conjugate by a word $X \equiv \theta_1(Y)$ over $S \cup T$. $\square$

THEOREM 3.2. *Let V and W be words over $S \cup S^{-1} \cup T$ with $\theta_2(V) \geq \theta_2(W)$. Then V is conjugate to W in $\mathscr{S}_M$ precisely when $\theta_1(V)$ is conjugate to $\zeta^{\theta_2(V)-\theta_2(W)}\theta_1(W)$ in $\mathscr{S}_M^+$. In particular, words over $S \cup T$ are conjugate in $\mathscr{S}_M$ precisely when they are conjugate in $\mathscr{S}_M^+$. Thus the conjugacy problem is solvable in $\mathscr{S}_M$.*

PROOF. Suppose $\theta_1(V)$ is conjugate to $\zeta^{\theta_2(V)-\theta_2(W)}\theta_1(W)$ in $\mathscr{S}_M^+$. Then there is a word X over $S \cup T$ such that $\theta_1(V)X \sim X\zeta^{\theta_2(V)-\theta_2(W)}\theta_1(W)$. Multiplying by $\zeta^{-\theta_2(V)}$, using the fact that $\zeta^{-\theta_2(V)}\theta_1(V) \approx V$, and the centrality of ζ, we have $VX \approx XW$.

On the other hand, suppose that V and W are conjugate in $\mathscr{S}_M$. By Lemma 3.1, there is a word X over $S \cup T$ such that $VX \approx XW$. Multiplying through by $\zeta^{\theta_2(V)}$ gives $\theta_1(V)X \approx X\zeta^{\theta_2(V)-\theta_2(W)}\theta_1(W)$, but, since $\theta_2(V) \geq \theta_2(W)$, all the words in the equation are over the alphabet $S \cup T$. By the Embedding Theorem of [9], this gives $\theta_1(V)X \sim X\zeta^{\theta_2(V)-\theta_2(W)}\theta_1(W)$, so $\theta_1(V)$ is conjugate to $\zeta^{\theta_2(V)-\theta_2(W)}\theta_1(W)$ in $\mathscr{S}_M$. $\square$

The remainder of this section deals with some results which we hope may explain our choice of the definition of conjugacy. There does not seem to be a general semigroup theoretic definition of conjugacy. Howie [18] introduces the following notion, which we call here 'swap conjugacy', in the context of a certain class of semigroups called equidivisibible semigroups. Say that V and W are *swap conjugate* if there exist X and Y such that $V \approx XY$ and $W \approx YX$. This notion is a natural one in the context of singular braids; these may be considered as geometric realisations of elements of singular Artin monoids of type A. This is discussed in more detail in Section 6.

Swap conjugacy is clearly reflexive and symmetric; though not necessarily transitive. In Howie's context, transitivity is easily shown to hold. However, even in the positive singular Artin monoid of type A_2, swap conjugacy fails to be transitive. For example, $\sigma_1\sigma_1\sigma_2$ is swap conjugate to $\sigma_1\sigma_2\sigma_1 \sim \sigma_2\sigma_1\sigma_2$, which is swap conjugate to $\sigma_1\sigma_2\sigma_2$. Using the fact that $\sigma_1\sigma_1\sigma_2$ is in a singleton equivalence class of words in $\mathscr{S}^+_{A_2}$, it is easy to see that the only words which are swap conjugate to it are $\sigma_1\sigma_1\sigma_2$, $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_1$. Thus $\sigma_1\sigma_1\sigma_2$ is not swap conjugate to $\sigma_1\sigma_2\sigma_2$ in $\mathscr{S}^+_{A_2}$.

We mention that, on the other hand, $\sigma_1\sigma_1\sigma_2 \approx (\sigma_1\sigma_2)(\sigma_1\sigma_2\sigma_1^{-1})$ *is* swap conjugate to $\sigma_1\sigma_2\sigma_2 \approx (\sigma_1\sigma_2\sigma_1^{-1})(\sigma_1\sigma_2)$ in $\mathscr{S}_{A_2}$. Thus this relation is coarser in $\mathscr{S}^+_M$ than in $\mathscr{S}_M$; preventing us from 'bootstrapping' our way from $\mathscr{S}^+_M$ up to $\mathscr{S}_M$ as has been the technique previously.

It turns out that provided we restrict our interest to the transitive closure of swap conjugacy in $\mathscr{S}_M$, we obtain the same relation as that of conjugacy as defined in this paper. (At the time of writing, the author does not know if swap conjugacy in $\mathscr{S}_M$ is transitive.) If V and W are swap conjugate then we write $V \rightleftharpoons W$. We remind the reader that M is assumed to be of finite type.

LEMMA 3.3. *Let V and W be words over $S \cup T$ such that $V \asymp_{\aleph} W$. Then $V \rightleftharpoons W$.*

PROOF. There exists $A \in \aleph = \mathscr{Q}\mathscr{F}(S) \cup T$ such that $VA \sim AW$. If $A \in \mathscr{Q}\mathscr{F}(S)$ then A is invertible, and so $V \approx (A)(A^{-1}V)$ and $W \approx (A^{-1}V)(A)$, whence $V \rightleftharpoons W$.

Otherwise, $A \equiv \tau_i$ for some i. Suppose that τ_i divides V. Then $V \sim \tau_i V'$ for some word V' over $S \cup T$, and we have $\tau_i V'\tau_i \sim \tau_i W$. After cancelling, we have $W \sim V'\tau_i$, and immediately $V \rightleftharpoons W$. Finally, suppose that τ_i does not divide V. However, τ_i does divide $V\tau_i$, so by (4) of Remark 1.3, $K_{\tau_i}(V)$ is a τ_i-chain to τ_i. Lemma 1.4 says this τ_i-chain must be preserving, and so $K_{\tau_i}(V)\tau_i \sim \tau_i K_{\tau_i}(V)$. Since $V \sim K_{\tau_i}(V)$, we have $\tau_i V \sim V\tau_i \sim \tau_i W$. After cancelling, we have $V \sim W$, so V and W are trivially swap conjugate. □

LEMMA 3.4. *If V and W are words over $S \cup S^{-1} \cup T$ such that $V \rightleftharpoons W$, then $\zeta^m V \rightleftharpoons \zeta^m W$ for all integers m.*

PROOF. This is immediate by the centrality of ζ. □

THEOREM 3.5. *Let V and W be words over $S \cup S^{-1} \cup T$.*

(1) *If $V \rightleftharpoons W$, then $V \asymp W$.*

(2) *If* $V \asymp W$, *then there exists an integer* $p \geq 0$ *and a sequence* $Z_0, Z_1, \dots, Z_p$ *such that* $V \equiv Z_0 \rightleftharpoons Z_1 \rightleftharpoons \cdots \rightleftharpoons Z_p \equiv W$.

PROOF. (1) If $V \rightleftharpoons W$, then there exist X and Y such that $V \approx XY$ and $W \approx YX$. Thus $VX \approx XYX \approx XW$, and so $V \asymp W$.

(2) Suppose $V \asymp W$. By Lemma 3.1 we may suppose that $VU \approx UW$ for some $U \in S \cup T$. There exists an integer $m = \max\{\theta_2(V), \theta_2(W)\}$ such that $V \approx \zeta^{-m} V^+$ and $W \approx \zeta^{-m} W^+$ for some words V^+ and W^+ over $S \cup T$. Multiplying through by ζ^m, and using the fact that ζ is central, we have $V^+ U \approx U W^+$. By the Embedding Theorem of [9], $V^+ U \sim U W^+$. By Lemma 2.1, there is an integer $p \geq 0$ and words $X_0, X_1, \dots, X_p$ with

$$V^+ \equiv X_0 \asymp_{\aleph} X_1 \asymp_{\aleph} \cdots \asymp_{\aleph} X_p \equiv W^+.$$

By Lemma 3.3, we have

$$V^+ \equiv X_0 \rightleftharpoons X_1 \rightleftharpoons \cdots \rightleftharpoons X_p \equiv W^+.$$

Multiplying through by ζ^{-m}, and invoking Lemma 3.4,

$$V \equiv Z_0 \rightleftharpoons Z_1 \rightleftharpoons \cdots \rightleftharpoons Z_p \equiv W,$$

where $Z_i \equiv \zeta^{-m} X_i$ for each $i = 0, 1, \dots, p$. □

4. Centralisers in singular Artin monoids of finite type

For generators a and b, denote by the set $Z(a, b)$ the set of words W over $S \cup S^{-1} \cup T$ such that $aW \approx Wb$. The centraliser of a generator a, which will be denoted $Z(a)$, is then the set $Z(a, a)$. Whenever $Z(a, b)$ is not empty, it is infinite in size—since, for example, whenever $P \in Z(a, b)$ then $a^k P$ is in $Z(a, b)$ also for arbitrary k. We assume throughout this section that M is of finite type. We will show (Proposition 4.6) that $Z(x_i, x_j)$ is not empty precisely when the vertices i and j of Γ_M are connected by a sequence of edges labelled by odd m_{ab} only.

LEMMA 4.1. *The set* $Z(a, b)$ *is empty if* a *and* b *are not both in* S, *not both in* S^{-1}, *and not both in* T.

PROOF. It can be seen from the relations that the number of letters of T appearing in a word is invariant under applications of the relations, since each relation either has no occurrences of τ or one on each side. Thus an equation of the form $aW \approx Wb$ means that either both a and b come from T, or neither a nor b comes from T.

Now suppose that $\sigma_i W \approx W\sigma_j^{-1}$. Since $W \approx \zeta^{-\theta_2(W)}\theta_1(W)$, multiplying both sides of the original equation by the central element $\zeta^{-\theta_2(W)}$ gives $\sigma_i\theta_1(W) \approx \theta_1(W)\sigma_j^{-1}$. Thus $\sigma_i\theta_1(W)\sigma_j \approx \theta_1(W)$, and so by [9, Embedding Theorem], $\sigma_i\theta_1(W)\sigma_j \sim \theta_1(W)$. But this cannot be the case, as equivalent words in $\mathscr{S}_M^+$ must have the same length. □

It is clear from the definition that $Z(\sigma_i^{-1}, \sigma_j^{-1}) = Z(\sigma_i, \sigma_j)$, since $\sigma_i W \approx W\sigma_j$ precisely when $W\sigma_j^{-1} \approx \sigma_i^{-1} W$. Thus we will momentarily restrict our consideration of $Z(a, b)$ to the case where a and b are both from $S \cup T$.

It turns out that preserving a-chains are important elements of these sets. Preserving chains are characterised by the property of Lemma 1.4 that a nonempty word W is equivalent to a preserving a-chain to b precisely when a does not divide W and $aW \sim Wb$. Thus preserving a-chains to b and all words equivalent to them are in $Z(a, b)$.

LEMMA 4.2. *Let a and b be in $S \cup T$ and $a \neq b$. A word W over $S \cup T$ is in $Z(a, b)$ precisely when it is equivalent to a word of the form $a^m P$ where m is a non negative integer and P is a preserving a-chain to b.*

A word W over $S \cup T$ is in $Z(a)$ precisely when it is equivalent to a word of the form $a^m P$ where m is a non negative integer and P is either a preserving a-chain to a or the empty word.

PROOF. In light of previous comments, we only need to prove the 'only if' direction. Suppose $W \in Z(a, b)$. There is some integer $m \geq 0$ such that a^m divides W but a^{m+1} does not divide W. Then $W \sim a^m V$ for some V which is not divisible by a. Since $aW \sim Wb$, then $a^{m+1} V \sim a^m Vb$, and so $aV \sim Vb$ by left cancellativity. By Lemma 1.4, V is equivalent to a preserving a-chain P with target b, and $W \sim a^m P$.

The argument for when $a = b$ is almost the same, except that P may be empty. □

Now take any word W over $S \cup S^{-1} \cup T$. Then $W \approx \zeta^{-k}\theta_1(W)$ where $k = \theta_2(W)$, a non-negative integer. Since ζ is central, then W is in $Z(a, b)$ precisely when $\theta_1(W)$ is in $Z(a, b)$. We know the form of such elements from the previous Lemma. Thus

PROPOSITION 4.3. $Z(a, b) = \left\{ W \,\middle|\, \begin{array}{l} W \approx \zeta^{-k} a^m P \textit{ where } k, m \geq 0 \textit{ and } P \textit{ is} \\ \textit{a preserving } a\textit{-chain to } b \textit{ or may be the} \\ \textit{empty word if } a = b \end{array} \right\}.$

LEMMA 4.4. *For any i and j, $Z(\sigma_i^{-1}, \sigma_j^{-1}) = Z(\sigma_i, \sigma_j) = Z(\tau_i, \tau_j)$.*

PROOF. That $Z(\sigma_i^{-1}, \sigma_j^{-1}) = Z(\sigma_i, \sigma_j)$ was noted earlier. A quick inspection of the relations $\mathscr{R}$ shows that the source and target of a simple preserving chain must be both in S or both in T. Further, if P is a simple preserving σ_i-chain to σ_j, then $\tau_i P \sim P\tau_j$, and if P is a simple preserving τ_i-chain to τ_j, then $\sigma_i P \sim P\sigma_j$.

Since compound preserving chains are just concatenations of suitably matching simple preserving chains, the same results hold there: if P is a preserving σ_i-chain to σ_j, then $\tau_i P \sim P\tau_j$, and if P is a preserving τ_i-chain to τ_j, then $\sigma_i P \sim P\sigma_j$.

Now suppose $W \in Z(\sigma_i, \sigma_j)$. Then Proposition 4.3 says $W \approx \zeta^{-k}\sigma_i^m P$ where P is a preserving σ_i-chain to σ_j and k and m are non-negative integers. So

$$\tau_i W \approx \tau_i\zeta^{-k}\sigma_i^m P \approx \zeta^{-k}\tau_i\sigma_i^m P \approx \zeta^{-k}\sigma_i^m\tau_i P \approx \zeta^{-k}\sigma_i^m P\tau_j \approx W\tau_j,$$

so $W \in Z(\tau_i, \tau_j)$. Thus we have that $Z(\sigma_i, \sigma_j) \subseteq Z(\tau_i, \tau_j)$. The reverse inclusion is analogous. $\square$

Thus it makes sense to denote the set $Z(\sigma_i^{-1}, \sigma_j^{-1}) = Z(\sigma_i, \sigma_j) = Z(\tau_i, \tau_j)$ by $Z(i, j)$.

PROPOSITION 4.5. *Whether or not a word W over $S \cup S^{-1} \cup T$ is in $Z(i, j)$ is calculable.*

PROOF. The word W is equivalent to $\zeta^{-\theta_2(W)}\theta_1(W)$, and since ζ is central, W is in $Z(i, j)$ precisely when $\theta_1(W)$ is. Calculate $(\theta_1(W)/\sigma_i^k)$ for $k = 0, 1, \ldots, p$ where p is the integer such that $U \equiv (\theta_1(W)/\sigma_i^p)$ is defined, but $(\theta_1(W)/\sigma_i^{p+1})$ is not. (At most $\ell(\theta_1(W))$ calculations need to be done.) Then we need to determine whether or not U is a preserving σ_i-chain to σ_j. According to Lemma 1.4, this is equivalent to determining whether $\sigma_i U \sim U\sigma_j$. This is the case precisely when $(U\sigma_j/\sigma_i)$ is defined and equivalent to U, or in other words when $((U\sigma_j/\sigma_i)/U)$ is defined and empty. $\square$

The next results of this section describe precisely when $Z(i, j)$ is not empty. It is clear that $Z(i, i)$ is never empty as ζ^n, σ_i^n, and τ_i^m are in $Z(i, i)$ for all integers n and non-negative integers m. If m_{ij} is odd then $\langle\sigma_j\sigma_i\rangle^{m_{ij}-1}$ is a preserving σ_i-chain to σ_j, so $Z(i, j)$ is not empty. Moreover if $Z(i, j)$ and $Z(j, k)$ are nonempty then $Z(i, k)$ is nonempty as $Z(i, j)Z(j, k) = \{VW \mid V \in Z(i, j) \text{ and } W \in Z(j, k)\} \subseteq Z(i, k)$.

PROPOSITION 4.6. *The set $Z(i, j)$ is nonempty precisely when there is an integer $r \geq 0$ and a sequence $i = i_0, \ldots, i_r = j$ such that $m_{i_{k-1}i_k}$ is odd for each $k = 1, \ldots, r$.*

PROOF. If there is such a sequence, then from the preceding comments it is clear that $Z(i_{k-1}, i_k)$ is nonempty for each $k = 1, \ldots, r$; and thus

$$Z(i, j) \supseteq Z(i, i_1)Z(i_1, i_2)\cdots Z(i_{r-1}, j)$$

is nonempty also.

On the other hand, suppose $Z(i, j)$ is nonempty. If $i = j$ then the result holds with $r = 0$. Suppose then that $i \neq j$, and take $W \in Z(i, j)$. From Proposition 4.3, there are non-negative integers k and n and P, a preserving σ_i-chain to σ_j, such that $W \approx \zeta^{-k}\sigma_i^n P$. Thus preserving σ_i-chains to σ_j exist.

Suppose $P \equiv P_1P_2\cdots P_p$ is such a σ_i-chain to σ_j, where each P_i is a simple preserving chain. By excluding the simple components whose source and target are the same, we obtain a subsequence $j_0, j_1, \ldots, j_r$ of $1, 2, \ldots, p$ such that $P' \equiv P_{j_0}P_{j_1}\cdots P_{j_r}$ is a preserving σ_i-chain to σ_j and each P_{j_l} is a simple preserving σ_{i_l}-chain to $\sigma_{i_{l+1}}$ where $i_l \neq i_{l+1}$. (So $i_0 = i$ and $i_r = j$). The only such words are $\langle\sigma_{i_{l+1}}\sigma_{i_l}\rangle^{m_{i_li_{l+1}}-1}$, with $m_{i_li_{l+1}}$ odd. (If m_{pq} is even, then $\langle\sigma_q\sigma_p\rangle^{m_{pq}-1}$ is a preserving σ_p-chain to σ_p). Thus the sequence $i = i_0, i_1, \ldots, i_{r-1}, i_r = j$ satisfies the requirements of the theorem entirely. □

A word is said to be τ*-free* if it has no occurrences of letters from T.

SCHOLIUM 4.7. *The set $Z(i,j)$ is nonempty if and only if there is a τ-free preserving σ_i-chain to σ_j.*

The group whose presentation is given by the generators $S \cup S^{-1}$ and relations all those of $\mathscr{S}_M$ with no occurrences of letters from T is called the *Artin group of type M*. It is a subgroup of the singular Artin monoid of type M. The Artin group of type A_n is often called the *braid group on $n+1$ strings*. Let

$$Z_S(i,j) = \{W \in Z(i,j) \mid W \text{ is } \tau\text{-free}\} = Z(i,j) \cap (S \cup S^{-1})^*.$$

From Scholium 4.7, we know this is nonempty precisely when $Z(i,j)$ is nonempty.

PROPOSITION 4.8. *The set $Z_S(i,j)$ consists precisely of those words W over $S \cup S^{-1}$ such that $W\sigma_i$ and $\sigma_j W$ represent the same element of the Artin group of type M. Membership of $Z_S(i,j)$ is calculable.*

The results of the next corollary follow immediately from Proposition 4.6 and an examination of the list of Coxeter graphs of finite types given in Figure 1.

COROLLARY 4.9. (1) *If i and j are not in the same connected component of Γ_M, then $Z(i,j)$ is empty.*

(2) *If i and j are in the same connected component of Γ_M and the connected component is not of type B_n, F_4, G_2 or $I_2(2m)$, then $Z(i,j)$ is always nonempty.*

(3) *If i and j are in a connected component of type B_n, then $Z(i,j)$ is empty precisely when $1 \in \{i,j\}$ and $i \neq j$.*

(4) *If i and j are in a connected component of type F_4, then $Z(i,j)$ is empty precisely when $\{i,j\}$ is either $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, or $\{2,4\}$.*

(5) *If i and j are in a connected component of type G_2 or $I_2(2m)$, then $Z(i,j)$ is nonempty precisely when $i = j$.*

COROLLARY 4.10. *The results of Corollary 4.9 hold with $Z_S(i, j)$ replacing $Z(i, j)$.*

Fenn, Rolfsen and Zhu [12] introduced the notions of *(j, k)-bands* and *singular (j, k)-bands* in the braid group and singular braid monoid—which in this article are called the Artin group of type A and the singular Artin monoid of type A respectively. These bands turn out to be geometric analogues to elements of $Z_S(j, k)$ and $Z(j, k)$ respectively. The relevant result, [12, Theorem 7.1], is the following, which shows that a singular braid with a (possibly singular) (j, k)-band has precisely the required property for belonging to $Z(j, k)$ in $\mathscr{S}_{A_n}$.

THEOREM 4.11 (Fenn, Rolfsen and Zhu). *For a singular braid x in the singular braid monoid, the following are equivalent*:

(a) $\sigma_j x = x\sigma_k$;
(b) $\sigma_j^r x = x\sigma_k^r$, *for some nonzero integer r*;
(c) $\sigma_j^r x = x\sigma_k^r$, *for every integer r*;
(d) $\tau_j x = x\tau_k$;
(e) $\tau_j^r x = x\tau_k^r$, *for some nonzero integer r*;
(f) *x has a (possibly singular) (j, k)-band.*

By the equivalence of (a) with (f) we deduce that a singular braid has a (j, k)-band if and only if it can be represented by an element of $Z(j, k)$.

The method of [12] to identify elements of the centraliser of a generator in the singular braid monoid relies on the geometric realisation of the Artin type A_n as braids. There are no known geometric realisations for types other than type A, so the method of (j, k)-bands cannot be extended in an obvious manner to the other types. The algebraic method provided here covers all types: By combining Theorem 1.5, Proposition 4.3, Lemma 4.4 and Proposition 4.5, we can extend this theorem of Fenn, Rolfsen and Zhu above to arbitrary type (that is, not just type A):

THEOREM 4.12. *An element x of $\mathscr{S}_M$ is in $Z(j, k)$ if and only if $\sigma_j^p x \approx x\sigma_k^p$ for some non-zero integer p, if and only if $\tau_j^r x \approx x\tau_k^r$ for some non-zero integer r. For any x, this is calculable.*

In [12], the authors remark that 'it may not be obvious, from a presentation as a word in the generators, whether a word has a (j, k)-band'. However, Proposition 4.5 and Proposition 4.8 provide ways of determining inclusion in $Z(j, k)$ and $Z_S(j, k)$ respectively.

Birman [5] conjectured that the monoid homomorphism η from the singular braid monoid to the group algebra of the braid group defined by $\eta(\sigma_i^{\pm 1}) = \sigma_i^{\pm 1}$ and $\eta(\tau_i) = \sigma_i - \sigma_i^{-1}$ is injective. Among many other interesting results about braids and singular braids, the Fenn *et al.*'s paper [12] confirms certain cases of Birman's conjecture using results about centralisers. It is hoped that the results of this paper may help in determining the truth or otherwise of Birman's conjecture in general. In the final section, the singular braid monoid and its relationship with singular links is discussed. Next, we define *parabolic submonoids* and describe some results about conjugacy of parabolics.

5. Parabolic submonoids of singular Artin monoids

Recall that M is a Coxeter matrix over I, a finite indexing set. For $J \subseteq I$, it is clear that the submatrix M_J of M containing the entries indexed by J is also a Coxeter matrix. We use the following notation:

$$S_J = \{\sigma_j \mid j \in J\}, \quad S_J^{-1} = \{\sigma_j^{-1} \mid j \in J\}, \quad T_J = \{\tau_j \mid j \in J\}.$$

Recall that $\mathscr{R}$ denotes the defining relations of $\mathscr{S}_M^+$. Denote by $\mathscr{R}_J$ the defining relations of $\mathscr{S}_{M_J}^+$. The next two observations follow from the definitions of these relations:

(1) $\mathscr{R}_J^\Sigma \subseteq \mathscr{R}^\Sigma$.

(2) If $(X, Y) \in \mathscr{R}^\Sigma$ and X is a word over $S_J \cup T_J$, then Y is a word over $S_J \cup T_J$ also, and $(X, Y) \in \mathscr{R}_J^\Sigma$.

Suppose that V and W are words over $S_J \cup T_J$. If V and W represent the same element of $\mathscr{S}_{M_J}^+$ write $V \sim_J W$. Note that the relations $\sim$ and $\sim_I$ are identical; in this section we will use $\sim_I$ for clarity. The observation (1) above ensures that if $V \sim_J W$, then $V \sim_I W$. Conversely, suppose that $V \sim_I W$. Since V and W only contain letters from $S_J \cup T_J$, the second observation says that the only relations from $\mathscr{R}^\Sigma$ which can be used to transform V into W are those which lie in $\mathscr{R}_J^\Sigma$ anyway; thus $V \sim_J W$. This proves

PROPOSITION 5.1. *$\mathscr{S}_{M_J}^+$ embeds in $\mathscr{S}_M^+$.*

Now suppose that V and W are words over $S_J \cup S_J^{-1} \cup T_J$. The notation $V \approx_J W$ (respectively, $V \approx_I W$) implies V and W represent the same element of $\mathscr{S}_{M_J}$ (respectively, $\mathscr{S}_M$). It is clear that if $V \approx_J W$, then $V \approx_I W$. However, to prove the stronger result that $V \approx_I W$ implies $V \approx_J W$, we use the Embedding Theorem of [9], which has only been proved for finite types.

Suppose that M is of finite type. Then Δ, the least common multiple of S_I, is defined, and is a common multiple of S_J, a subset of S_I. Thus a least common multiple of S_J exists, so M_J is of finite type. (Or alternatively, one may observe that the list of Coxeter graphs of Figure 1 is closed under taking full subgraphs). We will denote by Δ_J the fundamental element of $\mathcal{S}_{M_J}$, and by ζ_J its corresponding central element (either Δ_J or Δ_J^2 according to (5) of Remark 1.8).

Let V and W be words over $S_J \cup S_J^{-1} \cup T_J$. Then there exists an integer m such that $V \approx_J \zeta_J^m V^+$ and $W \approx_J \zeta_J^m W^+$ where V^+ and W^+ are words over $S_J \cup T_J$. By the comments following the proposition, we have that $V \approx_I \zeta_J^m V^+$ and $W \approx_I \zeta_J^m W^+$.

Suppose that $V \approx_I W$. Multiplying through by ζ_J^{-m}, we have $V^+ \approx_I W^+$. By the Embedding Theorem ([9, Theorem 20]), $V^+ \sim_I W^+$. Proposition 5.1 then ensures $V^+ \sim_J W^+$, and so $V^+ \approx_J W^+$. Finally, multiplying through by ζ_J^m results in $V \approx_J W$. Thus

PROPOSITION 5.2. *Suppose that M is of finite type. Then $\mathcal{S}_{M_J}$ embeds in $\mathcal{S}_M$.*

The submonoid P_J of $\mathcal{S}_M$ generated by $S_J \cup S_J^{-1} \cup T_J$ is called the *parabolic submonoid defined by J*. The image of the natural embedding described in the previous proposition is precisely P_J. Thus we have

THEOREM 5.3. *Parabolic submonoids of singular Artin monoids of finite type are (isomorphic to) singular Artin monoids.*

The *Artin group of type M* is the subgroup of $\mathcal{S}_M$ generated by $S \cup S^{-1}$. The previous result was first proved for parabolic subgroups of finite type Artin groups in [7] and [10]. The set of types for which it holds for Artin groups was gradually extended via various techniques of proof, eventually to include all types in [11]. Paris [22] provides an alternative proof via CW-complexes. It is not immediately clear how to generalise this to singular Artin monoids; although the author suspects that Theorem 5.3 does hold for arbitrary types of singular Artin monoids.

The rest of this section is devoted to investigating when parabolics are conjugate. Our first step is to generalise Paris' definition of *conjugators* of parabolics of Artin groups ([22]), and to do so we must introduce some notation. Let Σ_1 and Σ_2 be sets of words. Then $\Sigma_1 \approx \Sigma_2$ means that the sets of $\approx$ equivalence classes of elements of Σ_1 and Σ_2 respectively coincide, $\Sigma_1\Sigma_2$ denotes the set $\{XY \mid X \in \Sigma_1, Y \in \Sigma_2\}$, and if W is a word then $W\Sigma_1 = \{W\}\Sigma_1$. Using the terminology of Paris, given subsets J and K of I, we define a (J, K)*-conjugator* to be a word V over $S \cup S^{-1} \cup T$ such that $VS_J \approx S_K V$. Thus $Z(i, j)$ is the set of all $(\{j\}, \{i\})$-conjugators.

Suppose that V is a (J, K)-conjugator. Then V defines a bijection $f_V : J \to K$ by $V\sigma_j \approx \sigma_{f_V(j)} V$. Well-definedness is assured by right cancellativity, and injectivity by left cancellativity. For each $j \in J$, V lies in $Z(f_V(j), j)$; and thus

$$V \in \bigcap_{j \in J} Z(f_V(j), j).$$

Conversely, if $f : J \to K$ is a bijection, and some word W lies in the intersection of $Z(f(j), j)$ for all $j \in J$, then $W\sigma_j \approx \sigma_{f(j)} W$ for all $j \in J$, and so W is a

(J, K)-conjugator. Since membership of $Z(i, j)$ is calculable (Proposition 4.5), we can determine when a word is a conjugator.

THEOREM 5.4. *Suppose that J and K are subsets of I. Let $F = \{$bijections $f : J \to K\}$. The set of all (J, K) conjugators is $\bigcup_{f \in F} \bigcap_{j \in J} Z(f(j), j)$. Membership of this set is calculable.*

If J is a subset of I, then we denote by Γ_{M_J} the full subgraph of Γ_M on the vertices labelled by J—that is, the graph whose vertices are J, and for which there is an edge between vertices j_1 and j_2 in Γ_{M_J} precisely when there is one between j_1 and j_2 in Γ_M.

PROPOSITION 5.5. *Suppose that J and K are subsets of I. If the set of (J, K)-conjugators is not empty, then Γ_{M_J} and Γ_{M_K} are isomorphic.*

PROOF. Suppose that V is a (J, K)-conjugator; so V defines a bijection $f : J \to K$, as described already. Thus $|J| = |K|$. It remains to show that $m_{ij} = m_{f(i)f(j)}$ for each i and j in J.

Fix i and j, and suppose that $m_{ij} \leq m_{f(i)f(j)}$. Then $V\sigma_i \approx \sigma_{f(i)} V$, and similarly for j, and so

$$\langle \sigma_{f(i)} \sigma_{f(j)} \rangle^{m_{ij}} V \approx V \langle \sigma_i \sigma_j \rangle^{m_{ij}} \approx V \langle \sigma_j \sigma_i \rangle^{m_{ij}} \approx \langle \sigma_{f(j)} \sigma_{f(i)} \rangle^{m_{ij}} V.$$

After cancelling V from the right, we obtain $\langle \sigma_{f(i)} \sigma_{f(j)} \rangle^{m_{ij}} \approx \langle \sigma_{f(j)} \sigma_{f(i)} \rangle^{m_{ij}}$, and so the Embedding Theorem implies $\langle \sigma_{f(i)} \sigma_{f(j)} \rangle^{m_{ij}} \sim \langle \sigma_{f(j)} \sigma_{f(i)} \rangle^{m_{ij}}$. By the Reduction Lemma (which is Lemma 1.1 of this paper), $\langle \sigma_{f(i)} \sigma_{f(j)} \rangle^{m_{f(i)f(j)}}$ divides $\langle \sigma_{f(i)} \sigma_{f(j)} \rangle^{m_{ij}}$; and so we have $m_{ij} = m_{f(i)f(j)}$. The case when $m_{ij} \geq m_{f(i)f(j)}$ is analogous. $\square$

We say that two submonoids Q_1 and Q_2 of $\mathscr{S}_M$ are *conjugate* if there exists a word V over $S \cup S^{-1} \cup T$ such that $VQ_1 \approx Q_2 V$. Suppose that J and K are subsets of I, and that V is a (J, K)-conjugator with corresponding bijection $f : J \to K$. Thus $V \in Z(f(j), j)$ for each $j \in J$, and so, by Lemma 4.4, $V\sigma_j \approx \sigma_{f(j)} V$, $V\sigma_j^{-1} \approx \sigma_{f(j)}^{-1} V$ and $V\tau_j \approx \tau_{f(j)} V$. Thus if U is any word over $S_J \cup S_J^{-1} \cup T_J$, then, since $K = f(J)$, $VU \approx U'V$, by pushing each letter of U through V one at a time, for some U' over $S_K \cup S_K^{-1} \cup T_K$. Similarly, for any word W over $S_K \cup S_K^{-1} \cup T_K$, there is a word W' over $S_J \cup S_J^{-1} \cup T_J$ satisfying $WV \approx VW'$. Thus $VP_J \approx P_K V$, and we have:

LEMMA 5.6. *If the set of (J, K) conjugators is not empty, then P_J and P_K are conjugate.*

In Theorem 5.8, we will show that the converse of this lemma holds, although the converse of the preceding proposition does not. To see why, we will exploit results about, and connections between, Coxeter and Artin groups and singular Artin monoids. Recall that throughout, M is a finite type Coxeter matrix over I.

The *Artin group of type M*, denoted $\mathscr{G}_M$, is the group generated by $S \cup S^{-1}$ subject to the *braid relations* $\langle \sigma_i \sigma_j \rangle^{m_{ij}} = \langle \sigma_j \sigma_i \rangle^{m_{ij}}$ and the free group relations $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1$.

Suppose that V and W are words over $S \cup S^{-1} \cup T$ such that $V \approx W$. If V may be transformed into W using only the relations from above, we will write $V \approx_S W$. In particular, if V and W are τ-free (that is, are words over $S \cup S^{-1}$) with $V \approx W$, then the only relations which can be used to transform V into W are the τ-free defining relations of $\mathscr{S}_M$, which are precisely those above, so $V \approx_S W$. Thus the Artin group $\mathscr{G}_M$ embeds in the singular Artin monoid $\mathscr{S}_M$.

The Artin group $\mathscr{G}_M$ is also a homomorphic image of $\mathscr{S}_M$, by a map $\tilde{}$ which we define first on generators, and then show it can be extended homomorphically. Define $\tilde{}$ from $S \cup S^{-1} \cup T$ to $S \cup S^{-1}$ to be the identity on $S \cup S^{-1}$, and such that $\tilde{\tau_i} = \sigma_i$ for each $i \in I$. Extend $\tilde{}$ to words over $S \cup S^{-1} \cup T$ in the obvious way. Observe that if $\rho_1 = \rho_2$ is any defining relation of $\mathscr{S}_M$, then $\tilde{\rho_1} = \tilde{\rho_2}$ is one of the relations above. Thus $\tilde{}$ is a well-defined map from $\approx$ equivalence classes of words in $\mathscr{S}_M$ to $\approx_S$ equivalence classes of words in $\mathscr{G}_M$; so $V \approx W$ implies $\tilde{V} \approx_S \tilde{W}$.

The *Coxeter group of type M*, denoted by $\overline{\mathscr{G}}_M$, is a quotient of $\mathscr{G}_M$ obtained by identifying each element of S with its inverse, that is, $\overline{\mathscr{G}}_M$ is generated by $S \cup S^{-1}$ subject to the braid relations and the relations $\sigma_i^2 = 1$ for each $i \in I$. If V and W are words over $S \cup S^{-1}$ such that V may be transformed into W using only these relations, then write $V \doteq W$. For any word V over $S \cup S^{-1}$, denote by $\overline{V}$ the word obtained by replacing each occurrence of σ_i^{-1} in V with σ_i. Then $V \doteq \overline{V}$, showing that every element of $\overline{\mathscr{G}}_M$ may be expressed as a word over S. (We could have chosen S as a set of monoid generators for $\overline{\mathscr{G}}_M$.) A word over S is said to be *reduced* if it is of shortest length amongst all words representing the same element of $\overline{\mathscr{G}}_M$. We will use the following standard results (see [6] or [20]):

(1) The set of all reduced words is precisely $\mathscr{Q}\mathscr{F}(S)$.

(2) If V and W are reduced words such that $V \doteq W$, then $V \sim W$ (that is, V may be transformed into W using only the braid relations).

If $J \subseteq I$, and $W_J = \overline{\mathscr{P}}_J$ is the subgroup of $\overline{\mathscr{G}}_M$ generated by S_J. Suppose that J and K are subsets of I for which there is a word V over S such that $VW_J \doteq W_K V$. There is always an element v of minimal length in the coset VW_J, and Howlett [19] has shown:

(3) If $vW_J \doteq W_K v$ with v a minimal length coset representative, then $vS_J \doteq S_K v$.

The properties (1), (2) and (3) above allow us to refer to Coxeter groups to determine conjugacy of parabolic submonoids of singular Artin monoids, due to the following:

THEOREM 5.7. *Suppose that M is a finite type Coxeter matrix over I, and that J*

and K are subsets of I. Then P_J and P_K are conjugate in $\mathscr{S}_M$ if and only if W_J and W_K are conjugate in $\overline{\mathscr{G}}_M$.

PROOF. Suppose that there is a word $\underline{V}$ over $S \cup S^{-1} \cup \underline{T}$ such that $VP_J \approx P_K V$. Then $\widetilde{V}\widetilde{P_J} \approx_S \widetilde{P}_K \widetilde{V}$, and so $\overline{\widetilde{V}}\,\overline{\widetilde{P}}_J = \overline{\widetilde{P}}_K \overline{\widetilde{V}}$. Let $U \equiv \overline{\widetilde{V}}$, and recall that W_J is precisely $\overline{\widetilde{P}}_J$; thus $UW_J = W_K U$, so W_J and W_K are conjugate in $\overline{\mathscr{G}}_M$.

Conversely, suppose that $VW_J = W_K V$ for some word V over S. Then there exists a minimal length word v (a word over S) in the coset VW_J, so for this element

(a) $vW_J = W_K v$,
(b) v is reduced, and
(c) v is not right divisible by any word in W_J.

Then by (3) above, $vS_J = S_K v$, so there is a bijection $f : J \to K$ defined by $v\sigma_j = \sigma_{f(j)} v$. The points (b) and (c) ensure that $v\sigma_j$ is reduced, and so by (2) above, we have $v\sigma_j \sim \sigma_{f(j)} v$, for all $j \in J$. Thus the word v is a (J, K)-conjugator, and so by Lemma 5.6, P_J and P_K are conjugate. $\square$

Notice, as a scholium, that if W_J and W_K are conjugate, then we can find a (J, K)-conjugator v which lies in $\mathscr{Q}\mathscr{F}(S)$.

THEOREM 5.8. *Suppose J and K are subsets of I. The following statements are equivalent.*

(1) *$\mathscr{Q}\mathscr{F}(S)$ contains a (J, K) conjugator.*
(2) *The set of (J, K) conjugators is not empty.*
(3) *P_J and P_K are conjugate.*

PROOF. Clearly (1) implies (2). Lemma 5.6 is precisely that (2) implies (3). Finally, (3) implies W_J and W_K are conjugate by the previous theorem, and then the comments immediately above imply (1). $\square$

The equivalence of (1) and (3) allows us to calculate whether P_J and P_K are conjugate. Let F denote the bijections from J to K. Recall that the set of (J, K)-conjugators is $\bigcup_{f \in F} \bigcap_{j \in J} Z(f(j), j)$, so P_J and P_K are conjugate if and only if for some $f \in F$, there exists a word in $\mathscr{Q}\mathscr{F}(S)$ which lies in $Z(f(j), j)$ for each $j \in J$. Since J, F and $\mathscr{Q}\mathscr{F}(S)$ are finite, and by Proposition 4.5, membership of each $Z(f(j), j)$ is calculable, we have only a finite number of possibilities which we need to check, to determine whether or not P_J and P_K are conjugate.

THEOREM 5.9. *Let P_J and P_K be parabolic submonoids of the singular Artin monoid of type M, where M is of finite type. It is calculable whether P_J and P_K are conjugate.*

Theorem 5.8 provides the desired converse of Lemma 5.6. The converse of Theorem 5.5 turns out not to hold, that is, there are occasions when the Coxeter graphs of J and K are isomorphic, but P_J and P_K are not conjugate. This fact comes directly from the same result of Howlett [19] for parabolics in Coxeter groups. For example, in type D_4, there are three distinct A_3 parabolics, none of which are conjugate to one another. The idea is, if Γ_{M_J} and Γ_{M_K} are isomorphic subgraphs of Γ_M, the corresponding parabolics are conjugate if and only if one subgraph may be shifted along Γ_M one edge at a time, via isomorphic subgraphs, to eventually coincide with the second subgraph. Since A_3-type subgraphs of Γ_{D_4} cannot be shifted in any way to produce an A_3 subgraph, the three distinct subgraphs correspond to non-conjugate parabolics. Howlett [19], and Brink and Howlett [8] give complete description of how parabolic subgroups of Coxeter groups with isomorphic graphs can fail to be conjugate, and by Theorem 5.7, their list for finite types completely describes the situation for singular Artin monoids of finite types also.

6. Markov moves and the singular braid monoid

Singular braids may be defined geometrically in a similar way to the geometric braids of Artin [3]. The important amendment to the definition is that the strings of a singular braid may intersect—although at most two strings may intersect at any one point, and there may be at most a finite number of such intersections. The notion of equivalence of singular braids is *rigid vertex isotopy* (see [5, Section 5]). The set of (equivalence classes of) singular braids on n strings forms a monoid under concatenation, often denoted SB_n. Baez [4] first presented this monoid, and Birman first showed that the relations suffice ([5, Lemma 3]). Their presentation is precisely that of the singular Artin monoid of type A_{n-1} defined above. See Figure 1 for the Coxeter graph of this type. For this section, we will use the notation:

$$S_k = \{\sigma_1, \sigma_2, \dots, \sigma_k\}, \quad S_k^{-1} = \{\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_k^{-1}\}, \quad T_k = \{\tau_1, \tau_2, \dots, \tau_k\}.$$

Thus the presentation of SB_n ([4, 5]) has generators $S_{n-1} \cup S_{n-1}^{-1} \cup T_{n-1}$, and relations:

$$\left.\begin{aligned} 1 = \sigma_i\sigma_i^{-1} &= \sigma_i^{-1}\sigma_i \\ \sigma_i\tau_i &= \tau_i\sigma_i \end{aligned}\right\} \quad \text{for all } i, \text{ with } 1 \le i \le n-1,$$

$$\left.\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \\ \sigma_i\tau_j &= \tau_j\sigma_i \end{aligned}\right\} \quad \text{whenever } |i-j| \ge 2,$$

$$\left.\begin{aligned} \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j \\ \sigma_i\sigma_j\tau_i &= \tau_j\sigma_i\sigma_j \end{aligned}\right\} \quad \text{whenever } |i-j| = 1\ .$$

As in Artin's well known presentation of the braid group, the generator σ_i represents the braid with just a single crossing of the ith string over the $(i+1)$th string, and σ_i^{-1} represents the braid with just the $(i+1)$th string crossing over the ith string. The generator τ_i represents the braid with no crossings but a single intersection between the ith string and the $(i+1)$th string. The permutation associated to each of these singular braids is the transposition $(i \quad i+1)$.

The singular braid monoid SB_n arose in the context of knot theory and Vassiliev invariants of knots. A singular braid β may be *closed* by associating the corresponding endpoints, to produce a singular link $\overline{\beta}$. Birman showed that every singular link is equivalent to a closed singular braid β on n strings, for some n ([5, Lemma 2]). (See also [2] for a generalisation of this result.) The question becomes, when do inequivalent singular braids close to produce equivalent singular links? Gemein [14] answered this with his 'singular version' of Markov's theorem, stated below. He uses the notation (β, n) to indicate that β is a singular braid on n strings.

Proposition 5.2 implies that there is an injective homomorphism from the singular Artin monoid of type A_{n-1} to the singular Artin monoid of type A_n, type A being in the list of finite types. This is a map from SB_n to SB_{n+1}, and may be described geometrically as follows: given a singular braid (β, n), append an extra string in the $(n+1)$th place which has no interaction with the other strings of the braid. The new braid is denoted $(\beta, n+1)$. A word over $S_{n-1} \cup S_{n-1}^{-1} \cup T_{n-1}$ representing (β, n) will also serve to represent $(\beta, n+1)$. The additional fact that this map is injective means that if β and γ are singular braids on n strings such that $(\beta, n+1) = (\gamma, n+1)$, then $(\beta, n) = (\gamma, n)$.

Given a singular braid (β, n), the notation $(\beta\sigma_n, n+1)$ now makes sense—first apply the map 'adding a string' to obtain $(\beta, n+1)$, and then concatenate this new braid with σ_n. We now state Gemein's result [14].

MARKOV'S THEOREM (SINGULAR VERSION). *Let (β, n) and (β', n') be two braids. Let K and K' be the two associated closed braids. K and K' are equivalent as links if and only if (β, n) and (β', n') are related by a sequence of the following algebraic operations, called Markov's moves*:

(1) (a) $(\sigma_i^{\pm 1}\beta\sigma_i^{\mp 1}, n) \leftrightsquigarrow (\beta, n)$;
(b) $(\tau_i\beta, n) \leftrightsquigarrow (\beta\tau_i, n)$;

(2) $(\beta, n) \leftrightsquigarrow (\beta\sigma_n^{\pm 1}, n+1)$.

Both parts of (1) may be combined into the one statement:

$$(x\beta, n) \leftrightsquigarrow (\beta x, n) \quad \text{for } x \in S_{n-1} \cup S_{n-1}^{-1} \cup T_{n-1}.$$

Suppose that V and W are words over $S_{n-1} \cup S_{n-1}^{-1} \cup T_{n-1}$ representing $x\beta$ and βx respectively. Then V is swap conjugate to W (defined in Section 3). Applying Theorem 3.5, we immediately obtain the following

PROPOSITION 6.1. *Let β and γ be singular braids on n strings, represented by the words V and W over $S_{n-1} \cup S_{n-1}^{-1} \cup T_{n-1}$ respectively. Then β and γ are related by an arbitrary sequence of singular Markov moves of type* (1) *if and only if* $V \asymp W$.

Turning now to the second type of Markov move, we make use of the normal form for the singular Artin monoid given in [9, Section 5]. An operator $\overline{N}_n$ on words over $S_n \cup S_n^{-1} \cup T_n$ is defined, which has the properties that for all words V and W over $S_n \cup S_n^{-1} \cup T_n$,

(1) $\overline{N}_n(W)$ is calculable,
(2) $\overline{N}_n(W) \approx W$, and
(3) $W \approx V$ if and only if $\overline{N}_n(W) \equiv \overline{N}_n(V)$ ([9, Theorem 23]).

Thus we get immediately from (2) and (3) the following:

PROPOSITION 6.2. *Let (β, n) and $(\gamma, n+1)$ be singular braids represented by the words V over $S_{n-1} \cup S_{n-1}^{-1} \cup T_{n-1}$ and W over $S_n \cup S_n^{-1} \cup T_n$ respectively. Then β and γ are related by a type (2) Markov move if and only if either $\overline{N}_n(W) \equiv \overline{N}_n(V\sigma_n)$ or $\overline{N}_n(W) \equiv \overline{N}_n(V\sigma_n^{-1})$.*

Combining this result with Proposition 6.1 and (1), we have

THEOREM 6.3. *It is calculable when two singular braids differ by an application of a Markov move.*

We can consider *positive braids*—those which may be represented by positive words, that is, words with no inverses. The *positive Markov moves* are defined to be:

(1) $(x\beta, n) \leftrightsquigarrow (\beta x, n)$ for $x \in S_{n-1} \cup T_{n-1}$, and
(2) $(\beta, n) \leftrightsquigarrow (\beta\sigma_n, n+1)$.

Given any positive braid β on n strings, and any positive integer N, we will calculate a set which contains all words representing positive braids which can be obtained from β by up to N applications of positive Markov moves. Define the set

$$\Omega = \{(W, n+1) \mid n \text{ is a non-negative integer, and } W \text{ is a word over } S_n \cup T_n\}.$$

Elements of Ω are said to be *allowable*.

Let K be a set of allowables. For each n, define $K_n = \{W \mid (W, n) \in K\}$. The set $\varphi(K_n)$ is the set of all words over $S_{n-1} \cup T_{n-1}$ which are $\aleph$-conjugate to elements of K_n. By Lemma 3.3, $\varphi(K_n)$ contains all words which are swap-conjugate to elements of K_n. Thus $\varphi(K_n)$ contains all words corresponding to the application of a positive Markov move of type (1) to words in K_n.

For any set K of allowables, we may define

$$K_+ = \{(W\sigma_n, n+1) \mid (W, n) \in K\}, \quad \text{and}$$
$$K_- = \{(W, n) \mid (W\sigma_n, n+1) \in K \text{ and } (W, n) \text{ is allowable}\}.$$

Then $K_+ \cup K_-$ contains all words corresponding to the application of a positive Markov move of type (2) to words in K. We now define $K' = K \cup K_+ \cup K_-$, and let

$$\mu K = \{(W, n) \mid W \in \varphi(K'_n)\}.$$

Then μK contains only allowables, and is finite if K finite. Moreover, since K' is calculable, $\varphi(K')$ is calculable (Theorem 2.4) and so μK is calculable.

THEOREM 6.4. *Let (β, n) and (γ, n') be positive singular braids represented by allowables (V, n) and (W, n') respectively. If (β, n) is related to (γ, n') by N positive Markov moves, then (V, n) lies in $\mu^N\{(W, n')\}$. Conversely, if (V, n) lies in $\mu^N\{(W, n')\}$ for some positive integer N, then (β, n) and (γ, n') are related by a sequence of positive Markov moves. Membership of $\mu^N\{(W, n')\}$ is calculable.*

Acknowledgements

The author thanks David Easdown, Bob Howlett and the referee for many helpful suggestions and discussions. In particular, the author would like to recognise the referee's improvements to the article in strengthening Lemma 1.7, and in the proofs of Lemma 1.7 and Theorem 1.5. The majority of the work in this article was undertaken while the author was a research student in the School of Mathematics and Statistics at the University of Sydney, supported by the Australian Postgraduate Award.

References

[1] N. Antony, 'On singular Artin monoids and contributions to Birman's conjecture', *Comm. Algebra*, to appear.
[2] D. Armand-Ugon, R. Gambini and P. Mora, 'Intersecting braids and intersecting knot theory', *J. Knot Theory Ramifications* **4** (1995), 1–12.
[3] E. Artin, 'Theorie der Zöpfe', *Hamburg Abh.* **4** (1925), 47–72.
[4] J. Baez, 'Link invariants and perturbation theory', *Lett. Math. Phys.* **2** (1992), 43–51.
[5] J. Birman, 'New points of view in knot theory', *Bull. Amer. Math. Soc. (N.S.)* **28** (1993), 253–287.
[6] N. Bourbaki, *Groupes et algèbres de Lie* (Hermann, Paris, 1968).
[7] E. Brieskorn and K. Saito, 'Artin-Gruppen und Coxeter-Gruppen', *Invent. Math.* **17** (1972), 245–271.

[8] B. Brink and R. B. Howlett, 'Normalizers of parabolic subgroups in Coxeter groups', *Invent. Math.* **136** (1999), 323–351.

[9] R. Corran, 'A normal form for a class of monoids including the singular braid monoids', *J. Algebra* **223** (2000), 256–282.

[10] P. Deligne, 'Les immeubles des groupes de tresses généralisés', *Invent. Math.* **17** (1972), 273–302.

[11] H. Van der Lek, *The homotopy type of complex hyperplane complements* (Ph.D. Thesis, Nijmegen, 1983).

[12] R. Fenn, D. Rolfsen and J. Zhu, 'Centralisers in the braid group and singular braid monoid', *l'Enseignement Mathématique* **42** (1996), 75–96.

[13] F. A. Garside, 'The braid group and other groups', *Quart. J. Math. Oxford (2)* **20** (1969), 235–254.

[14] B. Gemein, 'Singular braids and Markov's theorem', *J. Knot Theory Ramifications* **6** (1997), 441–454.

[15] E. Godelle, 'Normalisateur et groupe d'Artin de type spherique', *J. Algebra* **269** (2003), 263–274.

[16] ———, 'Parabolic subgroups of Artin groups of type FC', *Pacific J. Math.* **208** (2003), 243–254.

[17] E. Godelle and L. Paris, 'On singular Artin monoids', in: *Proceedings of the special session Geometric Methods in Group Theory*, First Joint Meeting RMSE and AMS, Sevilla, 2003, *Contemporary Math.*, to appear.

[18] J. M. Howie, *Fundamentals of semigroup theory*, London Math. Soc. Monogr., New Series 12 (Oxford University Press, Oxford, 1995).

[19] R. B. Howlett, 'Normalizers of parabolic subgroups of reflection groups', *J. London Math. Soc.* **21** (1980), 62–80.

[20] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Adv. Math. 29 (Cambridge Univ. Press, Cambridge, UK, 1990).

[21] L. Paris, 'The proof of Birman's conjecture on singular braid monoids', *Geom. Topol.* **8** (2004), 1281–1300.

[22] ———, 'Parabolic subgroups of Artin groups', *J. Algebra* **196** (1997), 369–399.

Institut de Géométrie, Algèbre et Topologie
Bâtiment BCH
École Polytechnique Fédérale de Lausanne
CH-1015
Switzerland
e-mail: ruth.corran@epfl.ch