G. Borot et al. (2015) "Large-N Expansion for Mean Field Models,"
International Mathematics Research Notices, Vol. 2015, No. 20, pp. 10451–10524
Advance Access Publication January 18, 2015
doi:10.1093/imrn/rnu260

Large-N Asymptotic Expansion for Mean Field Models with Coulomb Gas Interaction

Gaëtan Borot[1,2,3], Alice Guionnet[3] and Karol K. Kozlowski[4]

[1]Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, 1211 Genève 4, Switzerland, [2]Max Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany, [3]Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA, and [4]Institut de Mathématiques de Bourgogne, CNRS UMR 5584, 9 avenue Alain Savary, BP 47870, 21078 Dijon Cedex, France

Correspondence to be sent to: gborot@mpim-bonn.mpg.de

We derive the large-N, all order asymptotic expansion for a system of N particles with mean field interactions on top of a Coulomb repulsion at temperature $1/\beta$, under the assumptions that the interactions are analytic, off-critical, and satisfy a local strict convexity assumption.

1 Introduction

This article aims at giving a basic framework to study the large-N expansion of the partition function and various observables in the mean field statistical mechanics of N repulsive particles in $1d$. This is one of the most simple form of interaction between particles and constitutes the first case to be fully understood before addressing the problem of more realistic interactions.

Received February 21, 2014; Revised September 3, 2014; Accepted November 27, 2014
Communicated by Prof. Kurt Johansson

An archetype of such models is provided by random $N \times N$ hermitian matrices, drawn from a measure $\mathrm{d}M\,\mathrm{e}^{-N\mathrm{Tr}\,V(M)}$ [30, 44]. The corresponding joint distribution of eigenvalues is

$$\prod_{i=1}^{N} \mathrm{d}\lambda_i\,\mathrm{e}^{-NV(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta}$$

with $\beta = 2$, that is, of the form $\mathrm{e}^{-E(\lambda)}$, where $E(\lambda)$ includes the energy of a $2d$ Coulomb interaction of eigenvalues, and the effect of an external potential V. The large-N behavior in those models—and for all values of $\beta > 0$—have been intensively studied: they are called β-*ensembles*. On top of contributions from physics [1, 3, 4, 18–20, 25], many rigorous results are available concerning the convergence of the empirical measure when N is large [21, 47], large deviation estimates [5], central limit theorems or their breakdown [15, 16, 33, 39, 46], and all order asymptotic expansion of the partition function and multilinear statistics [2, 15, 16, 24, 48]. The nature of the expansion depends on the topology of the locus of condensation S of the eigenvalues. Besides, the asymptotic expansion up to $O(N^{-\infty})$ is fully determined by a universal recursion, called "topological recursion" [25, 29] (or "topological recursion with nodes" in the multi-cut case [26]), taking as initial data the large-N spectral density and the large-N spectral covariance.

The models we propose to study are generalizations of β ensembles, with an arbitrary interaction between eigenvalues (not only pairwise), but assuming pairwise repulsion at short distance approximated by the Coulomb interaction already present in β ensembles. Then, combining tools of complex and functional analysis, we provide techniques showing that the theory for the all order large-N asymptotic expansion is very similar to the one developed for β ensembles.

1.1 The model

1.1.1 *The unconstrained model*

Let $\mathsf{A} = \dot{\bigcup}_{h=0}^{g} \mathsf{A}_h$ be a closed subset of $\mathbb{R}$ realized as the disjoint union of g intervals A_h—possibly semi-infinite or infinite. In this paper, we focus on the probability measure on A^N defined by

$$\mathrm{d}\mu_{\mathsf{A}^N} = \frac{1}{Z_{\mathsf{A}^N}} \prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^{\beta} \cdot \exp\left\{ \frac{N^{2-r}}{r!} \sum_{1 \le i_1, \ldots, i_r \le N} T(\lambda_{i_1}, \ldots, \lambda_{i_r}) \right\}. \qquad (1.1)$$

We assume $\beta > 0$ and Z_{A^N} is the partition function which ensures that $\int_{\mathsf{A}^N} \mathrm{d}\mu_{\mathsf{A}^N} = 1$. The function T represents an r-body interaction. Without loss of generality, we can assume

T to be symmetric in its r variables; we call it the *r-linear potential*. The scaling in N ensures that it contributes to the same order that the 2-body repulsive Coulomb interaction when N is large. A common case is $r = 2$, that is, the eigenvalues undergo a pairwise repulsion, which is approximated at short distance by a Coulomb repulsion. The r-linear potential can possibly admit a large-N asymptotic expansion of the type

$$T(x_1, \ldots, x_r) = \sum_{p\geq 0} N^{-p} T^{[p]}(x_1, \ldots, x_r),$$

where $T^{[p]}$ are symmetric functions on A^r not depending on N. These functions have the same regularity as T.

We do stress that the *r-linear potential* contains, as a specific example, the case of growing r-body interactions, namely the substitution

$$T(x_1, \ldots, x_r) = \sum_{J\subseteq [\![1\,;\,r]\!]} (r - |J|)!\,T_{|J|}(\boldsymbol{x}_J) \quad \text{with} \begin{cases} J = \{j_1, \ldots, j_{|J|}\} \\ \boldsymbol{x}_J = (x_{j_1}, \ldots, x_{j_{|J|}}) \end{cases} \tag{1.2}$$

recast the *r-linear* interaction term as

$$\sum_{k=1}^{r} \frac{N^{2-k}}{k!} \sum_{1\leq i_1,\ldots,i_k\leq N} T_k(\lambda_{i_1}, \ldots, \lambda_{i_k}).$$

The latter expression has a clear interpretation of a concatenation of $1, 2, \ldots, r$ body interactions. In the latter case, it is convenient to include the 2-body Coulomb repulsion in a total 2-body interaction:

$$T_2^{\mathrm{tot}}(x, y) = \beta \ln|x - y| + T_2(x, y).$$

For $\beta = 2$, sending $r \to \infty$ would allow the description of a quite general form of a $U(N)$ invariant measure on the space of $N \times N$ hermitian matrices. Indeed, for $\beta = 2$, (1.1) corresponds to the law of eigenvalues of a $N \times N$ random hermitian matrix Λ drawn with (unnormalized) distribution:

$$\mathrm{d}\Lambda \exp\left\{\frac{N^{2-r}}{r!}\,\mathrm{Tr}\,T(\Lambda^{(1)}, \ldots, \Lambda^{(r)})\right\},$$

where $\mathrm{d}\Lambda$ is the Lebesgue measure on the space of Hermitian matrices, and $\Lambda^{(i)}$ is the tensor product of r matrices, in which the ith factor is Λ and all other factors are identity matrices. In particular, in any model of several random and coupled $N \times N$ hermitian random matrices $\Lambda_1, \ldots, \Lambda_s$ which is invariant by simultaneous conjugation of all Λ_i by the same unitary matrix, the marginal distribution of Λ_i is $U(N)$ invariant, thus of the form (1.1) with, possibly, $r=\infty$ and a different dependence in N. Further, for simplicity, we shall restrict ourselves to r-linear interactions between eigenvalues with $r<\infty$.

1.1.2 *The model with fixed filling fractions*

In the process of studying the unconstrained model in the multi-cut regime, we need to deal with the so-called fixed filling fraction model. Let $g\geq 1$, and recall that $\mathsf{A}=\dot{\bigcup}_{h=0}^{g}\mathsf{A}_h$. Let $N=\sum_{h=0}^{g}N_h$ be a partition of N into $g+1$ integers and $\mathbf{N}=(N_0,\ldots,N_g)$ a vector built out of the entries of this partition. One can associate to such a partition a vector $\boldsymbol{\lambda}\in\mathsf{A}_{\mathbf{N}}\equiv\prod_{h=0}^{g}\mathsf{A}_h^{N_h}$ with entries ordered according to the lexicographic order on $\mathbb{N}^2$

$$\boldsymbol{\lambda}=(\lambda_{0,1},\ldots,\lambda_{0,N_0},\lambda_{2,1}\ldots,\lambda_{g,1},\ldots,\lambda_{g,N_g}).$$

The measure on $\mathsf{A}_{\mathbf{N}}$ associated with this partition reads

$$\mathrm{d}\mu_{\mathsf{A}_{\mathbf{N}}}=\frac{1}{Z_{\mathsf{A}_{\mathbf{N}}}}\prod_{\boldsymbol{i}\in\mathcal{I}}\{\mathbf{1}_{\mathsf{A}_{\mathrm{pr}(\boldsymbol{i})}}(\lambda_{\boldsymbol{i}})\mathrm{d}\lambda_{\boldsymbol{i}}\}\cdot\prod_{\boldsymbol{i}_1<\boldsymbol{i}_2}|\lambda_{\boldsymbol{i}_1}-\lambda_{\boldsymbol{i}_2}|^{\beta}\exp\left\{\frac{N^{2-r}}{r!}\sum_{\boldsymbol{i}_1,\ldots,\boldsymbol{i}_r\in\mathcal{I}}T(\lambda_{\boldsymbol{i}_1},\ldots,\lambda_{\boldsymbol{i}_r})\right\},\qquad(1.3)$$

where $\boldsymbol{i}$ and $\boldsymbol{i}_k$ are elements of

$$\mathcal{I}=\{(a,N_a)\,:\,a\in[\![\,0\,;\,g\,]\!]\},$$

$<$ is the lexicographic order on $\mathcal{I}$ and pr is the projection on the first coordinate. Note that the relation between the partition function of the unconstrained model and the fixed filling fraction model is

$$Z_{\mathsf{A}^N}=\sum_{N_0+\cdots+N_g=N}\frac{N!}{\prod_{h=0}^{g}N_h!}Z_{\mathsf{A}_{\mathbf{N}}}.$$

1.1.3 *Observables*

In this section, $\mu_{\mathcal{S}}$ denotes the measure and $Z_{\mathcal{S}}$ the partition function in any of the two models, *viz.* $\mathcal{S}=\mathsf{A}^N$ or $\mathsf{A}_{\mathbf{N}}$. The empirical measure is the random probability measure:

$$L_N=\frac{1}{N}\sum_{\mathbf{i}\in\mathcal{I}_{\mathcal{S}}}\delta_{\lambda_i}\quad\text{with } \mathcal{I}_{\mathsf{A}^N}=[\![\,1\,;\,N\,]\!]\text{ and }\mathcal{I}_{\mathsf{A}_{\mathbf{N}}}=\mathcal{I}.$$

We introduce the Stieltjes transform of the nth-order moments of the unnormalized empirical measure, called *disconnected correlators*:

$$\tilde{W}_n(x_1,\ldots,x_n)=N^n\mu_{\mathcal{S}}\left[\prod_{i=1}^{n}\int\frac{\mathrm{d}L_N(s_i)}{x_i-s_i}\right].\qquad(1.4)$$

They are holomorphic functions of $x_i \in \mathbb{C} \setminus \mathsf{A}$. When N is large, for reasons related to concentration of measures, it is more convenient to consider the Stieltjes transform of the nth-order cumulants of the unnormalized empirical measure, called *correlators*:

$$W_n(x_1, \ldots, x_n) = \partial_{t_1} \cdots \partial_{t_n} \ln Z_{\mathcal{S}}[T \to \widetilde{T}_{t_1,\ldots,t_n}]_{|t_i=0} \tag{1.5}$$

with

$$\widetilde{T}_{t_1,\ldots,t_n}(\xi_1, \ldots, \xi_r) = T(\xi_1, \ldots, \xi_r) + \frac{(r-1)!}{N} \sum_{i=1}^{n} \sum_{a=1}^{r} \frac{t_i}{x_i - \xi_a}, \tag{1.6}$$

and we have explicitly insisted on the functional dependence of $Z_{\mathcal{S}}$ on the r-linear potential.

If J is a set, exactly as in (1.2) we denote by $\boldsymbol{x}_J$ the $|J|$-dimensional vector whose components are labeled by the elements of J. The above two types of correlators are related by

$$\widetilde{W}_n(x_1, \ldots, x_n) = \sum_{s=1}^{n} \sum_{\substack{[\![1\,;\, n]\!] = \\ J_1 \dot\cup \cdots \dot\cup J_s}} \prod_{i=1}^{s} W_{|J_i|}(\boldsymbol{x}_{J_i}).$$

Above, the sum runs through all partitions of the set $[\![1\,;\, n]\!]$ into s nonempty, disjoint sets J_ℓ.

We do stress that the knowledge of the correlators for a smooth family of potentials $\{T_t\}$ indexed by some continuous variable t determines the partition function up to an integration constant. Indeed, let $\mu_{\mathcal{S}}^{T_t}$ denote the probability measure in any of the two models and in the presence of the r-linear interaction T_t. Then, one has

$$\partial_t \ln Z_{\mathcal{S}}[T \to T_t] = \frac{N^2}{r!} \mu_{\mathcal{S}}^{T_t} \left[\int \partial_t T_t(s_1, \ldots, s_r) \prod_{i=1}^{r} \mathrm{d}L_N(s_i) \right].$$

If $\partial_t T_t$ is analytic in a neighborhood of A^r, we can rewrite

$$\partial_t \ln Z_{\mathcal{S}}[T \to T_t] = \frac{N^{2-r}}{r!} \oint_{\mathsf{A}^r} \partial_t T_t(\xi_1, \ldots, \xi_r) \widetilde{W}_r[T \to T_t](\xi_1, \ldots, \xi_r) \prod_{i=1}^{r} \frac{\mathrm{d}\xi_i}{2\mathrm{i}\pi}.$$

In both cases, the superscript T_t denotes the replacement of the r-linear potential by the t-dependent one T_t.

1.2 Motivation

In the context of formal integrals for $\beta = 2$, which is accurate for combinatorics, (1.1) describes the generating series of discrete surfaces obtained by gluing along edges discrete surfaces of any topology and with up to r polygonal boundaries. As special cases, one obtains the enumeration of maps carrying any of the classical statistical physics models: self-avoiding loop configurations [36], spanning forests [17], the Potts model [10, 35, 50], the Ising model [34], the 6-vertex model [37], ... It was shown in [13] for $r = 2$, and [12] for arbitrary r that the correlators have a formal $1/N$ expansion, whose coefficients are given by the topological recursion "with initial conditions". Formal integrals with $\beta \neq 2$ are related to the combinatorics of nonorientable discrete surfaces and has been much less studied. In that case, the coefficients of the $1/N$ expansion are given for $r = 1$ by the β-topological recursion of [20], and this should extend to any r upon similar additions of "initial conditions". In the context of convergent integrals, the present article gives conditions under which the existence of a large-N expansion asymptotic expansion can be established. When this is $1/N$ asymptotic expansion, the coefficients are the same as those of the formal integrals.

The models we are studying are encountered for instance in three-dimensional topology: the computation of torus knot invariants and of the SU(N) Chern-Simons partition function in certain Seifert manifolds is of the form (1.1) for $r = 2$ [9, 43]. The results of this article was used in [14] to establish a large N expansion of these partition function, as well as some analyticity results on perturbative knot invariants. Besides, it was claimed in [31] that the SU(N) Chern-Simons partition function of 3-manifolds obtained by filling of a knot in $\mathbb{S}_3$ should be described by (1.1) for $r = \infty$. For related reasons, (1.1) with $r = 2$ is also relevant in topological strings and supersymmetric gauge theories, see, for example, [49], although our results would have to be generalized to complex-valued T in order to be applied to such problems.

The strict convexity property (Hypothesis 3.2) of the interaction plays a central role for our results: it requires that the energy functional (2.1) has a unique maximizer, and its Hessian at the maximizer is definite negative. It is therefore a basic framework where the maximizer is nondegenerate; beyond leading order when $N \to \infty$, more interesting phenomena could occur when the maximizer is not unique, or when the Hessian at maximizer is degenerate, but they are beyond the scope of this article. For $r = 2$, if the singular operator with kernel $T_2^{\mathrm{tot}}(x, y)$ is definite negative, then Hypothesis 3.2 is satisfied. Very often, one encounters translation invariant interactions $T_2^{\mathrm{tot}}(x, y) = R(x - y)$, so this operator is diagonalized in Fourier space, and it suffices to check that the Fourier transform of $\ln R$ is negative. It is well known that the Fourier transform of

$\ln R(u) = \beta \ln |u|$ is negative [21], thus Hypothesis 3.2 holds for pure Coulomb interaction. Here are a few other examples $r = 2$ where Hypothesis 3.2 also hold

- If f is a diffeomorphism,

$$T_2^{\mathrm{tot}}(x, y) = (\beta/2)\{\ln |x - y| + \ln |f(x) - f(y)|\}.$$

  For $\beta = 2$, the corresponding random matrix model is determinantal, and called "biorthogonal ensemble". For $f(x) = x^{\theta}$, they were studied in detail in [11, 45]. Since the second term is the pushforward of $\ln |x - y|$ by f, it also has negative Fourier transform, and so does $T_2^{\mathrm{tot}}(x, y)$ as a sum of two terms with negative Fourier transform. We remark that some results about the equilibrium measure and the zeroes of the biorthogonal polynomials suitable to those ensembles are established in [40], with some overlap with ours of Section 2.
- The sinh interactions: for $\psi(x) = \sinh(x/2)$,

$$T_2^{\mathrm{tot}}(x, y) = \beta \ln |\psi(x - y)|.$$

- The $U(N)$ Chern-Simons partition function of Seifert spaces [9, 43]: for a d-uple of positive integers $(p_1, \ldots, p_d)$,

$$T_2^{\mathrm{tot}}(x, y) = (2 - d) \ln |\psi(x - y)| + \sum_{i=1}^{d} \ln |\psi[(x - y)/p_i]|.$$

  The Fourier transform is computed in [14, Appendix A.1] and found to be negative (thus Hypothesis 3.2 holds) under the condition $2 - d + \sum_{i=1}^{d} 1/p_i \geq 0$. This includes as a particular case the sinh interaction above.
- The (q, t)-deformed interactions. For $q \neq 0$ and t are real parameters between -1 and 1, set

$$T_2^{\mathrm{tot}}(x, y) = R_{q,t}(x - y) = \frac{\left(e^{2i\pi(x-y)}, q\right)_{\infty}\left(e^{2i\pi(y-x)}, q\right)_{\infty}}{\left(t\, e^{2i\pi(x-y)}, q\right)_{\infty}\left(t\, e^{2i\pi(y-x)}, q\right)_{\infty}}$$
$$= \exp\left(-\sum_{k\geq 1} \frac{1 - t^k}{1 - q^k} \frac{2\cos(2\pi k(x - y))}{k}\right)$$

  $R_{q,t}(u)$ is a 1-periodic function of u. As can be seen from the last expression, the operator with kernel $\ln R_{q,t}(x - y)$ is diagonalized in Fourier space on $[0, 1]$, and the eigenvalues are all negative. Thus, Hypothesis 3.2 holds for $\mathsf{A} \subseteq [0, 1]$.

- The $O(n)$ model: for $|n| \leq 2$ and $\mathsf{A} \subseteq \mathbb{R}_+$,

$$T_2^{\mathrm{tot}}(x, y) = \beta\{2\ln|x-y| - n\ln(x+y)\}.$$

  The previous examples were related to A_{N-1} root systems. The analog of the simplest Coulomb interaction for BC_N root systems is the $O(-2)$ model, and the sinh, q and (q,t) cases also have a natural $O(n)$ deformation. As a matter of fact, one can define (q,t)-deformed interactions associated to any pair of root systems, and they intervene in the orthogonality measures of Macdonald polynomials [41]. They are relevant in $\mathrm{SO}(N)$ or $\mathrm{Sp}(2N)$ Chern-Simons theory, but also in condensed matter. For instance, the two-body interaction:

$$T_2^{\mathrm{tot}}(x, y) = (\beta/2)\{\ln|x-y| + \ln|\psi(x-y)| + \ln|x+y| + \ln|\psi(x+y)|\}$$

  has been shown to occur between transmission eigenvalues in metallic wires with disorder [6]. Hypothesis 3.2 for those type of models follows from [13, Lemma A.1].

For $r \geq 3$, it is in general complicated to check Hypothesis 3.2; yet, one can a priori use the fact that, if the interactions are a small enough perturbation of better-known strictly convex interactions, like the ones mentioned above, then Hypothesis 3.2 is satisfied.

The models cited above fall into the scope of our methods. This means that, to apply our main Theorem 1.1 to a concrete example where T_2^{tot} satisfies Hypothesis 3.2, one is left with the problem of determining the topology of the support of the equilibrium measure, and checking off-criticality assumptions. For $r = 2$, computing explicitly the equilibrium measure usually requires solving a linear, scalar, nonlocal Riemann–Hilbert problem (RHP) with jumps on an unknown support. It is a difficult problem which is interesting *per se*, and only a handful of cases have been treated so far—like a complete solution for the $O(n)$ model with the methods of [27, 28], or a partial solution for a more general class of nonlocal RHP with the methods of [14]. For $r \geq 3$, the problem even becomes nonlinear, and thus explicit solutions seems beyond hope. So, very often for $r \geq 2$, one has to rely on general potential theory/perturbative approach to obtain general qualitative information on the equilibrium measure to check if the other assumptions of our main theorem hold. Monte-Carlo simulations of the equilibrium measure can provide some help in checking those assumptions numerically, see, for example, [14, Section 9].

1.3 Main result

Our main result Theorem 8.1 is an all order expansion for the partition function of our model that we emphasize below. This in particular allows the study of fluctuations of linear statistics in Section 8.2.

Theorem 1.1. Assume Hypothesis 2.1, T holomorphic in a neighborhood of A^r, and μ_{eq} in the $(g+1)$-cut regime and off-critical. Then, the partition function in the A^N model admits the asymptotic expansion:

$$Z_{\mathsf{A}^N} = N^{(\beta/2)N+\gamma} \exp\Big(\sum_{k\geq -2} N^{-k} F^{[k]}_{\boldsymbol{\epsilon}^\star}\Big)$$
$$\times \left\{ \sum_{m\geq 0} \sum_{\substack{\ell_1,\ldots,\ell_m\geq 1 \\ k_1,\ldots,k_m\geq -2 \\ \sum_{i=1}^m \ell_i+k_i>0}} \frac{N^{-\sum_{i=1}^m(\ell_i+k_i)}}{m!} \left(\bigotimes_{i=1}^m \frac{F^{[k_i],(\ell_i)}_{\boldsymbol{\epsilon}^\star}}{\ell_i!}\right) \cdot \nabla_{\boldsymbol{v}}^{\otimes(\sum_{i=1}^m \ell_i)} \right\} \Theta_{-N\boldsymbol{\epsilon}^\star}\big(F^{[-1],(1)}_{\boldsymbol{\epsilon}^\star} \,|\, F^{[-2],(2)}_{\boldsymbol{\epsilon}^\star}\big). \tag{1.7}$$

□

The various terms appearing in Theorem 1.1 are defined in Sections 7 and 8. Here, we briefly comment on the structure of the asymptotic expansion. $F^{[k],(\ell)} \in (\mathbb{R}^g)^{\otimes \ell}$ are tensors independent of N. $\Theta_{\boldsymbol{\nu}}(\boldsymbol{w}|\boldsymbol{T})$ is the Siegel theta function depending on a g-dimensional vector $\boldsymbol{w}$ and $\boldsymbol{T}$ is a definite positive quadratic form in $\mathbb{R}^g$. ∇ is the gradient operator acting on the variable $\boldsymbol{w}$. This Theta function is $\mathbb{Z}^g$-periodic function of the vector $\boldsymbol{\nu}$. Since it is evaluated to $\boldsymbol{\nu} = -N\boldsymbol{\epsilon}^\star$ in (1.7), the partition function enjoys a pseudo-periodic behavior in N at each order in $1/N$. We mention that the definite positive quadratic form in the Theta function is evaluated at

$$\boldsymbol{T} = F^{[-2],(2)}_{\boldsymbol{\epsilon}^\star} = -\mathrm{Hessian}_{\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^\star} \mathcal{E}[\mu^{\epsilon}_{\mathrm{eq}}],$$

where $\mathcal{E}$ and $\mu^{\boldsymbol{\epsilon}}_{\mathrm{eq}}$ are defined at the beginning of Section 2. The exponent $\gamma = \sum_{h=0}^{g} \gamma_h$ only depends on β and the nature of the edges, it was already determined in [15]:

- $\gamma_h = \frac{3+\beta/2+2/\beta}{12}$ if the component S_h of the support has two soft edges;
- $\gamma_h = \frac{\beta/2+2/\beta}{6}$ if it has one soft edge and one hard edge;
- $\gamma_h = \frac{-1+\beta/2+2/\beta}{4}$ if it has two hard edges.

Note that, in the 1-cut regime ($g = 0$), the Theta function is absent and we retrieve a $1/N$ expansion (established in Corollary 7.2).

1.4 Method and outline

We stress that in general (1.1) is not an exactly solvable model even for $\beta = 2$—with the exception of the aforementioned biorthogonal ensembles—so the powerful techniques of orthogonal polynomials and integrable systems cannot be used. In principle, at $\beta = 2$ one could analyze the integral within the method developed in [38]. For such a purpose, one should first carry out the Riemann–Hilbert analysis of a general multiple integral with $T = T_1$ (cf. (1.2)) and then implement the multideformation procedure developed in [38]. Here, we rather rely on a priori concentration of measures properties, and the analysis of the Schwinger-Dyson equations of the model what allows us, in particular, to treat uniformly the case of general β.

In Section 2, we establish the convergence of the empirical measure $L_N = \frac{1}{N}\sum_{i\in\mathcal{I}_S}\delta_{\lambda_i}$ in the unconstrained model ($\mathcal{S} = \mathsf{A}^N$), and in the model with fixed filling fractions ($\mathcal{S} = \mathsf{A}_{\mathbf{N}}$), to an equilibrium measure μ_{eq}. We study the properties of μ_{eq} in Section 2.3, showing that the results of regularity of μ_{eq} compared with the Lebesgue measure, and squareroot behaviors at the edges—which are well known for pure Coulomb two-body repulsion—continue to hold in the general setting. We give a large deviation principle in Section 3.1 allowing, as a particular case, a restriction to A compact.

We prove in Section 3 the concentration of the empirical measure around μ_{eq}, modulo the existence of an adapted functional space $\mathcal{H}$. Such adapted spaces are constructed in Section 4, thanks to the existence of the inverse of a linear operator $\mathcal{T}$. This inverse is explicitly constructed in Appendix A.4 by invoking functional analysis arguments. All these handlings lead to rough a priori bounds on the correlators. In Section 5, we improve those bounds by a bootstrap method using the Schwinger-Dyson equations in the fixed filling fraction model, and obtain the asymptotic expansion of the correlators in this model. The bootstrap method is based on the existence of a continuous inverse for a linear operator $\mathcal{K}$, which relies on basic results of Fredholm theory reminded in Appendix A.3. In Section 6, we deduce the asymptotic expansion of the partition function in the fixed filling fraction model by performing an interpolation to a model with $r = 1$, for which we can use the result of [15] relating the partition function to asymptotics of Selberg β integrals, again by interpolation. In the one-cut regime, this concludes the proof. In the multi-cut regime, we prove that the coefficients of expansion depend smoothly on the filling fractions. This allows in Section 7 to establish the asymptotic expansion of the partition function for the unconstrained model in the multi-cut regime, and to study the convergence in law of fluctuations of linear statistics in Section 8.2.

1.5 Notations and basic facts

1.5.1 *Functional analysis*

- $L^p(\mathsf{X})$ is the space of real-valued measurable functions φ on X such that $|\varphi|^p$ is integrable. Unless specified otherwise, the space X is endowed with its canonical measure (Lebesgue measure for a subset of $\mathbb{R}$, curvilinear measure for a Jordan curve, etc.).
- $\mathcal{F}$ denotes the Fourier transform which, defined on $L^1(\mathsf{A})$, reads $\mathcal{F}[\varphi](\boldsymbol{k}) = \int_{\mathsf{A}} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\varphi(\boldsymbol{x})\,\mathrm{d}^r\boldsymbol{x}$.
- $H^s(\mathbb{R}^r)$ is the Sobolev space of functions $\varphi \in L^2(\mathbb{R}^r)$ such that

$$\|\varphi\|_{H^s} = \int_{\mathbb{R}^r} |\mathcal{F}[\varphi](\mathbf{k})|^2 \left(1 + \sum_{i=1}^{r} k_i^2\right)^s \mathrm{d}^r\mathbf{k} < +\infty.$$

- More generally, $W^{1;p}(\mathsf{A})$ denotes the space of measurable functions φ on A such that

$$\|\varphi\|_p = \|\varphi\|_{L^p(\mathsf{A})} + \|\varphi'\|_{L^p(\mathsf{A})} < +\infty.$$

- If $b > 0$ and φ is a real-valued function defined on a subset X of a normed vector space, we agree upon:

$$\kappa_b[\varphi] = \sup_{x,y\in\mathsf{X}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^b} \in [0, +\infty].$$

In the case when $\mathsf{X} \subset \mathbb{C}^p$, $|\cdot|$ stands for the sup-norm. The space of b-Hölder functions corresponds to

$$\mathrm{Ho}_b(\mathsf{X}) = \{\varphi \in \mathcal{C}^0(\mathsf{X}),\ \kappa_b[\varphi] < +\infty\}$$

and the space of Lipschitz functions to $\mathrm{Ho}_1(\mathsf{X})$.

1.5.2 *Complex analysis*

- If A is a compact of $\mathbb{R}$ and $m \geq 1$, $\mathscr{H}^m(\mathsf{A})$ denotes the space of holomorphic functions f in $\mathbb{C} \setminus \mathsf{A}$, so that $f(x) \in O(1/x^m)$ when $x \to \infty$. If f is a function in $\mathbb{C} \setminus \mathsf{A}$, we denote $f \cdot \mathscr{H}^m(\mathsf{A}) = \{f \cdot \varphi, \quad \varphi \in \mathscr{H}^m(\mathsf{A})\}$.
- We can define similarly a space $\mathscr{H}^m(\mathsf{A}, r)$ for functions of r variables. In that case, the asymptotics in each variables take the form $f(x_1, \ldots, x_r) \in O(1/x_p^m)$, with a O that is uniform with respect to the other variables satisfying $d(x_k, A) > \eta$, for some $\eta > 0$.

- If A is a collection of segments, we denote $\oint_{\mathsf{A}}$, the integral over a counterclockwise contour surrounding each connected component of A exactly once.
- If Γ is a Jordan curve (hereafter called contour) surrounding A in $\mathbb{C} \setminus \mathsf{A}$, we denote $\mathrm{Ext}(\Gamma) \subseteq \mathbb{C} \setminus \mathsf{A}$ the unbounded connected component of $\mathbb{C} \setminus \Gamma$, and $\mathrm{Int}(\Gamma)$ the other connected component. If Γ and Γ' are two contours, we say that Γ' is exterior to Γ if $\Gamma' \subseteq \mathrm{Ext}(\Gamma)$ and we denote $\Gamma \subset \Gamma'$. We denote $\Gamma[1]$ an arbitrary contour in $\mathbb{C} \setminus \mathsf{A}$ exterior to Γ, and more generally $(\Gamma[i])_{i \geq 0}$ with $\Gamma[0] = \Gamma$ an arbitrary sequence of contours in $\mathbb{C} \setminus \mathsf{A}$ so that $\Gamma[i+1]$ is exterior to $\Gamma[i]$. $\Gamma[-1]$ denotes a contour interior to Γ, etc.
- We can equip $\mathscr{H}^m(\mathsf{A})$ (respectively, $\mathscr{H}^m(\mathsf{A}, r)$) with the norm:

$$\|\varphi\|_{\Gamma} = \sup_{x \in \Gamma} |\varphi(x)| = \sup_{x \in \mathrm{Ext}(\Gamma)} |\varphi(x)| \qquad \left(\text{respectively, } \|\varphi\|_{\Gamma^r} = \sup_{x \in \Gamma^r} |\varphi(x)|\right).$$

- $\mathscr{O}(\mathsf{A})$ is the space of holomorphic functions in a neighborhood of A.
- Given a contour Γ in $\mathbb{C}$ and a holomorphic function f on $\mathbb{C} \setminus \Gamma$, we denote by $f_{\pm}$ its boundary values (if they exist) when a point $z \in \mathbb{C} \setminus \Gamma$ approaches a point $x \in \Gamma$ from the $+$ (i.e., left) side or $-$ (i.e., right) side of Γ and nontangentially to Γ. The convergences of $f(z)$ to $f_{\pm}(x)$ are given in terms of a norm $(L^p, \mathcal{C}^0, \ldots)$ appropriate to the nature of $f_{\pm}$.

1.5.3 *Probability*

- $\mathbf{1}_{\mathsf{X}}$ denotes the indicator function of a set X.
- $\mathcal{M}^1(\mathsf{A})$ denotes the space of probability measures on A. $\mathcal{M}^0(\mathsf{A})$ denotes the set of differences of finite positive measure with same mass.
- $\mathcal{C}_b^0(\mathsf{A})$ denotes the space of bounded continuous functions on A. $\mathcal{M}^1(\mathsf{A})$ and $\mathcal{M}^0(\mathsf{A})$ are endowed with the weak-* topology, which means that

$$\lim \mu_n = \mu_\infty \iff \forall f \in \mathcal{C}_b^0(\mathsf{A}), \quad \lim_{n \to \infty} \left(\int_{\mathsf{A}} f(x)\, \mathrm{d}\mu_n(x)\right) = \int_{\mathsf{A}} f(x)\, \mathrm{d}\mu_\infty(x).$$

  If A is compact, Prokhorov theorem ensures that $\mathcal{M}^1(\mathsf{A})$ is compact for this topology.
- If $\nu \in \mathcal{M}^0(\mathsf{A})$, the Vasershtein norm is defined as:

$$\|\nu\| = \sup_{\substack{\varphi \in \mathrm{Ho}_1(\mathsf{A}) \\ \kappa_1[\varphi] \leq 1}} \left|\int_{\mathsf{A}} \varphi(x)\, \mathrm{d}\nu(x)\right|.$$

- Given the representation as a disjoint union $\mathsf{A} = \dot{\bigcup}_{h=0}^{g} \mathsf{A}_h$ and $\boldsymbol{\epsilon} = (\epsilon_0, \ldots, \epsilon_g)$ a $g+1$-dimensional vector with entries consisting of nonnegative real numbers summing up to 1, we denote $\mathcal{M}^{\boldsymbol{\epsilon}}(\mathbf{A})$ the set of probability measures μ on A such that $\mu[\mathsf{A}_h] = \epsilon_h$, $h = 0, \ldots, g$. We recall that $\mathcal{M}^{\boldsymbol{\epsilon}}(\mathbf{A})$ is a closed, convex subset of $\mathcal{M}^1(\mathsf{A})$.
- If X is a union of segments or a Jordan curve, $\ell(\mathsf{X})$ denotes its length.
- The notation $O(N^{-\infty})$ stands for $O(N^{-k})$ for any $k \geq 0$.
- c, C denote constants whose values may change from line to line.

2 The Equilibrium Measure

In this section, we assume the following:

Hypothesis 2.1.

- (Regularity) $T \in \mathcal{C}^0(\mathsf{A}^r)$.
- (Confinement) If $\pm\infty \in \mathsf{A}$, we assume the existence of a function f so that $T(x_1, \ldots, x_r) \leq -(r-1)! \sum_{i=1}^{r} f(x_i)$, when $|\boldsymbol{x}|$ is large enough, and

$$\liminf_{x \to \pm\infty} \frac{f(x)}{\beta \ln |x|} > 1.$$

- In the fixed filling fraction model, let $\boldsymbol{\epsilon} = (\epsilon_0, \ldots, \epsilon_g) \in [0, 1]^{g+1}$ be such that $\sum_{h=0}^{g} \epsilon_h = 1$, and $\boldsymbol{N} = (N_0, \ldots, N_g)$ be a vector of integers whose components depend on N and satisfy the constraint $\sum_{h=0}^{g} N_h = N$ and $N_h/N \to \epsilon_h$.
- (Uniqueness of the minimum) The energy functional $\mathcal{E}$—defined in (2.1)—has a unique global minimum on $\mathcal{M}^1(\mathsf{A})$ (in the unconstrained model, denoted μ_{eq}), or on $\mathcal{M}^{\boldsymbol{\epsilon}}(\mathbf{A})$ (in the fixed filling fraction model, denoted $\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$). □

2.1 Energy functional

We would like to consider the energy functional:

$$\mathcal{E}[\mu] = -\int_{\mathsf{A}^r} \left(\frac{T(x_1, \ldots, x_r)}{r!} + \frac{\beta}{r(r-1)} \sum_{1 \leq i \neq j \leq r} \ln |x_i - x_j| \right) \prod_{i=1}^{r} \mathrm{d}\mu(x_i). \tag{2.1}$$

Because of the singularity of the logarithm, $\mathcal{E}$ assumes value in $\mathbb{R} \cup \{+\infty\}$, and it is well known that $\mathcal{E}$ is lower semi-continuous. Let us introduce the level sets:

$$E_M = \{\mu \in \mathcal{M}^1(\mathsf{A}),\ \mathcal{E}[\mu] \leq M\}, \quad E_{<\infty} = \{\mu \in \mathcal{M}^1(\mathsf{A}),\ \mathcal{E}[\mu] < \infty\}.$$

We know that $E_{<\infty}$ is not empty. Let $\mathcal{M}'$ be a closed subset of $\mathcal{M}^1(\mathsf{A})$ which intersect $E_{<\infty}$. By standard arguments [7, 21, 47], $\mathcal{E}$ has compact level sets E_M in $\mathcal{M}'$, has a minimizing measure μ^* on $\mathcal{M}'$, and $\mathcal{E}[\mu^*]$ is finite. $\mathcal{M}'$ can be either $\mathcal{M}^1(\mathsf{A})$ or $\mathcal{M}^\epsilon(\mathbf{A})$, and Hypothesis 2.1 guarantees in either case that μ^* is unique. Following [7] (see also [5]), we can prove the following large deviation principle.

Theorem 2.1. Assume Hypothesis 2.1. Then, the law of $L_N = \frac{1}{N}\sum_{i\in\mathcal{I}_S}\delta_{\lambda_i}$ under the probability measure (1.1) (respectively, (1.3)) satisfies a large deviation principle on $\mathcal{M}^1(\mathsf{A})$ (respectively, $\mathcal{M}^\epsilon(\mathbf{A})$) with speed N^2 and good rate function $\mathcal{J} = \mathcal{E} - \inf_{\mathcal{M}^1(\mathsf{A})}\mathcal{E}$ (respectively, $\mathcal{E} - \inf_{\mathcal{M}^\epsilon(\mathbf{A})}\mathcal{E}$) . □

Proof. The case where $r=1$ and $\mathsf{A}=\mathbb{R}$ was proved in [7] (see also [5, Section 2.6.1]). The case $r\geq 1$, $\mathsf{A}=\mathbb{R}$ and $T-T_1$ bounded then follows by Varadhan's Lemma [22, Theorem 4.3.1] as $\mu\to\int(T-T_1)(x_1,\ldots,x_r)\,\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_r)$ is then bounded continuous on $\mathcal{M}^1(\mathbb{R})$. In fact, the proof of the general case is very close to the case $r=1$ given in [5, 7]. Let us briefly outline the arguments. First, one notes that $\mathcal{E}$ is a good rate function, that is, it has compact level sets. Indeed, one can write $\mathcal{E}(\mu) = \mathcal{E}_k(\mu) := \int\cdots\int k(x_1,\ldots,x_r)\,\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_r)$ with the function

$$K(x_1,\ldots,x_r) = -\frac{1}{r!}T(x_1,\ldots,x_r) - \frac{\beta}{r(r-1)}\sum_{i\neq j}\ln|x_i - x_j|.$$

By the monotone function theorem, K is the increasing limit of bounded continuous functions K_M—obtained by replacing $\ln|x|$ by $\max(-M,\ln|x|)$—hence $\mathcal{E}$ is the increasing limit of bounded continuous functions $\mathcal{E}_M$. Therefore, $\mathcal{E}$ is lower semi-continuous, that is, has closed level sets. Moreover, as K goes to infinity when any of the x_i goes to infinity—since the potential goes to minus infinity faster than $\beta\ln|x|$—we deduce that its level sets are included in compact sets, hence are compact. For the same reason, one can check that the law of L_N is exponentially tight, so that it is enough to prove a weak large deviation principle [22, Lemma 1.2.18]. Namely, recalling that $\|\cdot\|$ denotes the Vasershtein norm, it is enough to prove that for any probability measure $\mu\in\mathcal{M}^1(\mathsf{A})$, if we let $\mathrm{d}\nu_S = Z_S\,\mathrm{d}\mu_S$, we have

$$\begin{aligned}-\mathcal{E}(\mu) &\leq \liminf_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\ln\nu_S\left(\|L_N-\mu\|\leq\delta\right)\\ &\leq \limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\ln\nu_S\left(\|L_N-\mu\|\leq\delta\right)\leq -\mathcal{E}(\mu).\end{aligned}$$

To prove the upper bound, it is enough to bound the term $\ln|\lambda_i-\lambda_j|$ by $\max(-M,\ln|\lambda_i-\lambda_j|)$. Up to an error e^{NM} created by the addition of the diagonal terms, we have bounded from above the density of $\nu_{\mathcal{S}}$ by $e^{-N^2\mathcal{E}_M(L_N)}$. As $\mathcal{E}_M$ is continuous, we obtain the upper bound $-\mathcal{E}_M(\mu)$. We finally can let M going to infinity to conclude. To derive the lower bound, the idea is to localize each λ_i in a tiny neighborhood around the ith quantile of μ. This amounts to repeating the proof of Lemma 3.11 when μ has a smooth density. The latter assumption can be removed by approximations, see, for example, [7]. ∎

2.2 Convergence of the empirical measure

As a consequence of the previous large deviation principle, we can state the following convergence.

Theorem 2.2. Assume Hypothesis 2.1 in the unconstrained model, that is, $\mathcal{S} = \mathsf{A}^N$. When $N \to \infty$, $L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$ under the law μ_{A^N} converges almost surely and in expectation to the unique minimizer of μ_{eq} of $\mathcal{E}$ on $\mathcal{M}^1(\mathsf{A})$. μ_{eq} has a compact support, denoted S. It is characterized by the existence of a constant C such that

$$\forall x \in \mathsf{A},\quad \beta\int_{\mathsf{A}} \ln|x-\xi|\,\mathrm{d}\mu_{\mathrm{eq}}(\xi) + \int_{\mathsf{A}^{r-1}} \frac{T(x,\xi_2,\ldots,\xi_r)}{(r-1)!}\prod_{i=2}^{r}\mathrm{d}\mu_{\mathrm{eq}}(\xi_i) \leq C, \tag{2.2}$$

with equality μ_{eq}-almost surely. □

Theorem 2.3. Assume Hypothesis 2.1 in the model with fixed filling fractions, that is, $\mathcal{S} = \mathsf{A}_{\mathbf{N}}$. Then, $L_N = \frac{1}{N}\sum_{i\in\mathcal{I}}^{N}\delta_{\lambda_i}$ under the law μ_{A_N} converges almost surely and in expectation to the unique minimizer μ_{eq} of $\mathcal{E}$ on $\mathcal{M}^{\epsilon}(\mathbf{A})$. μ_{eq} has a compact support, denoted by S. It is characterized by the existence of constants C_h^{ϵ} such that

$$\forall h \in [\![0\,;\,g]\!],\quad \forall x \in \mathsf{A}_h,\quad \beta\int_{\mathsf{A}} \ln|x-\xi|\,\mathrm{d}\mu_{\mathrm{eq}}(\xi) + \int_{\mathsf{A}^{r-1}} \frac{T(x,\xi_2,\ldots,\xi_r)}{(r-1)!}\prod_{i=2}^{r}\mathrm{d}\mu_{\mathrm{eq}}(\xi_i) \leq C_h^{\epsilon}, \tag{2.3}$$

with equality μ_{eq}-almost surely. □

In either of the two models, we define the *effective potential* as

$$T_{\mathrm{eff}}(x) = \beta\int_{\mathsf{A}} \ln|x-\xi|\,\mathrm{d}\mu_{\mathrm{eq}}(\xi) + \int_{\mathsf{A}^{r-1}} \frac{T(x,\xi_2,\ldots,\xi_r)}{(r-1)!}\prod_{i=2}^{r}\mathrm{d}\mu_{\mathrm{eq}}(\xi_i) - \begin{cases} C, \\ \mathbf{1}_{\mathsf{A}_h}(x)C_h^{\epsilon}, \end{cases} \tag{2.4}$$

if $x \in \mathsf{A}$, and $T_{\mathrm{eff}}(x) = -\infty$ otherwise. It is thus nonpositive and vanishes μ_{eq}-almost surely.

We wait until Section 7.3 and Proposition 7.3 to establish that if $\mathcal{E}$ has a unique global minimum on $\mathcal{M}^1(\mathsf{A})$, and if we denote $\epsilon_h^\star = \mu_{\mathrm{eq}}[\mathsf{A}_h]$, then for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}^\star$, $\mathcal{E}$ has a unique minimizer over $\mathcal{M}^{\boldsymbol{\epsilon}}(\mathbf{A})$. In other words, Hypothesis 2.1 for the unconstrained model implies Hypothesis 2.1 for the model with fixed filling fractions close to $\boldsymbol{\epsilon}^\star$. Although the full Proposition 7.3 is stated for T holomorphic, the aforementioned statement is valid under weaker regularity, for example, $T \in \mathcal{C}^m(\mathsf{A}^r)$ with $m > \min(3, 2r)$.

2.3 Regularity of the equilibrium measure

In this section, we shall be more precise about the regularity of equilibrium measures, using the first Schwinger-Dyson equation.

Lemma 2.4. Assume Hypothesis 2.1 and $T \in \mathcal{C}^m(\mathsf{A})$ with $m \geq 2$. Then, μ_{eq} has a $\mathcal{C}^{m-2}$ density on $\mathring{\mathsf{S}}$. Let $\alpha \in \partial\mathsf{S}$.

(i) If $\alpha \in \partial\mathsf{A}$ (hard edge), then $\frac{\mathrm{d}\mu_{\mathrm{eq}}}{\mathrm{d}x}(x) \in O(|x-\alpha|^{-1/2})$ when $x \to \alpha$.
(ii) If $\alpha \notin \partial\mathsf{A}$ (soft edge) and $T \in \mathcal{C}^3(\mathsf{A}^r)$, then $\frac{\mathrm{d}\mu_{\mathrm{eq}}}{\mathrm{d}x}(x) \in O(|x-\alpha|^{1/2})$ when $x \to \alpha$. □

Lemma 2.5. Assume Hypothesis 2.1 and T holomorphic in a neighborhood of A in $\mathbb{C}$. Then, S is a finite union of segments which are not reduced to a point, and the equilibrium measure takes the form:

$$\mathrm{d}\mu_{\mathrm{eq}}(x) = \frac{\mathbf{1}_{\mathsf{S}}(x)\,\mathrm{d}x}{2\pi} M(x)\sigma_0(x) \prod_{\alpha \in \partial\mathsf{S}\setminus\partial\mathsf{A}} |x-\alpha|^{1/2} \prod_{\alpha \in \partial\mathsf{S}\cap\partial\mathsf{A}} |x-\alpha|^{-1/2}, \tag{2.5}$$

where M is holomorphic and positive (a fortiori nowhere vanishing) on A, and $\sigma_0(x)$ is a polynomial assuming nonnegative values on S. □

Proof of Lemma 2.4. As soon as $T \in \mathcal{C}^1(\mathsf{A}^r)$, we can derive a Schwinger-Dyson equation for the model $\mu_{\mathcal{S}}$. It is an exact equation, which can be proved by integration by parts, or by expressing the invariance of the integral $Z_{\mathcal{S}}$ by change of variables preserving A. It can be written: for any $x \in \mathbb{C} \setminus \mathsf{A}$,

$$\begin{aligned} \mu_{\mathcal{S}}\Bigg[& N\int_{\mathsf{A}} \partial_\xi \left(\frac{(1-\beta/2)\sigma_{\mathsf{A}}(\xi)}{x-\xi}\right) \mathrm{d}L_N(\xi) + N^2 \int_{\mathsf{A}^2} \frac{\beta}{2(\xi_1-\xi_2)} \left(\frac{\sigma_{\mathsf{A}}(\xi_1)}{x-\xi_1} - \frac{\sigma_{\mathsf{A}}(\xi_2)}{x-\xi_2}\right) \mathrm{d}L_N(\xi_1)\,\mathrm{d}L_N(\xi_2) \\ & + N^2 \int_{\mathsf{A}^r} \frac{\partial_{\xi_1} T(\xi_1, \xi_2, \ldots, \xi_r)}{(r-1)!} \frac{\sigma_{\mathsf{A}}(\xi_1)}{x-\xi_1} \prod_{i=1}^r \mathrm{d}L_N(\xi_i)\Bigg] = 0. \end{aligned} \tag{2.6}$$

Here, we have defined

$$\sigma_{\mathsf{A}}(x) = \prod_{a \in \partial \mathsf{A}} (x-a).$$

We insist that it takes the same form for the unconstrained model $\mathcal{S} = \mathsf{A}^N$ and in the model with fixed filling fractions $\mathcal{S} = \mathsf{A}_{\mathbf{N}}$, see Section 1.1 for the definitions. We do not attempt to recast this Schwinger-Dyson equation in the most elegant form; this is the matter of Section 5.

For any fixed $x \in \mathbb{C} \setminus \mathsf{A}$, the functions against which the empirical measure are integrated are continuous. Therefore, since L_N converges to μ_{eq} (Theorems 2.2 or 2.3), the first term is negligible in the large N limit, and we obtain

$$\begin{aligned} &\int_{\mathsf{S}^2} \frac{\beta}{2(\xi_1-\xi_2)} \left(\frac{\sigma_{\mathsf{A}}(\xi_1)}{x-\xi_1} - \frac{\sigma_{\mathsf{A}}(\xi_2)}{x-\xi_2}\right) \mathrm{d}\mu_{\mathrm{eq}}(\xi_1)\, \mathrm{d}\mu_{\mathrm{eq}}(\xi_2) \\ &\quad + \int_{\mathsf{S}^r} \frac{\partial_{\xi_1} T(\xi_1,\ldots,\xi_r)}{(r-1)!} \frac{\sigma_{\mathsf{A}}(\xi_1)}{x-\xi_1} \prod_{i=1}^{r} \mathrm{d}\mu_{\mathrm{eq}}(\xi_i) = 0. \end{aligned} \tag{2.7}$$

This equality only involves analytic functions of $x \in \mathbb{C} \setminus \mathsf{S}$ and is established for $x \in \mathbb{C} \setminus \mathsf{A}$. Thus, it is also valid for $x \in \mathbb{C} \setminus \mathsf{S}$. The first term can be rewritten partly in terms of the Stieltjes transform of the equilibrium measure:

$$\frac{\beta}{2} \sigma_{\mathsf{A}}(x) W_{\mathrm{eq}}^2(x) + U(x) + P(x) = 0, \tag{2.8}$$

with:

$$\begin{aligned} W_{\mathrm{eq}}(x) &= \int_{\mathsf{S}} \frac{\mathrm{d}\mu_{\mathrm{eq}}(\xi)}{x-\xi}, \\ U(x) &= \int_{\mathsf{S}^r} \sigma_{\mathsf{A}}(\xi_1) \frac{\partial_{\xi_1} T(\xi_1,\ldots,\xi_r)}{(r-1)!(x-\xi_1)} \prod_{i=1}^{r} \mathrm{d}\mu_{\mathrm{eq}}(\xi_i), \\ P(x) &= \int_{\mathsf{S}^2} \frac{\beta}{2(\xi_1-\xi_2)} \left(\frac{\sigma_{\mathsf{A}}(\xi_1)-\sigma_{\mathsf{A}}(x)}{x-\xi_1} - \frac{\sigma_{\mathsf{A}}(\xi_2)-\sigma_{\mathsf{A}}(x)}{x-\xi_2}\right) \mathrm{d}\mu_{\mathrm{eq}}(\xi_1)\, \mathrm{d}\mu_{\mathrm{eq}}(\xi_2). \end{aligned}$$

Since $\sigma_{\mathsf{A}}(x)$ is a polynomial (of degree $g+1$), $P(x)$ is also a polynomial. Since $T \in \mathcal{C}^2(\mathsf{A}^r)$, $U(x)$ admits continuous $\pm$ boundary values when $x \in \mathring{\mathsf{S}}$. Therefore, $\sigma_{\mathsf{A}}(x) W_{\mathrm{eq}}^2(x)$—and a fortiori $W_{\mathrm{eq}}(x)$—also admits continuous $\pm$ boundary values when $x \in \mathring{\mathsf{S}}$. Then, (2.8) at $x \in \mathring{\mathsf{S}}$ leads to

$$\sigma_{\mathsf{A}}(x) \left(W_{\mathrm{eq};\pm}^2(x) - V'(x) W_{\mathrm{eq};\pm}(x) + \frac{\tilde{P}(x)}{\sigma_{\mathsf{A}}(x)} \right) = 0 \tag{2.9}$$

with:

$$V(x) = -\frac{2}{\beta}\int_{\mathsf{S}^{r-1}} \frac{T(x,\xi_2,\ldots,\xi_r)}{(r-1)!}\prod_{i=2}^{r}\mathrm{d}\mu_{\mathrm{eq}}(\xi_i), \tag{2.10}$$

$$\tilde{P}(x) = \frac{2}{\beta}P(x) - \int_{\mathsf{S}} \frac{\sigma_{\mathsf{A}}(x)\,V'(x) - \sigma_{\mathsf{A}}(\xi)\,V'(\xi)}{(x-\xi)}\,\mathrm{d}\mu_{\mathrm{eq}}(\xi). \tag{2.11}$$

Since we assume $T \in \mathcal{C}^2(\mathsf{A}^r)$, we also find $V \in \mathcal{C}^2(\mathsf{S})$, hence $\tilde{P} \in \mathcal{C}^0(\mathsf{S})$. We also remind that the equilibrium measure is given in terms of its Stieltjes transform by

$$2\mathrm{i}\pi\,\frac{\mathrm{d}\mu_{\mathrm{eq}}}{\mathrm{d}x}(x) = W_{\mathrm{eq};-}(x) - W_{\mathrm{eq};+}(x). \tag{2.12}$$

Therefore, solving the quadratic equations (2.9) for $W_{\mathrm{eq};\pm}(x)$, we find

$$\frac{\mathrm{d}\mu_{\mathrm{eq}}}{\mathrm{d}x}(x) = \frac{\mathbf{1}_{\mathsf{S}}(x)}{2\pi}\sqrt{\frac{4\tilde{P}(x) - \sigma_{\mathsf{A}}(x)\,(V'(x))^2}{\sigma_{\mathsf{A}}(x)}}. \tag{2.13}$$

From (2.13), we see that the only possible divergence of $\mathrm{d}\mu_{\mathrm{eq}}/\mathrm{d}x$ is at $\alpha \in \mathsf{S} \cap \partial\mathsf{A}$, and the divergence is at most a $O((x-\alpha)^{-1/2})$, hence (i). If $\alpha \in \partial\mathsf{S} \setminus \partial\mathsf{A}$, we have $\sigma_{\mathsf{A}}(\alpha) \neq 0$ but the density of the equilibrium measure must vanish at α. If $T \in \mathcal{C}^3(\mathsf{A}^r)$, we find that $\tilde{P} \in \mathcal{C}^1(\mathsf{A})$ and thus the quantity inside the squareroot is $\mathcal{C}^1$. So, it must vanish at least linearly in α, which entails (ii). ∎

Proof of Lemma 2.5. Let Ω be an open neighborhood of A such that T is holomorphic in Ω^r. Then, $V(x)$ and $\tilde{P}(x)$ defined in (2.10)–(2.11) are well-defined, holomorphic functions of $x \in \Omega$. So, the limiting Schwinger-Dyson equation (2.7) can be directly recast for any $x \in \Omega \setminus \mathsf{A}$:

$$W_{\mathrm{eq}}^2(x) - V'(x)\,W_{\mathrm{eq}}(x) + \frac{\tilde{P}(x)}{\sigma_{\mathsf{A}}(x)} = 0.$$

Its solution is

$$W_{\mathrm{eq}}(x) = \frac{V'(x)}{2} \pm \frac{1}{2}\sqrt{\frac{\sigma_{\mathsf{A}}(x)\,(V'(x))^2 - 4\tilde{P}(x)}{\sigma_{\mathsf{A}}(x)}}. \tag{2.14}$$

By continuity of $W_{\mathrm{eq}}(x)$, the sign is uniformly $+$ or uniformly $-$ in each connected component of Ω. From (2.12), the equilibrium measure reads

$$\frac{\mathrm{d}\mu_{\mathrm{eq}}}{\mathrm{d}x}(x) = \pm\frac{\mathbf{1}_{\mathsf{S}}(x)\,\mathrm{d}x}{2\pi}\sqrt{R(x)},\qquad R(x) = \frac{\sigma_{\mathsf{A}}(x)\,(V'(x))^2 - 4\tilde{P}(x)}{-\sigma_{\mathsf{A}}(x)}. \tag{2.15}$$

The support S is the closure of the set of $x \in \mathsf{A}$ for which the right-hand side is positive. The function R is meromorphic in $\Omega \cup \mathsf{A}$ and real-valued on A ; further, its only poles are simple and all located in ∂A. Hence, given a compact $\Omega' \subseteq \Omega$ neighborhood of A, R can be recast as $R = R_0 \cdot M^2$. In such a factorization, $R_0(x)$ is a rational function having the same poles and zeroes as $R(x)$ on Ω' while M^2 is a holomorphic function on Ω that is nowhere vanishing on Ω' and that keeps a constant sign on A. We shall denote its square root by M. According to the formula (2.15), $R_0(x)$ can only have simple poles that occur at the edges of A. Thence, the edges of S must be either its poles or its zeroes. Therefore, we may factorize further the zeroes of even order and write

$$\frac{\mathrm{d}\mu_{\mathrm{eq}}}{\mathrm{d}x}(x) = \frac{\mathbf{1}_{\mathsf{S}}(x)\, M(x)}{2\pi}\,\sigma_0(x) \prod_{\alpha \in \partial\mathsf{S}\setminus\partial\mathsf{A}} |x-\alpha|^{1/2} \prod_{\alpha\in\partial\mathsf{S}\cap\partial\mathsf{A}} |x-\alpha|^{-1/2}$$

for some polynomial $\sigma_0(x)$. Since $\mathrm{d}\mu_{\mathrm{eq}}/\mathrm{d}x$ is a density, $\sigma_0(x)$ has constant sign on S. If we require that it is nonnegative and has dominant coefficient ± 1, $\sigma_0(x)$ is uniquely determined. ∎

Definition 2.6. We speak of a (g_0+1)-cut regime when S is the disjoint union of g_0+1 segments, and we write

$$\mathsf{S} = \dot{\bigcup_{h=0}^{g_0}} \mathsf{S}_h, \qquad \mathsf{S}_h = [\alpha_h^-, \alpha_h^+].$$

We speak of an off-critical regime when $\sigma_0(x) = 1$. □

3 Concentration Around Equilibrium Measures

3.1 Large deviation for the support of the spectrum

As in [15, 16], we can prove the following.

Lemma 3.1. Assume Hypothesis 2.1. We have large deviation estimates: for any $\mathsf{F} \subseteq \mathsf{A}$ closed and $\Omega \subseteq \mathsf{A}$ open,

$$\limsup_{N\to\infty} \frac{1}{N} \ln \mu_{\mathcal{S}}[\exists i \quad \lambda_i \in \mathsf{F}] \leq \sup_{x\in\mathsf{F}} T_{\mathrm{eff}}(x),$$

$$\liminf_{N\to\infty} \frac{1}{N} \ln \mu_{\mathcal{S}}[\exists i \quad \lambda_i \in \Omega] \geq \sup_{x\in\Omega} T_{\mathrm{eff}}(x).$$

$-T_{\mathrm{eff}}(x)$ defined in (2.4) is thus the rate function. □

Proof. We simply outline the arguments, as the main points are discussed in details in [15, 16]. Let us consider a subset B of A_h and let $\Lambda = \{\lambda_1, \ldots, \lambda_{N_h}\}$ be the eigenvalues in

A_h. Then, as $\mu_{\mathcal{S}}$ is symmetric in the eigenvalues Λ,

$$\mu_{\mathcal{S}}[\lambda_1 \in \mathsf{B}] \leq \mu_{\mathcal{S}}[\exists i \quad \lambda_i \in \mathsf{B}] \leq N_h \mu_{\mathcal{S}}[\lambda_1 \in \mathsf{B}].$$

On the other hand

$$\mu_{\mathcal{S}}[\lambda_1 \in \mathsf{B}] = C_N \int \mathrm{d}\lambda_1 \int \mathrm{d}\tilde{\mu}_{\mathcal{S}}(\lambda_2, \cdots, \mathrm{d}\lambda_N)\, \mathbf{1}_{\mathsf{B}}(\lambda_1)$$
$$\times \prod_{j=2}^{N} |\lambda_1 - \lambda_j|^{\beta}\, \mathrm{e}^{\frac{N^{2-r}}{(r-1)!} \int T(\lambda_1, x_2, \ldots, x_r) \mathrm{d}L_{N-1}(x_2) \cdots \mathrm{d}L_{N-1}(x_r)},$$

where $\tilde{\mu}_{\mathcal{S}}$ is the law $\mu_{\mathcal{S}}$ with $N-1$ particles and potential $[(N-1)/N]^r\, T$ instead of T, $L_{N-1} = \frac{1}{N-1} \sum_{i=2}^{N} \delta_{\lambda_i}$, C_N is a normalization constant. Noting that under $\tilde{\mu}_{\mathcal{S}}$, L_{N-1} is very close to μ_{eq} with overwhelming probability with respect to the exponential scale, we see that if we neglect the singularity of the logarithm, the density of the law of λ_1 is approximately $C_N \exp\{N T_{\mathrm{eff}}(\lambda_1)\}$. The result then follows from Laplace method. The main point is therefore to deal with the singularity of the logarithm, which can be done exactly as in [15, 16]. ■

It is natural to supplement the conclusion of this lemma with an extra assumption:

Hypothesis 3.1 (Control of large deviations). $T_{\mathrm{eff}}(x) < 0$ outside $\mathsf{S} = \mathrm{supp}\, \mu_{\mathrm{eq}}$. □

Lemma 3.1 along with Hypothesis 3.1 allows one the simplification of the form of A. First of all, we can always assume the domain of integration A to be compact. Indeed, a noncompact domain A would only alter the answer obtained for the correlators or the partition function in the case of the compact domain $\mathsf{A}_{[M]} = \mathsf{A} \cap [-M\,;\, M]$ with M sufficiently large, by exponentially small in N terms. Secondly, when the control of large deviations holds, for the price of the same type of exponentially small in N corrections (see [16, Proposition 2.2 and 2.3] for more precise statements), we may restrict further the domain of integration to any fixed $\mathsf{A}' \subseteq \mathsf{A}$ such that $\mathsf{A}' \setminus \mathsf{S}$ is as small as desired. For instance, in the (g_0+1) cut regime, one can always restrict A to be a disjoint union of $(g+1) = (g_0+1)$ closed compact intervals $\mathsf{A}'_h \cap \mathsf{A}$, such that A'_h contains an open neighborhood of S_h in A_h for any $h \in [\![0\,;\, g_0]\!]$.

Therefore, from now on, we shall always assume A to be a disjoint union of $(g+1)$ closed compact intervals A_h, such that $\mathsf{S}_h \subseteq \mathsf{A}_h$ for any $h \in [\![0\,;\, g]\!]$ as above. In particular, we do not continue distinguishing g from g_0.

3.2 Pseudo-distance and adapted spaces

In view of showing concentration of the empirical measure around the equilibrium measure μ_{eq} in either of the two models, we add two assumptions:

Hypothesis 3.2 (Local strict convexity). For any $\nu \in \mathcal{M}^0(\mathsf{A})$,

$$\mathcal{Q}[\nu] = -\beta \int_{\mathsf{A}^2} \ln|x-y| \mathrm{d}\nu(x)\, \mathrm{d}\nu(y) - \int_{\mathsf{A}^r} \frac{T(x_1,\ldots,x_r)}{(r-2)!} \mathrm{d}\nu(x_1)\, \mathrm{d}\nu(x_2) \prod_{i=3}^{r} \mathrm{d}\mu_{\mathrm{eq}}(x_i) \tag{3.1}$$

is nonnegative, and vanishes iff $\nu = 0$. □

We observe that for any measure with zero mass:

$$\mathcal{Q}[\nu] = \beta\, \mathcal{Q}_C[\nu] + \mathcal{Q}_T[\nu], \quad \mathcal{Q}_C[\nu] = -\int_{\mathsf{A}^2} \ln|x_1 - x_2| \mathrm{d}\nu(x_1)\, \mathrm{d}\nu(x_2) = \int_0^{\infty} \frac{|\mathcal{F}[\nu](k)|^2}{k} \mathrm{d}k \in [0,+\infty], \tag{3.2}$$

whereas the other part $\mathcal{Q}_T$ is always finite since $T \in \mathcal{C}^0(\mathsf{A}^r)$ and A is compact. Therefore, $\mathcal{Q}[\nu]$ is well-defined and takes its values in $\mathbb{R} \cup \{+\infty\}$. Hypothesis 3.2 requires it to assume values in $[0,+\infty]$.

The model gives us a quadratic functional $\mathcal{Q}$, which defines a natural pseudo-distance. We will control deviations of the empirical measure from the equilibrium measure with respect to this pseudo-distance. This control does not imply a control of the multilinear statistics for arbitrary test functions. The following notion of adapted space of test functions summarizes our minimal needs in order to derive bounds for multilinear statistics.

Definition 3.2. A vector subspace $\mathcal{H} \subseteq \mathcal{C}^0(\mathbb{R})$ is then called an adapted space if there exists a norm $\|\cdot\|_{\mathcal{H}}$ on $\mathcal{H}$ and a continuous function χ_{A} which assumes values 1 on A and 0 outside of a compact, and such that

- there exists $c_0 > 0$ such that

$$\forall \nu \in \mathcal{M}^0(\mathsf{A}), \quad \forall \varphi \in \mathcal{H}, \qquad \left| \int_{\mathsf{A}} \varphi(x)\, \mathrm{d}\nu(x) \right| \leq c_0\, \mathcal{Q}^{1/2}[\nu]\, \|\varphi\|_{\mathcal{H}}.$$

- there exists $c_1 > 0$ and an integer $m \geq 0$ called the growth index such that, for any $k \in \mathbb{R}$, the function $\mathrm{e}_k(x) = \chi_{\mathsf{A}}(x)\, \mathrm{e}^{\mathrm{i}kx}$ belongs to $\mathcal{H}$ and one has $\|\mathrm{e}_k\|_{\mathcal{H}} \leq c_1\, (|k|^m + 1)$. □

We show in Section 4 how to construct an adapted space $\mathcal{H}$ provided Hypothesis 3.2 holds. We show that adapted spaces can be at least found among Sobolev spaces, but in that construction we are not optimal, and later on we do not use the particularities of Sobolev spaces but only the properties of Definition 3.2. Depending on the structure of $\mathcal{Q}$ in a given example, one may construct by a refined analysis larger spaces of test functions adapted to $\mathcal{Q}$.

We will often encounter multilinear statistics, and we use both Vasershtein norm and $\mathcal{Q}$ in their estimation. The following technical lemma appears useful.

Lemma 3.3. Let $l, l' \geq 0$ be integers, $l'' \leq l'$ be another integer, and $m' > (m-1)l'' + 2l' + l$. Then, given an adapted space $\mathcal{H}$, for any $\varphi \in \mathcal{C}^{m'}(\mathsf{A}^{l+l'})$, any $\nu_1, \ldots, \nu_k \in \mathcal{M}^0(\mathsf{A})$ and $\mu_1, \ldots, \mu_l \in \mathcal{M}^1(\mathsf{A})$, one has the bounds:

$$\left| \int_{\mathsf{A}^{l+l'}} \varphi(x_1, \ldots, x_{l+l'}) \prod_{i=1}^{l'} \mathrm{d}\nu_i(x_i) \prod_{j=l'+1}^{l+l'} \mathrm{d}\mu_j(x_j) \right| \leq c_0\, C_{l,l',l''}[\varphi] \prod_{i=1}^{l''} \mathcal{Q}^{1/2}[\nu_i] \prod_{i=l''+1}^{l'} \|\nu_i\| \tag{3.3}$$

for some finite nonnegative constant $C[\varphi]$. □

Proof. We may extend $\varphi \in \mathcal{C}^{m'}(\mathsf{A}^{l'+l})$ to a function $\tilde{\varphi} \in \mathcal{C}^{m'}(\mathbb{R}^{l'+l})$. Then, we may write

$$\begin{aligned} X[\varphi] &= \int_{\mathsf{A}^{l'+l}} \varphi(x_1, \ldots, x_{l'+l}) \prod_{i=1}^{l'} \mathrm{d}\nu_i(x_i) \prod_{j=l'+1}^{l'+l} \mathrm{d}\mu_j(x_j) \\ &= \int_{\mathbb{R}^{l'+l}} \tilde{\varphi}(x_1, \ldots, x_{l'+l}) \prod_{i=1}^{l'} \mathrm{d}\nu_i(x_i) \prod_{j=l'+1}^{l'+l} \mathrm{d}\mu_j(x_j) \\ &= \int_{\mathbb{R}^{l'+l}} \frac{\mathrm{d}^{l'+l}\boldsymbol{k}}{(2\pi)^{l'+l}}\, \mathcal{F}[\tilde{\varphi}](\boldsymbol{k}) \prod_{i=1}^{l''} \left(\int_{\mathbb{R}} \mathrm{e}_{k_i}(x_i)\, \mathrm{d}\nu_i(x_i) \right) \prod_{i=l''+1}^{l'} \left(\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k_j x_j}\, \mathrm{d}\nu(x_j) \right) \prod_{j=l'+1}^{l'+l} \\ &\quad \times \left(\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}k_j x_j}\, \mathrm{d}\mu_j(x_j) \right), \end{aligned}$$

where we could introduce the function $\mathrm{e}_k(x) = \chi_{\mathsf{A}}(x)\, \mathrm{e}^{\mathrm{i}kx}$ since ν_i are supported on A. We bound the first group of integrals for $i \in [\![1\,;\, l'']\!]$ by $c_0\, \mathcal{Q}^{1/2}[\nu_i] \cdot \|\mathrm{e}_{k_i}\|_{\mathcal{H}}$, the second group for $i \in [\![l''+1\,;\, l']\!]$ with the Vasershtein norm by $|k_i| \cdot \|\nu_i\|$, and the remaining by the obvious bound 1 since μ_j are probability measures.

$$|X[\varphi]| \leq c_0^{l''} \left(\prod_{i=1}^{l''} \mathcal{Q}^{1/2}[\nu_i] \prod_{i=l''+1}^{l'} \|\nu_i\| \right) \int_{\mathbb{R}^{l'+l}} \frac{\mathrm{d}^{l+l'}\boldsymbol{k}}{(2\pi)^{l+l'}} |\mathcal{F}[\tilde{\varphi}](\boldsymbol{k})| \prod_{i=1}^{l''} \|\mathrm{e}_{k_i}\|_{\mathcal{H}} \prod_{i=l''+1}^{l'} |k_i|.$$

Since $\tilde{\varphi}$ is $\mathcal{C}^{m'}$, the integral in the right-hand side converges at least for $m' > l + 2l' + (m-1)l''$, and gives the constant $C_{l,l',l''}[\varphi]$ in (3.3). ■

3.3 Concentration results

The next paragraphs are devoted to the proof of:

Theorem 3.4. Assume Hypothesis 2.1, 3.1, and 3.2, an adapted space with growth index m, and $T \in \mathcal{C}^{m'}(\mathsf{A}^r)$ with $m' > 2m + r + 1$. Denote μ_{eq} the equilibrium measure in one of the two models (1.1) or (1.3) and $\tilde{L}_N^u$ the regularization of L_N defined in Section 3.4. There exists constants $c > 0$ and C, C', such that

$$\mu_{\mathcal{S}}[\mathcal{Q}[\tilde{L}_N^u - \mu_{\mathrm{eq}}]^{\frac{1}{2}} \geq t] \leq \exp\left\{C\, N \ln N - \frac{N^2 t^2}{4}\right\} + C' \exp\{-cN^2\}. \qquad \square$$

As in [15], we easily derive in Section 3.6.

Corollary 3.5. Under the same assumptions, let $b > 0$. There exists finite constants C, C' and $c > 0$ such that, for N large enough, for any $\varphi \in \mathcal{H} \cap \mathrm{Ho}_b(\mathsf{A})$, we have

$$\mu_{\mathcal{S}}\left[\left|\int \varphi(\xi)\, \mathrm{d}(L_N - \mu_{\mathrm{eq}})(\xi)\right| \geq \frac{c\kappa_b[\varphi]}{N^{2b}} + c_0 t \|\varphi\|_{\mathcal{H}}\right] \leq \exp\left\{C\, N \ln N - \frac{1}{4} N^2 t^2\right\} + C' \exp\{-c' N^2\}.$$

$\square$

As a special case, we can obtain a rough a priori control on the correlators:

Corollary 3.6. Let $w_N = \sqrt{N \ln N}$ and $\psi_x(\xi) = \mathbf{1}_{\mathsf{A}}(\xi)/(x - \xi)$. For N large enough, and there exists $c, c_1 > 0$ such that

$$|W_1(x) - N W_{\mathrm{eq}}(x)| \leq \frac{c}{N d(x, \mathsf{A})^2} + c_1 \|\psi_x\|_{\mathcal{H}} w_N.$$

Similarly, for any $n \geq 2$ and N large enough, there exists $c_n > 0$ such that

$$|W_n(x_1, \ldots, x_n)| \leq c_n \prod_{i=1}^{n} \left[\frac{c}{N d(x, \mathsf{A})^2} + c_1 \|\psi_x\|_{\mathcal{H}} w_N\right]. \tag{3.4}$$

$\square$

We recall that we are in a $(g+1)$-cut regime with $g \geq 1$ and that A_h is a partition of A in $(g+1)$ segments so that A_h is a neighborhood of S_h in A. For any configuration $(\lambda_1, \ldots, \lambda_N) \in \mathsf{A}^N$, we denote $\widetilde{N}_h$ the number of λ_i's in A_h, and $\widetilde{\mathbf{N}} = (\widetilde{N}_0, \ldots, \widetilde{N}_g)$. Let

$$N\boldsymbol{\epsilon}^{\star} = (N\epsilon_0^{\star}, \ldots, N\epsilon_g^{\star}) \quad \text{with } \epsilon_{\mathrm{eq}}^{\star} = \int_{\mathsf{S}_h} \mathrm{d}\mu_{\mathrm{eq}}(\xi).$$

We can derive an estimate for large deviations of $\mathbf{N}'$ away from $N\boldsymbol{\epsilon}^{\star}$:

Corollary 3.7. Assume a $(g+1)$-cut regime with $g \geq 1$, and let $\widetilde{\mathbf{N}}$ be as above. Then, there exists a positive constant C such that, for N large enough and uniformly in t:

$$\mu_{\mathsf{A}^N}[|\widetilde{\mathbf{N}} - N\boldsymbol{\epsilon}^\star| > t\sqrt{N \ln N}] \leq \exp\{N \ln N(C - t^2)\}. \qquad \square$$

3.4 Regularization of L_N

We cannot compare directly L_N to μ_{eq} with $\mathcal{Q}$, because of the logarithmic singularity in $\mathcal{Q}_C$ and the atoms in L_N. Following an idea of Maïda and Maurel-Segala [42], we associate to any configurations of points $\lambda_1 < \cdots < \lambda_N$ in A, another configuration $\tilde{\lambda}_1 < \cdots < \tilde{\lambda}_N$ by the formula:

$$\tilde{\lambda}_1 = \lambda_1, \quad \tilde{\lambda}_{i+1} = \tilde{\lambda}_i + \max(\lambda_{i+1} - \lambda_i, N^{-3}).$$

It has the following properties:

$$\forall i \neq j, \quad |\tilde{\lambda}_i - \tilde{\lambda}_j| \geq N^{-3}, \quad |\lambda_i - \lambda_j| \leq |\tilde{\lambda}_i - \tilde{\lambda}_j|, \quad |\tilde{\lambda}_i - \lambda_i| \leq (i-1)N^{-3}.$$

Let us denote $\tilde{L}_N = \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{\lambda}_i}$ the new counting measure, and $\tilde{L}_N^u$ its convolution with the uniform measure on $[0, N^{-7/2}]$. Let us define a regularized version of the energy functional $\mathcal{E}^\Delta = (\beta/2)\,\mathcal{E}_C^\Delta + \mathcal{E}_T$ with:

$$\mathcal{E}_C^\Delta[\mu] = -\int_{x_1 \neq x_2} \ln|x_1 - x_2| \mathrm{d}\mu(x_1)\,\mathrm{d}\mu(x_2),$$

$$\mathcal{E}_T[\mu] = -\int_{\mathsf{A}^r} \frac{T_r(x_1, \ldots, x_r)}{r!} \prod_{i=1}^r \mathrm{d}\mu(x_i).$$

As in [42] (see also [15]), we have the following.

Lemma 3.8.

$$\mathcal{E}_C^\Delta[L_N] - \mathcal{E}_C^\Delta[\tilde{L}_N^u] \geq \frac{1}{3N^4} - \frac{7\ln N}{2N}. \qquad \square$$

It is then straightforward to deduce the following.

Corollary 3.9. There exists constants $c, c' > 0$ such that

$$\mathcal{E}^\Delta[L_N] - \mathcal{E}[\tilde{L}_N^u] \geq c\frac{\ln N}{N} + c'\frac{\kappa_1[T]}{N^3}. \qquad \square$$

We also have the following:

Lemma 3.10. There exists $c>0$ such that, for any $f\in \mathrm{Ho}_b(\mathsf{A})$, we have

$$\left|\int_{\mathsf{A}} f(\xi)\,\mathrm{d}(L_N-\tilde{L}_N^u)(\xi)\right| \leq \frac{c\kappa_b[f]}{N^{2b}}. \qquad \square$$

3.5 Concentration of $\tilde{L}_N^u$ (Proof of Theorem 3.4)

We would like to estimate the probability of large deviations of $\tilde{L}_N^u$ from the equilibrium measure μ_{eq}. We first need a lower bound on $Z_{\mathcal{S}}$ similar to that of [7], obtained by localizing the ordered eigenvalues at distance N^{-3} of the quantiles λ_i^{cl} of the equilibrium measure μ_{eq}, which are defined by

$$\lambda_i^{\mathrm{cl}}=\inf\left\{x\in\mathsf{A},\ \int_{-\infty}^{x}\mathrm{d}\mu_{\mathrm{eq}}(x)\geq i/N\right\}.$$

Lemma 3.11. Assume Hypothesis 3.2 with $T\in\mathcal{C}^3(\mathsf{A})$. Then, there exists a finite constant c so that

$$Z_{\mathcal{S}}\geq \exp\{-cN\ln N-N^2\mathcal{E}[\mu_{\mathrm{eq}}]\}. \qquad \square$$

Proof. According to Lemma 2.4, $T\in\mathcal{C}^3(\mathsf{A})$ implies that μ_{eq} has a $\mathcal{C}^1$ density in the interior of S, and behaves at most like the inverse of a squareroot at $\partial\mathsf{S}$. This ensures the existence of $c_0>0$ such that

$$|\lambda_{i+1}^{\mathrm{cl}}-\lambda_i^{\mathrm{cl}}|\geq c_0N^{-2} \tag{3.5}$$

for any $i\in[\![0\,;\,N]\!]$, where by convention $\lambda_0^{\mathrm{cl}}=\min\{x\,:\,x\in\mathsf{S}\}$ and $\lambda_{N+1}^{\mathrm{cl}}=\max\{x\,:\,x\in\mathsf{A}\}$. The proof of the lower bound for $Z_{\mathcal{S}}$ is similar to [15], we redo it here for sake of being self-contained. It can be obtained by restricting the integration over the configurations $\{\boldsymbol{\lambda}\in\mathcal{S},\quad |\lambda_i-\lambda_i^{\mathrm{cl}}|\leq N^{-3}\}$, where $\mathcal{S}=\mathsf{A}^N$ or $=\mathsf{A}_{\mathsf{N}}$ depending on the model. For any such $\boldsymbol{\lambda}$, one has

$$|\lambda_i-\lambda_j|\geq|\lambda_i^{cl}-\lambda_j^{cl}|\left(1-\frac{2}{c_0N}\right)\qquad |T(\lambda_{i_1},\ldots,\lambda_{i_r})-T(\lambda_{i_1}^{\mathrm{cl}},\ldots,\lambda_{i_r}^{\mathrm{cl}})|\leq\frac{\kappa_1[T]}{N^3}$$

and this implies that

$$\begin{aligned} Z_N\geq{}&(1-N^{-1})^{N(N-1)\beta/2}\exp\left(\frac{\kappa_1[T]}{r!\,N}\right)N^{-3N}\prod_{1\leq i<j\leq N}|\lambda_i^{\mathrm{cl}}-\lambda_j^{\mathrm{cl}}|^{\beta}\exp\\ &\times\left(\frac{N^{2-r}}{r!}\sum_{1\leq i_1,\ldots,i_r\leq N}T(\lambda_{i_1}^{\mathrm{cl}},\ldots,\lambda_{i_r}^{\mathrm{cl}})\right). \end{aligned} \tag{3.6}$$

Then, for any i, j such that $j+1 \leq i-1$, we have, by monotonicity of the logarithm:

$$\ln|\lambda_i^{\mathrm{cl}} - \lambda_j^{\mathrm{cl}}| \geq N^2 \int_{\lambda_{i-1}^{\mathrm{cl}}}^{\lambda_i^{\mathrm{cl}}} \int_{\lambda_j^{\mathrm{cl}}}^{\lambda_{j+1}^{\mathrm{cl}}} \ln|\xi_1 - \xi_2| \, \mathrm{d}\mu_{\mathrm{eq}}(\xi_1) \, \mathrm{d}\mu_{\mathrm{eq}}(\xi_2).$$

For the remaining pairs $\{i, j\}$, we rather use the lower bound (3.5), and we find after summing over pairs:

$$\beta \sum_{i<j} \ln|\lambda_i^{\mathrm{cl}} - \lambda_j^{\mathrm{cl}}| \geq -\beta N^2 \mathcal{E}_C[\mu_{\mathrm{eq}}] + c_1 \, N \ln N. \tag{3.7}$$

If $\varphi : \mathsf{A} \to \mathbb{R}$ is a function with finite total variation $\mathrm{TV}[\varphi]$, we can always decompose it as the difference of two increasing functions, the total variation of each of them being $\mathrm{TV}[\varphi]$. And, if $\varphi_>$ is an increasing function:

$$\frac{1}{N} \sum_{i=0}^{N-1} \varphi_>(\lambda_i^{\mathrm{cl}}) \leq \int_{\mathsf{A}} \varphi_>(\xi) \, \mathrm{d}\mu_{\mathrm{eq}}(\xi) \leq \frac{1}{N} \sum_{i=1}^{N} \varphi_>(\lambda_i^{\mathrm{cl}}). \tag{3.8}$$

Therefore, we deduce that

$$\left| \frac{1}{N} \sum_{i=1}^{N} \varphi(\lambda_i^{\mathrm{cl}}) - \int_{\mathsf{A}} \varphi(\xi) \, \mathrm{d}\mu_{\mathrm{eq}}(\xi) \right| \leq \frac{2\mathrm{TV}[f]}{N}.$$

This can be generalized for functions defined in A^r by recursion, and we apply the result to T, which is $\mathcal{C}^1$, hence is of bounded total variation with $\mathrm{TV}[T] \leq \ell(\mathsf{A}) \, \kappa_1[T]$:

$$\left| \frac{1}{N^r} \sum_{1 \leq i_1, \ldots, i_r \leq N} T(\lambda_{i_1}^{\mathrm{cl}}, \ldots, \lambda_{i_r}^{\mathrm{cl}}) - \int_{\mathsf{A}^r} T(\xi_1, \ldots, \xi_r) \prod_{i=1}^{r} \mathrm{d}\mu_{\mathrm{eq}}(\xi_i) \right| \leq \frac{c_2 \, \kappa_1[T]}{N}. \tag{3.9}$$

Combining (3.7)–(3.9) with (3.6), we find the desired result. ■

Now, the density of probability measure in either of the models (1.1) or (1.3) can be written

$$\mathrm{d}\mu_{\mathcal{S}} = \left[\prod_{i \in \mathcal{I}_{\mathcal{S}}} \mathrm{d}\lambda_i \right] \exp\{-N^2 \mathcal{E}^{\Delta}[L_N]\}.$$

With the comparison of Corollary 3.9, we find that, for N large enough:

$$\mathrm{d}\mu_{\mathcal{S}} \leq \left[\prod_{i \in \mathcal{I}_{\mathcal{S}}} \mathrm{d}\lambda_i \right] \exp\{cN \ln N - N^2 \mathcal{E}[\tilde{L}_N^u]\}.$$

We can then compare the value of the energy functional at $\tilde{L}_N^u$ and μ_{eq} by a Taylor-Lagrange formula to order 3. The existence of the order 3 Fréchet derivative of $\mathcal{E}$ is here guaranteed since $\mathcal{E}$ is polynomial. Setting $\nu_N = \tilde{L}_N^u - \mu_{\mathrm{eq}}$, we find

$$\mathcal{E}[\tilde{L}_N^u] = \mathcal{E}[\mu_{\mathrm{eq}}] - \int_{\mathsf{A}} T_{\mathrm{eff}}(\xi)\,\mathrm{d}\nu_N(\xi) + \frac{1}{2}\mathcal{Q}[\nu_N] + \mathcal{R}_3[\nu_N] \tag{3.10}$$

and we compute from the definition of $\mathcal{E}$:

$$\begin{aligned} \mathcal{R}_3[\nu_N] &= \int_0^1 \frac{\mathrm{d}t\,(1-t)^2}{2}\mathcal{E}^{(3)}[(1-t)\mu_{\mathrm{eq}} + t\tilde{L}_N^u]\cdot(\nu_N,\nu_N,\nu_N), \\ \mathcal{E}^{(3)}[\mu]\cdot(\nu^1,\nu^2,\nu^3) &= -\int_{\mathsf{A}^{r-3}} \frac{T(\xi_1,\ldots,\xi_r)}{(r-3)!}\prod_{i=1}^{3}\mathrm{d}\nu^i(\xi_i)\prod_{j=4}^{r}\mathrm{d}\mu(\xi_j). \end{aligned} \tag{3.11}$$

Since the ν^i have zero masses, $\mathcal{E}^{(3)}[\mu]\cdot(\nu^1,\nu^2,\nu^3)$ vanishes if there are only 1 or 2 body interactions. In other words, the remainder $\mathcal{R}_3[\nu]$ is only present in the case where there are at least $r \geq 3$ body interactions. Since $T_{\mathrm{eff}}(x)$ is nonpositive and vanishes μ_{eq}-almost surely, we have for the linear term:

$$-\int_{\mathsf{A}} T_{\mathrm{eff}}(\xi)\,\mathrm{d}\nu_N(\xi) = -\int_{\mathsf{A}} T_{\mathrm{eff}}(\xi)\,\mathrm{d}\tilde{L}_N^u(\xi) \geq 0.$$

Therefore, combining with the lower bound of Lemma 3.11, we find

$$\frac{\mathrm{d}\mu_{\mathcal{S}}}{Z_{\mathcal{S}}} \leq \left[\prod_{i=1}^{N}\mathrm{d}\lambda_i\right]\exp\left\{cN\ln N - \frac{N^2}{2}(\mathcal{Q}[\nu_N] + 2\mathcal{R}_3[\nu_N])\right\}. \tag{3.12}$$

By using Lemma 3.3 with $(l,l',l'') = (r-3,3,2)$ and the fact that T is $\mathcal{C}^{m'}$ for $m' > 2m + r + 1$, we obtain, for some T-dependent constant C:

$$|\mathcal{R}_3[\nu_N]| \leq C[T]\,\mathcal{Q}[\nu_N]\,\|\nu_N\|.$$

Note that, in the above bound, we have used the existence of an adapted space, as is inferred in Section 4. So, if we restrict to configurations realizing the event $\{\|\nu_N\| \leq \varepsilon\}$ for some fixed but small enough $\varepsilon > 0$, we have $|\mathcal{R}_3[\nu_N]| \leq \mathcal{Q}[\nu_N]/4$. Integrating (3.12) on this event, we find

$$\mu_{\mathcal{S}}[\{\mathcal{Q}^{1/2}[\nu_N] \geq t\} \cap \{\|\nu_N\| \leq \varepsilon\}] \leq \exp\left\{C\,N\ln N - \frac{N^2t^2}{4}\right\}.$$

On the other end, since $\{\nu \in \mathcal{M}^0(\mathsf{A}),\ \|\nu - \mu_{\mathrm{eq}}\| \geq \varepsilon\}$ is a closed set which does not contain μ_{eq}, and since $L_N - \tilde{L}_N^u$ converges to zero uniformly for the weak-* topology as N goes to infinity, uniformly on configurations of λ_i's according to Lemma 3.10, we find by the large deviation principle of Theorem 2.1 that there exists a positive constant c_ε such that

$$\mu_{\mathcal{S}}[\{\|\nu_N\| \geq \varepsilon\}] \leq \mathrm{e}^{-c_\varepsilon N^2}.$$

This concludes the proof of Theorem 3.4.

3.6 Proof of Corollaries 3.5–3.7

Proof of Corollary 3.5. We decompose

$$\int_{\mathsf{A}} \varphi(\xi)\,\mathrm{d}(L_N - \mu_{\mathrm{eq}})(\xi) = \int_{\mathsf{A}} \varphi(\xi)\,\mathrm{d}\nu_N(\xi) + \int_{\mathsf{A}} \varphi(\xi)\,\mathrm{d}(L_N - \tilde{L}_N^u)(\xi).$$

As shown in Section 4, Hypothesis 3.2 ensures the existence of an adapted space, *viz.*

$$\left|\int_{\mathsf{A}} \varphi(\xi)\,\mathrm{d}\nu_N(\xi)\right| \leq c_0 \mathcal{Q}^{1/2}[\nu_N]\|\varphi\|_{\mathcal{H}} \quad \text{with } \nu_N = \widetilde{L}_N^u - \mu_{\mathrm{eq}}$$

and the second term is bounded by Lemma 3.10. Therefore,

$$\mu_{\mathcal{S}}\left[\left|\int_{\mathsf{A}} \varphi(\xi)\,\mathrm{d}(L_N - \mu_{\mathrm{eq}})(\xi)\right| \geq \frac{c\kappa_b[\varphi]}{N^{2b}} + c_0 t\|\varphi\|_{\mathcal{H}}\right] \leq \mu_{\mathcal{S}}[\mathcal{Q}[\nu_N]^{1/2} \geq t] \tag{3.13}$$

so that the conclusion follows from Theorem 3.4. ■

Proof of Corollary 3.6. We have set $\psi_x(\xi) = \mathbf{1}_{\mathsf{A}}(\xi)/(x-\xi)$, and we have

$$\begin{aligned} N^{-1}\,W_1(x) - W_{\mathrm{eq}}(x) &= \mu_{\mathcal{S}}\left[\int \psi_x(\xi)\mathrm{d}(L_N - \mathrm{d}\mu_{\mathrm{eq}})(\xi)\right] \\ &= \mu_{\mathcal{S}}\left[\int \psi_x(\xi)\,\mathrm{d}\nu_N(\xi) + \int \psi_x(\xi)\,\mathrm{d}(L_N - \tilde{L}_N^u)(\xi)\right], \end{aligned} \tag{3.14}$$

where ν_N is as defined in (3.13). The function ψ_x is Lipschitz with constant $\kappa_1[\psi_x] = d^{-2}(x, \mathsf{A})$. In virtue of Lemma 3.10, it follows that

$$\left|\int_{\mathsf{A}} \psi_x(\xi)\,\mathrm{d}(L_N - \tilde{L}_N^u)(\xi)\right| \leq \frac{c}{N^2 d^2(x, \mathsf{A})}. \tag{3.15}$$

In what concerns the first term in (3.14), we set $t_N = 4\sqrt{|C|\ln N/N}$ in Theorem 3.4 to find, for N large enough:

$$\begin{aligned} \mu_{\mathcal{S}}\left[\left|\int \psi_x(\xi)\,\mathrm{d}(\tilde{L}_N^u - \mu_{\mathrm{eq}})(\xi)\right|\right] &\leq c_0 t_N\|\psi_x\|_{\mathcal{H}} + \frac{2\mu_{\mathcal{S}}[\mathcal{Q}^{1/2}[\nu_N] \geq t_N]}{d(x, \mathsf{A})} \\ &\leq c_0 t_N\|\psi_x\|_{\mathcal{H}} + \frac{c''\,\mathrm{e}^{-3|C|\,N\ln N}}{d(x, \mathsf{A})}. \end{aligned} \tag{3.16}$$

And, for x bounded independently of N, and N large enough, the last term in (3.16) is $o(N^{-2}d^{-2}(x,\mathsf{A}))$. So, combining (3.15) and (3.16), we obtain the existence of constants $c, c_1 > 0$ such that

$$|N^{-1}W_1(x) - W_{\mathrm{eq}}(x)| \leq \frac{c}{N^2 d^2(x,\mathsf{A})} + c_1\,\|\psi_x\|_{\mathcal{H}}\sqrt{\frac{\ln N}{N}}.$$

Multiplying by N entails the result.

For $n \geq 2$, $N^{-n}W_n(x_1,\ldots,x_n)$ is the $\mu_{\mathcal{S}}$ expectation value of a homogeneous polynomial of degree n having a partial degree at most 1 in the quantities $\int_{\mathsf{A}}\psi_{x_i}(\xi_i)\,\mathrm{d}(L_N-\mu_{\mathrm{eq}})(\xi_i)$ and $\mu_{\mathcal{S}}[\int_{\mathsf{A}}\psi_{x_i}(\xi_i)\,\mathrm{d}(L_N-\mu_{\mathrm{eq}})(\xi_i)]$. The coefficients of this polynomial are independent of N. A similar reasoning shows that, for any subset I of $[\![1\,;\,n]\!]$, and if x_i is bounded independently of N and N is large enough:

$$\mu_{\mathcal{S}}\left[\prod_{i\in I}\int_{\mathsf{A}}\psi_{x_i}(\xi_i)\,\mathrm{d}(L_N-\mu_{\mathrm{eq}})(\xi_i)\right] \leq \prod_{i\in I}\left[\frac{c}{N^2 d^2(x_i,\mathsf{A})} + c_1\,\|\psi_x\|_{\mathcal{H}}\sqrt{\frac{\ln N}{N}}\right].$$

Multiplying back by $N^{|I|}$ gives the desired result. ■

Proof of Corollary 3.7. We have $\widetilde{N}_h - N\epsilon_h^{\star} = N\int_{\mathsf{A}}\mathbf{1}_{\mathsf{A}_h}(\xi)\,\mathrm{d}(L_N-\mu_{\mathrm{eq}})(\xi)$. Let us choose $(\mathsf{A}'_h)_{0\leq h\leq g}$ to be a collection of pairwise disjoint segments, such that A'_h is a neighborhood of S_h in A_h, and denote $\mathsf{A}' = \bigcup_{h=0}^{g} A'_h$. We would like to consider the model $\mu_{\mathcal{S}}$ or $\mu_{\mathcal{S}'}$ where eigenvalues are integrated over A, or over A'. More precisely,

- in the unconstrained model, $\mathcal{S} = \mathsf{A}^N$ and $\mathcal{S}' = (\mathsf{A}')^N$;
- in model with fixed filling fractions subordinate to the vector $\mathbf{N} = (N_0,\ldots,N_g)$, A is already partitioned as $\bigcup_{h=0}^{g}\mathsf{A}_h$, and this induces a new partition $\mathsf{A}' = (\mathsf{A}_0\cap\mathsf{A}',\ldots,\mathsf{A}_g\cap\mathsf{A}')$, and we define $\mathcal{S} = \mathsf{A}_{\mathbf{N}}$ and $\mathcal{S}' = \mathsf{A}'_{\mathbf{N}}$.

In either case, we stress that $\widetilde{N}_h = \widetilde{N}_h^{\mathcal{S}}$ is computed for the model $\mathcal{S}$. We also observe that W_{eq} is the same in the models $\mu_{\mathcal{S}}$ and $\mu_{\mathcal{S}'}$. We write

$$\widetilde{N}_h - N\epsilon_h^{\star} = \oint_{\Gamma'_h}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\left(\sum_{i=1}^{N}\frac{1}{\xi-\lambda_i} - N\,W_{\mathrm{eq}}(\xi)\right),$$

where Γ'_h is a contour surrounding A_h, and observe that if there is no eigenvalues in $\mathsf{A}\setminus\mathring{\mathsf{A}}'$ the function $\varphi(x) = \mathbf{1}_{\mathsf{A}}(x)\oint\frac{\mathrm{d}\xi}{2\mathrm{i}\pi(\xi-x)}$ with Γ'_h as integration contour is Lipschitz and with finite $\|\cdot\|_{\mathcal{H}}$ norm. Therefore, we can apply Corollary 3.5 and the large deviations of

Lemma 3.1 to deduce that if $\Lambda = \{\exists\, \boldsymbol{i},\ \lambda_{\boldsymbol{i}} \in \mathsf{B} = \mathsf{A} \setminus \mathring{\mathsf{A}}'\}$,

$$\mu_{\mathcal{S}}[|\widetilde{N}_h - N\epsilon_h^{\star}| \geq t] \leq \mu_{\mathcal{S}}[\Lambda] + \mu_{\mathcal{S}}\left[\left\{\left|N\int_{\mathsf{A}} \varphi(\xi)\,\mathrm{d}(L_N - \mu_{\mathrm{eq}})(\xi)\right| \geq t\right\} \cap \Lambda^c\right]$$
$$\leq \exp\left\{N \sup_{x \in \mathsf{A}\setminus\mathring{\mathsf{A}}'} T_{\mathrm{eff}}(x)\right\} + \exp\left\{C N \ln N - \frac{1}{4}N^2 s^2\right\} + C' \exp\{-c' N^2\}, \tag{3.17}$$

where we have set $t = N(c\kappa_b[\varphi]/N^{2b} + c_0 s\|\varphi\|_{\mathcal{H}})$. By construction of A'_h and Hypothesis 3.1, the sup is negative. Therefore, after rescaling t, we obtain the existence of $C' > 0$ so that, for N large enough:

$$\mu_{\mathcal{S}}[|\widetilde{N}_h - N\epsilon_h^{\star}| \geq t\sqrt{N \ln N}] \leq \mathrm{e}^{N \ln N (C' - t^2)}. \qquad \blacksquare$$

4 Construction of Adapted Spaces

4.1 Example: translation invariant two-body interaction

The construction of adapted spaces as described in Definition 3.2 can be easily addressed in the case of two-body interactions ($r = 2$) depending only of the distance. We shall consider in this paragraph:

$$T(x, y) = u(x - y) + \frac{v(x) + v(y)}{2}.$$

The functional $\mathcal{Q}$ introduced in Section 3.2 takes the form:

$$\mathcal{Q}[\nu] = \int q(x - y)\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y) = \int_{\mathbb{R}} \mathcal{F}[q](k)\,|\mathcal{F}[\nu](k)|^2\,\mathrm{d}k, \quad \text{with } q(x) = -\beta\,\ln|x| - u(x).$$

Lemma 4.1. Assume $u \in \mathcal{C}^1(\mathbb{R})$ is such that $k \mapsto \mathcal{F}[q](k)$, with $\mathcal{F}[q]$ understood in the sense of distributions is continuous on $\mathbb{R}^*$ and positive everywhere, and $|k|^b\,\mathcal{F}[q](k) \geq c$ for some $c > 0$ and $b \geq 1$ when $|k|$ is small enough. Then, $\mathcal{H} = \iota_{\mathsf{A}}(H^{b/2}(\mathbb{R}))$ equipped with its Sobolev norm, and growth index $m = 0$ is an adapted space. Here, ι_{A} is the operation of restriction of the domain of definition to A. $\square$

Proof. Let $\nu \in \mathcal{M}^0(\mathsf{A})$ and $\varphi \in H^{b/2}(\mathbb{R})$. Then

$$\begin{aligned}\left|\int_{\mathsf{A}} \varphi(x)\,\mathrm{d}\nu(x)\right| &= \left|\int_{\mathbb{R}} \mathcal{F}[\varphi](-k)\cdot\mathcal{F}[\nu](k)\,\mathrm{d}k\right| \\ &\leq \left(\int_{\mathbb{R}} \mathcal{F}[q](k)\,|\mathcal{F}[\nu](k)|^2\,\mathrm{d}k\right)^{1/2}\left(\int_{\mathbb{R}} \frac{|\mathcal{F}[\varphi](k)|^2}{\mathcal{F}[q](k)}\,\mathrm{d}k\right)^{1/2} \\ &\leq \mathcal{Q}^{1/2}[\nu]\left(\int_{\mathbb{R}} |k|^b\,|\mathcal{F}[\varphi](k)|^2\,\frac{\mathrm{d}k}{|k|^b\,\mathcal{F}[q](k)}\right)^{1/2}.\end{aligned}$$

We observe that $|k|^b\,\mathcal{F}[q](k) = \beta|k|^{b-1} - |k|^b\,\mathcal{F}[u](k)$. Since u is $\mathcal{C}^1$ and $b \geq 1$, we have $|k|^b\,\mathcal{F}[q](k) \geq 1$ when $|k| \to \infty$. And, by assumption, we have $|k|^b\,\mathcal{F}[q](k) > c$ when $|k|$ is small enough. Since, furthermore, $\mathcal{F}[q](k) > 0$, there exists $c' > 0$ so that $|k|^b\,\mathcal{F}[q](k) > c'$ for any k, and

$$\left|\int_{\mathsf{A}} \varphi(x)\,\mathrm{d}\nu(x)\right| \leq \frac{1}{\sqrt{c'}}\,\mathcal{Q}^{1/2}[\nu]\,\|\varphi\|_{H^{b/2}}.$$

Eventually, for some $\varepsilon > 0$, we can always find a function $\chi_{\mathsf{A}} \in \mathcal{C}^{(b+1+\varepsilon)/2}(\mathbb{R})$ with compact support and assuming values 1 on A. Then

$$\mathcal{F}[\mathrm{e}_{k_0}] = \mathcal{F}[\chi_{\mathsf{A}}](k+k_0), \qquad \lim_{|k|\to\infty} |k|^{(b+1+\varepsilon)/2}\,\mathcal{F}[\chi_{\mathsf{A}}](k) = 0.$$

Therefore, $\|\mathrm{e}_k\|_{H^{b/2}}$ remains bounded when $k \in \mathbb{R}$. ■

In many applications, $\mathcal{F}[q](k)$ can be explicitly computed. In such cases, it is relatively easy to check its positivity and extract its $k \to 0$ behavior, which is related to the growth of $q(x)$ when $|x| \to \infty$.

4.2 Space adapted to $\mathcal{Q}$

We now present a general construction of adapted spaces thanks to functional analysis arguments. It applies in particular to 2-body interactions which are not translation invariant (i.e., do not have a simple representation in Fourier space) or to more general r-body interactions.

Theorem 4.2. Let $q > 2$ and assume that $\mathcal{Q}$ defined as in (3.1) satisfies to Hypothesis 3.2. Then, the space $W^{1;q}(\mathsf{A})$ equipped with its Sobolev norm $\|\cdot\|_q$ is an adapted space. It has a growth index $m = 1$. □

Proof. Let $2 > p \geq 1$ be conjugate to q (viz. $1/p + 1/q = 1$) and define the integral operator $\underline{\mathcal{T}} : L_0^p(\mathsf{A}) \mapsto L_0^p(\mathsf{A})$ with

$$L_0^p(\mathsf{A}) = \left\{ f \in L^p(\mathsf{A}) \ : \ \int_{\mathsf{A}} f(x)\,\mathrm{d}x = 0 \right\}$$

by

$$\underline{\mathcal{T}}[f](x) = -\beta \int_{\mathsf{A}} \ln|x-y|\, f(y)\,\mathrm{d}y - \int_{\mathsf{A}} \tau(x,y)\, f(y)\,\mathrm{d}y + \int_{\mathsf{A}^2} (\beta \ln|x-y| + \tau(x,y)) f(y)\,\mathrm{d}y\,\mathrm{d}x,$$

where

$$\tau(x,y) = \int_{\mathsf{A}^{r-2}} \frac{T(x,y,x_1,\ldots,x_{r-2})}{(r-2)!} \prod_{a=1}^{r-2} \mathrm{d}\mu_{\mathrm{eq}}(x_a).$$

The functional $\mathcal{Q}$ is strictly positive definite by Hypothesis 3.2. Given $\varphi \in L_0^r(\mathsf{A})$, $\varphi \neq 0$ and $r > 1$, it is clear from the fact that A is compact that $\underline{\mathcal{T}}[\varphi] \in L_0^r(\mathsf{A})$ as well. Hence,

$$\int_A \varphi(x) \underline{\mathcal{T}}[\varphi](x)\,\mathrm{d}x = \mathcal{Q}[\nu_\varphi] > 0 \quad \text{with } \mathrm{d}\nu_\varphi = \varphi(x)\,\mathrm{d}x \in \mathcal{M}^0(\mathsf{A}).$$

In particular, $\underline{\mathcal{T}}$ defines a continuous positive definite self-adjoint operator on $L_0^2(\mathsf{A})$. By functional calculus on its spectrum, one can define any power of the operator $\underline{\mathcal{T}}$ as an operator on $L_0^2(\mathsf{A})$. Further, observe that given $\varphi \in L_0^2(\mathsf{A})$ and any $1 \leq p < 2$

$$\int_{\mathsf{A}} \underline{\mathcal{T}}^{\frac{1}{2}}[\varphi](x)\, \underline{\mathcal{T}}^{\frac{1}{2}}[\varphi](x)\,\mathrm{d}x = \int_{\mathsf{A}} \varphi(x)\, \underline{\mathcal{T}}[\varphi](x)\,\mathrm{d}x \leq \|\varphi\|_{L^p(\mathsf{A})}\, \|\underline{\mathcal{T}}[\varphi]\|_{L^q(\mathsf{A})} \leq C\, \|\varphi\|^2_{L^p(\mathsf{A})}, \tag{4.1}$$

where q is conjugated to p. To obtain the last inequality, we have used that $\|\mathcal{T}[\varphi]\|_{L^q(\mathsf{A})} \leq C\, \|\mathcal{T}[\varphi]\|_{L^\infty(\mathsf{A})}$ since A is compact. And, since the integral kernel $\mathscr{T}$ of $\mathcal{T}$ is in L^q uniformly—provided $q < \infty$—for p given by $1/p + 1/q = 1$ we can bound:

$$\|\,\underline{\mathcal{T}}[\varphi]\,\|_{L^\infty(\mathsf{A})} \leq C \sup_{x \in \mathsf{A}} \|\mathscr{T}(x,\bullet)\|_{L^q(\mathsf{A})}\, \|\varphi\|_{L^p(\mathsf{A})}.$$

From (4.1), we deduce that $\underline{\mathcal{T}}^{\frac{1}{2}}$ extends into a continuous operator $\underline{\mathcal{T}}^{\frac{1}{2}} = L_0^p(\mathsf{A}) \mapsto L_0^2(\mathsf{A})$ for any $1 \leq p < 2$. Further, it is established in the appendix, see Proposition A.3, that $\underline{\mathcal{T}}^{-1} : W^{1;q}(\mathsf{A}) \mapsto L_0^p(\mathsf{A})$ is continuous. Thus, given $\varphi_1 \in L_0^2(\mathsf{A})$ and $\varphi_2 \in W^{1;q}(\mathsf{A})$, one has that

$$\left|\int_{\mathsf{A}} \varphi_1(x)\cdot\varphi_2(x)\,\mathrm{d}x\right| = \left|\int_{\mathsf{A}} \varphi_1(x)\cdot \underline{\mathcal{T}}^{\frac{1}{2}} \circ \underline{\mathcal{T}}^{\frac{1}{2}}[\underline{\mathcal{T}}^{-1}[\varphi_2]](x)\,\mathrm{d}x\right| = \left|\int_{\mathsf{A}} \underline{\mathcal{T}}^{\frac{1}{2}}[\varphi_1](x)\cdot \underline{\mathcal{T}}^{\frac{1}{2}}[\underline{\mathcal{T}}^{-1}[\varphi_2]](x)\,\mathrm{d}x\right|$$
$$\leq (\mathcal{Q}[\nu_{\varphi_1}])^{\frac{1}{2}} \left(\int_{\mathsf{A}} \underline{\mathcal{T}}^{\frac{1}{2}}[\underline{\mathcal{T}}^{-1}[\varphi_2]](x)\cdot \underline{\mathcal{T}}^{\frac{1}{2}}[\underline{\mathcal{T}}^{-1}[\varphi_2]](x)\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

In the first line, we have used that $\underline{\mathcal{T}}^{-1}\circ\underline{\mathcal{T}}^{-1}=\mathrm{id}_{W^{1;q}(\mathsf{A})}$. In order to obtain the second equality, we have used that $\underline{\mathcal{T}}^{\frac{1}{2}}[\underline{\mathcal{T}}^{-1}[\varphi_2]]\in L^2(\mathsf{A})$, $\underline{\mathcal{T}}^{\frac{1}{2}}$ is a continuous self-adjoint operator on $L^2(\mathsf{A})$ and that $\varphi_1\in L^2(\mathsf{A})$.

For $x\notin\mathsf{A}$, let $\sigma_{\mathsf{A}}(x)=(\prod_{a\in\partial\mathsf{A}}(x-a))^{1/2}$, where the square root is taken so that $\sigma_{\mathsf{A}}(x)\sim x^{g+1}$ when $x\to\infty$. It is readily checked on the basis of the explicit expression given in (A.17), that for any $\varepsilon>0$, the function

$$\Phi_\varepsilon(x)=|\sigma_{\mathsf{A},+}^{1/2}|^{\varepsilon}(x)\,\underline{\mathcal{T}}^{-1}[\varphi_2](x)-\int_{\mathsf{A}}|\sigma_{\mathsf{A},+}^{1/2}|^{\varepsilon}(\xi)\,\underline{\mathcal{T}}^{-1}[\varphi_2](\xi)\,\mathrm{d}\xi$$

belongs to $L^2_0(\mathsf{A})$ and converges in $L^p_0(\mathsf{A})$, $1\le p<2$, to $\underline{\mathcal{T}}^{-1}[\phi]$. Thus,

$$\begin{aligned}\int_{\mathsf{A}}\underline{\mathcal{T}}^{\frac{1}{2}}[\underline{\mathcal{T}}^{-1}[\varphi_2]](x)\cdot\underline{\mathcal{T}}^{\frac{1}{2}}[\underline{\mathcal{T}}^{-1}[\varphi_2]](x)\,\mathrm{d}x&=\int_{\mathsf{A}}\lim_{\varepsilon\to0^+}\{\underline{\mathcal{T}}^{\frac{1}{2}}[\Phi_\varepsilon](x)\cdot\underline{\mathcal{T}}^{\frac{1}{2}}[\Phi_\varepsilon](x)\}\,\mathrm{d}x\\&=\lim_{\varepsilon\to0^+}\left\{\int_{\mathsf{A}}\underline{\mathcal{T}}^{\frac{1}{2}}[\Phi_\varepsilon](x)\cdot\underline{\mathcal{T}}^{\frac{1}{2}}[\Phi_\varepsilon](x)\,\mathrm{d}x\right\}\\&=\lim_{\varepsilon\to0^+}\left\{\int_{\mathsf{A}}\Phi_\varepsilon(x)\cdot\underline{\mathcal{T}}[\Phi_\varepsilon](x)\,\mathrm{d}x\right\}\\&=\int_{\mathsf{A}}\Phi_0(x)\cdot\underline{\mathcal{T}}[\Phi_0](x)\,\mathrm{d}x.\end{aligned}$$

Above we have used the continuity of $\underline{\mathcal{T}}^{\frac{1}{2}}$ on $L^p_0(\mathsf{A})$, the dominated convergence, the self-adjointness of $\underline{\mathcal{T}}^{\frac{1}{2}}:L^2_0(\mathsf{A})\to L^2_0(\mathsf{A})$ and, finally, dominated convergence and the fact that $\underline{\mathcal{T}}[\Phi_\varepsilon](x)\in L^\infty_0(\mathsf{A})$ uniformly in ϵ.

Thus, all in all, we have shown that

$$\left|\int_{\mathsf{A}}\varphi_1(x)\cdot\varphi_2(s)\,\mathrm{d}x\right|\le(\mathcal{Q}[\nu_{\varphi_1}])^{\frac{1}{2}}\left(\int_{\mathsf{A}}\varphi_2(x)\cdot\underline{\mathcal{T}}^{-1}[\varphi_2](x)\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

For any $\varphi_2=\phi\in W^{1;q}(\mathsf{A})$ and $\nu\in\mathcal{M}^0(\mathsf{A})$ such that $\mathcal{Q}^{1/2}[\nu]$, we apply this bound to $\varphi_{1;m}=(\nu*G_m)/\mathrm{d}x\in L^2_0(\mathsf{A})$ for G_m a centered Gaussian distribution with variance $1/m$. Then, noting that

$$\lim_{m\to\infty}\mathcal{Q}^{1/2}[\varphi_{1;m}]=\mathcal{Q}^{1/2}[\nu],$$

by an argument similar to [8, Lemma 2.2], we obtain

$$\left|\int_{\mathsf{A}}\phi(x)\,\mathrm{d}\nu(x)\right|\le\mathcal{Q}^{1/2}[\nu]\left(\int_{\mathsf{A}}\phi(x)\cdot\underline{\mathcal{T}}^{-1}[\phi](x)\,\mathrm{d}s\right)^{\frac{1}{2}}.$$

The claim follows from Hölder's inequality and invoking the continuity of $\underline{\mathcal{T}}^{-1}$.

5 Schwinger-Dyson Equations and Linear Operators

5.1 Hierarchy of Schwinger-Dyson equations

To write the Schwinger-Dyson equations in a way amenable to asymptotic analysis, we require

Hypothesis 5.1. Hypothesis 2.1 and T is holomorphic in a neighborhood of A. □

Lemma 2.5 is therefore applicable. It is convenient for the asymptotic analysis to introduce:

$$\sigma_{\mathrm{hd}}(x) = \prod_{\alpha \in \partial\mathsf{S} \cap \partial\mathsf{A}} (x-\alpha), \qquad \sigma_{\mathsf{S}}(x) = \prod_{\alpha \in \partial\mathsf{S}} (x-\alpha), \tag{5.1}$$

and

$$\sigma_{\mathrm{hd}}^{[1]}(x,\xi) = \frac{\sigma_{\mathrm{hd}}(x) - \sigma_{\mathrm{hd}}(\xi)}{x-\xi}, \qquad \sigma_{\mathrm{hd}}^{[2]}(x;\xi_1,\xi_2) = \frac{\sigma_{\mathrm{hd}}^{[1]}(x,\xi_1) - \sigma_{\mathrm{hd}}^{[1]}(x,\xi_2)}{\xi_1-\xi_2}. \tag{5.2}$$

Then, for any $n \geq 1$ and I a set of cardinality $n-1$, we derive the Schwinger-Dyson equations in Appendix 1, and they take the form

$$\begin{aligned}
&\Big(1-\frac{2}{\beta}\Big)\,\partial_x W_n(x,\boldsymbol{x}_I) + W_{n+1}(x,x,\boldsymbol{x}_I) + \sum_{J\subseteq I} W_{|J|+1}(x,\boldsymbol{x}_J)\,W_{n-|J|}(x,\boldsymbol{x}_{I\setminus J}) \\
&\quad - \frac{2}{\beta}\sum_{\substack{a\in\partial\mathsf{A}\\ \setminus\partial\mathsf{S}}} \frac{\sigma_{\mathrm{hd}}(a)}{\sigma_{\mathrm{hd}}(x)}\,\frac{\partial_a W_{n-1}(\boldsymbol{x}_I)}{x-a} + \frac{2}{\beta}N^{2-r}\oint_{\mathsf{A}^r}\frac{\mathrm{d}^r\boldsymbol{\xi}}{(2\mathrm{i}\pi)^r}\,\frac{\sigma_{\mathrm{hd}}(\xi_1)}{\sigma_{\mathrm{hd}}(x)}\,\frac{\partial_{\xi_1}T(\xi_1,\ldots,\xi_r)}{(r-1)!\,(x-\xi_1)} \\
&\quad \times \overline{W}_{r;n-1}(\xi_1,\ldots,\xi_r\mid\boldsymbol{x}_I) \\
&\quad + \Big(1-\frac{2}{\beta}\Big)\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\sigma_{\mathrm{hd}}^{[2]}(x;\xi,\xi)}{\sigma_{\mathrm{hd}}(x)}\,W_n(\xi,\boldsymbol{x}_I) + \frac{2}{\beta}\sum_{i\in I}\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\sigma_{\mathrm{hd}}(\xi)}{\sigma_{\mathrm{hd}}(x)}\,\frac{W_{n-1}(\xi,\boldsymbol{x}_{I\setminus\{i\}})}{(x-\xi)(x_i-\xi)^2} \\
&\quad - \oint_{\mathsf{A}}\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\mathrm{i}\pi)^2}\,\frac{\sigma_{\mathrm{hd}}^{[2]}(x;\xi_1,\xi_2)}{\sigma_{\mathrm{hd}}(x)}\Big\{W_{n+1}(\xi_1,\xi_2,\boldsymbol{x}_I) + \sum_{J\subseteq I} W_{|J|+1}(\xi_1,\boldsymbol{x}_J)\,W_{n-|J|}(\xi_2,\boldsymbol{x}_{I\setminus J})\Big\} = 0.
\end{aligned} \tag{5.3}$$

There, $\boldsymbol{x}_I$ is as defined in (1.2) and we have made use of the semi-connected correlators:

$$\overline{W}_{r;n}(\xi_1,\ldots,\xi_r\mid x_1,\ldots,x_n) = \partial_{t_1}\ldots\partial_{t_n}\widetilde{W}_r(\xi_1,\ldots,\xi_r)[T\to\widetilde{T}_{t_1\ldots t_n}]|_{t_1,\ldots,t_n=0},$$

where $\widetilde{T}_{t_1 \dots t_n}$ is as defined in (1.6). For instance

$$\overline{W}_{2;2}(\xi_1, \xi_2 | x_1, x_2) = W_4(\xi_1, \xi_2, x_1, x_2) + W_3(\xi_1, x_1, x_2) W_1(\xi_2)$$
$$+ W_2(\xi_1, x_1) W_2(\xi_2, x_2) + W_2(\xi_1, x_2) W_2(\xi_2, x_1) + W_1(\xi_1) W_3(\xi_2, x_1, x_2). \tag{5.4}$$

We also use the convention $W_0 = \ln Z$.

5.2 The master operator

Upon a naive expansion of the Schwinger-Dyson equation around $W_1 = N W_{\mathrm{eq}} + o(N)$, there arises a linear operator $\mathcal{K} : \mathscr{H}^m(\mathsf{A}) \to \mathscr{H}^1(\mathsf{A})$. This operator depends on the Stieltjes transform W_{eq} of the equilibrium measure and on T and is given by

$$\mathcal{K}[\varphi](x) = 2 W_{\mathrm{eq}}(x)\, \varphi(x) - 2 \oint_{\mathsf{A}^2} \frac{\mathrm{d}^2 \boldsymbol{\xi}}{(2i\pi)^2} \frac{\sigma_{\mathrm{hd}}^{[2]}(x; \xi_1, \xi_2)}{\sigma_{\mathrm{hd}}(x)}\, \varphi(\xi_1) W_{\mathrm{eq}}(\xi_2)$$
$$+ \frac{2}{\beta} \oint_{\mathsf{A}^r} \frac{\mathrm{d}^r \boldsymbol{\xi}}{(2\mathrm{i}\pi)^r} \frac{\sigma_{\mathrm{hd}}(\xi_1)}{\sigma_{\mathrm{hd}}(x)} \frac{\partial_{\xi_1} T(\xi_1, \dots, \xi_r)}{(r-1)!\,(x - \xi_1)} \{\varphi(\xi_1) W_{\mathrm{eq}}(\xi_2) + (r-1) W_{\mathrm{eq}}(\xi_1) \varphi(\xi_2)\}$$
$$\times \left[\prod_{i=3}^{r} W_{\mathrm{eq}}(\xi_i)\right]. \tag{5.5}$$

It is then necessary to invert $\mathcal{K}$ in a continuous way in order to study the corrections to the leading order of the correlators via the Schwinger-Dyson equations. The two lemmata below answer this question, and are the key to the bootstrap analysis of Section 5. Let us introduce the period map $\Pi : \mathscr{H}^m(\mathsf{A}) \to \mathbb{C}^{g+1}$ as

$$\Pi[\varphi] = \left(\oint_{\mathsf{A}_0} \frac{\mathrm{d}\xi\, \varphi(\xi)}{2\mathrm{i}\pi}, \dots, \oint_{\mathsf{A}_g} \frac{\mathrm{d}\xi\, \varphi(\xi)}{2\mathrm{i}\pi} \right).$$

We denote $\mathscr{H}_0^m(\mathsf{A}) = \mathrm{Ker}\, \Pi$.

Lemma 5.1. Assume the local strict convexity of Hypothesis 3.2, the analyticity of Hypothesis 5.1, and that μ_{eq} is off-critical (Definition 2.6). Let $m \geq 1$. Then, the restriction of $\mathcal{K}$ to $\mathscr{H}_0^m(\mathsf{A})$ is invertible on its image $\mathscr{J}^m(\mathsf{A}) = \mathcal{K}[\mathscr{H}_0^m(\mathsf{A})]$. □

Lemma 5.2. $\mathscr{J}^2(\mathsf{A})$ is a closed subspace of $\mathscr{H}^1(\mathsf{A})$, and for any contour Γ surrounding A in $\mathbb{C} \setminus \mathsf{A}$, and $\Gamma[1]$ exterior to Γ, there exists a constant $c > 0$ so that

$$\forall \psi \in \mathscr{J}^2(\mathsf{A}), \quad \|\mathcal{K}^{-1}[\psi]\|_{\Gamma[1]} \leq c \|\psi\|_{\Gamma}. \qquad \square$$

The remaining of this section is devoted to the proof of these results.

5.3 Preliminaries

We remind that, in the off-critical regime, according to the definitions given in (5.1) and in virtue of Lemma 2.5, one has the representation:

$$\frac{\mathrm{d}\mu_{\mathrm{eq}}}{\mathrm{d}x}(x) = \frac{\mathbf{1}_{\mathsf{S}}(x)}{2\pi}\,M(x)\left|\frac{\sigma_{\mathsf{S}}^{1/2}(x)}{\sigma_{\mathrm{hd}}(x)}\right|$$

with M holomorphic and nowhere vanishing in a neighborhood of A. Equivalently, in terms of the Stieltjes transform:

$$W_{\mathrm{eq};-}(x) - W_{\mathrm{eq};+}(x) = \frac{M(x)\sigma_{\mathsf{S};-}^{1/2}(x)}{\sigma_{\mathrm{hd}}(x)}. \tag{5.6}$$

And, from the formula (2.14) for the Stieltjes transform:

$$2W_{\mathrm{eq}}(x) - V'(x) = M(x)\,\frac{\sigma_{\mathsf{S}}^{1/2}(x)}{\sigma_{\mathrm{hd}}(x)}, \tag{5.7}$$

$$V'(x) = -\frac{2}{\beta}\oint_{\mathsf{S}^{r-1}}\frac{\mathrm{d}^{r-1}\boldsymbol{\xi}}{(2\mathrm{i}\pi)^{r-1}}\,\frac{\partial_x T(x,\xi_2,\ldots,\xi_r)}{(r-1)!}\prod_{i=2}^{r}W_{\mathrm{eq}}(\xi_i), \tag{5.8}$$

There $\sigma_{\mathsf{S}}^{1/2}(x)$ is the square root such that $\sigma_{\mathsf{S}}^{1/2}(x)\sim x^{g+1}$ when $x\to\infty$. To rewrite $\mathcal{K}$ in a more convenient form, we introduce four auxiliary operators. We refer to Section 1.5 for the notations of functional spaces. Let $m\geq 1$:

- $\mathcal{O}:\mathscr{H}^m(\mathsf{A})\to\mathscr{O}(\mathsf{A})$ is defined by

$$\mathcal{O}[\varphi](x) = \frac{2}{\beta}\oint_{\mathsf{A}^{r-1}}\frac{\mathrm{d}^{r-1}\boldsymbol{\xi}}{(2\mathrm{i}\pi)^{r-1}}\,\frac{\partial_x T(x,\xi_2,\ldots,\xi_r)}{(r-2)!}\,\varphi(\xi_2)\left[\prod_{i=3}^{r}W_{\mathrm{eq}}(\xi_i)\right].$$

- $\mathcal{L}:\mathscr{H}^m(\mathsf{A})\to\mathscr{H}^{g+2}(\mathsf{A})$ is defined by

$$\mathcal{L}[\varphi](x) = \oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\sigma_{\mathsf{S}}^{1/2}(\xi)}{\sigma_{\mathsf{S}}^{1/2}(x)}\,\frac{\mathcal{O}[\varphi](\xi)}{2(x-\xi)}.$$

  As a matter of fact, since $\mathcal{O}[\varphi]$ is holomorphic in a neighborhood of A, the contour integral in the formula above can be squeezed to S, and $\mathrm{Im}\,\mathcal{L}\subseteq\mathscr{H}^{g+2}(\mathsf{S})$.
- $\mathcal{P}:\mathscr{H}^m(\mathsf{A})\to\mathscr{H}^m(\mathsf{A})$ is defined by

$$\mathcal{P}[\varphi](x) = \operatorname*{Res}_{\xi\to\infty}\frac{\sigma_{\mathsf{S}}^{1/2}(\xi)\,\varphi(\xi)\,\mathrm{d}\xi}{\sigma_{\mathsf{S}}^{1/2}(x)\,(x-\xi)}.$$

By construction, $\mathcal{P}$ is a projector with:

$$\mathrm{Ker}\,\mathcal{P} = \mathscr{H}^{g+2}(\mathsf{A}), \qquad \mathrm{Im}\,\mathcal{P} = \sigma_{\mathsf{S}}^{-1/2} \cdot \mathbb{C}_{g+1-m}[x] \subseteq \mathscr{H}^{m}(\mathsf{S}).$$

- $\mathcal{I} : \mathscr{H}^1(\mathsf{A}) \to \mathscr{H}^1(\mathsf{A})$ is defined by

$$\mathcal{I}[\psi](x) = \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{\sigma_{\mathrm{hd}}(\xi)\, \psi(\xi)}{M(\xi)\,(x-\xi)}.$$

Its kernel is the space of rational functions with at most simple poles at hard edges (that is the zeroes of σ_{hd}). Its pseudo-inverse $\mathcal{I}^{-1} : \mathscr{H}^1(\mathsf{A}) \to \mathscr{H}^1(\mathsf{A})/\mathrm{Ker}\,\mathcal{I}$ can be readily described

$$\mathcal{I}^{-1}[\varphi](x) = \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{M(\xi)\, \varphi(\xi)}{\sigma_{\mathrm{hd}}(\xi)\,(x-\xi)}.$$

Lemma 5.3. We have the factorization between operators in $\mathscr{H}^m(\mathsf{A})$:

$$\mathrm{id} + \mathcal{L} - \mathcal{P} = \sigma_{\mathsf{S}}^{-1/2} \cdot (\mathcal{I} \circ \mathcal{K}). \tag{5.9}$$

□

Proof. A sequence of elementary manipulations allows one to recast $\mathcal{K}$ in the form

$$\mathcal{K}[\varphi](x) = \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{\sigma_{\mathrm{hd}}(\xi)}{\sigma_{\mathrm{hd}}(x)}\, \frac{1}{x-\xi}\{[2W_{\mathrm{eq}}(\xi) - V'(\xi)]\varphi(\xi) + W_{\mathrm{eq}}(\xi)\mathcal{O}[\varphi](\xi)\}, \tag{5.10}$$

where the definition of $V(x)$ was given in (5.8). Using (5.7) and the fact that M is holomorphic and nowhere vanishing in a neighborhood of A, we find

$$(\mathcal{I} \circ \mathcal{K})[\varphi](x) = \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{\sigma_{\mathsf{S}}^{1/2}(\xi)\varphi(\xi)}{x-\xi} + \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{\sigma_{\mathrm{hd}}(\xi)\, W_{\mathrm{eq}}(\xi)\, \mathcal{O}[\varphi](\xi)}{M(\xi)(x-\xi)}.$$

The first integral can be computed by taking the residues outside of the integration contour, whereas the second integral can be simplified by squeezing the integration contour to S and then using (5.6). Coming back to a contour integral, we obtain

$$(\mathcal{I} \circ \mathcal{K})[\varphi](x) = \sigma_{\mathsf{S}}^{1/2}(x)\, \varphi(x) - \sigma_{\mathsf{S}}^{1/2}(x)\, \mathcal{P}[\varphi](x) + \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{\sigma_{\mathsf{S}}^{1/2}(\xi)\mathcal{O}[\varphi](\xi)}{2(x-\xi)},$$

which takes the desired form. ■

5.4 Kernel of the master operator

The factorization property of Lemma 5.3 implies

Corollary 5.4.

$$\mathrm{Ker}\,\mathcal{K} \subseteq \mathrm{Ker}(\mathrm{id} + \mathcal{L} - \mathcal{P}),$$

with equality when there is no hard edge. □

We may give an alternative description of the kernel of $\mathcal{I} \circ \mathcal{K}$.

Lemma 5.5. The three properties are equivalent:

(i) $\mathcal{K}[\varphi] \in \mathrm{Ker}\,\mathcal{I}$.

(ii) $\varphi \in \mathscr{H}^m(\mathsf{S})$, the function $\sigma_{\mathsf{S}}^{1/2} \cdot \varphi$ has continuous $\pm$ boundary values on S, and for any $x \in \mathring{\mathsf{S}}$:

$$\varphi_+(x) + \varphi_-(x) + \mathcal{O}[\varphi](x) = 0.$$

(iii) The expression:

$$\mathrm{d}\nu_\varphi(x) = (\varphi_-(x) - \varphi_+(x)) \cdot \frac{\mathrm{d}x}{2\mathrm{i}\pi}$$

defines a complex measure supported in S with density in $L^p(\mathsf{S})$ for $p < 2$. It satisfies the singular integral equation: for any $x \in \mathring{\mathsf{S}}$,

$$\beta \fint_{\mathsf{S}} \frac{\mathrm{d}\nu_\varphi(\xi)}{x - \xi} + \int_{\mathsf{S}^{r-1}} \frac{\partial_x T(x, \xi_2, \ldots, \xi_r)}{(r-2)!}\,\mathrm{d}\nu_\varphi(\xi_2) \prod_{i=3}^{r} \mathrm{d}\mu_{\mathrm{eq}}(\xi_i) = 0.$$ □

Proof. • (i) $\Rightarrow$ (ii) : If φ satisfies (i), then

$$\varphi(x) = (\mathcal{P} - \mathcal{L})[\varphi](x). \tag{5.11}$$

From the definition of our operators, $\sigma_{\mathsf{S}}^{1/2}(x)\,(\mathcal{P} - \mathcal{L})[\varphi](x)$ is holomorphic on $\mathbb{C} \setminus \mathsf{S}$, and admits continuous $\pm$ boundary values on S. So, Equation (5.11) ensures that $\varphi(x)\sigma_{\mathsf{S}}^{1/2}(x)$ admits continuous $\pm$ boundary values on S. Given the definition of $\mathcal{L}$, we have

$$\forall x \in \mathsf{S}, \quad \mathcal{L}[\varphi]_+(x) + \mathcal{L}[\varphi]_-(x) = \mathcal{O}[\varphi](x). \tag{5.12}$$

Hence, the claim follows upon computing the sum of the + and − boundary values of $\varphi(x)$ expressed by (5.11).

• (ii) $\Rightarrow$ (i) : Conversely, assume φ satisfies (ii). Then, the definition of $\mathcal{K}$ implies that $\sigma_{\mathrm{hd}}(x)\,\mathcal{K}[\varphi](x)$ has continuous $\pm$ boundary values on S. Let us compute the difference of those boundary values using (5.10). For $x \in \mathsf{S}$:

$$\mathcal{K}[\varphi]_-(x) - \mathcal{K}[\varphi]_+(x) = (W_{\mathrm{eq};-}(x) - W_{\mathrm{eq};+}(x))(\varphi_+(x) + \varphi_-(x) + \mathcal{O}[\varphi](x)) \\ + (W_{\mathrm{eq};+}(x) + W_{\mathrm{eq};-}(x) - V'(x))(\varphi_-(x) - \varphi_+(x)).$$

Since φ satisfies (ii), for any $x \in \mathsf{S}$, the second factor in the first line vanishes. Moreover, the first factor in the second line vanishes as well by the characterization of the equilibrium measure. Hence, $\sigma_{\mathrm{hd}}(x)\,\mathcal{K}[\varphi](x)$ has continuous and equal $\pm$ boundary values on S. As a consequence $\sigma_{\mathrm{hd}}(x)\,\mathcal{K}[\varphi](x)$ is an entire function. Since $\mathcal{K}[\varphi](x)$ behaves as $O(1/x)$ when $x \to \infty$, we deduce that $\mathcal{K}[\varphi](x)$ is a rational function with at most simple poles at hard edges, that is, that $\mathcal{K}[\varphi]$ belongs to $\mathrm{Ker}\,\mathcal{I}$.

• (iii) $\Leftrightarrow$ (ii) : (iii) is stronger than (ii). Conversely, assume φ satisfies (ii). Since σ_{S} has simple zeroes, the information in (ii) imply that $\mathrm{d}\nu_{\varphi}$ is an integrable, complex measure, which has a density which is $L^p(\mathsf{S})$ for any $p < 2$. By construction

$$\varphi(x) = \int_{\mathsf{S}} \frac{\mathrm{d}\nu_{\varphi}(\xi)}{x - \xi}.$$

Then, the equation for the $\pm$ boundary values of φ in (ii) translates into the singular integral equation for the measure $\mathrm{d}\nu_{\varphi}$. ∎

Proof of Lemma 5.1. We need to show that the restriction of $\mathcal{K}$ to $\mathrm{Ker}\,\Pi$ is injective. Let $\varphi \in \mathrm{Ker}\,\mathcal{K} \cap \mathrm{Ker}\,\Pi$. The singular integral equation of (iii) holds since $\mathcal{K}[\varphi] = 0 \in \mathrm{Ker}\,\mathcal{I}$. Let us integrate it: there exist constants $c_0, \ldots, c_g$ such that,

$$\forall x \in \mathsf{S}_h \quad \beta \int_{\mathsf{S}} \ln|x - \xi|\,\mathrm{d}\nu_{\varphi}(\xi) + \int_{\mathsf{S}^{r-1}} \frac{T(x, \xi_2, \ldots, \xi_r)}{(r-2)!}\,\mathrm{d}\nu_{\varphi}(\xi_2) \prod_{i=3}^{r} \mathrm{d}\mu_{\mathrm{eq}}(\xi_i) = c_h. \tag{5.13}$$

Now, let us integrate under $\mathrm{d}\nu_{\varphi}^*(x)$ (here $*$ denotes the complex conjugate) and integrate over $x \in \mathsf{S}$. This last operation is licit since every term in (5.13) belongs to $L^{\infty}(\mathsf{S}, \mathrm{d}x)$. The right-hand side vanishes since:

$$\int_{\mathsf{S}_h} \mathrm{d}\nu_{\varphi}(x) = \oint_{\mathsf{S}_h} \frac{\varphi(x)\,\mathrm{d}x}{2\mathrm{i}\pi} = 0. \tag{5.14}$$

We find

$$\mathcal{Q}[\mathrm{Re}\,\nu_{\varphi}] + \mathcal{Q}[\mathrm{Im}\,\nu_{\varphi}] = 0, \tag{5.15}$$

where $\mathcal{Q}$ is the quadratic form of Hypothesis 3.2. The vanishing of periods (5.14) implies *a fortiori* that $\mathrm{Re}\,\nu_{\varphi}$ and $\mathrm{Im}\,\nu_{\varphi}$ are signed measures supported on $\mathsf{S} \subseteq \mathsf{A}$ with total mass zero. The assumption of local strict convexity (Hypothesis 3.2) states that for any such measure ν supported on A, $\mathcal{Q}[\nu] \geq 0$, with equality iff $\nu = 0$. So, (5.15) implies $\mathrm{Re}\,\nu_\varphi = \mathrm{Im}\,\nu_\varphi = 0$, that is, $\varphi = 0$. ■

5.5 Continuity of the inverse

We prove Lemma 5.2 by a detour via Fredholm theory in L^2 spaces. We fix nonintersecting contours Γ_h surrounding A_h in $\mathbb{C} \setminus \mathsf{A}$, all lying in Ω such that T is holomorphic on Ω^r. We denote $\Gamma = \bigcup_{h=0}^{g} \Gamma_h$. Then, $\mathcal{L}$ can be interpreted as an integral operator on $L^2(\Gamma)$

$$\mathcal{L}[\varphi](x) = \oint_\Gamma \frac{\mathrm{d}y}{2\mathrm{i}\pi} \mathscr{L}(x, y)\varphi(y),$$

where the integral kernel $\mathscr{L}(x, y)$ is smooth on $\Gamma \times \Gamma$:

$$\mathscr{L}(x, y) = -\frac{2}{\beta \sigma_{\mathsf{S}}^{1/2}(x)} \oint_{(\Gamma[-1])^{r-1}} \frac{\mathrm{d}^{r-1}\boldsymbol{\xi}}{(2\mathrm{i}\pi)^{r-1}} \frac{\sigma_{\mathsf{S}}^{1/2}(\xi_1)\,\partial_{\xi_1} T(\xi_1, y, \xi_2, \ldots, \xi_{r-1})}{(r-2)!\,(\xi_1 - x)} \prod_{i=2}^{r-1} W_{\mathrm{eq}}(\xi_i).$$

Similarly, the operator $\mathcal{P}$ can be recast as

$$\mathcal{P}[\varphi](x) = \oint_\Gamma \frac{\mathrm{d}y}{2\mathrm{i}\pi} \mathscr{P}(x, y)\,\varphi(y)\,, \quad \text{with } \mathscr{P}(x, y) = \frac{1}{\sigma_{\mathsf{S}}^{1/2}(x)} \operatorname*{Res}_{\xi \to \infty} \frac{\sigma_{\mathsf{S}}^{1/2}(\xi)\,\mathrm{d}\xi}{(\xi - y)(x - \xi)}.$$

This last kernel is smooth on $\Gamma \times \Gamma$ and of finite rank $g + 1$.

Since $\mathcal{L}$ and $\mathcal{P}$, as operators on $L^2(\Gamma)$, have smooth kernels and Γ is compact, the operator $(\mathcal{L} - \mathcal{P}) : L^2(\Gamma) \to L^2(\Gamma)$ is compact and trace class in virtue of the condition established in [23]. Finally, let p_k be the unique polynomial of degree at most g such that

$$\forall h, k \in [\![0 ;\ g]\!], \qquad \oint_{\Gamma_h} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \frac{p_k(\xi)}{\sigma_{\mathsf{S}}^{1/2}(\xi)} = \delta_{k,h}.$$

Now consider the measure space $\mathsf{X} = [\![0 ;\ g]\!] \cup \Gamma$ endowed with the measure $\mathrm{d}s$ that is the atomic measure on $[\![0 ;\ g]\!]$ and a curvilinear measure on Γ. We shall constantly make the identification $L^2(\mathsf{X}) \simeq \mathbb{C}^{g+1} \oplus L^2(\mathsf{A})$. It is then readily seen that the operator

$$\mathcal{N} : \begin{array}{ccc} \mathbb{C}^{g+1} \oplus L^2(\Gamma) & \longrightarrow & \mathbb{C}^{g+1} \oplus L^2(\Gamma) \\ (\boldsymbol{v}, \varphi) & \longmapsto & \left(-\boldsymbol{v} + \Pi[\varphi],\, (\mathcal{L} - \mathcal{P})[\varphi] + \sigma_{\mathsf{S}}^{-1/2} \cdot \left(\sum_{h=0}^{g} v_h\, p_h \right) \right) \end{array}$$

is compact and trace class when considered as an integral operator $L^2(\mathsf{X}) \mapsto L^2(\mathsf{X})$. The matrix integral kernel $\mathscr{N}$ of $\mathcal{N}$ has a block decomposition:

$$\mathscr{N} = \begin{pmatrix} \mathscr{N}(h, k) & \mathscr{N}(h, y) \\ \mathscr{N}(x, k) & \mathscr{N}(x, y) \end{pmatrix} = \begin{pmatrix} -\delta_{j,k} & \mathbf{1}_{\Gamma_h}(y)/2\mathrm{i}\pi \\ \sigma_{\mathsf{S}}^{-1/2}(x) \cdot p_k(x) & (\mathscr{L} - \mathscr{P})(x, y) \end{pmatrix}.$$

The operator $\mathrm{id}+\mathcal{N}$ is injective: indeed, if $(\boldsymbol{v},\varphi)\in\ker(\mathrm{id}+\mathcal{N})$, then

$$\forall h\in[\![\,0\,;\,g\,]\!],\quad \oint_{\Gamma_h}\frac{\mathrm{d}\xi\,\varphi(\xi)}{2\mathrm{i}\pi}=0\quad\text{and}\quad \varphi(x)=-\frac{\sum_{h=0}^{g}v_h\,p_h(x)}{\sigma_{\mathsf{S}}^{1/2}(x)}+(\mathcal{P}-\mathcal{L})[\varphi](x). \tag{5.16}$$

The second equation implies that, in fact, $\varphi\in\mathscr{H}^1(\mathsf{A})$. Further, since $\oint_{\Gamma}\mathrm{d}\xi\,\varphi(\xi)=0$, it follows that $\varphi\in\mathscr{H}^2(\mathsf{A})$. Thus, $\varphi(x)\,\mathrm{d}x$ is a holomorphic differential all of whose Γ_h periods are zero. Hence $\varphi=0$. Then (5.16) implies that $\boldsymbol{v}=0$ as well. The Fredholm alternative thus ensures that $\mathrm{id}+\mathcal{N}$ is continuously invertible. Furthermore, its inverse $\mathrm{id}-\mathcal{R}_{\mathcal{N}}$ is given in terms of the resolvent kernel defined as in (A.16). The integral kernel $\mathcal{N}$ falls into the class discussed in Appendix A.3 with $f=1$. Hence, the integral kernel $\mathscr{R}_{\mathcal{N}}$ of $\mathcal{R}_{\mathcal{N}}$ belongs to $L^{\infty}(\mathsf{X}^2)$.

We are now in position to establish the continuity of its inverse. The representation (5.10) and the explicit form of the operators appearing there make it clear that $\mathcal{K}$ is a continuous operator for any norm $\|\cdot\|_{\Gamma}$ in the sense that

$$\|\mathcal{K}[\varphi]\|_{\Gamma[1]}\leq\|\varphi\|_{\Gamma}.$$

We have already proved in Lemma 5.1 that the map

$$\begin{aligned}\widehat{\mathcal{K}}:\quad \mathscr{H}^1(\mathsf{A})\ &\longmapsto\ \mathbb{C}^{g+1}\oplus\mathscr{H}^1(\mathsf{A}),\\ \varphi\ &\longrightarrow\ (\Pi[\varphi],\mathcal{K}[\varphi])\end{aligned}$$

is injective. So, for any $\psi\in\mathscr{J}^1(\mathsf{A})$, there exists there exists a unique $\varphi\in\mathscr{H}^1(\mathsf{A})$ such that

$$\mathcal{K}[\varphi]=\psi\quad\text{and}\quad\oint_{\mathsf{A}_h}\frac{\mathrm{d}\xi\,\varphi(\xi)}{2\mathrm{i}\pi}=0.$$

In other words, $(\mathbf{0},\varphi)\in\mathbb{C}^{g+1}\oplus\mathscr{H}^1(\mathsf{A})$ does provide one with the unique solution to

$$(\mathrm{id}+\mathcal{N})[(\mathbf{0},\varphi)]=(\mathbf{0},\mathcal{I}[\psi]).$$

Then, it readily follows that for any $\psi\in\mathscr{J}^1(\mathsf{A})$

$$\mathcal{K}^{-1}[\psi](x)=\mathcal{I}[\psi](x)-\oint_{\Gamma}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\mathscr{R}_{\mathcal{N}}(x,\xi)\,\sigma_{\mathrm{hd}}(\xi)}{M(\xi)}\,\psi(\xi),$$

where we have used that the resolvent kernel $\mathscr{R}_{\mathcal{N}}(x,\xi)$ is an analytic function on $(\mathbb{C}\setminus\mathsf{A})\times\Omega$, with Ω the open neighborhood of A such that T is analytic on Ω^r. It is straightforward to establish the continuity of the inverse on the basis of the formula above. As a consequence, it follows that $\mathscr{J}^1(A)$ (respectively, $\mathscr{J}^2(A)$) is a closed subspace of $\mathscr{H}^1(\mathsf{A})$ (respectively, $\mathscr{H}^2(\mathsf{A})$). ■

6 Asymptotics of Correlators in the Fixed Filling Fraction Model

6.1 More linear operators

Let us decompose

$$W_1 = N(W_{\mathrm{eq}} + \Delta_{-1}W_1).$$

We define, with the notations of (5.2),

$$\mathcal{D}_1[\varphi](x_1,x_2) = \frac{2}{\beta}\oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\sigma_{\mathrm{hd}}(\xi)}{\sigma_{\mathrm{hd}}(x_1)}\,\frac{\varphi(\xi)}{(x_1-\xi)(x_2-\xi)^2},$$

$$\mathcal{D}_2[\varphi](x) = \varphi(x,x) - \oint_{\mathsf{A}^2}\frac{\mathrm{d}\xi_1\mathrm{d}\xi_2}{(2\mathrm{i}\pi)^2}\,\frac{\sigma_{\mathrm{hd}}^{[2]}(x;\xi_1,\xi_2)}{\sigma_{\mathrm{hd}}(x)}\,\varphi(\xi_1,\xi_2),$$

and

$$\mathcal{T}[\varphi](x) = \frac{2}{\beta}\oint_{\mathsf{A}^r}\frac{\mathrm{d}^r\boldsymbol{\xi}}{(2\mathrm{i}\pi)^r}\,\frac{\partial_{\xi_1}T(\boldsymbol{\xi})\,\varphi(\boldsymbol{\xi})}{(r-1)!\,(x-\xi_1)},$$

$$\Delta\mathcal{K}[\varphi](x) = \frac{1-2/\beta}{N}\left(\partial_x\varphi(x) + \oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\sigma_{\mathrm{hd}}^{[2]}(x;\xi,\xi)}{\sigma_{\mathrm{hd}}(x)}\,\varphi(\xi)\right) + 2\,\mathcal{D}_2[(\Delta_{-1}W_1)(\bullet_1)\,\varphi(\bullet_2)](x)$$
$$+\sum_{i=1}^{r}\sum_{\substack{J\subseteq[\![1\,;\,r]\!]\setminus\{i\}\\ J\neq\emptyset}}\mathcal{T}\left[\varphi(\bullet_i)\prod_{j\in J}(\Delta_{-1}W_1)(\bullet_j)\prod_{j\notin J}W_{\mathrm{eq}}(\bullet_j)\right].$$

Above and in the following the notation $\bullet_j$ inside of the action of an operator denotes the jth running variable of the function on which the given operator acts. We also remind that if $\Gamma=\Gamma[0]$ is a contour surrounding A, then we denote by $(\Gamma[i])_{i\geq 0}$ a family of nested contours such that $\Gamma[i+1]$ is exterior to $\Gamma[i]$ for any i. There exists positive constants $c_1, c_2, \ldots$ which depend on the model and on the contours, so that

$$\begin{aligned}
\|\mathcal{D}_1[\varphi]\|_{(\Gamma[1])^2} &\leq c_1\,\|\varphi\|_{\Gamma},\\
\|\mathcal{D}_2[\varphi]\|_{\Gamma} &\leq c_2\,\|\varphi\|_{\Gamma^2},\\
\|\mathcal{T}[\varphi]\|_{\Gamma[1]} &\leq c_3\,\|\varphi\|_{\Gamma^r},\\
\|\Delta\mathcal{K}[\varphi]\|_{\Gamma[1]} &\leq (c_4/N)\,\|\varphi\|_{\Gamma} + c_5\|(\Delta_{-1}W_1)\|_{\Gamma}\,\|\varphi\|_{\Gamma[1]}.
\end{aligned}\tag{6.1}$$

Above, φ belongs to the domain of definition of the respective operators. Note that we have to push the contour toward the exterior in order to control the effect of the singular factors in $\mathcal{D}_1$ and $\mathcal{T}$. Further, we have also used the continuity of the derivation operator $\|\partial_x\varphi\|_{\Gamma[1]} \leq c\|\varphi\|_{\Gamma}$. In order to gather all of the relevant operator bounds in one place, we remind the continuity of $\mathcal{K}^{-1}$: for $\varphi \in \mathscr{J}^2(\mathsf{A})$

$$\|\mathcal{K}^{-1}[\varphi]\|_{\Gamma[1]} \leq c_6\, \|\varphi\|_{\Gamma}.$$

Besides, if φ is holomorphic in $\mathbb{C} \setminus \mathsf{S}$ instead of $\mathbb{C} \setminus \mathsf{A}$, then $\mathcal{K}^{-1}[\varphi]$ is also holomorphic in $\mathbb{C} \setminus \mathsf{S}$.

6.2 Improving concentration bounds using SD equations

In order to improve the a priori control on the correlators which follows from the concentration bounds

$$\|N\,\Delta_{-1} W_1\|_{\Gamma} \leq c_1\, \eta_N, \qquad \|W_n\|_{\Gamma} \leq c_n\, \eta_N^n \ \text{ with } \eta_N = (N \ln N)^{1/2}, \tag{6.2}$$

it is convenient to recast the Schwinger-Dyson equations.

The Schwinger-Dyson equation relative to W_1, that is, (5.3) with $n = 1$ takes the form:

$$\mathcal{K}[N\Delta_{-1} W_1](x) = A_1(x) + B_1(x) - (\Delta\mathcal{K})[N\Delta_{-1} W_1](x) \tag{6.3}$$

with:

$$\begin{aligned} A_1(x) = & -N^{-1}\mathcal{D}_2[W_2](x) + N\mathcal{D}_2[(\Delta_{-1} W_1)(\bullet_1)\,(\Delta_{-1} W_1)(\bullet_2)] \\ & - \sum_{\substack{J \vdash [\![1\,;\,r]\!] \\ [J] \leq r-1}} N^{1-r}\mathcal{T}\left[\prod_{i=1}^{[J]} W_{|J_i|}(\bullet_{J_i})\right](x) \\ & + N \sum_{\substack{J \subseteq [\![1\,;\,r]\!] \\ |J| \geq 2}} (|J|-1)\mathcal{T}\left[\prod_{j \in J} (\Delta_{-1} W_1)(\bullet_j) \prod_{j \notin J} W_{\mathrm{eq}}(\bullet_j)\right](x) \\ & - (1 - 2/\beta)\cdot\left\{\partial_x W_{\mathrm{eq}}(x) + \oint_{\mathsf{S}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{\sigma_{\mathrm{hd}}^{[2]}(x;\xi,\xi)}{\sigma_{\mathrm{hd}}(x)}\, W_{\mathrm{eq}}(\xi)\right\}, \\ B_1(x) = & \frac{2}{N\beta} \sum_{a \in \partial\mathsf{A}\setminus\partial\mathsf{S}} \frac{\sigma_{\mathrm{hd}}(a)}{\sigma_{\mathrm{hd}}(x)}\, \frac{\partial_a \ln Z_{\mathcal{S}}}{x-a}. \end{aligned} \tag{6.4}$$

Also, some notational clarifications are in order. The symbol $J \vdash [\![1\,;\,r]\!]$ refers to a sum over all partitions of the set $[\![1\,;\,r]\!]$ into $[J]$ disjoint subsets $J_1, \ldots, J_{[J]}$, with $[J]$ going from 1 to r. In particular, the above sum is empty when $r = 1$. Finally, the summation arising in the second line of (6.4) corresponds to a summation over all subsets J of $[\![1\,;\,r]\!]$ whose cardinality $|J|$ varies from 2 to r.

More generally, for $n \geq 2$, the nth Schwinger-Dyson equation takes the form:

$$\mathcal{K}[W_n(\bullet, \boldsymbol{x}_I)](x) = A_n(x; \boldsymbol{x}_I) + B_n(x; \boldsymbol{x}_I) - (\Delta\mathcal{K})[W_n(\bullet, \boldsymbol{x}_I)](x) \tag{6.5}$$

with:

$$\begin{aligned} A_n(x; \boldsymbol{x}_I) = & -N^{-1}\mathcal{D}_2\left[W_{n+1}(\bullet_1, \bullet_2, \boldsymbol{x}_I) + \sum_{\substack{I' \subseteq I \\ I' \neq \emptyset, I}} W_{|I'|+1}(\bullet_1, \boldsymbol{x}_{I'})W_{n-|I'|}(\bullet_2, \boldsymbol{x}_{I\setminus I'})\right](x) \\ & - \sum_{i \in I} N^{-1}\mathcal{D}_1[W_{n-1}(\bullet, \boldsymbol{x}_{I\setminus\{i\}})](x, x_i) - \sideset{}{^*}\sum_{\substack{J \vdash [\![1\,;\,r]\!] \\ I_1 \sqcup \cdots \sqcup I_{[J]} = I}} N^{1-r}\mathcal{T}\left[\prod_{i=1}^{[J]} W_{|J_i|+|I_i|}(\bullet_{J_i}, \boldsymbol{x}_{I_i})\right](x) \end{aligned} \tag{6.6}$$

and

$$B_n(x; \boldsymbol{x}_I) = \frac{2}{N\beta}\sum_{a \in \partial\mathsf{A}\setminus\partial\mathsf{S}} \frac{\sigma_{\mathrm{hd}}(a)}{\sigma_{\mathrm{hd}}(x)}\frac{\partial_a W_{n-1}(\boldsymbol{x}_I)}{x-a}.$$

Finally, one has

$$\begin{aligned} & \sideset{}{^*}\sum_{\substack{J \vdash [\![1\,;\,r]\!] \\ I_1 \sqcup \cdots \sqcup I_{[J]} = I}} \prod_{k=1}^{[J]} W_{|J_k|+|I_k|}(\boldsymbol{\xi}_{J_k}, \boldsymbol{x}_{I_k}) \\ & = \sum_{\substack{I_1 \sqcup \cdots \sqcup I_{[J]} = I \\ |I_k| < |I|}} \prod_{k=1}^{r} W_{1+|I_k|}(\xi_k, \boldsymbol{x}_{I_k}) + \sum_{\substack{J \vdash [\![1\,;\,r]\!] \\ \exists \ell\,:\,|J_\ell| \geq 2}} \sum_{I_1 \sqcup \cdots \sqcup I_{[J]} = I} \prod_{k=1}^{[J]} W_{|J_k|+|I_k|}(\boldsymbol{\xi}_{J_k}, \boldsymbol{x}_{I_k}). \end{aligned}$$

The $\sqcup$ means that one should sum up over all decompositions of I into $[J]$ disjoint subsets $I_1, \ldots, I_{[J]}$ some of which can be empty. We stress that the order of dispatching the elements does count, *viz*. the decompositions $I \sqcup \{\emptyset\}$ and $\{\emptyset\} \sqcup I$ differ. In other words, the $*$ label means that one excludes all terms of the form $W_n(\bullet_i, \boldsymbol{x}_I)\prod_{j \neq i} W_1(\bullet_j)$.

Schematically, the reason for this decomposition is that if we want to know W_n—so it is put in the left-hand side—A_n should contain the leading contribution, B_n an exponentially small contribution since it is a boundary term, and the $\Delta\mathcal{K}$ term a negligible contribution compared with the $\mathcal{K}$ term. Let us explain how (6.5) for $n \geq 2$ is obtained from the Schwinger-Dyson equations (5.3). We first substitute in (5.3) $W_1(x) = N(W_{\mathrm{eq}}(x) + \Delta_{-1}W(x))$. We have collected in the left-hand side of (6.3) the terms which are linear in W_n and involve W_{eq} only, and the operator $\mathcal{K}$ was introduced in (5.5) for this purpose. There also exist linear terms in W_n which is small, either because they are originally of order $1/N$ compared with the $\mathcal{K}[W_n]$ terms (like the derivative term), or because they came in the substitution with a factor of $\Delta_{-1} W_1$ instead of W_{eq}. Recollecting all those terms makes appear the operator $\Delta\mathcal{K}$, thought as a "small deformation" of the master operator $\mathcal{K}$. Then, we collect the boundary terms in B_n, and all the remaining terms in A_n. Thus, the origin of $\sum^*$ in A_n is that terms which are linear in W_n should be excluded, since they already appeared in $\mathcal{K}[W_n]$ and $\Delta\mathcal{K}[W_n]$.

The decomposition (6.3) is obtained from the $n=1$ Schwinger-Dyson equation by a similar scheme. The quantity we would like to know is now $N\Delta_{-1} W_1$. All leading linear terms in $\Delta_{-1} W_1$ appear in the left-hand side of (6.3) in the form of $\mathcal{K}[N\Delta_{-1} W_1]$. Then, we do isolate some terms $\Delta\mathcal{K}[N\Delta_{-1} W_1]$, and A_1 collect all the remaining terms which do not come from the boundary. However, we remark that the expression (6.1) of the operator $\Delta\mathcal{K}$ itself is polynomial $\Delta_{-1} W_1$—which in principle makes it small—so $\Delta\mathcal{K}[N\Delta_{-1} W_1]$ contains nonlinear terms in the function $N\Delta_{-1} W_1$ sought for, and this is the main difference with $n \geq 2$. Since we insisted to isolate $\Delta\mathcal{K}[N\Delta_{-1} W_1]$, the expression (6.6) is not valid for A_1, since it would lead to overcounting of nonlinear terms in $\Delta_{-1} W_1$. The correct expression of A_1 is (6.4): it differs from (6.6) at $n=1$ by the sign of the second term, and the appearance of the first term of the second line, that precisely avoid overcounting.

The above rewriting in basically enough so as to prove that the Schwinger-Dyson are rigid, in the sense that even a very rough a priori control on the correlators enables establishing that $W_n \in O(N^{2-n})$. Indeed,

Proposition 6.1. There exist integers p_n and positive constants c'_n so that

$$\|\Delta_{-1} W_1\|_{\Gamma[p_1]} \leq c'_1\, N^{-1}, \quad \|W_n\|_{(\Gamma[p_n])^n} \leq c'_n\, N^{2-n}. \qquad \square$$

In order to prove the above proposition we, however, first need to establish a technical lemma emphasizing a one-step improvement of bounds.

Lemma 6.2. Assume there exist positive constants c_n so that

$$\|N\,\Delta_{-1} W_1\|_{\Gamma} \leq c_1(\eta_N \kappa_N + 1), \tag{6.7}$$

$$\|W_n\|_{\Gamma^n} \leq c_n(\eta_N^n \kappa_N + N^{2-n}), \quad (n \geq 2) \tag{6.8}$$

for $\eta_N \to \infty$ so that $\eta_N/N \to 0$, and $\kappa_N \leq 1$. Then, there exist positive constants c'_n so that

$$\|N\,\Delta_{-1} W_1\|_{\Gamma[2]} \leq c'_1(\eta_N\, \kappa_N\, (\eta_N/N) + 1),$$

$$\|W_n\|_{(\Gamma[2])^n} \leq c'_n(\eta_N^n \cdot \eta_N/N \cdot \kappa_N + N^{2-n}). \qquad \square$$

Proof. Hereafter, the values of the positive constants $c_1, c_2, \ldots$ may vary from line to line, and we use repeatedly the continuity of the auxiliary operators introduced in Section 6.1. We remind that in the fixed filling fraction model, we have a priori:

$$\Delta_{-1} W_1 \in \mathscr{H}_0^2(\mathsf{A}),$$

and for any fixed $\boldsymbol{x}_I = (x_2, \ldots, x_r) \in (\mathbb{C} \setminus \mathsf{A})^{n-1}$:

$$W_n(\bullet, \boldsymbol{x}_I) \in \mathscr{H}_0^2(\mathsf{A}).$$

Therefore, the right-hand side of (6.3) or (6.5) (seen as a function of x) belongs to $\mathscr{J}^2(\mathsf{A})$ and we can apply the inverse of $\mathcal{K}$:

$$\Delta_{-1} W_1(x) = \mathcal{K}^{-1}[A_1 + B_1 - (\Delta\mathcal{K})[\Delta_{-1} W_1]](x), \tag{6.9}$$

$$W_n(x, \boldsymbol{x}_I) = \mathcal{K}^{-1}[A_n(\bullet, \boldsymbol{x}_I) + B_n(\bullet, \boldsymbol{x}_I) - (\Delta\mathcal{K})[W_n(\cdot, \boldsymbol{x}_I)](\bullet)](x). \tag{6.10}$$

We start by estimating the various terms present in A_1. The terms not associated with the sum over J in the first line are readily estimated by using the control on the correlators and the continuity of the various operators introduced at the beginning of the section. In what concerns the terms present in the sum, we bound them by using that the bound (6.8) trivially holds for $n = 1$. All in all this leads to

$$\|A_1\|_{\Gamma[1]} \leq \frac{c_2 \tilde{c}}{N}(\eta_N^2 \kappa_N + 1) + \frac{c_1^2 \tilde{c}}{N}(\eta_N \kappa_N + 1)^2 + c' \sum_{\substack{J \vdash [\![1\,;\, r]\!] \\ [J] \leq r-1}} N^{1-r} \prod_{a=1}^{[J]} (\eta_N^{|J_a|} \kappa_N + N^{2-|J_a|})$$
$$+ \tilde{c} c_1^2 (\eta_N \kappa_N + 1)(\eta_N \kappa_N / N + 1/N) + c''.$$

Thus, it solely remains to obtain an optimal bound for the product

$$\Pi_1([J]) = N^{1-r} \prod_{a=1}^{[J]} (\eta_N^{|J_a|} \kappa_N + N^{2-|J_a|}) = \sum_{\alpha \sqcup \overline{\alpha} \;=[\![1\,;\, [J]]\!]} \kappa_N^{|\alpha|} N^{1-r} \prod_{a \in \alpha} \eta_N^{|J_a|} \prod_{a \in \overline{\alpha}} N^{2-|J_a|}. \tag{6.11}$$

Note that, because of the structure of the sum, there exists at least one ℓ such that $|J_\ell| \geq 2$. There are two scenarii then. Either, $\overline{\alpha} = [\![1\,;\, [J]]\!]$ or $|\overline{\alpha}| < [J]$. In the second case, the sum is maximized by the choice of partitions α such that $\ell \in \alpha$ since $\eta_N^2 > 1$ for N large enough. In such partitions, one bounds

$$\prod_{a \in \alpha} \eta_N^{|J_a|} \prod_{a \in \overline{\alpha}} N^{2-|J_a|} \leq \eta_N^2 \prod_{a \in \alpha \setminus \{\ell\}} N^{|J_a|} \prod_{a \in \overline{\alpha}} N^{2-|J_a|} \leq \eta_N^2 N^{r-2}.$$

So, taking this into account, one obtains

$$\Pi_1([J]) \leq c_1 N^{1-r+[J]} + c_2 \frac{\eta_N^2}{N} \kappa_N.$$

It remains to recall that $[J] \leq r-1$ so as to obtain

$$\| A_1 \|_{\Gamma[1]} \leq c_1' \left(\frac{\eta_N^2}{N} \kappa_N + 1 \right).$$

It follows from the large deviations of single eigenvalues, Lemma 3.1, that $\|B_1\|_{\Gamma} = \mathrm{O}(N^{-\infty})$. Finally, the bound (6.7) for $\Delta_{-1} W_1$ leads to

$$\| \Delta\mathcal{K}[\varphi] \|_{\Gamma[1]} \leq c \frac{\eta_N}{N} \|\varphi\|_{\Gamma}, \quad \text{i.e. } \|(\Delta\mathcal{K})[N \Delta_{-1} W_1]\|_{\Gamma[1]} \leq c' \left(\eta_N \cdot \frac{\eta_N}{N} \kappa_N + \frac{\eta_N}{N} \right), \tag{6.12}$$

where we have used (6.7) to obtain the last bound. For the second term of the last inequality, we remind that $\eta_N/N \to 0$, a fortiori is bounded by 1 for N large enough. Note that, above, the shift of contour was necessary because of (6.1). It solely remains to invoke the continuity of $\mathcal{K}^{-1}$—which, however, demands one additional shift of contour—so as to obtain the claimed improvement of bounds relative to $N \Delta_{-1} W_1$.

We can now repeat the chain of bounds for the Schwinger-Dyson equation associated with the nth correlator with $n \geq 2$. Since

$$\max_{j \in [\![1\,;\, n-2]\!]} \{ (\eta_N^{j+1} \kappa_N + N^{2-(j+1)})(\kappa_N \eta_N^{n-j} + N^{2-n+j}) \} \leq c(\eta_N^n \cdot \frac{\eta_N}{N} \kappa_N + N^{2-n}),$$

and

$$\frac{1}{N}(\eta_N^{n-1} \kappa_N + N^{-(n-1)}) \leq \eta_N^n \cdot \frac{\eta_N}{N} \kappa_N + N^{2-n} \tag{6.13}$$

for N large enough, one obtains

$$\|A_n\|_{(\Gamma[1])^n} \leq c(\eta_N^n \frac{\eta_N}{N} \kappa_N + N^{2-n}) + c' \delta_N^{(1)} + c'' \delta_N^{(2)}, \tag{6.14}$$

where

$$\delta_N^{(1)} = \sum_{\substack{I_1 \sqcup \cdots \sqcup I_{[J]} = I \\ |I_k| < |I|}} N^{1-r} \prod_{a=1}^{[J]} (\eta_N^{|I_a|} \kappa_N + N^{1-|I_a|}),$$

$$\delta_N^{(2)} = \sum_{\substack{J \vdash [\![1\,;\, r]\!] \\ \exists \ell : |J_\ell| \geq 2}} \sum_{I_1 \sqcup \cdots \sqcup I_{[J]} = I} \prod_{a=1}^{[J]} (\eta_N^{|I_a| + |J_a|} \kappa_N + N^{2-|I_a|-|J_a|})$$

issue from bounding the last term in (6.6). It is readily seen that

$$\delta_N^{(1)} \leq \sum_{\substack{I_1\sqcup\cdots\sqcup I_{[J]}=I \\ |I_k|<|I|}} \sum_{\substack{\alpha\sqcup\overline{\alpha} \\ =[\![1\,;r]\!]}} \left(\frac{\eta_N}{N}\kappa_N\right)^{|\alpha|} N^{2-n}(\eta_N N)^{m_\alpha} \quad \text{with } m_\alpha = \sum_{a\in\alpha}|I_a|.$$

Thus, the right-hand side is maximized by choosing $|\alpha|$ minimal and m_α maximal. However, if $|\alpha|=0$, then $\ell_\alpha=0$ and one obtain N^{2-n} as a bound. When $|\alpha|=1$, due to the constraint $|I_k|<|I|=n-1$, one obtain that $\max m_\alpha < n-2$. Finally, for $\alpha\geq 2$ one has that $\max m_\alpha = n-1$. A short calculation then shows that

$$\delta_N^{(1)} \leq c\left(N^{2-n} + \eta_N^n \cdot \frac{\eta_N}{N}\kappa_N\right).$$

Likewise, one obtains,

$$\delta_N^{(2)} = \sum_{\substack{J\vdash[\![1\,;r]\!] \\ \exists a:|J_a|\geq 2}} \sum_{I_1\sqcup\cdots\sqcup I_{[J]}=I} \sum_{\substack{\alpha\cup\overline{\alpha} \\ =[\![1\,;[J]]\!]}} \kappa_N^{|\alpha|} N^{2-n}(\eta_N N)^{\ell_\alpha+m_\alpha} N^{2([J]-r-|\alpha|)} \quad \text{with } \begin{cases} m_\alpha = \sum_{a\in\alpha}|I_a|, \\ \ell_\alpha = \sum_{a\in\alpha}|J_a|. \end{cases}$$

Thus, the above summand is maximized by taking the smallest possible value for $|\alpha|$ and the largest possible ones for ℓ_α, m_α and $[J]$. Note, however, that $[J]\leq r-1$ due to the constraint $\exists a : |J_a|\geq 2$. If $|\alpha|=0$, then $\ell_\alpha = m_\alpha = 0$ and one obtains a bound by $N^{-n}\leq N^{2-n}$. If $|\alpha|>0$, then one has $\ell_\alpha \leq r-[J]+1+(|\alpha|-1)$, thus leading to

$$\kappa_N^{|\alpha|} N^{2-n}(\eta_N N)^{\ell_\alpha+m_\alpha} N^{2([J]-r-|\alpha|)} \leq \kappa_N^{|\alpha|} N^{2-n}(\eta_N N)^{|\alpha|+m_\alpha}\left(\frac{\eta_N}{N}\right)^{r-[J]}.$$

The right-hand side is maximized for $m_\alpha=n-1$, $[J]=r-1$ and $|\alpha|=1$, what leads to a bound by $(\eta_N^{n+1}/N)\kappa_N$. Hence, $\|A_n\|_{(\Gamma[1])^n} \leq c(\eta_N^n\cdot(\eta_N/N)\cdot\kappa_N + N^{2-n})$. The remaining terms in (6.6) are bounded analogously to the $n=1$ case. Repeating then the steps of this derivation one, eventually, obtains the sought bounds on W_n. ■

Proof of Proposition 6.1. The concentration results (6.2) provide us the bounds (6.7)–(6.8) with $\eta_N=(N\ln N)^{1/2}$ and $\kappa_N=1$. From Lemma 6.2, we can replace κ_N by $(\eta_N/N)^m = (\ln N/N)^{m/2}$ provided Γ is replaced by $\Gamma[2m]$. In particular, for $n=1$, we may choose $m=2$ to have $\eta_N\,(\eta_N/N)^m\in o(1)$, and for $n\geq 2$, we may choose $m=4n+4$, so that $\eta_N^n(\eta_N/N)^m\in o(N^{2-n})$. At this point, the remainder N^{2-n} in (6.7)–(6.8) gives us the desired bounds. ■

6.3 Recursive asymptotic analysis using SD equations

Lemma 6.3. There exist $W_n^{[k]} \in \mathscr{H}_0^2(\mathsf{S}, n)$, positive integers $p_n^{[k]}$ and positive constants $c_n^{[k]}$, indexed by integers $n \geq 1$ and $k \geq n-2$, so that, for any $k_0 \geq -1$:

$$W_n = \sum_{k=n-2}^{k_0} N^{-k}\, W_n^{[k]} + N^{-k_0}\, \Delta_{k_0} W_n, \quad \|\Delta_{k_0} W_n\|_{\Gamma[p_n^{[k]}]} \leq c_n^{[k]}/N.$$

By convention, the first sum is empty whenever $k_0 < n-2$. □

Proof. The proof goes by recursion. Our recursion hypothesis at step k_0 is that we have a decomposition for any $n \geq 1$:

$$W_n = \sum_{k=n-2}^{k_0} N^{-k}\, W_n^{[k]} + N^{-k_0}\, \Delta_{k_0} W_n, \quad \|\Delta_{k_0} W_n\|_{\Gamma} \to 0,$$

where $W_n^{[k]} \in \mathscr{H}_0^2(\mathsf{S}, n)$ is known and the convergence holds without uniformity in Γ and n. From Proposition 6.1, we know the recursion hypothesis is true for $k_0 = -1$. We choose not to specify anymore the shift of the contours which are necessary at each step of inversion of $\mathcal{K}^{-1}$, since this mechanism of shifting is clear from the Proof of Proposition 6.1.

The recursion hypothesis induces a decomposition:

$$A_n(x, \mathbf{x}_I) = \sum_{k=n-2}^{k_0+1} N^{-k}\, A_n^{[k]}(x, \mathbf{x}_I) + N^{-(k_0+1)}\, \Delta_{(k_0+1)} A_n(x, \mathbf{x}_I),$$

$$(\Delta\mathcal{K})[\varphi](x) = \sum_{k=1}^{k_0+1} N^{-k}\, \mathcal{K}^{[k]}[\varphi](x) + N^{-(k_0+1)} (\Delta_{(k_0+1)}\mathcal{K})[\varphi](x).$$

We give below the expressions of those quantities for $k \in [\![1\,;\, k_0+1]\!]$ (this set is empty if $k_0 = -1$).

$$\mathcal{K}^{[k]}[\varphi](x) = \delta_{k,1}(1 - 2/\beta)\left(\partial_x \varphi(x) + \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{\sigma_{\mathrm{hd}}^{[2]}(x;\xi,\xi)}{\sigma_{\mathrm{hd}}(x)}\, \varphi(\xi)\right) + 2\,\mathcal{D}_2[W_1^{[k-1]}(\bullet_1)\varphi(\bullet_2)](x)$$
$$+ \sum_{i=1}^{r} \sum_{\substack{J \subseteq [\![1\,;\,r]\!] \setminus \{i\} \\ J \neq \emptyset}} \sum_{\substack{k_1,\ldots,k_{|J|} \geq 0 \\ (\sum_j k_j) + |J| = k}} \mathcal{T}\left[\varphi(\bullet_i) \prod_{j \in J} W_1^{[k_j]}(\bullet_j) \prod_{j \notin J} W_{\mathrm{eq}}(\bullet_j)\right](x). \tag{6.15}$$

For $n \geq 2$, we have for $k \leq k_0 + 1$:

$$A_n^{[k]}(x, \boldsymbol{x}_I) = -\mathcal{D}_2\left[W_{n+1}^{[k-1]}(\bullet_1, \bullet_2, \boldsymbol{x}_I) + \sum_{\substack{I' \subseteq I \\ I' \neq \emptyset, I}} \sum_{0 \leq k' \leq k-1} W_{|I'|+1}^{[k']}(\bullet_1, \boldsymbol{x}_{I'})\, W_{n-|I'|}^{[k-k'-1]}(\bullet_2, \boldsymbol{x}_{I \setminus I'})\right]$$
$$- \sum_{i \in I} \mathcal{D}_1[W_{n-1}^{[k-1]}(\bullet, \boldsymbol{x}_{I \setminus \{i\}})](x, x_i) - \sum^{*}_{\substack{J \vdash [\![1\,;\,r]\!] \\ I_1 \sqcup \cdots \sqcup I_{[J]} = I}} \sum_{\substack{k_1, \ldots, k_{[J]} \geq -1 \\ (\sum_i k_i) + r - 1 = k}} \mathcal{T}\left[\prod_{i=1}^{[J]} W_{|J_i|+|I_i|}^{[k_i]}(\bullet_{J_i}, \boldsymbol{x}_{I_i})\right](x). \tag{6.16}$$

In the above expression, we agree that $W_m^{[\ell]} = 0$ whenever $\ell < m - 2$. The main point is that both $A_n^{[k]}(x, \boldsymbol{x}_I)$ and $\mathcal{K}^{[k]}[\varphi](x)$ only involve $W_m^{[k']}$ with $k' \leq k - 1 \leq k_0$. This is a matter of simple reading off in the case of the expression for $\mathcal{K}^{[k]}[\varphi](x)$. Likewise, in the case of $A_n^{[k]}(x, \boldsymbol{x}_I)$ this is clear in what concerns the first three terms, but the last one ought to be discussed. So, for a given term of the sum, let $J^{[-1]}$ be the collection of singletons $\{i\}$ such that $k_i = -1$. Then, the k_i associated with this precise term of the sum satisfy

$$\left(\sum_{p \notin J^{[-1]}} k_p\right) + r - 1 - |J^{[-1]}| = k.$$

The restriction $\sum^*$ ensures that, in the nonvanishing terms, $|J^{[-1]}| \leq r - 2$ that is to say that there is at most $r - 2$ k_p's which can be equal to -1. Since $k_p \geq 0$ for $p \notin J^{[-1]}$, this implies $k_p \leq k - 1$ for any $p \notin J^{[-1]}$. We now discuss the error terms at order N^{-k_0-1}. These take the form:

$$\begin{aligned}(\Delta_{(k_0+1)} A_n)(x, \boldsymbol{x}_I) = &-\mathcal{D}_2[(\Delta_{k_0} W_{n+1})(\bullet_1, \bullet_2, \boldsymbol{x}_I) \\ &+ \sum_{\substack{I' \subseteq I \\ I' \neq \emptyset, I}} \sum_{k'=0}^{k_0} (\Delta_{k'} W_{|I'|+1})(\bullet_1, \boldsymbol{x}_{I'})\, (\Delta_{(k_0-k')} W_{n-|I'|})(\bullet_2, \boldsymbol{x}_{I \setminus I'})] \\ &- \sum_{i \in I} \mathcal{D}_1[(\Delta_{k_0} W_{n-1})(\bullet, \boldsymbol{x}_{I \setminus \{i\}})](x, x_i) \\ &- \sum^{*}_{\substack{J \vdash [\![1\,;\,r]\!] \\ I_1 \sqcup \cdots \sqcup I_{[J]} = I}} \sum_{\substack{k_1, \ldots, k_{[J]} \geq -1 \\ (\sum_i k_i) + r - 1 = k_0 + 1}} \mathcal{T}\left[\prod_{i=1}^{[J]} (\Delta_{k_i} W_{|J_i|+|I_i|})(\bullet_{J_i}, \boldsymbol{x}_{I_i})\right](x)\end{aligned} \tag{6.17}$$

and

$$(\Delta_{(k_0+1)}\mathcal{K})[\varphi](x) = 2\,\mathcal{D}_2[(\Delta_{k_0} W_1)(\bullet_1)\,\varphi(\bullet_2)](x)$$
$$+\sum_{i=1}^{r}\sum_{\substack{J\subseteq [\![1\,;\,r]\!]\setminus\{i\}\\ |J|\geq 1}}\sum_{\substack{k_1,\ldots,k_{|J|}\geq -1\\ (\sum_j k_j)+|J|=k_0+1}} \mathcal{T}\left[\varphi(\bullet_i)\prod_{j\in J}(\Delta_{k_j}W_1)(\bullet_j)\prod_{j\notin J}W_{\mathrm{eq}}(\bullet_j)\right](x).$$

For the reason invoked above, these expressions only involve $\Delta_k W_\ell$ with $k\leq k_0$. Note that, for $r=1$, the $\sum^*$ is empty. Further, one readily checks that the recursion hypothesis implies

$$\|\Delta_{(k_0+1)}A_n\|_{\Gamma^n}\leq \frac{c}{N}\quad\text{and}\quad \|\Delta_{(k_0+1)}\mathcal{K}[\varphi]\|_{\Gamma[1]}\leq\frac{c'}{N}\,\|\varphi\|_{\Gamma}.$$

One likewise obtains similar expressions at $n=1$, namely

$$A_1^{[k]}(x) = -\delta_{0,k}(1-2/\beta)\left\{\partial_x W_{\mathrm{eq}}(x)+\oint_S\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\sigma_{\mathrm{hd}}^{[2]}(x;\xi,\xi)}{\sigma_{\mathrm{hd}}(x)}\,W_{\mathrm{eq}}(\xi)\right\}-\mathcal{D}_2[W_2^{[k-1]}](x)$$
$$+\sum_{\substack{k_1,k_2\geq 0\\ k_1+k_2=k-1}}\mathcal{D}_2[W_1^{[k_1]}(\bullet_1)\,W_1^{[k_2]}(\bullet_2)]$$
$$+\sum_{\substack{J\subseteq [\![1\,;\,r]\!]\\ |J|\geq 2}}\sum_{\substack{k_1,\ldots,k_{|J|}\geq -1\\ (\sum_j k_j)+|J|=k+1}}(|J|-1)\,\mathcal{T}\left[\prod_{j\in J}W_1^{[k_j]}(\bullet_j)\prod_{j\notin J}W_{\mathrm{eq}}(\bullet_j)\right](x) \tag{6.18}$$

and

$$(\Delta_{(k_0+1)}A_1)(x) = -\mathcal{D}_2[(\Delta_{k_0}W_2)](x)+\sum_{\substack{k_1,k_2\geq 0\\ k_1+k_2=k_0}}\mathcal{D}_2[(\Delta_{k_1}W_1)(\bullet_1)\,(\Delta_{k_2}W_1)(\bullet_2)]$$
$$+\sum_{\substack{J\subseteq [\![1\,;\,r]\!]\\ |J|\geq 2}}\sum_{\substack{k_1,\ldots,k_{|J|}\geq -1\\ (\sum_j k_j)+|J|=k_0+2}}(|J|-1)\,\mathcal{T}\left[\prod_{j\in J}(\Delta_{k_j}W_1)(\bullet_j)\prod_{j\notin J}W_{\mathrm{eq}}(\bullet_j)\right](x)$$
$$-\sum_{J\vdash [\![1\,;\,r]\!]}^{*}\sum_{\substack{k_1,\ldots,k_{[J]}\geq -1\\ (\sum_i k_i)+r-1=k_0+1}}\mathcal{T}\left[\prod_{i=1}^{[J]}\Delta_{k_i}W_{|J_i|}(\bullet_{J_i})\right](x).$$

One checks that likewise, $A_1^{[k]}$ only involves $W_1^{[k']}$ with $k'<k$, and $W_2^{[k-1]}$. Similarly, $\Delta_{(k_0+1)}A_1$ only involves $\Delta_k W_1$ and $\Delta_k W_2$ with $k\leq k_0$ and satisfies to the bounds $\|\Delta_{(k_0+1)}A_1\|_{\Gamma}\leq c/N$.

By inserting the obtained expansions into the appropriate Schwinger-Dyson equations (6.3)–(6.5) one obtains for $n \geq 2$:

$$\mathcal{K}[W_n(\bullet,\boldsymbol{x}_I)](x) = \sum_{k=n-2}^{k_0+1} N^{-k}\Big(A_n^{[k]}(x;\boldsymbol{x}_I) - \sum_{\ell=n-2}^{k-1} \mathcal{K}^{[k-\ell]}[W_n^{[\ell]}(\bullet,\boldsymbol{x}_I)](x)\Big) + B_n(x;\boldsymbol{x}_I)$$

$$- N^{-(k_0+1)}\mathcal{K}[\Delta_{(k_0+1)}W_n(\bullet,\boldsymbol{x}_I)](x) - N^{-(k_0+1)}\sum_{\ell=n-2}^{k_0} \mathcal{K}^{[k_0+1-\ell]}[\Delta_\ell W_n(\bullet,\boldsymbol{x}_I)](x)$$

$$+ N^{-(k_0+1)}\Delta_{(k_0+1)}A_n^{[k]}(x;\boldsymbol{x}_I).$$

For $n=1$, the right-hand side is similar but the left-hand side should be replaced by $\mathcal{K}[N\Delta_{-1}W_1]$.

Using the continuity of $\mathcal{K}^{-1}$ and of the other operators involved the above equations yields a system of equations which determine the $W_n^{[k]}$ recursively on k. In particular, this implies that W_n admits an asymptotic expansion up to order k_0+1. By the recursion hypothesis at step k_0, the function to which we apply $\mathcal{K}^{-1}$ to obtain $W_n^{[k_0+1]}$ is holomorphic on $\mathbb{C}\setminus\mathsf{S}$. Therefore, $W_n^{[k_0+1]}$ is also holomorphic in $\mathbb{C}\setminus\mathsf{S}$, so that $W_n^{[k_0+1]} \in \mathscr{H}_0^2(\mathsf{S},n)$. We just proved that the recursion hypothesis holds at step k_0+1, so we can conclude by induction. To summarize, the recursive formula for the coefficient of expansion of the correlators is

$$W_n^{[k]} = \mathcal{K}^{-1}[A_n^{[k]}(\bullet;\boldsymbol{x}_I) - \sum_{\ell=n-2}^{k-1}\mathcal{K}^{[k-\ell]}[W_n^{[\ell]}(\cdot,\boldsymbol{x}_I)](\bullet)](x) \tag{6.19}$$

with $A_n^{[k]}$ given by (6.16)–(6.18) and $\mathcal{K}^{[\ell]}$ by (6.15). ■

7 Partition Function in the Fixed Filling Fraction Model

7.1 Asymptotic expansion

Lemma 7.1. Assume the local strict convexity of Hypothesis 3.2, the analyticity of Hypothesis 5.1, and that $\mu_{\mathrm{eq},\epsilon}$ is off-critical. There exists a 1-linear potential $\hat{T}$ satisfying the same assumptions and so that, for any k_0, we have an asymptotic expansion of the form:

$$\frac{Z^{T}_{\mathsf{A}_{\mathsf{N}}}}{Z^{\hat{T}}_{\mathsf{A}_{\mathsf{N}}}} = \exp\left(\sum_{k=-2}^{k_0} N^{-k}\,G_{\epsilon}^{[k]} + N^{-k_0}\Delta_{k_0}G_{\epsilon}\right).$$

$G_{\epsilon}^{[k]}$ are smooth functions of the filling fractions ϵ in a small enough domain, where $\mu_{\mathrm{eq}}^{\epsilon}$ remains off-critical, and for any fixed k_0, the error is uniform in ϵ in such a compact domain. □

Since the asymptotic expansion for 1-linear potentials and its smoothness with respect to ϵ have been established in [15] under the same assumptions as here, we obtain automatically:

Corollary 7.2. Assume the local strict convexity of Hypothesis 3.2, the analyticity of Hypothesis 5.1, and that $\mu_{\mathrm{eq}}^{\epsilon}$ is off-critical. The partition function with fixed filling fractions has an asymptotic expansion of the form

$$Z_{\mathsf{A}_{\mathsf{N}}}^{T} = N^{(\beta/2)N+\gamma} \exp\left(\sum_{k=-2}^{k_0} N^{-k} F_{\epsilon}^{[k]} + o(N^{-k_0})\right).$$

γ is a universal exponent depending only on β and the nature of the edges and reminded in Section 1.3. $F_{\epsilon}^{[k]}$ are smooth functions of ϵ and for any fixed k_0, the error is uniform in ϵ as explained in Lemma 7.1. □

7.2 Proof of Lemma 7.1: except regularity

The characterization (2.3) of equilibrium measure can be rephrased by saying that $\mu_{\mathrm{eq}}^{\epsilon,T} = \mu_{\mathrm{eq}}^{\epsilon,\hat{T}}$ where

$$\hat{T}(x_1, \ldots, x_r) = (r-1)! \sum_{j=1}^{r} \hat{T}_1(x_j)$$

is the 1-body interaction defined, if $x \in \mathsf{A}_h$, by

$$\hat{T}_1(x) = \sum_{\substack{h'=0 \\ h' \neq h}}^{g} \beta \int_{\mathsf{S}_{h'}} \ln|x-\xi|\, \mathrm{d}\mu_{\mathrm{eq}}^{\epsilon,T}(\xi) \mathbf{1}_{\mathsf{S}_{h'}}(\xi) + \int_{\mathsf{S}^{r-1}} \frac{T(x, \xi_2, \ldots, \xi_r)}{(r-1)!} \prod_{i=2}^{r} \mathrm{d}\mu_{\mathrm{eq}}^{\epsilon,T}(\xi_i). \tag{7.1}$$

Since A_h and $\mathsf{S}_{h'}$ are disjoint if $h' \neq h$, $(x-y)$ keeps the same sign for $x \in \mathsf{S}_{h'}$, $\hat{T}_1(x)$ has actually an analytic continuation for x in a neighborhood of each A_h. Besides, the characterization of the equilibrium measure implies that

$$\forall t \in [0,1], \quad \mu_{\mathrm{eq}}^{\epsilon, tT+(1-t)\hat{T}} = \mu_{\mathrm{eq}}^{\epsilon,T}.$$

So, if T satisfies our assumptions, $tT + (1-t)\hat{T}$ satisfies it uniformly for $t \in [0, 1]$. And, we have the general formula:

$$\partial_t \ln Z_{\mathsf{A}_{\mathsf{N}}}^{tT+(1-t)\hat{T}} = \frac{N^{2-r}}{r!} \oint_{\mathsf{A}^r} \frac{\mathrm{d}^r \boldsymbol{\xi}}{(2\mathrm{i}\pi)^r} \left(T(\boldsymbol{\xi}) - \hat{T}(\boldsymbol{\xi})\right) \tilde{W}_r^{tT+(1-t)\hat{T}}(\boldsymbol{\xi}) \tag{7.2}$$

in terms of the disconnected correlators introduced in (1.4). By Lemma 6.3, $\tilde{W}_r^{tT+(1-t)\hat{T}}(\boldsymbol{\xi})$ has an asymptotic expansion in $1/N$, starting at order N^r, and if we truncate it to an order N^{-k_0}, it is uniform in $\boldsymbol{\xi}$ on the contour of integration of (7.2) and in $t \in [0, 1]$. So, we can integrate (7.2) over $t \in [0, 1]$, exchange the expansion and the integrations to obtain the expansion of the partition function. The smoothness of the coefficients of expansion of the correlators with respect to filling fractions is a consequence of Proposition 7.4.

7.3 Lipschitz dependence in filling fractions

We first show that the equilibrium measures depend on the filling fractions in a Lipschitz way.

Lemma 7.3. Assume Hypothesis 2.1 in the unconstrained model and T holomorphic in a neighborhood of A in $\mathbb{C}$. Then, for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}^\star$, $\mathcal{E}$ has a unique minimizer over $\mathcal{M}^{\boldsymbol{\epsilon}}(\mathbf{A})$, denoted $\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$. Let $(g+1)$ be the number of cuts of μ_{eq}, and $\epsilon^\star = \mu_{\mathrm{eq}}[\mathsf{A}_h]$ its masses. Assume μ_{eq} is off-critical. Then, for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}^\star$, $\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$ still has $g+1$ cuts, is off-critical, has the edges of the same nature which are Lipschitz functions of $\boldsymbol{\epsilon}$, and the density of $\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$ is a Lipschitz function of $\boldsymbol{\epsilon}$ away from its edges. □

Proof. We remind (Section 2.1) that the level sets $E_M = \{\mu \in \mathcal{M}^1(\mathsf{A}),\ \mathcal{E}[\mu] \leq M\}$ are compact. Therefore, $\mathcal{E}$ achieves its minimum on $\mathcal{M}^{\boldsymbol{\epsilon}}(\mathbf{A})$, and we denote $\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$ any such minimizer. It must satisfy the saddle point equation (2.3). By assumption, the minimizer over $\mathcal{M}^1(\mathsf{A})$ is unique, it is denoted by μ_{eq}, and its partial masses $\epsilon_h^\star = \mu_{\mathrm{eq}}[\mathsf{A}_h]$. In other words, the minimizer $\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}^\star}$ is unique and equal to μ_{eq}. We must prove that the minimizer is unique for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}^\star$.

- We first show that any $\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$ must belong to a ball $B(\mu_{\mathrm{eq}}, \delta_{\boldsymbol{\epsilon}})$ around μ_{eq} for the Vasershtein distance, with $\delta_{\boldsymbol{\epsilon}}$ going to zero when $\boldsymbol{\epsilon}$ goes to $\boldsymbol{\epsilon}^\star$.

  Let us first prove that

$$\mathcal{E}^{\boldsymbol{\epsilon}} := \inf_{\mu \in \mathcal{M}^{\boldsymbol{\epsilon}}(\mathbf{A})} \mathcal{E}(\mu) \to \mathcal{E}^{\boldsymbol{\epsilon}^\star} \quad \text{as } \boldsymbol{\epsilon} \to \boldsymbol{\epsilon}^\star.$$

In fact, if we denote $\mu_{\rm eq}^h$ the probability on A_h so that $\mu_{\rm eq} = \sum_h \epsilon_h^\star \mu_{\rm eq}^h$, we have

$$\mathcal{E}^{\boldsymbol{\epsilon}^\star} \leq \mathcal{E}^{\boldsymbol{\epsilon}} \leq \mathcal{E}\left(\sum_h \epsilon_h \mu_{\rm eq}^h\right).$$

But we have seen that $\int \log|x-y| \mathrm{d}\mu_{\rm eq}^h(y)$ is uniformly bounded on A for all i and therefore one easily sees that there exists a finite constant C such that

$$\mathcal{E}\left(\sum_h \epsilon_h \mu_{\rm eq}^h\right) \leq \mathcal{E}\left(\sum_h \epsilon_h^\star \mu_{\rm eq}^h\right) + C \max|\epsilon_h - \epsilon_h^\star|$$

from which the announced continuity follows.

Let us deduce by contradiction that there exists a sequence δ_ϵ so that $\mu_{\rm eq}^{\boldsymbol{\epsilon}} \in B(\mu_{\rm eq}, \delta_\epsilon)$ for $|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^\star|$ small enough. Otherwise, we can find a $\delta > 0$ and a sequence $\mu_{\rm eq}^{\boldsymbol{\epsilon}_n} \notin B(\mu_{\rm eq}, \delta)$ with $\boldsymbol{\epsilon}_n$ converging to $\boldsymbol{\epsilon}^\star$. As we can assume by the above continuity that this sequence belongs to the level set $E_{\mathcal{E}^{\boldsymbol{\epsilon}^\star}+1}$, this sequence is tight and we can consider a limit point μ. But by lower semi-continuity of $\mathcal{E}$, we must have

$$\liminf_{n\to\infty} \mathcal{E}(\mu_{\rm eq}^{\boldsymbol{\epsilon}_n}) \geq \mathcal{E}(\mu),$$

whereas the previous continuity shows that the left-hand side is actually equal to $\mathcal{E}^{\boldsymbol{\epsilon}^\star}$. Hence μ minimizes $\mathcal{E}$ on $\mathcal{M}^1(\mathsf{A})$ which implies by Hypothesis 2.1 that $\mu = \mu_{\rm eq}$ hence yielding the announced contradiction.

- We now show uniqueness of the minimizer of $\mathcal{E}$ on $\mathcal{M}^{\boldsymbol{\epsilon}}(\mathbf{A})$ for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}^\star$ by showing that the interaction keeps the property of local strict convexity.

  Let us define, for any $\nu \in \mathcal{M}^0(\mathsf{A})$:

$$\mathcal{Q}^{\boldsymbol{\epsilon}}[\nu] = \beta\,\mathcal{Q}_C[\nu] - \int_{\mathsf{A}^{r-2}} \frac{T(x_1, \ldots, x_r)}{(r-2)!} \mathrm{d}\nu(x_1)\,\mathrm{d}\nu(x_2) \prod_{j=3}^r \mathrm{d}\mu_{\rm eq}^{\boldsymbol{\epsilon}}(x_j),$$

where $\mathcal{Q}_C$ was defined in (3.2). For any probability measure μ, we can write using Taylor-Lagrange formula at order 3 around $\mu_{\rm eq}^{\boldsymbol{\epsilon}}$:

$$\mathcal{E}[\mu] = \mathcal{E}[\mu_{\rm eq}^{\boldsymbol{\epsilon}}] - \int T_{\rm eff}^{\boldsymbol{\epsilon}}(x)\,\mathrm{d}\mu(x) + \frac{1}{2}\,\mathcal{Q}^{\boldsymbol{\epsilon}}[\mu - \mu_{\rm eq}^{\boldsymbol{\epsilon}}] + \mathcal{R}_3^{\boldsymbol{\epsilon}}[\mu - \mu_{\rm eq}^{\boldsymbol{\epsilon}}], \tag{7.3}$$

where $T_{\rm eff}^{\boldsymbol{\epsilon}}$ is the effective potential (2.4) for $\mu_{\rm eq}^{\boldsymbol{\epsilon}}$. The remainder is

$$\mathcal{R}_3^{\boldsymbol{\epsilon}}[\nu] = \int_0^1 \frac{\mathrm{d}t(1-t)^2}{2} \mathcal{E}^{(3)}[(1-t)\mu_{\rm eq}^{\boldsymbol{\epsilon}} + t\mu]\cdot(\nu, \nu, \nu), \tag{7.4}$$

where $\mathcal{E}^{(3)}$ was already defined in (3.11). If $\boldsymbol{\kappa} \in \mathbb{R}_+^{g+1}$ so that $\sum_h \kappa_h = 1$, for any measure $\mu^{\kappa} \in \mathcal{M}^{\kappa}(\mathbf{A})$, we have

$$\mathcal{E}[\mu_{\mathrm{eq}}^{\kappa}] \leq \mathcal{E}[\mu^{\kappa}].$$

We now use the equality (7.3) for both sides, and assume the support of μ^{κ} is included in the support of $\mu_{\mathrm{eq}}^{\epsilon}$. Since $T_{\mathrm{eff}}^{\epsilon}$ is nonpositive, and equal to 0 on the support of $\mu_{\mathrm{eq}}^{\epsilon}$, we find

$$\frac{1}{2}\,\mathcal{Q}^{\epsilon}[\mu_{\mathrm{eq}}^{\kappa} - \mu_{\mathrm{eq}}^{\epsilon}] \leq \frac{1}{2}\,\mathcal{Q}^{\epsilon}[\mu^{\kappa} - \mu_{\mathrm{eq}}^{\kappa}] + \mathcal{R}_3^{\epsilon}[\mu^{\kappa} - \mu_{\mathrm{eq}}^{\epsilon}] - \mathcal{R}_3^{\epsilon}[\mu_{\mathrm{eq}}^{\kappa} - \mu_{\mathrm{eq}}^{\epsilon}]. \tag{7.5}$$

We assume $\kappa_h \in [0, 2\epsilon_h]$, and put $\mu^{\kappa} = t\mu_{\mathrm{eq}}^{\epsilon} + (1-t)\mu_{\mathrm{ref}}$ with the choice

$$1 - t = \max_{0 \leq h \leq g} \frac{|\kappa_h - \epsilon_h|}{\epsilon_h} \in [0, 1], \tag{7.6}$$

and the choice of a probability measure μ_{ref}, whose support is included in that of $\mu_{\mathrm{eq}}^{\epsilon}$, and with masses satisfying

$$t\epsilon_h + (1-t)\mu_{\mathrm{ref}}[\mathsf{A}_h] = \kappa_h.$$

Note that we can assume $\boldsymbol{\epsilon}_h \neq 0$ since $\boldsymbol{\epsilon}_h^{\star} \neq 0$ follows from the assumption that μ_{eq} is off-critical. We also require that μ_{ref} is such that $\mathcal{Q}^{\epsilon}[\mu_{\mathrm{ref}} - \mu_{\mathrm{eq}}^{\epsilon}] < +\infty$, which is always possible, for instance by taking for μ_{ref} the renormalized Lebesgue measure on the support of $\mu_{\mathrm{eq}}^{\epsilon}$.

Using $\mathcal{Q} = \mathcal{Q}^{\epsilon^{\star}}$, we know from Lemma 3.3 that the remainder can be bounded as

$$|\mathcal{R}_3^{\epsilon}[\nu]| \leq C^{\epsilon}\,\|\nu\|\,\mathcal{Q}[\nu].$$

Therefore, we have

$$\begin{aligned} &\mathcal{Q}^{\epsilon}[\mu_{\mathrm{eq}}^{\kappa} - \mu_{\mathrm{eq}}^{\epsilon}] - C^{\epsilon}\,\|\mu_{\mathrm{eq}}^{\kappa} - \mu_{\mathrm{eq}}^{\epsilon}\|\,\mathcal{Q}[\mu_{\mathrm{eq}}^{\kappa} - \mu_{\mathrm{eq}}^{\epsilon}] \\ &\quad \leq (1-t)^2\,\mathcal{Q}^{\epsilon}[\mu_{\mathrm{ref}} - \mu_{\mathrm{eq}}^{\epsilon}] + (1-t)^3\,C^{\epsilon}\,\|\mu_{\mathrm{ref}} - \mu_{\mathrm{eq}}^{\epsilon}\|\,\mathcal{Q}[\mu_{\mathrm{ref}} - \mu_{\mathrm{eq}}^{\epsilon}]. \end{aligned} \tag{7.7}$$

If we apply this inequality to $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\star}$, and $\boldsymbol{\kappa}$ close enough to $\boldsymbol{\epsilon}^{\star}$ so that

$$C^{\epsilon^{\star}}\,\|\mu_{\mathrm{eq}}^{\kappa} - \mu_{\mathrm{eq}}^{\epsilon^{\star}}\| \leq \tfrac{1}{2}$$

(this is possible by the continuity previously established), we deduce from (7.6) that for $\max_h |\kappa_h - \epsilon_h| < c'$ for some $c' > 0$ independent of $\boldsymbol{\epsilon}$:

$$\mathcal{Q}[\mu_{\mathrm{eq}}^{\boldsymbol{\kappa}} - \mu_{\mathrm{eq}}] \leq C \, \max_h |\kappa_h - \epsilon_h^{\star}|^2.$$

Besides, we may compare $\mathcal{Q}^{\boldsymbol{\epsilon}}$ and $\mathcal{Q} = \mathcal{Q}^{\boldsymbol{\epsilon}^{\star}}$ by writing

$$\mathcal{Q}^{\boldsymbol{\epsilon}}[\nu] - \mathcal{Q}[\nu] = \sum_{m=3}^{r} \int_{\mathsf{A}^r} \frac{T(\xi_1, \ldots, \xi_r)}{(r-2)!} \, \mathrm{d}\nu(\xi_1)\, \mathrm{d}\nu(\xi_2)\, \mathrm{d}(\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}} - \mu_{\mathrm{eq}})(\xi_3) \prod_{i=4}^{m} \mathrm{d}\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}(\xi_i) \times \prod_{j=m+1}^{r} \mathrm{d}\mu_{\mathrm{eq}}(\xi_j).$$

Hence, from Lemma 3.3, there exists a constant $C > 0$, independent of $\boldsymbol{\epsilon}$ but depending on T, so that

$$|\mathcal{Q}^{\boldsymbol{\epsilon}}[\nu] - \mathcal{Q}[\nu]| \leq C\, \mathcal{Q}[\nu]\, \mathcal{Q}^{1/2}[\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}} - \mu_{\mathrm{eq}}] \leq C' \mathcal{Q}[\nu] \, \max_h |\epsilon_h - \epsilon_h^{\star}|.$$

Therefore, for $\max_h C'|\epsilon_h - \epsilon_h^{\star}| < 1$, there exists constants $c_1, c_2 > 0$ so that

$$\forall \nu \in \mathcal{M}^0(\mathsf{A}), \qquad c_1\, \mathcal{Q}[\nu] \leq \mathcal{Q}^{\boldsymbol{\epsilon}}[\nu] \leq c_2\, \mathcal{Q}[\nu], \tag{7.8}$$

in particular $\mathcal{Q}^{\boldsymbol{\epsilon}}[\nu] \geq 0$ with equality iff $\nu = 0$. So, coming back to (7.7), we deduce for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}^{\star}$ and $\boldsymbol{\kappa}$ close enough to $\boldsymbol{\epsilon}$ that there exists a constant c' such that

$$\mathcal{Q}[\mu_{\mathrm{eq}}^{\boldsymbol{\kappa}} - \mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}] \leq c'(1-t)^2\, \mathcal{Q}[\mu_{\mathrm{ref}} - \mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}]$$

with t as in (7.6). This entails

$$\mathcal{Q}[\mu_{\mathrm{eq}}^{\boldsymbol{\kappa}} - \mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}] \leq C \, \max_h |\kappa_h - \epsilon_h|^2. \tag{7.9}$$

We can apply this relation with $\boldsymbol{\epsilon} = \boldsymbol{\kappa}$ but $\mu_{\mathrm{eq}}^{\boldsymbol{\kappa}}$ another minimizer of $\mathcal{E}$ over $\mathcal{M}^{\boldsymbol{\epsilon}}(\mathbf{A})$, which would tell us that the $\mathcal{Q}$-distance between two minimizers is 0. So, the minimizer is unique for any $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}^{\star}$.

- We finally prove smoothness of the minimizing measures and related quantities. As a second consequence of (7.8) and (7.9), for any $m \geq 1$ and any smooth test function φ of m variables, we have a finite constant C_{φ} so that

$$\left|\int \varphi(\xi_1,\ldots,\xi_m)\left(\prod_{j=1}^{m}\mathrm{d}\mu_{\mathrm{eq}}^{\kappa}(\xi_j)-\prod_{j=1}^{m}\mathrm{d}\mu_{\mathrm{eq}}^{\epsilon}(\xi_j)\right)\right| \leq C_{\varphi}\,\mathcal{Q}^{1/2}[\mu_{\mathrm{eq}}^{\kappa}-\mu_{\mathrm{eq}}^{\epsilon}]$$
$$\leq C\max_{h}|\kappa_h-\epsilon_h|. \qquad (7.10)$$

In other words, the integrals of m-variables test functions in $\mathcal{H}(m)$ against $\mu_{\mathrm{eq}}^{\epsilon}$ (called below m-linear statistics) are Lipschitz in the variable $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}^{\star}$.

To extend this regularity result to the equilibrium measure, we consider the expression (2.13) for its density:

$$\frac{\mathrm{d}\mu_{\mathrm{eq}}}{\mathrm{d}x}(x)=\frac{\mathbf{1}_{\mathsf{S}_{\epsilon}}(x)}{2\pi}\sqrt{R_{\boldsymbol{\epsilon}}(x)},\qquad R_{\boldsymbol{\epsilon}}(x)=\frac{4\tilde{P}_{\boldsymbol{\epsilon}}(x)-\sigma_{\mathsf{A}}(x)(V_{\boldsymbol{\epsilon}}'(x))^2}{\sigma_{\mathsf{A}}(x)}. \qquad (7.11)$$

The important feature of this formula is that $V_{\boldsymbol{\epsilon}}'(x)$ (respectively, $\tilde{P}_{\boldsymbol{\epsilon}}(x)$) defined in (2.10)–(2.11) are integrals against $\mu_{\mathrm{eq}}^{\epsilon}$ of a holomorphic test function in a neighborhood of A^{r-1} (respectively, A^{r}) which depends holomorphically in x in a neighborhood of A. Thanks to (7.10), $x\mapsto R_{\boldsymbol{\epsilon}}(x)$ is a holomorphic function when x belongs to a compact neighborhood K (independent of $\boldsymbol{\epsilon}$) of A avoiding the hard edges, which has a Lipschitz dependence in $\boldsymbol{\epsilon}$. Thus, the density itself is a Lipschitz function of $\boldsymbol{\epsilon}$ away from the edges. The edges of the support of $\mu_{\mathrm{eq}}^{\epsilon}$ are precisely the zeroes and the poles of $R_{\boldsymbol{\epsilon}}(x)$ in K. If we assume that $\mu_{\mathrm{eq}}^{\epsilon}$ is off-critical, then these zeroes and poles are simple. So, they must remain simple zeroes (respectively, simple poles) for $\boldsymbol{\epsilon}'$ close enough to $\boldsymbol{\epsilon}$, and their dependence in $\boldsymbol{\epsilon}'$ is Lipschitz. ■

7.4 Smooth dependence in filling fractions

Proposition 7.4. Lemma 7.3 holds with $\mathcal{C}^{\infty}$ dependence in $\boldsymbol{\epsilon}$. □

Corollary 7.5. Under the same assumptions, the coefficients of expansion of the correlators $W_n^{[k];\boldsymbol{\epsilon}}(x_1,\ldots,x_n)$ depends smoothly on $\boldsymbol{\epsilon}$ for $x_1,\ldots,x_n$ uniformly in any compact of $\mathbb{C}\setminus\mathsf{A}$. □

Proof. The idea of the proof is again very similar to [15, Appendix A.2]. Let E be an open neighborhood of $\boldsymbol{\epsilon}^{\star}$ in

$$\left\{\boldsymbol{\epsilon}\in[0,1]^{g+1}\ \sum_{h}\epsilon_h=1\right\}$$

so that the result of Lemma 7.3 holds. For any given x in a compact neighborhood K of A avoiding the edges, $R_{\boldsymbol{\epsilon}}(x)$ is Lipschitz function of $\boldsymbol{\epsilon}$, therefore differentiable for $\boldsymbol{\epsilon}$ in a subset E_x whose complement in E has measure 0. By Baire theory, $\mathsf{E}_{\infty} = \bigcap_{x \in \mathsf{K}} \mathsf{E}_x$ has still a complement of measure 0, therefore is dense in E. For any $\boldsymbol{\epsilon} \in \mathsf{E}_{\infty}$ and any $\boldsymbol{\eta} \in \mathbb{R}^{g+1}$ so that $\sum_h \eta_h = 0$, we can then study the effect of differentiation at $\boldsymbol{\epsilon}$ in a direction $\boldsymbol{\eta}$ in the characterization and properties of the equilibrium measure. We find that $\mathrm{d}\nu_{\boldsymbol{\eta}}^{\boldsymbol{\epsilon}} = \partial_{t=0}\, \mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}+t\boldsymbol{\eta}}$ defines a signed measure on $\mathsf{S}_{\boldsymbol{\epsilon}}$ which is integrable (its density behaves atmost like the inverse of a squareroot at the edges), gives a mass η_h to $\mathsf{S}_{\boldsymbol{\epsilon},h}$, and satisfies

$$\forall x \in \mathring{\mathsf{S}}_{\boldsymbol{\epsilon}}, \qquad \beta \fint_{\mathsf{S}_{\boldsymbol{\epsilon}}} \frac{\mathrm{d}\nu_{\boldsymbol{\eta}}^{\boldsymbol{\epsilon}}(\xi)}{x-\xi} + \int_{\mathsf{S}_{\boldsymbol{\epsilon}}^{r-1}} \frac{\partial_x T(x,\xi_2,\ldots,\xi_r)}{(r-2)!}\, \mathrm{d}\nu_{\boldsymbol{\eta}}^{\boldsymbol{\epsilon}}(\xi_2) \prod_{j=3}^{r} \mathrm{d}\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}(\xi_j) = 0.$$

We have seen in the proof of Lemma 5.1 that there is a unique solution to this problem. As a matter of fact, if we introduce the Stieltjes transform:

$$\varphi_{\boldsymbol{\eta}}^{\boldsymbol{\epsilon}}(x) = \int_{\mathsf{S}_{\boldsymbol{\epsilon}}} \frac{\mathrm{d}\nu_{\boldsymbol{\eta}}^{\boldsymbol{\epsilon}}(\xi)}{x-\xi},$$

by construction of the operator $\mathcal{K}$ we have $\mathcal{K}[\varphi_{\boldsymbol{\eta}}^{\boldsymbol{\epsilon}}] = 0$ and the condition on masses is $\oint_{\mathsf{S}_{\boldsymbol{\epsilon},h}} \varphi_{\boldsymbol{\eta}}^{\boldsymbol{\epsilon}}(\xi)\, \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} = \eta_h$ for any h. So, the invertibility of $\mathcal{K}$ used toward the end of the proof of Lemma 5.1 gives

$$\varphi_{\boldsymbol{\eta}}^{\boldsymbol{\epsilon}}(x) = -\sum_{h=0}^{g} \eta_h\, \mathscr{R}_{\mathcal{N}}(x,h), \tag{7.12}$$

where $\mathscr{R}_{\mathcal{N}}(x,h)$ is one of the blocks of the resolvent kernel of $\mathcal{N}$. Eventually, we observe that $\mathcal{K}$ depends on $\boldsymbol{\epsilon}$ only via the Stieltjes transform of $\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$, therefore is Lipschitz in $\boldsymbol{\epsilon}$. The expression (A.15) for the resolvent kernel implies that if $\mathcal{K}$ depends on a parameter (here $\boldsymbol{\epsilon}$) with a certain regularity, its inverse depends on the parameter with the same regularity. Therefore, the right-hand side of (7.12) is Lipschitz in $\boldsymbol{\epsilon}$, a fortiori continuous. To summarize, we have obtained that

$$W_{\mathrm{eq}}^{\boldsymbol{\epsilon}}(x) = \int_{\mathsf{S}_{\boldsymbol{\epsilon}}} \frac{\mathrm{d}\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}(\xi)}{x-\xi}$$

is differentiable at a dense subset of $\boldsymbol{\epsilon}$, and that the differential happens to be a continuous function of $\boldsymbol{\epsilon}$. Therefore, $W_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$ is differentiable everywhere, and (7.12) can be considered as a differential equation, where the right-hand side is differentiable. Hence $W_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$ is twice differentiable. This regularity then carries to the right-hand side, and by induction, this entails the $\mathcal{C}^{\infty}$ regularity of $W_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$—and thus the density of $\mu_{\mathrm{eq}}^{\boldsymbol{\epsilon}}$—for any x away from the edges of $\mathsf{S}_{\boldsymbol{\epsilon}}$. Therefore, $R_{\boldsymbol{\epsilon}}$ in (7.11) was $\mathcal{C}^{\infty}$ in $\boldsymbol{\epsilon}$, and the result of Lemma 7.3 is improved to $\mathcal{C}^{\infty}$ regularity in $\boldsymbol{\epsilon}$. ■

Proof of Corollary 7.5. From the Proof of Section 6.3, the coefficient of expansion of correlators $W_n^{[k]}(x_1, \ldots, x_n)$ (cf. (6.19)) and the errors $\Delta_k W_n(x_1, \ldots, x_n)$ can be computed recursively, by successive applications of $\mathcal{K}^{-1}$ to combinations involving $W_{\mathrm{eq}}^{\epsilon} = W_1^{[-1],\epsilon}(x)$, T and the $W_n^{[k'],\epsilon}$ computed at the previous steps. As we have seen, $\mathcal{K}^{-1}$ and $W_{\mathrm{eq}}^{\epsilon}$ depend smoothly on ϵ under the conditions stated above, so the $W_n^{[k]}$ enjoy the same property, and similarly one can show that the bounds on the errors $\Delta_k W_n$ are uniform with respect to ϵ. ■

End of the proof of Lemma 7.1. After the proof of the corollary, we just have to check that $T = \hat{T}$ given by (7.1) depends smoothly on ϵ. Since it is expressed as the integration of an analytic function against (several copies of) μ_{eq}, this follows from the proof of Section 7.3 and its improvement in Proposition (7.4). ■

7.5 Strict convexity of the energy

We show that the value of the energy functional at $\mu_{\mathrm{eq}}^{\epsilon}$ is a strictly convex function of the filling fraction in a neighborhood of $\epsilon^{\star}$. This property is useful in the analysis of the unconstrained model in the multi-cut regime.

Proposition 7.6. Assume Hypothesis 2.1 in the unconstrained model, T holomorphic in a neighborhood of A in $\mathbb{C}$, μ_{eq} is off-critical. Denote $\epsilon_h^{\star} = \mu_{\mathrm{eq}}[\mathsf{A}_h]$. Then, for ϵ in a neighborhood of $\epsilon^{\star}$, $\mathcal{E}[\mu_{\mathrm{eq}}^{\epsilon}]$ is $\mathcal{C}^2$ and its Hessian is definite positive. □

Proof. Proposition 7.3 ensures the existence of $\mu_{\mathrm{eq}}^{\epsilon}$ for ϵ close enough to $\epsilon^{\star}$. It is characterized, for any $h \in [\![0\,;\,g]\!]$ and $x \in \mathsf{S}_{\epsilon,h}$, by

$$\beta \int_{\mathsf{S}_\epsilon} \ln|x-\xi| \mathrm{d}\mu_{\mathrm{eq}}^{\epsilon}(\xi) + \int_{\mathsf{S}_\epsilon^{r-1}} \frac{T(x,\xi_2,\ldots,\xi_r)}{(r-1)!} \prod_{j=2}^{r} \mathrm{d}\mu_{\mathrm{eq}}^{\epsilon}(\xi_j) = C_h^{\epsilon}.$$

For any $x \in \mathsf{A}$, the left-hand side minus the right-hand side defines the effective potential $T_{\mathrm{eff}}^{\epsilon}$, which is therefore 0 for $x \in \mathsf{S}_{\epsilon}$. The proof of Proposition 7.4 provides us, for any $\eta \in \mathbb{R}^{g+1}$ so that $\sum_{h=0}^{g} \eta_h = 0$, with the existence of:

$$\nu_{\eta}^{\epsilon} = \partial_{t=0}\, \mu_{\mathrm{eq}}^{\epsilon + t\eta},$$

as a signed, integrable measure on A, so that $\nu_{\eta}^{\epsilon}[\mathsf{A}_h] = \eta_h$ for any h. It satisfies, for any $h \in [\![0\,;\,g]\!]$ and $x \in \mathsf{S}_{\epsilon,h}$:

$$\beta \int_{\mathsf{S}_\epsilon} \ln|x-\xi| \mathrm{d}\nu_{\eta}^{\epsilon}(\xi) + \int_{\mathsf{S}_\epsilon^{r-1}} \frac{T(x,\xi_2,\ldots,\xi_r)}{(r-2)!} \mathrm{d}\nu_{\eta}^{\epsilon}(\xi_2) \prod_{j=3}^{r} \mathrm{d}\mu_{\mathrm{eq}}^{\epsilon} = \partial_{t=0}\, C_h^{\epsilon+t\eta}. \tag{7.13}$$

Therefore, $\mathcal{E}[\mu_{\mathrm{eq}}^{\epsilon}]$ is $\mathcal{C}^1$ and we have

$$\partial_{t=0}\,\mathcal{E}[\mu_{\mathrm{eq}}^{\epsilon+t\eta}] = -\int_{\mathsf{S}_{\epsilon}}\left(T_{\mathrm{eff}}^{\epsilon}(x) + \sum_{h=0}^{g} C_{\epsilon,h}\,\mathbf{1}_{\mathsf{S}_{\epsilon,h}}(x)\right)\mathrm{d}\nu_{\eta}^{\epsilon}(x) = -\sum_{h=0}^{g} C_{h}^{\epsilon}\,\eta_h.$$

We can differentiate the result once more:

$$\mathrm{Hessian}_{\epsilon}\,\mathcal{E}[\mu_{\mathrm{eq}}^{\epsilon}]\cdot(\boldsymbol{\eta},\boldsymbol{\eta}') = \partial_{t'=0}\,\partial_{t=0}\,\mathcal{E}[\mu_{\mathrm{eq}}^{\epsilon+t\eta+t'\eta'}] = -\sum_{h=0}^{g}(\partial_{t'=0}\,C_{h}^{\epsilon+t'\eta'})\,\eta_h. \tag{7.14}$$

Since the right-hand side of (7.13) is constant on each $\mathsf{S}_{\epsilon,h}$, we integrate it against $-\mathrm{d}\nu_{\eta}^{\epsilon}$ and find a result equal to the right-hand side of (7.14):

$$\mathrm{Hessian}_{\epsilon}\,\mathcal{E}[\mu_{\mathrm{eq}}^{\epsilon}]\cdot(\boldsymbol{\eta},\boldsymbol{\eta}') = \mathcal{Q}^{\epsilon}[\nu_{\eta}^{\epsilon},\nu_{\eta'}^{\epsilon}], \tag{7.15}$$

where we have recognized the bilinear functional $\mathcal{Q}^{\epsilon}$ introduced in Section 3.2. By Hypothesis 3.2, we deduce that the Hessian at $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^{\star}$ is definite positive, and (7.8) actually shows that this remains true for $\boldsymbol{\epsilon}$ in a vicinity of $\boldsymbol{\epsilon}^{\star}$. ■

8 Asymptotics in the Multi-Cut Regime

8.1 Partition function

We have gathered all the ingredients needed to analyze the partition function in the $(g+1)$ regime when $g \geq 1$, decomposed as

$$Z_{\mathsf{A}^N} = \sum_{N_0+\cdots+N_g=N}\frac{N!}{\prod_{h=0}^{g} N_h!}\,Z_{\mathsf{A}_{\mathbf{N}}}.$$

Given the large deviations of filling fractions (Corollary 3.7), the expansion in $1/N$ of the partition function at fixed filling fractions $\boldsymbol{\epsilon}$ close to $\boldsymbol{\epsilon}^{\star}$ (Corollary 7.2), the smooth dependence of the coefficients in $\boldsymbol{\epsilon}$ (Proposition 7.4) and the positivity of the Hessian of $F^{[-2]} = -\mathcal{E}[\mu_{\mathrm{eq}}^{\epsilon}]$ (Lemma 7.6), the proof of the asymptotic expansion of Z_{A^N} is identical to [15, Section 8]. When necessary, we identify $\boldsymbol{\epsilon}^{\star}$ with an element of $\mathbb{R}^g$ by forgetting the component $\epsilon_0^{\star}$. To summarize the idea of the proof, one first restricts the sum in (8.1) over vectors $\boldsymbol{N} = (N_1,\ldots,N_g)$ such that $|\boldsymbol{N} - N\boldsymbol{\epsilon}^{\star}| \leq \sqrt{N\ln N}$ up to exponentially small corrections thanks to the large deviations of filling fractions, cf. Corollary 3.7. Since the filling fractions $\boldsymbol{\epsilon}$ kept in this sum are close to $\boldsymbol{\epsilon}^{\star}$, one can write down the $1/N$-expansion of each term in the sum. Then, one performs a Taylor expansion around $\boldsymbol{\epsilon}^{\star}$ of the coefficients of the latter expansion, and one can exchange the finite (although large) sum over $\boldsymbol{N}$ with the Taylor expansion while controlling the error terms. Eventually, one recognizes the answer as the general term (in $\boldsymbol{N}$) of an exponentially fast converging series, so we can actually lift the restriction $\boldsymbol{N} - N\boldsymbol{\epsilon}^\star$ to sum over all $\boldsymbol{N} \in \mathbb{Z}^g$ up to an error $o(\mathrm{e}^{-cN})$. The result can be expressed in terms of:

- The Theta function:

$$\Theta_{\boldsymbol{\gamma}}(\boldsymbol{v}|\boldsymbol{T}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp\left(-\frac{1}{2}\,(\boldsymbol{m}+\boldsymbol{\gamma})\cdot\boldsymbol{T}\cdot(\boldsymbol{m}+\boldsymbol{\gamma}) + \boldsymbol{v}\cdot(\boldsymbol{m}+\gamma)\right),$$

  where $\boldsymbol{T}$ is a symmetric definite positive $g\times g$ matrix, $\boldsymbol{v}\in\mathbb{C}^g$ and $\boldsymbol{\gamma}\in\mathbb{C}^g \bmod \mathbb{Z}^g$.
- The ℓth-order derivative of $F^{[k]}_{\boldsymbol{\epsilon}}$ with respect to the filling fractions. For a precise definition, we consider the canonical basis $(\boldsymbol{e}^h)_{0\le h\le g}$ of $\mathbb{R}^{g+1}$, and introduce $\boldsymbol{\eta}^h = \boldsymbol{e}^h - \boldsymbol{e}^0$ for $h\in [\![1\,;\, g]\!]$. Then, we can define the tensor of ℓth-order derivatives as an element of $(\mathbb{R}^g)^{\otimes \ell}$:

$$F^{[k],(\ell)}_{\boldsymbol{\epsilon}} = \sum_{1\le h_1,\ldots,h_\ell\le g} (\partial_{t_1=0}\cdots\partial_{t_\ell=0} F^{[k]}_{\boldsymbol{\epsilon}+\sum_{i=1}^{\ell} t_i\,\boldsymbol{\eta}^{h_i}}) \bigotimes_{i=1}^{\ell} \boldsymbol{e}^{h_i}.$$

When necessary, we identify $\boldsymbol{\epsilon}^\star$ with an element of $\mathbb{R}^g$ by forgetting the component ϵ_0.

Theorem 8.1. Assume Hypothesis 2.1, T holomorphic in a neighborhood of A^r, and μ_{eq} off-critical. Then, for any k_0, we have an asymptotic expansion of the form:

$$Z_{\mathsf{A}^N} = N^{(\beta/2)N+\gamma}\,\exp\left(\sum_{k=-2}^{k_0} N^{-k}\,F^{[k]}_{\boldsymbol{\epsilon}^\star} + o(N^{-k_0})\right)$$
$$\times\left\{\sum_{m\ge 0}\ \sum_{\substack{\ell_1,\ldots,\ell_m\ge 1\\ k_1,\ldots,k_r\ge -2\\ \sum_{i=1}^m \ell_i+k_i>0}} \frac{N^{-\sum_{i=1}^m(\ell_i+k_i)}}{m!}\left(\bigotimes_{i=1}^{m}\frac{F^{[k_i],(\ell_i)}_{\boldsymbol{\epsilon}^\star}}{\ell_i!}\right)\cdot\nabla_{\boldsymbol{v}}^{\otimes(\sum_{i=1}^m \ell_i)}\right\}\Theta_{-N\boldsymbol{\epsilon}^\star}(\boldsymbol{F}^{[-1],(1)}_{\boldsymbol{\epsilon}^\star}\,|\,\boldsymbol{F}^{[-2],(2)}_{\boldsymbol{\epsilon}^\star}). \tag{8.1}$$

□

8.2 Fluctuations of linear statistics

We mention that, along the line of [15, Corollary 6.4; 48], it is possible to show Theorem 8.1 while allowing T to contain $1/N$ complex-valued contribution on A—still under the assumptions that T is analytic. Then, for any test function φ holomorphic is a neighborhood of A, it follows for the fluctuations of the linear statistics:

$$X_N[\varphi] = \sum_{i=1}^{N} \varphi(\lambda_i) - N \int \varphi(x)\, \mathrm{d}\mu_{\mathrm{eq}}(x)$$

that for any $s \in \mathbb{R}$, we have

$$\mu_{\mathsf{A}_N}[\mathrm{e}^{\mathrm{i}s X_N[\varphi]}] \underset{N\to\infty}{=} \mathrm{e}^{\mathrm{i}s\, M_1[\varphi] - s^2\, M_2[\varphi]}\, \frac{\Theta_{-N\epsilon^\star}(F_{\epsilon^\star}^{[-1],(1)} + \mathrm{i}s\, \boldsymbol{w}[\varphi] \,|\, F_{\epsilon^\star}^{[-2],(2)})}{\Theta_{-N\epsilon^\star}(F_{\epsilon^\star}^{[-1],(1)} \,|\, F_{\epsilon^\star}^{[-2],(2)})}$$

with:

$$\begin{aligned} M_1[\varphi] &= \oint_{\mathsf{S}} \frac{\mathrm{d}x}{2\mathrm{i}\pi}\, \varphi(x)\, W_{1;\epsilon^\star}^{[0]}(x), \\ M_2[\varphi] &= \oint_{\mathsf{S}^2} \frac{\mathrm{d}x_1\, \mathrm{d}x_2}{(2\mathrm{i}\pi)^2}\, \varphi(\xi_1)\varphi(\xi_2)\, W_{2;\epsilon^\star}^{[0]}(x_1, x_2), \\ \boldsymbol{w}[\varphi] &= \sum_{h=1}^{g} \left(\int_{\mathsf{S}} \varphi(x)\, \mathrm{d}\nu_{\eta^h}^{\epsilon^\star}(x) \right) \boldsymbol{e}^h = \sum_{h=1}^{g} \partial_{t=0} \left(\int \varphi(x)\, \mathrm{d}\mu_{\mathrm{eq}}^{\epsilon + t\eta^h}(x) \right) \boldsymbol{e}^h, \end{aligned} \tag{8.2}$$

where we recall that $\eta^h = \boldsymbol{e}^h - \boldsymbol{e}^0$ and $(\boldsymbol{e}^h)_{0\le h\le g}$ is the canonical basis of $\mathbb{R}^{g+1}$. At this point, the regularity of φ can be weakened by going to Fourier space, see [15, Section 6.1] for details. We deduce a central limit theorem when the contribution of the Theta function vanishes, namely the following.

Proposition 8.2. For the codimension g space of test functions φ satisfying $\boldsymbol{w}[\varphi] = \boldsymbol{0}$, $X_N[\varphi]$ converges in law to a random Gaussian $\mathscr{G}(M_1[\varphi], M_2[\varphi])$ with mean $M_1[\varphi]$ and covariance $M_2[\varphi]$. □

For test functions so that $\boldsymbol{w}[\varphi] \neq \boldsymbol{0}$, the ratio of Theta functions is present, and we recognize it to be the Fourier transform of the law of a random variable which is the scalar product of a deterministic vector $\boldsymbol{w}[\varphi]$ with $\mathscr{D}(\boldsymbol{\gamma}_N, \boldsymbol{T}^{-1}[\boldsymbol{v}], \boldsymbol{T}^{-1})$, where $\mathscr{D}$ is the sampling on $\boldsymbol{\gamma}_N + \mathbb{Z}^g$ of a random Gaussian vector with g components, with covariance matrix $\boldsymbol{T}^{-1}$, and mean $\boldsymbol{T}^{-1}[\boldsymbol{v}]$. The values of the various parameters appearing here is

$$\boldsymbol{T} = F_{\epsilon^\star}^{[-2],(2)}, \qquad \boldsymbol{\gamma}_N = -N\boldsymbol{\epsilon}^\star \bmod \mathbb{Z}^g, \qquad \boldsymbol{v} = F_{\epsilon^\star}^{[-1],(1)}.$$

Therefore, we can only say that, along subsequences of N so that $-N\boldsymbol{\epsilon}^\star \bmod \mathbb{Z}^g$ converges to a limit $\boldsymbol{\gamma}^*$, $X_N[\varphi]$ converges in law to the independent sum

$$\mathscr{G}(M_1[\varphi], M_2[\varphi]) + \boldsymbol{w}[\varphi] \cdot \mathscr{D}(\boldsymbol{\gamma}^*, \boldsymbol{T}^{-1}[\boldsymbol{v}], \boldsymbol{T}^{-1}).$$

Acknowledgement

We thank Tom Claeys for providing references on biorthogonal ensembles. K.K.K would like to thank Katedra Method Matematycznych dla Fizyki, Warsaw University, for their hospitality.

Funding

The work of G.B. is supported by Swiss NSF 200021-143434, Fonds Européen S16905 (UE7 - CONFRA), a Forschungsstipendium of the Max-Planck-Gesellschaft, and the Simons Foundation. The work of A.G. is supported by the Simons Foundation and the NSF 1307704. K.K.K. is supported by CNRS, the ANR grant "DIADEMS", the Burgundy region PARI 2013 FABER grant "Structures et asymptotiques d'intégrales multiples" and PEPS-PTI "Asymptotique d'intégrales multiples" grant.

Appendix 1. Derivation of the Schwinger-Dyson Equations

Let $\mathsf{A} = \bigcup_{h=0}^{g}[a_h^-, a_h^+]$ be a disjoint union of closed segments of $\mathbb{R}$. We consider an r-body interaction T satisfying Hypothesis 5.1. In this section, we derive the Schwinger-Dyson equation given in (5.3).

A.1 Diffeomorphism invariance

Let $f : \mathsf{A} \to \mathbb{R}$ be a smooth function. For $\varepsilon > 0$ small enough, the function $\psi_{f,\varepsilon}(\lambda) = \lambda + \varepsilon\, f(\lambda)$ is a diffeomorphism sending $[\psi_{f,\varepsilon}^{-1}(a_h^-), \psi_{f,\varepsilon}^{-1}(a_h^+)]$ to $[a_h^-, a_h^+]$. Let us express the invariance of the integral computing the partition function under change of variables:

$$Z_{\mathsf{A}^N} = \int_{\mathsf{A}^N} \mathrm{d}\mu(\lambda_1, \ldots, \lambda_N) = \int_{\psi_{f,\varepsilon}^{-1}(\mathsf{A})^N} \mathrm{d}\mu(\psi_{f,\varepsilon}(\lambda_1), \ldots, \psi_{f,\varepsilon}(\lambda_N)). \tag{A.1}$$

Therefore, expanding when $\varepsilon \to 0$ and collecting the term of order ε, we should find 0. We compute

$$\begin{aligned} &\frac{\partial_\varepsilon\big[\prod_{i=1}^{N} \mathrm{d}\psi_{f,\varepsilon}(\lambda_i) \cdot \prod_{1\leq i<j\leq N} |\psi_{f,\varepsilon}(\lambda_i) - \psi_{f,\varepsilon}(\lambda_j)|^{\beta}\big]_{\varepsilon=0}}{\prod_{i=1}^{N} \mathrm{d}\lambda_i \cdot \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^{\beta}} \\ &= \sum_{i=1}^{N} f'(\lambda_i) + \beta \sum_{1\leq i<j\leq N} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \\ &= \sum_{i=1}^{N} \left(1 - \frac{\beta}{2}\right) f'(\lambda_i) + \frac{\beta}{2} \sum_{1\leq i,j\leq N} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \end{aligned} \tag{A.2}$$

and for the r-body interaction:

$$\frac{\partial_\varepsilon[\exp(\frac{N^{2-r}}{r!}\sum_{1\leq i_1,\ldots,i_r\leq N} T(\psi_{f,\varepsilon}(\lambda_{i_1}),\ldots,\psi_{f,\varepsilon}(\lambda_{i_r})))]_{\varepsilon=0}}{\exp(\frac{N^{2-r}}{r!}\sum_{1\leq i_1,\ldots,i_r\leq N} T(\lambda_{i_1},\ldots,\lambda_{i_r}))}$$
$$= \frac{N^{2-r}}{r!}\sum_{j=1}^{r}\sum_{1\leq i_1,\ldots,i_r\leq N} \partial_j T(\lambda_{i_1},\ldots,\lambda_{i_r})\, f(\lambda_{i_j}), \tag{A.3}$$

where ∂_j denotes the derivation with respect to the jth variable. We also need to take into account the variation of the range of integration:

$$\partial_\varepsilon \psi_{f,\varepsilon}^{-1}(a_h^{\pm})|_{\varepsilon=0} = -f(a_h^{\pm}). \tag{A.4}$$

Combining all the terms, we find the first Schwinger-Dyson equation:

$$\mu_{\mathsf{A}^N}\left[\left(1-\frac{\beta}{2}\right)\sum_{i=1}^{N} f'(\lambda_i) + \sum_{1\leq i,j\leq N}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} + \frac{N^{2-r}}{(r-1)!}\sum_{1\leq i_1,\ldots,i_r\leq N} f(\lambda_{i_1})\,\partial_1 T(\lambda_{i_1},\ldots,\lambda_{i_r})\right]$$
$$-\sum_{\alpha\in\partial\mathsf{A}} f(\alpha)\,\partial_\alpha \ln Z_{\mathsf{A}^N} = 0. \tag{A.5}$$

We exploited the symmetry of the measure under permutation of the λ_i's to rewrite the third term. Since this equation is linear in f, it is also valid for f complex-valued by decomposing into f into real and imaginary part.

A.2 In terms of correlators

We recall that the essential properties of the disconnected correlators $\overline{W}_n(\xi_1,\ldots,\xi_n)$ is that, for any holomorphic function φ in a neighborhood of A^n, we have

$$\mu_{\mathsf{A}^N}\left[\sum_{1\leq i_1,\ldots,i_n\leq N}\varphi(\lambda_{i_1},\ldots,\lambda_{i_n})\right] = \oint_{\mathsf{A}^n}\left[\prod_{i=1}^{n}\frac{\mathrm{d}\xi_i}{2i\pi}\right]\varphi(\xi_1,\ldots,\xi_n)\,\overline{W}_n(\xi_1,\ldots,\xi_n). \tag{A.6}$$

Let H be a subset of $\partial\mathsf{A}$—in Section 5.1 we choose H to be the subset of hard edges. Let us define

$$\sigma_{\mathsf{H}}(x) = \prod_{\alpha\in\mathsf{H}}(x-\alpha),\quad \sigma_{\mathsf{H}}^{[1]}(x;\xi) = \frac{\sigma_{\mathsf{H}}(x)-\sigma_{\mathsf{H}}(\xi)}{x-\xi},\quad \sigma_{\mathsf{H}}^{[2]}(x;\xi_1,\xi_2) = \frac{\sigma_{\mathsf{H}}^{[1]}(x;\xi_1)-\sigma_{\mathsf{H}}^{[1]}(x;\xi_2)}{\xi_1-\xi_2}. \tag{A.7}$$

If $x\in\mathbb{C}\setminus\mathsf{A}$, let us choose $f_x(\lambda) = \frac{\sigma_{\mathsf{H}}(\lambda)}{x-\lambda}$. It is an analytic function for which we can write the Schwinger-Dyson equations (A.5). It has the property that $f_x(\alpha) = 0$ for any edge $\alpha\in\mathsf{H}$, so there is no boundary term issuing from those points. Using (A.6), we find

$$\left(\frac{\beta}{2}-1\right)\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\partial_\xi\,f_x(\xi)W_1(\xi)+\oint_{\mathsf{A}^2}\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\mathrm{i}\pi)^2}\,\frac{f_x(\xi_1)-f_x(\xi_2)}{\xi_1-\xi_2}\,\overline{W}_2(\xi_1,\xi_2)$$
$$+\frac{N^{2-r}}{(r-1)!}\oint_{\mathsf{A}^2}\frac{\mathrm{d}^r\boldsymbol{\xi}}{(2\mathrm{i}\pi)^r}\,f_x(\xi_1)\,\partial_{\xi_1}T(\xi_1,\ldots,\xi_r)-\sum_{\alpha\in\partial\mathsf{A}\setminus\mathsf{H}}f_x(\alpha)\,\partial_\alpha\ln Z_{\mathsf{A}^N}=0. \tag{A.8}$$

For the first term, we need to compute

$$\partial_\xi\left(\frac{\sigma_{\mathsf{H}}(\xi)}{x-\xi}\right)=-\frac{\sigma_{\mathsf{H}}^{[2]}(x;\xi,\xi)}{x-\xi}+\frac{\sigma_{\mathsf{H}}(\xi)}{(x-\xi)^2} \tag{A.9}$$

therefore, by moving the contour to ∞ when integrating the second term in (A.9), the contribution to (A.8) is

$$\left(\frac{\beta}{2}-1\right)\left(\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\sigma_{\mathsf{H}}^{[2]}(x;\xi,\xi)\,W_1(\xi)+\sigma_{\mathsf{H}}(x)\,\partial_x W_1(x)\right). \tag{A.10}$$

For the second term in (A.8), we need

$$\frac{f_x(\xi_1)-f_x(\xi_2)}{\xi_1-\xi_2}=-\sigma_{\mathsf{H}}^{[2]}(x;\xi_1,\xi_2)+\frac{\sigma_{\mathsf{H}}(x)}{(x-\xi_1)(x-\xi_2)} \tag{A.11}$$

and moving the contours at ∞ when integrating the second term in (A.11), the contribution to (A.8) reads

$$\frac{\beta}{2}\left(-\oint_{\mathsf{A}^2}\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\mathrm{i}\pi)^2}\,\sigma_{\mathsf{H}}^{[2]}(x;\xi_1,\xi_2)\,\overline{W}_2(\xi_1,\xi_2)+\sigma(x)\,\overline{W}_2(x,x)\right). \tag{A.12}$$

The third term in (A.8) is just:

$$\frac{N^{2-r}}{(r-1)!}\oint_{\mathsf{A}^r}\frac{\mathrm{d}^r\boldsymbol{\xi}}{(2\mathrm{i}\pi)^r}\,\frac{\sigma_{\mathsf{H}}(\xi_1)}{x-\xi_1}\,\partial_{\xi_1}T(\xi_1,\ldots,\xi_r)\,\overline{W}_r(\xi_1,\ldots,\xi_r). \tag{A.13}$$

Combining all the terms, dividing by $\sigma_{\mathsf{H}}(x)$ and $\beta/2$, and replacing $\overline{W}_2(x,x)=W_2(x,x)+W_1(x)^2$, one obtains the $n=1$ Schwinger-Dyson equation:

$$\begin{aligned}&\left(1-\frac{2}{\beta}\right)\partial_x W_1(x)+W_2(x,x)+W_1^2(x)\\&\quad-\frac{2}{\beta}\sum_{\alpha\in\partial\mathsf{A}\setminus\mathsf{H}}\frac{\sigma_{\mathsf{H}}(\alpha)}{\sigma_{\mathsf{H}}(x)}\,\frac{\partial_\alpha\ln Z_{\mathsf{A}^N}}{x-\alpha}+\frac{2}{\beta}N^{2-r}\oint_{\mathsf{A}^r}\frac{\mathrm{d}^r\boldsymbol{\xi}}{(2\mathrm{i}\pi)^r}\,\frac{\sigma_{\mathsf{H}}(\xi_1)}{\sigma_{\mathsf{H}}(x)}\,\frac{\partial_{\xi_1}T(\xi_1,\ldots,\xi_r)}{(r-1)!\,(x-\xi_1)}\,\overline{W}_r(\xi_1,\ldots,\xi_r)\\&\quad+\left(1-\frac{2}{\beta}\right)\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\sigma_{\mathsf{H}}^{[2]}(x;\xi,\xi)}{\sigma_{\mathsf{H}}(x)}\,W_1(\xi)-\oint_{\mathsf{A}^2}\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\mathrm{i}\pi)^2}\,\frac{\sigma_{\mathsf{H}}^{[2]}(x;\xi_1,\xi_2)}{\sigma_{\mathsf{H}}(x)}\{W_2(\xi_1,\xi_2)+W_1(\xi_1)W_1(\xi_2)\}\\&\quad=0.\end{aligned} \tag{A.14}$$

This is the equation announced in (5.3) for $n=1$, provided one chooses H to be the set of hard edges.

We can apply this equation to infinitesimal perturbation $\tilde{T}_{t_1,\ldots,t_{n-1}}$ of the interaction T. Then, collecting the term of order $t_1\cdots t_{n-1}$ with $t_i\to 0$ yields the nth Schwinger-Dyson equation in the form (5.3).

Appendix 2. Inversion of Integral Operators

In this section, we study integral operators on the real line which parallel the operators defined on the complex plane defined in Section 5. This is necessary to obtain the concentration bounds of Section 3.5.

A.3 Reminder of Fredholm theory

Let $(\mathsf{X}, \mathrm{d}s)$ be a measured space, so that $|s|(\mathsf{X})<+\infty$. Let K be an integral operator on $L^p(\mathsf{X}, \mathrm{d}s)$, $p\geq 1$ with a kernel $\mathscr{K}(x,y)=f(x)\widetilde{\mathscr{K}}(x,y)$ such that $\widetilde{\mathscr{K}}\in L^\infty(\mathsf{X}\times\mathsf{X}, \mathrm{d}^2s)$ and $f\in L^p(\mathsf{X},\mathrm{d}s)$. Then, the series of multiple integrals

$$\det[\mathrm{id}+K]=\sum_{n\geq 0}\frac{1}{n!}\int_{\mathsf{X}}\det_n[\mathscr{K}(\lambda_a,\lambda_b)]\prod_{a=1}^{n}\mathrm{d}s(\lambda_a) \tag{A.15}$$

converges uniformly and defines the so-called Fredholm determinant associated with the integral operator $\mathrm{id}+K$. The convergence follows by means of an application of Hadamard's inequality

$$\left|\int_X \det_n[\mathscr{K}(\lambda_a,\lambda_b)]\,\mathrm{d}^n s(\lambda)\right|\leq n^{\frac{n}{2}}\cdot\|\widetilde{\mathscr{K}}\|^n_{L^\infty(X\times X,\mathrm{d}^2s)}\cdot\|f\|^n_{L^1(\mathsf{X},\mathrm{d}s)}.$$

This operator is invertible if and only if $\det[\mathrm{id}+K]\neq 0$ and its inverse operator $\mathrm{id}-\mathcal{R}_K$ is described in terms of the resolvent kernel given by the absolutely convergent series of multiple integrals:

$$\mathscr{R}_K(x,y)=\frac{1}{\det[\mathrm{id}+K]}\sum_{n\geq 0}\frac{1}{n!}\int_{\mathsf{X}}\det_{n+1}\begin{bmatrix}\mathscr{K}(x,y) & \mathscr{K}(x,\lambda_b)\\ \mathscr{K}(\lambda_a,y) & \mathscr{K}(\lambda_a,\lambda_b)\end{bmatrix}\cdot\prod_{a=1}^{n}\mathrm{d}\mu(\lambda_a). \tag{A.16}$$

In particular, such a description ensures that the inverse operator is continuous as soon as the operator $\mathrm{id}+K$ is injective. See [32] for a more detailed discussion. Note that the resolvent kernel $\mathscr{R}_K(x, y)$ satisfies to the bounds

$$|\mathscr{R}_K(x, y)| \leq |f(x)| \cdot \sum_{n\geq 0} \frac{(n+1)^{\frac{n+1}{2}}}{n!} \cdot \frac{\|\widetilde{\mathscr{K}}\|^{n+1}_{L^\infty(\mathsf{X}\times\mathsf{X},\mathrm{d}^2s)} \cdot \|f\|^n_{L^1(\mathsf{X},\mathrm{d}s)}}{\det[\mathrm{id}+K]} \leq c_K \cdot |f(x)|,$$

with c_K a kernel K-dependent constant.

A.4 Inversion of $\underline{\mathcal{T}}$

Let $\underline{\mathcal{T}}$ be the integral operator

$$\underline{\mathcal{T}}[\phi](x) = -\int_{\mathsf{A}} [\beta \ln|x-y| + \tau(x, y)]\phi(y)\,\mathrm{d}y + \int_{\mathsf{A}^2} [\beta \ln|x-y| + \tau(x, y)]\phi(y)\,\mathrm{d}x\,\mathrm{d}y$$

with τ defined by

$$\tau(x, y) = \int \frac{T(x, y, \xi_3, \ldots, \xi_r)}{(r-2)!} \prod_{i=3}^{r-2} \mathrm{d}\mu_{\mathrm{eq}}(\xi_i).$$

Let $\underline{\mathcal{L}}$ and $\underline{\mathcal{P}}$ be the integral operators on $L^p(\mathsf{A}, \mathrm{d}x)$ for $1 < p < 2$, with respective integral kernels

$$\underline{\mathscr{L}}(x, y) = \frac{1}{\beta\pi^2} \fint_{\mathsf{A}} \mathrm{d}\xi\, \frac{\sigma^{1/2}_{\mathsf{A};+}(\xi)}{\sigma^{1/2}_{\mathsf{A};+}(x)}\, \frac{\partial_\xi \tau(\xi, y)}{x-\xi}, \qquad \underline{\mathscr{P}}(x, y) = \frac{1}{\mathrm{i}\pi\sigma^{1/2}_{\mathsf{A};+}(x)} \operatorname*{Res}_{\xi\to\infty}\left(\frac{\sigma^{1/2}_{\mathsf{A}}(\xi)}{(x-\xi)(\xi-y)}\right),$$

where we remind $\sigma_{\mathsf{A}}(x) = \prod_{h=0}^{g}(x-a_h^-)(x-a_h^+)$. Let $\mathfrak{X}$ be the set $[\![1\,;\,g]\!] \cup \mathsf{A}$ endowed with the measure $\mathrm{d}s$ given by the atomic measure on $[\![1\,;\,g]\!]$ and the Lebesgue measure on A. We shall make the identification $L^p_0(\mathfrak{X}, \mathrm{d}s) \simeq \mathbb{C}^g \oplus L^p_0(\mathsf{A}, \mathrm{d}x)$, where

$$L^p_0(\mathsf{A}, \mathrm{d}x) = \left\{\phi \in L^p(\mathsf{A}, \mathrm{d}x)\ :\ \int_{\mathsf{A}} \phi(x)\,\mathrm{d}x = 0\right\}.$$

We define similarly a subspace $W^{1,q}_0(\mathsf{A})$ of the Sobolev space $W^{1,q}(\mathsf{A}) \subseteq L^q(\mathsf{A})$, and introduce the space:

$$W^{1,q}_0(\mathfrak{X}) = \left\{(\boldsymbol{v}, \phi) \in L^q_0(\mathfrak{X}, \mathrm{d}s),\ \left(\max_{1\leq k\leq g} |v_k|\right) + \|\phi\|_q < +\infty\right\}.$$

Let $\underline{\mathcal{N}}$ be integral operator

$$\underline{\mathcal{N}}\ :\ \begin{cases} \mathbb{C}^g \oplus L^p_0(\mathsf{A}, \mathrm{d}x) \longrightarrow \mathbb{C}^g \oplus L^p_0(\mathsf{A}, \mathrm{d}x), \\ (\boldsymbol{v}, \phi) \longmapsto (\underline{\Pi}[\phi] - \boldsymbol{v}, (\underline{\mathcal{L}} - \underline{\mathcal{P}})[f] + \sigma^{-1/2}_{\mathsf{A};+} \cdot Q_{\boldsymbol{v}}), \end{cases}$$

where

$$\underline{\Pi}\ :\ \begin{cases} L^p(\mathsf{A}, \mathrm{d}x) \longrightarrow \mathbb{C}^g, \\ \phi \longmapsto \left(\int_{\mathsf{A}_1} \underline{\mathcal{T}}[\phi](\xi)\,\mathrm{d}\xi, \ldots, \int_{\mathsf{A}_g} \underline{\mathcal{T}}[\phi](\xi)\,\mathrm{d}\xi\right), \end{cases}$$

and Q_v is the unique polynomial of degree $g-1$ such that $\int_{\mathsf{A}_k} \mathrm{d}\xi\, \sigma_{\mathsf{A};+}^{-1/2}(\xi)\, Q_v(\xi) = v_k$ for any $k \in [\![1\,;\, g]\!]$. The operators in underline letters—like $\underline{\mathcal{L}}$—are the analog on the real axis of the operators—like $\mathcal{L}$—defined in Section 5.3 on spaces of analytic functions, the correspondence being given by the Stieltjes transform. It should therefore not be surprising that the computations in this Appendix are parallel to those of Section 5.

Proposition A.3. Let $1 < p < 2$ and $q > 2p/(2-p)$. The integral operator $\underline{\mathcal{N}} : L_0^p(\mathfrak{X}, \mathrm{d}s) \to W_0^{1,q}(\mathfrak{X}, \mathrm{d}s)$ is compact. The operator $\mathrm{id} + \underline{\mathcal{N}}$ is bi-continuous with inverse $\mathrm{id} - \mathcal{R}_{\underline{\mathcal{N}}}$. Furthermore, the inverse of $\underline{\mathcal{T}}$ is expressed by

$$\underline{\mathcal{T}}^{-1}[f](x) = \Xi[f'](\xi) - \sum_{k=1}^{g} \mathcal{R}_{\mathcal{N}}(x,k) \int_{\mathsf{A}_k} f(\xi)\,\mathrm{d}\xi \; - \int_{\mathsf{A}} \mathcal{R}_{\underline{\mathcal{N}}}(x,\xi)\cdot \Xi[f'](\xi)\,\mathrm{d}\xi, \tag{A.17}$$

where

$$\Xi[f](x) = \frac{1}{\beta\pi^2} \fint_{\mathsf{A}} \frac{\sigma_{\mathsf{A};+}^{1/2}(x)\, f(\xi)}{\sigma_{\mathsf{A};+}^{1/2}(\xi)\,(x-\xi)}\,\mathrm{d}\xi.$$

As a consequence, $\underline{\mathcal{T}}^{-1}$ extends to a continuous operator $\underline{\mathcal{T}}^{-1} : W_0^{1,q}(\mathsf{A}, \mathrm{d}x) \to L_0^p(\mathsf{A}, \mathrm{d}x)$. □

Proof. We first establish that $\underline{\mathcal{L}} = L_0^p(\mathsf{A}, \mathrm{d}x) \to L_0^p(\mathsf{A}, \mathrm{d}x)$, defined for $1 < p < 2$, is compact. It follows from

$$\underline{\mathscr{L}}(x,y) = \frac{1}{\beta\pi^2}\int_{\mathsf{A}} \mathrm{d}\xi\, \frac{\sigma_{\mathsf{A};+}^{1/2}(\xi)}{\sigma_{\mathsf{A};+}^{1/2}(x)}\, \frac{\partial_\xi \tau(\xi,y) - \partial_\xi \tau(x,y)}{x-\xi} + \frac{\partial_x \tau(x,y)}{2\beta\pi^2\,\sigma_{\mathsf{A};+}^{1/2}(x)} \oint_{\Gamma(\mathsf{A})} \frac{\mathrm{d}\eta\,\sigma_{\mathsf{A}}^{1/2}(\eta)}{x-\eta},$$

where $\Gamma(\mathsf{A})$ is a contour surrounding A with positive orientation, that $\underline{\mathscr{L}}(x,y) = \underline{\widetilde{\mathscr{L}}}(x,y)\,\sigma_{\mathsf{A};+}^{-1/2}(x)$ for a continuous function $\underline{\widetilde{\mathscr{L}}}(x,y)$ on A^2. Furthermore, the relation

$$\fint_{\mathsf{A}} \frac{\mathrm{d}\xi}{\sigma_{\mathsf{A};+}^{1/2}(\xi)(x-\xi)} = 0$$

ensures that $\underline{\mathcal{L}}$ stabilizes $L_0^p(\mathsf{A}, \mathrm{d}x)$. Taking into account that A is compact, there exists a sequence of continuous functions $(\Phi_n, \Psi_n)_{n\geq 1}$ on A such that

$$\underline{\widetilde{\mathscr{L}}}^{[n]}(x,y) = \sum_{m=1}^{n} \Phi_m(x)\Psi_m(y)$$

converges uniformly on A^2 to $\underline{\widetilde{\mathscr{L}}}(x,y)$. Let $\underline{\mathcal{L}}^{[n]}$ be the integral operator on $L_0^p(\mathsf{A}, \mathrm{d}x)$ with the integral kernel $\underline{\mathscr{L}}^{[n]}(x,y) = \sigma_{\mathsf{A};+}^{-1/2}(x)\,\underline{\widetilde{\mathscr{L}}}^{[n]}(x,y)$. It follows from Hölder inequality that the $L^p_0(\mathsf{A}, \mathrm{d}x)$ operator norm satisfies

$$|||\, \underline{\mathcal{L}} - \underline{\mathcal{L}}^{[n]}\, ||| \leq \ell(\mathsf{A})^{\frac{p-1}{p}} \, \|\sigma_{\mathsf{A};+}^{-1/2}\|_{L^p(\mathsf{A})} \, \|\widetilde{\underline{\mathcal{L}}} - \widetilde{\underline{\mathcal{L}}}^{[n]}\|_{L^\infty(\mathsf{A}^2)}.$$

As a consequence, $\underline{\mathcal{L}}$ is indeed compact. An analogous statement is readily established for $\underline{\mathcal{P}}$ and hence $\underline{\mathcal{N}}$. We now establish that $\mathrm{id} + \underline{\mathcal{N}}$ is injective. Let $(\boldsymbol{v}, \phi) \in \ker(\mathrm{id} + \underline{\mathcal{N}})$. Then one has

$$\fint_{\mathsf{A}} \int_{\mathsf{A}} \frac{\underline{\underline{\mathscr{L}}}(x, y)\phi(y)}{s - x} \mathrm{d}y = 0 \quad \Longrightarrow \quad \int_{\mathsf{A}} \phi(s) \cdot \mathrm{d}s = 0.$$

By going back to the definition of a principal value integral, one obtains

$$\begin{aligned} -\beta \fint_{\mathsf{A}} \frac{\Xi[\phi](\xi)}{x - \xi} \mathrm{d}\xi = & \int_{\mathsf{A}} \mathrm{d}\eta \frac{\phi(\eta)\,\sigma_{\mathsf{A};+}^{1/2}(\eta)}{(2\mathrm{i}\pi)^2} \oint_{\Gamma(\mathsf{A})} \frac{2\,\mathrm{d}\xi}{\sigma_{\mathsf{A}}^{1/2}(\xi)(x - \xi)(\eta - \xi)} \\ & - \lim_{\epsilon_1, \epsilon_2 \to 0^+} \left\{ \int_{\mathsf{A}^2} \frac{\phi(\eta)\,\sigma_{\mathsf{A};+}^{1/2}(\eta)}{\sigma_{\mathsf{A};+}^{1/2}(\xi)} \left[\frac{1}{x - \xi + \mathrm{i}\epsilon_1} \left(\frac{1}{\eta - \xi - \mathrm{i}\epsilon_2} - \frac{1}{\eta - \xi + \mathrm{i}\epsilon_2} \right) \right.\right. \\ & \left.\left. + \frac{1}{x - \xi + \mathrm{i}\epsilon_1} \left(\frac{1}{\eta - \xi - \mathrm{i}\epsilon_2} - \frac{1}{t - \xi + \mathrm{i}\epsilon_2} \right) \right] \mathrm{d}\eta\, \mathrm{d}\xi \right\} = \phi(x). \end{aligned}$$

This ensures that

$$-\beta \fint_{\mathsf{A}} \frac{\underline{\underline{\mathscr{L}}}(\xi, y)}{x - \xi}\, \mathrm{d}\xi = (\partial_x \tau)(x, y).$$

In its turn this leads to $\partial_\xi \underline{\mathcal{T}}[f](\xi) = 0$ by acting with the principal value operator on $(\mathrm{id} + \underline{\mathcal{L}})[f]$ and using that

$$\fint_{\mathsf{A}} \frac{\mathrm{d}\xi\ Q(\xi)}{\sigma_{\mathsf{A};+}^{1/2}(\xi) \cdot (x - \xi)} = 0$$

for any polynomial Q of degree at most g. In other words, there exist constants c_k such that $\underline{\mathcal{T}}[\phi](\xi) = c_k$ on A_k, $k = 0, \ldots, g$. Since, furthermore,

$$\int_{\mathsf{A}_k} \underline{\mathcal{T}}[\phi](x)\, \mathrm{d}x = 0, \quad k = 1, \ldots, g \quad \text{and by definition} \quad \int_{\mathsf{A}} \underline{\mathcal{T}}[\phi](x)\, \mathrm{d}x = 0$$

it follows that, in fact, $\underline{\mathcal{T}}[\phi](x) = 0$. Therefore,

$$\int_{\mathsf{A}} \phi(x) \cdot \underline{\mathcal{T}}[\phi](x)\, \mathrm{d}x = \mathcal{Q}[\nu_\phi] = 0 \quad \text{with } \nu_\phi = \phi(x)\, \mathrm{d}x \in \mathcal{M}^0(\mathsf{A}).$$

In virtue of the strict positivity of the functional $\mathcal{Q}$, it follows that $\nu_\phi = 0$, *viz.* $\phi = 0$. This implies, in its turn, that $Q_{\boldsymbol{v}} = 0$, that is, $\boldsymbol{v} = 0$.

We now focus on the invertibility of the operator $\underline{\mathcal{T}}$. Hence, assume that one is given $f \in \underline{\mathcal{T}}[L^p_0(\mathsf{A}, \mathrm{d}x)] \cap W^{1;q}_0(\mathsf{A}, \mathrm{d}x)$ with $1 < p < 2$ and $q > 2p/(2 - p)$. In other words that the function $\phi \in L^p_0(\mathsf{A}, \mathrm{d}x)$, $1 < p < 2$ is a solution to $\underline{\mathcal{T}}[\phi] = f$ for the given $f \in W^{1;q}_0(\mathsf{A}, \mathrm{d}x)$. Since, the principal value operator is continuous on $L^p(\mathsf{A}, \mathrm{d}x)$, $1 < p < 2$, it follows that one can differentiate both sides of the equality leading to

$$\fint_{\mathsf{A}} \frac{\phi(s)}{x - \xi}\,\mathrm{d}\xi = -\frac{f'(x)}{\beta} + \frac{1}{\beta}\int_{\mathsf{A}} \partial_x \tau(x, \xi)\,\phi(\xi)\,\mathrm{d}\xi \equiv F(x).$$

The function

$$\kappa[\phi](z) = \sigma_{\mathsf{A}}^{1/2}(z) \cdot \int_{\mathsf{A}} \frac{\phi(y)}{z - y} \cdot \frac{\mathrm{d}y}{2\mathrm{i}\pi}$$

is holomorphic on $\mathbb{C} \setminus \mathsf{A}$, admits $L^p(\mathsf{A})$ $\pm$ boundary values on A, and has the asymptotic behavior at infinity

$$\kappa[\phi](z) = \underbrace{\operatorname*{Res}_{\xi \to \infty} \frac{\sigma_{\mathsf{A}}^{1/2}(\xi)}{x - \xi}\left(\int_{\mathsf{A}} \frac{\mathrm{d}y}{2\mathrm{i}\pi}\,\frac{\phi(y)}{\xi - y}\right)}_{P[\phi](z)/2} + O(1/z).$$

Furthermore, it satisfies to the jump conditions

$$\kappa[\phi]_+(x) - \kappa[\phi]_-(x) = \sigma_{\mathsf{A};+}^{1/2}(x)\,\frac{F(x)}{\mathrm{i}\pi}, \quad x \in \mathring{\mathsf{A}}.$$

Thus,

$$\kappa[\phi](z) = \frac{P[\phi](z)}{2} + \int_{\mathsf{A}} \frac{\mathrm{d}\xi\, F(\xi)\,\sigma_{\mathsf{A};+}(\xi)}{2\pi^2\,(z - \xi)}.$$

Note that $P[\phi]$ is at most of degree $g - 1$ since $\phi \in L^p_0(\mathsf{A}, \mathrm{d}x)$. Finally, it follows from the equation

$$\kappa[\phi]_+(x) + \kappa[\phi]_-(x) = -\sigma_{\mathsf{A};+}^{1/2}(x)\phi(x)$$

that ϕ solves the regular integral equation

$$\Xi[f'](x) = (\mathrm{id} + \underline{\mathcal{L}} - \underline{\mathcal{P}})[\phi](x).$$

As a consequence, for any $f \in \underline{\mathcal{T}}[L^p_0(\mathsf{A}, \mathrm{d}x)] \cap W^{1;q}(\mathsf{A})$, there exists ϕ solving

$$(\mathrm{id} + \underline{\mathcal{N}})[(\mathbf{0}, \phi)] = \left(\int_{\mathsf{A}_1} f(x)\,\mathrm{d}x, \ldots, \int_{\mathsf{A}_g} f(x)\,\mathrm{d}x,\, \Xi[f']\right).$$

Since $(\mathrm{id} + \underline{\mathcal{N}})$ is bijective, ϕ is necessarily unique and given by (A.17). The continuity of $\underline{\mathcal{T}}^{-1}$ is then obvious. ■

References

[1] Akemann, G. “Higher genus correlators for the hermitian matrix model with multiple cuts.” *Nuclear Physics B* 482, no. 1 (1996): 403–30.

[2] Albeverio, S., L. Pastur, and M. Shcherbina. “On the $1/N$ expansion for some unitary invariant ensembles of random matrices.” *Communications in Mathematical Physics* 224, no. 1 (2001): 271–305.

[3] Ambjørn, J., L. O. Chekhov, C. Kristjansen, and Yu. Makeenko. “Matrix model calculations beyond the spherical limit.” *Nuclear Physics B* 449, no. 1–2 (1995): 681, [erratum], hep-th/9302014.

[4] Ambjørn, J., L. O. Chekhov, and Yu. Makeenko. “Higher genus correlators from the hermitian one-matrix model.” *Physics Letters B* 282, no. 3–4 (1992): 341–48.

[5] Anderson, G. W., A. Guionnet, and O. Zeitouni. *An Introduction to Random Matrices*. Cambridge: Cambridge University Press, 2010.

[6] Beenakker, C. W. J. and B. Rejaei. “Nonlogarithmic repulsion of transmission eigenvalues in a disordered wire.” *Physical Review Letters* 71, no. 22 (1993): 3689.

[7] Ben Arous, G. and A. Guionnet. “Large deviations for Wigner’s law and Voiculescu’s non-commutative entropy.” *Probability Theory and Related Fields* 108, no. 4 (1997): 517–42.

[8] Ben Arous, G. and O. Zeitouni. “Large deviations from the circular law.” *ESAIM Probability and Statistics* 2 (1998): 123–34.

[9] Blau, M. and G. Thompson. “Chern–Simons theory on Seifert manifolds.” *JHEP* 33 (2013).

[10] Bonnet, G. and B. Eynard. “The Potts-q random matrix model: loop equations, critical exponents, and rational case.” *Physics Letters B* 463, no. 2–4 (1999): 273–9.

[11] Borodin, A. “Biorthogonal ensembles.” *Nuclear Physics B* 536, no. 3 (1999): 704–32.

[12] Borot, G. “Formal multidimensional integrals, stuffed maps, and topological recursion.” *Annales Institut Henri Poincare D* 1, no. 2: 225–64.

[13] Borot, G., B. Eynard, and N. Orantin. “Abstract loop equations, topological recursion, and applications.” *Communications in Number Theory and Physics*, to appear.

[14] Borot, G., B. Eynard, and A. Weiße. “Root systems, spectral curves, and analysis of a Chern–Simons matrix model for Seifert fibered spaces.” math-ph/1407.4500.

[15] Borot, G. and A. Guionnet. “Asymptotic expansion of beta matrix models in the multi-cut regime.” math-ph/1303.1045.

[16] Borot, G. and A. Guionnet. “Asymptotic expansion of β matrix models in the one-cut regime.” *Communications in Mathematical Physics* 317, no. 2 (2013): 447–83.

[17] Caracciolo, S. and A. Sportiello. “Spanning forests on random planar lattices.” *Journal of Statistical Physics* 135, no. 5–6 (2009): 1063–104.

[18] Chekhov, L. O. “Hermitean matrix model free energy: Feynman graph technique for all genera.” *JHEP* no. 0603:014 (2006).

[19] Chekhov, L. O. “Matrix models with hard walls: geometry and solutions.” *Journal of Physics A* 39 (2006): 8857–94.

[20] Chekhov, L. O. and B. Eynard. “Matrix eigenvalue model: Feynman graph technique for all genera.” *JHEP* no. 0612:026 (2006).

[21] Deift, P. *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*. New York: AMS, 1998. Courant Institute of Mathematical Sciences.

[22] Dembo, A. and O. Zeitouni. *Large Deviations Techniques and Applications*. Stochastic Modelling and Applied Probability 38. Berlin: Springer, 2010.

[23] Dudley, R.-M. and J. M. Gonzalez-Barrios. "Metric entropy conditions for an operator to be of trace class." *Proceedings of the American Mathematical Society* 118 (1993).

[24] Ercolani, N. M. and K. T.-R. McLaughlin. "Asymptotics of the partition function for random matrices via Riemann–Hilbert techniques, and applications to graphical enumeration." *International Mathematics Research Notices* 14 (2003): 755–820.

[25] Eynard, B. "All genus correlation functions for the hermitian 1-matrix model." *JHEP* no. 0411:031 (2004).

[26] Eynard, B. "Large N expansion of convergent matrix integrals, holomorphic anomalies, and background independence." *JHEP* no. 0903:003 (2009).

[27] Eynard, B. and C. Kristjansen. "Exact solution of the $O(n)$ model on a random lattice." *Nuclear Physics B* 455 (1995): 577–618.

[28] Eynard, B. and C. Kristjansen. "More on the exact solution of the $O(n)$ model on a random lattice and an investigation of the case $|n| > 2$." *Nuclear Physics B* 466 (1996): 463–87.

[29] Eynard, B. and N. Orantin. "Topological recursion in random matrices and enumerative geometry." *Journal of Physics A: Mathematical and Theoretical* 42, no. 29 (2009).

[30] Forrester, P. J. *Log-gases and Random Matrices*. Princeton, NJ: Princeton University Press, 2010.

[31] Garoufalidis, S. and M. Mariño. "On Chern–Simons matrix models." math.GT/0601390.

[32] Gohberg, I., S. Goldberg, and N. Krupnik. *Traces and Determinants of Linear Operators*. Operator Theory Advances and Applications 116. Basel: Birkhäuser, 2000.

[33] Johansson, K. "On fluctuations of eigenvalues of random Hermitian matrices." *Duke Mathematical Journal* 91, no. 1 (1998): 151–204.

[34] Kazakov, V. "Ising model on a dynamical planar random lattice: exact solution." *Physics Letters. A* 119, no. 3 (1986): 140–4.

[35] Kazakov, V. "Exactly solvable Potts models, bond- and tree-like percolation on dynamical (random) planar lattice." *Nuclear Physics B. Proceedings Supplement* 4 (1988): 93–7.

[36] Kostov, I. K. "$O(n)$ vector model on a planar random lattice: spectrum of anomalous dimensions." *Modern Physics Letters A* 4, no. 217 (1989): 946–55.

[37] Kostov, I. K. "Exact solution of the six-vertex model on a random lattice." *Nuclear Physics B* 575, no. 3 (2000): 513–34.

[38] Kozlowski, K. K. "Large-distance and long-time asymptotic behavior of the reduced density matrix in the nonlinear Schrödinger model." *Annales Henri Poincar* (2014): 1–98.

[39] Kriecherbauer, T. and M. Shcherbina. "Fluctuations of eigenvalues of matrix models and their applications." math-ph/1003.6121.

[40] Lubinsky, D., A. Sidi and H. Stahl. "Asymptotic zero distribution of biorthogonal polynomials." *Journal of Approximation Theory*, to appear.

[41] Macdonald, I. G. "Orthogonal polynomials associated with root systems." *Séminaire Lotharingien de Combinatoire* 45 (2000).

[42] Maïda, M. and E. Maurel-Segala. “Free transport-entropy inequalities for non-convex potentials and application to concentration for random matrices.” *Probability Theory and Related Fields* 159, no. 1–2 (2014): 329–56.

[43] Mariño, M. “Chern–Simons theory, matrix integrals, and perturbative three-manifold invariants.” *Communications in Mathematical Physics* 253, no. 1 (2004): 25–49.

[44] Mehta, M. L. *Random Matrices*, 3rd ed., Pure and Applied Mathematics 142. Amsterdam: Elsevier/Academic, 2004.

[45] Muttalib, K. A. “Random matrix models with additional interactions.” *Journal of Physics A. Mathematical and General* 28, no. L159 (1995): 159–64.

[46] Pastur, L. “Limiting laws of linear eigenvalue statistics for Hermitian matrix models.” *Journal of Mathematical Physics* 47, no. 10 (2006).

[47] Saff, E. B. and V. Totik. *Logarithmic Potentials with External Fields*. Comprehensive Studies in Mathematics. Berlin: Springer, 1997.

[48] Shcherbina, M. “Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime.” *Journal of Statistical Physics* 151, no. 6 (2013): 1004–34.

[49] Sulkowski, P. “Matrix models for β-ensembles from Nekrasov partition functions.” *JHEP* no. 1004:063 (2010).

[50] Zinn-Justin, P. “The dilute Potts model on random surfaces.” *Journal of Statistical Physics* 98 (2000): 245–64.