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Convexity of certain integrals of the calculus of variations

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Synopsis

In this paper we study the convexity of the integral $I(u)=\int_0^1 f(x, u(x), u'(x))\,dx$ over the space $W_0^{1,\infty}(0,1)$. We isolate a necessary condition on f and we find necessary and sufficient conditions in the case where $f(x, u, u')=a(u)u'^{2n}$ or $g(u)+h(u')$.

1. Introduction

In this paper we are concerned with integrals of the calculus of variations of the type

$$I(u)=\int_0^1 f(x, u(x), u'(x))\,dx, \tag{1}$$

where $f\colon (0,1)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is C^2. We study the conditions on f under which the integral I is convex over the space $W_0^{1,\infty}(0,1)$, which denotes the space of Lipschitz functions vanishing at 0 and 1.

We first give a necessary condition on f, which is that $f(x, u, \cdot)$ is convex. We then give examples showing that no implication can be inferred *a priori* on the convexity of f with respect to the variable u. We then study two examples

(i) $f(x, u, \xi)=a(u)\xi^{2n}$

with $n\geqq 1$, n an integer, and we show in this case that

$$I \text{ convex over } W_0^{1,\infty}(0,1)\Leftrightarrow a(u)=\text{constant}.$$

(ii) $f(x, u, \xi)=g(u)+h(\xi)$ and we show that if

$$g_0=\inf\{g''(u)\colon u\in\mathbb{R}\}$$
$$h_0=\inf\{h''(\xi)\colon \xi\in\mathbb{R}\},$$

then

$$I \text{ convex over } W_0^{1,\infty}(0,1)\Leftrightarrow \pi^2h_0+g_0\geqq 0 \quad\text{and}\quad h_0\geqq 0.$$

In this last example we show that even if $f(x, u, \xi)$ is not convex in the variables (u, ξ), while I is convex over $W_0^{1,\infty}(0,1)$, there exists $\tilde f\colon (0,1)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $\tilde f(x, .\,, .)$ is convex and

$$I(u)=\int_0^1 \tilde f(x, u(x), u'(x))\,dx=\int_0^1 (g(u(x))+h(u'(x)))\,dx$$

for every $u\in W_0^{1,\infty}(0,1)$.

The question of the convexity of the integral I is important in the sense that one can then apply the abstract results of convex analysis to I; in particular a solution of the Euler equation must then be a minimiser of I.

Usually in the direct methods of the calculus of variations one studies the weak lower semicontinuity of I in a Sobolev space $W^{1,p}$ and we have the following result

(i) I convex $\Rightarrow I$ weakly lower semicontinuous;

(ii) I weakly lower semicontinuous $\Leftrightarrow f(x, u, .)$ is convex.

So, in particular, if

$$f(x, u, \xi) = \xi^4 + (u^2 - 1)^2,$$

then, in view of the above results, we have that the associated I is weakly lower semicontinuous but not convex.

2. Main results

We start with a necessary condition.

THEOREM 1. *Let $f: (0, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy*

$$|f(x, u, \xi)| \leqq a(x, |u|, |\xi|),$$

where a is increasing with respect to $|u|$ and $|\xi|$ and locally integrable in x. If I is convex over $W_0^{1,\infty}(0, 1)$, then $f(x, u, .)$ is convex.

Proof. Since f is continuous and I is convex over $W_0^{1,\infty}(0, 1)$ then I is weak* lower semicontinuous in $W^{1,\infty}$ (this is a direct application of Mazur's lemma, see for example [**1**]). However, it is well known that under the above hypotheses on f and if I is weak* lower semicontinuous in $W^{1,\infty}$, then $f(x, u, .)$ is convex (see for example [**3**] and the references quoted therein). □

Remark. The above result is still true for multiple integrals of the type

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u: \Omega \subset \mathbb{R}^n \to \mathbb{R}$. However, it is false if $u: \mathbb{R}^n \to \mathbb{R}^m$ with $n,m > 1$; for example, if $m = n = 2$ and

$$f(x, u, \xi) = \det \xi,$$

then f is obviously not convex, while $I(u) \equiv 0$ for every $u \in W_0^{1,\infty}(\Omega)$ and hence I is convex.

We now turn our attention to sufficient conditions in some particular cases. The most important and the simplest is, of course, the case with no dependence on u, i.e.

$$f(x, u, \xi) \equiv f(x, \xi).$$

We then have, trivially, the following:

PROPOSITION 2. *I is convex over $W_0^{1,\infty}(0, 1)$ if and only if $f(x, .)$ is convex.*

We now give a trivial example showing that no convexity on the variable u can in general be inferred from the convexity of I.

PROPOSITION 3. *Let* $g: \mathbb{R} \to \mathbb{R}$ *be continuous and let*

$$f(x, u, \xi) = g(u)\xi.$$

Then

$$I(u) \equiv 0 \text{ for every } u \in W_0^{1,\infty}(0, 1).$$

Remark. Note, however, in the above example that there exists $\tilde{f}: (0, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, namely $\tilde{f} \equiv 0$, convex in the last two variables such that

$$I(u) = \int_0^1 \tilde{f}(x, u(x), u'(x))\, dx$$

for every $u \in W_0^{1,\infty}(0, 1)$.

We now turn our attention to the last two cases.

PROPOSITION 4. *Let* $a \in C^\infty(\mathbb{R})$ *be such that*

$$a(u) \geqq a_0 > 0 \quad \text{for every} \quad u \in \mathbb{R}$$

and for $n \geqq 1$, *n an integer, let*

$$f(x, u, \xi) = a(u)\xi^{2n}.$$

Then I is convex over $W_0^{1,\infty}(0, 1)$ *if and only if a is constant.*

PROPOSITION 5. *Let* $g, h \in C^\infty(\mathbb{R})$ *and*

$$f(x, u, \xi) = g(u) + h(\xi)$$

and let

$$g_0 = \inf \{g''(u): u \in \mathbb{R}\},\ h_0 = \inf \{h''(\xi): \xi \in \mathbb{R}\}.$$

Then

(i) *There exist g nonconvex and h convex such that I is convex over* $W_0^{1,\infty}(0, 1)$, *for example*

$$g(u) = \tfrac{1}{2}(u^2 - 1)^2,\ h(\xi) = \xi^2.$$

(ii) *I is convex over* $W_0^{1,\infty}(0, 1)$ *if and only if* $h_0 \geqq 0$ *and*

$$\pi^2 h_0 + g_0 \geqq 0. \tag{2}$$

(iii) *Case* 1. *If* $g_0 \geqq 0$ *and* $h_0 \geqq 0$, *then* $f(x, u, \xi) = g(u) + h(\xi)$ *is convex in the variables* (u, ξ).

Case 2. *If* $g_0 < 0$ *and* $\pi^2 h_0 + g_0 > 0$, *then let*

$$\varphi(x, u, \xi) = \sqrt{-g_0 h_0} \tan\left[\sqrt{\frac{-g_0}{h_0}}(x - \tfrac{1}{2})\right]u\xi - \frac{g_0}{2}\left(1 + \tan^2\left[\sqrt{\frac{-g_0}{h_0}}(x - \tfrac{1}{2})\right]\right)u^2. \tag{3}$$

If

$$\Phi(x, u) = \tfrac{1}{2}\sqrt{-g_0 h_0}\left(\tan\left[\sqrt{\frac{-g_0}{h_0}}(x-\tfrac{1}{2})\right]\right)u^2, \tag{4}$$

then

$$\frac{d}{dx}(\Phi(x, u(x)) = \varphi(x, u(x), u'(x)) \text{ almost everywhere} \tag{5}$$

for every $u \in W^{1,\infty}(0, 1)$ *and*

$$\tilde{f}(x, u, \xi) = g(u) + h(\xi) + \varphi(x, u, \xi) \tag{6}$$

is convex in the variables (u, ξ) *for every* $x \in [0, 1]$ *and satisfies*

$$I(u) = \int_0^1 \tilde{f}(x, u(x), u'(x))\, dx \tag{7}$$

for every $u \in W^{1,\infty}(0, 1)$.

Case 3. If $g_0 \leqq 0$ *and* $\pi^2 h_0 + g_0 = 0$ *then* $\tilde{f}$ *defined by* (6) *is convex in* (u, ξ) *for every* $x \in (0, 1)$ *and* (7) *holds if* $u \in \mathscr{D}(0, 1) = \{u \in C^\infty(0, 1)\colon \operatorname{supp} u \subset (0, 1)\}$.

Remarks. (i) Note that the function $\varphi(x, u, \xi)$ in (3) is linear in ξ and it is such that

$$\int_0^1 \varphi(x, u(x), u'(x))\, dx \equiv 0$$

for every $u \in W_0^{1,\infty}(0, 1)$ if $\pi^2 h_0 + g_0 > 0$; such an integral is called an invariant integral in the field theories in the calculus of variations.

(ii) Note also that if $\pi^2 h_0 + g_0 = 0$, then the function φ is not defined at the boundary points $x = 0$ and 1.

Before proceeding with the proof, we quote a lemma whose proof is obvious.

Lemma 6. *Let* $f\colon (0, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be* C^2, *let*

$$I(u) = \int_0^1 f(x, u(x), u'(x))\, dx$$

and for $\lambda \in [0, 1]$, *for* $u, v \in W^{1,\infty}(0, 1)$ *let*

$$\psi(\lambda) = I(\lambda(u - v) + v) - \lambda I(u) - (1 - \lambda)I(v).$$

Then the three following assertions are equivalent:

(i) *I is convex over* $W_0^{1,\infty}(0, 1)$.

(ii) *ψ is convex for every* $u, w \in W_0^{1,\infty}(0, 1)$.

(iii) $\psi''(\lambda) \geqq 0$ *for every* $\lambda \in [0, 1]$, $u, v \in W_0^{1,\infty}(0, 1)$, *where*

$$\psi''(\lambda) = \int_0^1 [(u - v)^2 f_{uu}(x, \lambda(u - v) + v, \lambda(u' - v') + v') + 2(u - v)(u' - v') f_{u\xi} + (u' - v')^2 f_{\xi\xi}]\, dx,$$

where

$$f_{uu}=\frac{\partial^2 f}{\partial u^2},\quad f_{u\xi}=\frac{\partial^2 f}{\partial u\,\partial\xi},\quad f_{\xi\xi}=\frac{\partial^2 f}{\partial\xi^2}.$$

Proof of Proposition 4. The fact that if a is constant then I is convex is trivial. We therefore prove the converse. We divide the proof into three steps.

Step 1. In the above lemma we let $w=u-v$ and $z=\lambda(u-v)+v$. We then have

$$\begin{aligned}0\leqq\psi''(\lambda)&=\int_0^1[w^2a''(z)z'^{2n}+4nww'a'(z)z'^{2n-1}+2n(2n-1)w'^2a(z)z'^{2n-2}]\,dx\\&=\int_0^1 2n(2n-1)a(z)z'^{2n-2}\Bigg[w'^2+\frac{2}{(2n-1)}ww'\frac{a'(z)}{a(z)}z'\\&\quad+\left(\frac{wa'(z)z'}{(2n-1)a(z)}\right)^2-\left(\frac{wa'(z)z'}{(2n-1)a(z)}\right)^2+\frac{w^2a''(z)z'^2}{2n(2n-1)a(z)}\Bigg]\,dx\\&=\int_0^1 2n(2n-1)a(z)z'^{2n-2}\Bigg[\left(w'+\frac{a'(z)z'}{(2n-1)a(z)}w\right)^2\\&\quad-\frac{w^2z'^2}{2n(2n-1)^2(a(z))^2}\cdot(2n(a'(z))^2-(2n-1)a''(z)a(z))\Bigg]\,dx\\&=\int_0^1 2n(2n-1)a(z)z'^{2n-2}\Bigg\{\Bigg[\left(\frac{1}{a(z)}\right)^{1/(2n-1)}((a(z))^{1/(2n-1)}w)'\Bigg]^2\\&\quad-\frac{w^2z'^2(a(z))^{1/(2n-1)}}{2n(2n-1)^2}\frac{2n(a'(z))^2-(2n-1)a''(z)a(z)}{(a(z))^{2+1/(2n-1)}}\Bigg\}\,dx.\end{aligned}\tag{8}$$

On letting

$$b(t)=(a(t))^{-1/(2n-1)},\tag{9}$$

we have

$$\begin{aligned}b''(t)&=-\frac{1}{2n-1}(a^{-1-1/(2n-1)}a')'\\&=+\frac{1}{(2n-1)^2}\frac{2n(a')^2-(2n-1)aa''}{a^{2+1/(2n-1)}}.\end{aligned}$$

Therefore, returning to (8), we have

$$0\leqq\psi''(\lambda)=\int_0^1 2n(2n-1)\frac{z'^{2n-2}}{(b(z))^{(2n-1)^{-1}}}\left\{\left[b(z)\left(\frac{w}{b(z)}\right)'\right]^2-\frac{w^2z'^2}{2nb(z)}b''(z)\right\}dx.\tag{10}$$

Step 2. We now show that (10) implies that

$$b''(t)\leqq 0\text{ for every }t\in\mathbb{R}.\tag{11}$$

Assume, for the sake of contradiction, that there exists $\alpha\in\mathbb{R}$ such that

$$b''(\alpha)>0.\tag{12}$$

By continuity of b'', we can choose $\alpha \neq 0$. We then construct z and w in the following way.

Construction of z. We define for N an integer

$$z(x)=\begin{cases} N\alpha x & \text{if } x\in\left(0,\dfrac{1}{N}\right) \\ \alpha+N\alpha\left(x-\dfrac{k}{N}-\dfrac{m}{N^2}\right) & \text{if } x\in\bigcup_{m=0}^{N-1}\left(\dfrac{k}{N}+\dfrac{m}{N^2},\dfrac{k}{N}+\dfrac{m}{N^2}+\dfrac{1}{2N^2}\right),\quad 1\leqq k\leqq N-2, \\ \alpha-N\alpha\left(x-\dfrac{k}{N}-\dfrac{m+1}{N^2}\right) & \text{if } x\in\bigcup_{m=0}^{N-1}\left(\dfrac{k}{N}+\dfrac{m}{N^2}+\dfrac{1}{2N^2},\dfrac{k}{N}+\dfrac{m+1}{N^2}\right),\quad 1\leqq k\leqq N-2 \\ \alpha-N\alpha\left(x-\dfrac{N-1}{N}\right) & \text{if } x\in\left(\dfrac{N-1}{N},1\right). \end{cases}$$

We then have that $z\in W_0^{1,\infty}(0,1)$ and

$$\begin{cases} |z(x)-\alpha|\leqq\dfrac{|\alpha|}{2N} & \text{if } x\in\left(\dfrac{1}{N},\dfrac{N-1}{N}\right), \\ |z'(x)|=N|\alpha| & \text{almost everywhere in } (0,1). \end{cases} \tag{13}$$

Therefore if $\varepsilon>0$ is fixed, there exists N sufficiently large that

$$|b''(z)-b''(\alpha)|,\quad |b'(z)-b'(\alpha)|,\quad |b(z)-b(\alpha)|\leqq\varepsilon \tag{14}$$

for every $x\in(1/N, N-1/N)$.

Construction of w. We choose w in such a way that

$$\frac{1}{b(z(x))}w(x)=\begin{cases} 0 & \text{if } x\in\left(0,\dfrac{1}{N}\right), \\ \sin\dfrac{\pi N}{N-2}\left(x-\dfrac{1}{N}\right) & \text{if } x\in\left(\dfrac{1}{N},\dfrac{N-1}{N}\right), \\ 0 & \text{if } x\in\left(\dfrac{N-1}{N},1\right). \end{cases} \tag{15}$$

Returning to (10) we have

$$\begin{aligned} 0\leqq\psi''(\lambda)=\int_{1/N}^{1-(1/N)} & 2n(2n-1)\frac{N^{2n-2}|\alpha|^{2n-2}}{(b(z))^{(2n-1)^{-1}}} \\ & \times\left\{\left[b(z)\frac{\pi N}{N-2}\cos\left(\frac{\pi N}{N-2}\left(x-\frac{1}{N}\right)\right)\right]^2\right. \\ & \left.-\frac{b(z)N^2\alpha^2}{2n}\sin^2\left(\frac{\pi N}{N-2}\left(x-\frac{1}{N}\right)\right)b''(z)\right\}dx. \end{aligned}$$

With K_1 and $K_2>0$ denoting constants depending on n, α and $b(\alpha)$, but not on N, and using (14), we have

$$0\leqq\psi''(\lambda)\leqq K_1\left(\varepsilon+\frac{1}{N}\right)+K_2(N^{2n-2}-b''(\alpha)N^{2n}). \tag{16}$$

By letting N tend to infinity and using (12), we have a contradiction with (16). Therefore (11) holds.

Step 3. The conclusion then follows immediately from (11), i.e. from the concavity of b. Recall that $a(t) \geqq a_0 > 0$, therefore

$$0 < b(t) = \left(\frac{1}{a(t)}\right)^{1/(2n-1)} \leqq \left(\frac{1}{a_0}\right)^{1/(2n-1)};$$

the fact that b is concave, and bounded, implies that b, and therefore a, is constant. □

We now conclude with the following proof.

Proof of Proposition 5. Recall that

$$f(x, u, \xi) = g(u) + h(\xi).$$

Recall also the Poincaré–Wirtinger inequality, that is

$$\int_0^1 (w(x))^2 \, dx \leqq \frac{1}{\pi^2} \int_0^1 (w'(x))^2 \, dx$$

for every $w \in W_0^{1,\infty}(0, 1)$ and that equality holds if $w(x) = \sin \pi x$ (see **[2]**).

(i) We now prove that if

$$h(\xi) = \xi^2 \text{ and } g(u) = \tfrac{1}{2}(u^2 - 1)^2,$$

i.e. g is not convex, then the associated I is convex. We use Lemma 6 and we have

$$\begin{aligned}\psi''(\lambda) &= \int_0^1 \{2(u' - v')^2 + [6(\lambda(u - v) + v)^2 - 2](u - v)^2\} \, dx \\ &\geqq 2 \int_0^1 [(u' - v')^2 - (u - v)^2] \, dx.\end{aligned}$$

The Poincaré–Wirtinger inequality then immediately implies the positivity of ψ'' and therefore the convexity of I over $W_0^{1,\infty}(0, 1)$.

(ii) We always have

$$\psi''(\lambda) = \int_0^1 [(u - v)^2 g''(\lambda(u - v) + v) + (u' - v')^2 h''(\lambda(u' - v') + v')] \, dx. \quad (17)$$

($\Leftarrow$) We now wish to show that if $h_0 = \inf\{h''(t) : t \in \mathbb{R}\} \geqq 0$ and $\pi^2 h_0 + g_0 \geqq 0$ where $g_0 = \inf\{g''(t) : t \in \mathbb{R}\}$ then I is convex over $W_0^{1,\infty}(0, 1)$.

It is clear that

$$\psi''(\lambda) \geqq \int_0^1 [(u' - v')^2 h_0 + (u - v)^2 g_0] \, dx.$$

By using the Poincaré–Wirtinger inequality, we have

$$\psi''(\lambda) \geqq \int_0^1 (\pi^2 h_0 + g_0)(u - v)^2 \geqq 0$$

and therefore from Lemma 6, I is convex.

($\Rightarrow$) We now assume that I is convex over $W_0^{1,\infty}(0,1)$ and we wish to show that $h_0 \geqq 0$ and $\pi^2 h_0 + g_0 \geqq 0$. First, as before, we let

$$w = u - v, \quad z = \lambda(u - v) + v.$$

Then $w, z \in W_0^{1,\infty}(0, 1)$ and (17) becomes

$$\psi''(\lambda) = \int_0^1 [w^2 g''(z) + (w')^2 h''(z')]\, dx \geqq 0. \tag{18}$$

Since I is convex, it then follows immediately from Theorem 1 that $h_0 \geqq 0$. It therefore remains to show that $\pi^2 h_0 + g_0 \geqq 0$. Observe that if $g_0 \geqq 0$, then the result is trivial; we therefore assume that $g_0 < 0$.

We now fix N an integer, then there exist ξ_0, $u_0 \in \mathbb{R}$ such that

$$\begin{cases} 0 \leqq h''(\xi_0) - h_0 \leqq \dfrac{1}{N}, \\ 0 \leqq g''(u_0) - g_0 \leqq \dfrac{1}{N}. \end{cases} \tag{19}$$

The aim of the following construction is to choose $w, z \in W_0^{1,\infty}(0, 1)$ such that the left-hand side of (18) is up to a multiplicative constant equal to $\pi^2 h_0 + g_0$, the positivity of $\psi''(\lambda)$ then implying the result.

Construction of z. We let

$$z(x) = \begin{cases} N u_0 x & \text{if } x \in \left(0, \dfrac{1}{N}\right) \\ u_0 + \xi_0\left(x - \dfrac{k}{N}\right) & \text{if } x \in \left(\dfrac{k}{N}, \dfrac{k+1}{N} - \dfrac{1}{N^2}\right), \quad 1 \leqq k \leqq N-3 \\ u_0 - \xi_0(N-1)\left(x - \dfrac{k+1}{N}\right) & \text{if } x \in \left(\dfrac{k+1}{N} - \dfrac{1}{N^2}, \dfrac{k+1}{N}\right), \quad 1 \leqq k \leqq N-3 \\ u_0 & \text{if } x \in \left(\dfrac{N-2}{N}, \dfrac{N-1}{N}\right), \\ -N u_0(x-1) & \text{if } x \in \left(\dfrac{N-1}{N}, 1\right). \end{cases}$$

We then obviously have that $z \in W_0^{1,\infty}(0, 1)$ and that

$$\begin{cases} |z(x) - u_0| \leqq |\xi_0| \left(\dfrac{1}{N} - \dfrac{1}{N^2}\right) & \text{if } x \in \left(\dfrac{1}{N}, \dfrac{N-1}{N}\right), \\ z'(x) = \xi_0 & \text{if } x \in \displaystyle\bigcup_{k=1}^{N-3} \left(\dfrac{k}{N}, \dfrac{k+1}{N} - \dfrac{1}{N^2}\right). \end{cases} \tag{20}$$

Hence for $\varepsilon > 0$ fixed we may choose N sufficiently large so that for $x \in (1/N, N - 1/N)$

$$|g''(z) - g''(u_0)| \leqq \varepsilon. \tag{21}$$

Construction of w. We let

$$w(x)=\begin{cases} 0 & \text{if } x\in\left(0,\dfrac{1}{N}\right),\\ \sin\dfrac{N\pi}{N-2}\left(x-\dfrac{1}{N}\right)+a_{2k-1} & \text{if } x\in\left(\dfrac{k}{N},\dfrac{k+1}{N}-\dfrac{1}{N^2}\right),\quad 1\leqq k\leqq N-3\\ a_{2k} & \text{if } x\in\left(\dfrac{k+1}{N}-\dfrac{1}{N^2},\dfrac{k+1}{N}\right),\quad 1\leqq k\leqq N-3,\\ -Na_{2(N-3)}\left(x-\dfrac{N-1}{N}\right) & \text{if } x\in\left(\dfrac{N-2}{N},\dfrac{N-1}{N}\right),\\ 0 & \text{if } x\in\left(\dfrac{N-1}{N},1\right),\end{cases}$$

where for $1\leqq k\leqq N-3$

$$\begin{cases} a_1=0\\ a_{2k}=a_{2k-1}+\sin\dfrac{Nk-1}{N(N-2)}\pi,\\ a_{2k+1}=a_{2k}-\sin\dfrac{k\pi}{N-2}.\end{cases}$$

Therefore $w\in W_0^{1,\infty}(0,1)$ and

$$\begin{aligned} a_{2k}&=\sum_{\nu=1}^{k}\sin\left[\frac{(N\nu-1)\pi}{N(N-2)}\right]-\sum_{\nu=1}^{k-1}\sin\left[\frac{\nu\pi}{N-2}\right]\\ &=\sin\left[\frac{Nk-1}{N(N-2)}\pi\right]-2\sin\left[\frac{\pi}{2N(N-2)}\right]\sum_{\nu=1}^{k-1}\cos\left[\frac{2N\nu-1}{2N(N-2)}\pi\right],\end{aligned}$$

and similarly

$$a_{2k+1}=-2\sin\left[\frac{\pi}{2N(N-2)}\right]\sum_{\nu=1}^{k}\cos\left[\frac{2N\nu-1}{2N(N-2)}\pi\right].$$

We then deduce that

$$\begin{cases} |a_{2k+1}|\to 0 \quad \text{as} \quad N\to\infty,\\ \left|a_{2k}-\sin\left[\dfrac{Nk-1}{N(N-2)}\pi\right]\right|\to 0 \quad \text{as} \quad N\to\infty,\end{cases}$$

and therefore

$$\left|w(x)-\sin\left[\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]\right|\to 0 \quad \text{as} \quad N\to\infty \quad x\in\left(\frac{1}{N},\frac{N-2}{N}\right). \tag{22}$$

More precisely if $k=N-3$ from (22) we have

$$|a_{2(N-3)}|\leqq\left|\sin\left[\frac{N(N-3)-1}{N(N-2)}\pi\right]\right|+\frac{K}{N}\leqq\frac{K'}{N},$$

where K and K' are constant independent of N and hence $Na_{2(N-3)}$ is uniformly bounded.

Summarising the results we have for $\varepsilon>0$ fixed that there exists N sufficiently large that

$$\begin{cases} |w'|\leqq K \quad \text{almost everywhere in } (0,1) \text{ uniformly in } N,\\ \left|w(x)-\sin\left[\dfrac{N\pi}{N-2}\left(x-\dfrac{1}{N}\right)\right]\right|\leqq \varepsilon,\ x\in(0,1),\\ w'(x)=\dfrac{N\pi}{N-2}\cos\left[\dfrac{N\pi}{N-2}\left(x-\dfrac{1}{N}\right)\right] \quad \text{if} \quad x\in\bigcup_{k=1}^{N-3}\left(\dfrac{k}{N},\dfrac{k+1}{N}-\dfrac{1}{N^2}\right),\\ w'(x)=0 \quad \text{if} \quad x\in\bigcup_{k=1}^{N-3}\left(\dfrac{k+1}{N}-\dfrac{1}{N^2},\dfrac{k+1}{N}\right). \end{cases} \tag{23}$$

Returning to (18), we have

$$\begin{aligned} 0\leqq \psi''(\lambda)&=\int_0^1 [w'^2h''(z')+w^2g''(z)]\,dx\\ &=\int_{1/N}^{(N-1)/N}[w'^2h''(z')+w^2g''(z)]\,dx\\ &=\int_{1/N}^{(N-1)/N}\left(\sin^2\left[\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]g''(z)\right)dx\\ &\quad+\int_{1/N}^{(N-1)/N}\left[w^2(x)-\sin^2\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]g''(z)\,dx\\ &\quad+\sum_{k=1}^{N-3}\int_{k/N}^{(k+1)/N-(1/N^2)}\left(\frac{N\pi}{N-2}\right)^2\cos^2\left[\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]h''(\xi_0)\,dx\\ &\quad+\int_{(N-2)/N}^{(N-1)/N}(Na_{2(N-3)})^2h''(0)\,dx. \end{aligned}$$

Using (20) and (23) we have, with K denoting a generic constant independent of N and ε, that

$$\begin{aligned} 0\leqq -K\left(\varepsilon+\frac{1}{N}\right)&+\int_{1/N}^{(N-1)/N}\sin^2\left[\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]g''(z)\,dx\\ &+h''(\xi_0)\left(\frac{N\pi}{N-2}\right)^2\sum_{k=1}^{N-3}\int_{k/N}^{(k+1)/N-(1/N^2)}\cos^2\left[\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]dx. \end{aligned}$$

Using (19) and (21), we have

$$\begin{aligned} 0\leqq -K\left(\varepsilon+\frac{1}{N}\right)&+g_0\int_{1/N}^{(N-1)/N}\sin^2\left[\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]dx\\ &+h_0\left(\frac{N\pi}{N-2}\right)^2\int_{1/N}^{(N-1)/N}\cos^2\left[\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]dx\\ &-h_0\left(\frac{N\pi}{N-2}\right)^2\left\{\sum_{k=1}^{N-3}\int_{(k+1)/N-(1/N^2)}^{(k+1)/N}\cos^2\left[\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]dx\right.\\ &\left.+\int_{(N-2)/N}^{(N-1)/N}\cos^2\left[\frac{N\pi}{N-2}\left(x-\frac{1}{N}\right)\right]dx\right\}. \end{aligned}$$

Finally, we have

$$0 \leqq K\left(\varepsilon + \frac{1}{N}\right) + (\pi^2 h_0 + g_0) \int_0^1 \sin^2 \pi y \, dy.$$

Letting $N \to \infty$ and using the arbitrariness of ε we have indeed obtained

$$\pi^2 h_0 + g_0 \geqq 0$$

and thus the result.

(iii) Case 1 is trivial and we now show that if $\tilde{f}$ is defined by

$$\tilde{f}(x, u, \xi) = g(u) + h(\xi) + \varphi(x, u, \xi),$$

then

(a) $\tilde{f}(x, ., .)$ is convex over $\mathbb{R}^2$ for every $x \in (0, 1)$.
(b) For every $u \in W_0^{1,\infty}(0, 1)$ we have

$$I(u) = \int_0^1 \tilde{f}(x, u(x), u'(x)) \, dx.$$

In (a), since $\pi^2 h_0 > -g_0$ (Case 2) and $g_0 < 0$, then

$$-\frac{\pi}{2} < \sqrt{\frac{-g_0}{h_0}}(x - \tfrac{1}{2}) < \frac{\pi}{2} \quad \text{if} \quad x \in [0, 1]$$

and if $\pi^2 h_0 = -g_0$ (Case 3), then the above inequality holds only if $x \in (0, 1)$, so that $\tilde{f}$ is well defined if $x \in (0, 1)$. In order to show the convexity of $\tilde{f}$ we show that, denoting by $\gamma = \sqrt{-g_0/h_0}\,(x - \frac{1}{2})$,

$$\nabla^2 \tilde{f} = \begin{pmatrix} \tilde{f}_{\xi\xi} \tilde{f}_{u\xi} \\ \tilde{f}_{u\xi} \tilde{f}_{uu} \end{pmatrix} = \begin{pmatrix} h''(\xi) & \sqrt{-h_0 g_0} \tan \gamma \\ \sqrt{-h_0 g_0} \tan \gamma & g''(u) - g_0(1 + \tan^2 \gamma) \end{pmatrix}$$

the above matrix is positive definite for every $(u, \xi) \in \mathbb{R}^2$. Since $h''(\xi) \geqq h_0$ and $g''(u) \geqq g_0$, and $g_0 < 0$ it remains to show that

$$\det \nabla^2 \tilde{f} = h''(\xi)(g''(u) - g_0(1 + \tan^2 \gamma)) + h_0 g_0 \tan^2 \gamma \geqq 0.$$

We have immediately that

$$\det \nabla^2 \tilde{f} \geqq h_0(g_0 - g_0(1 + \tan^2 \gamma)) + h_0 g_0 \tan^2 \gamma = 0,$$

and thus $\tilde{f}$ is convex.

In (b), we observe that if $u \in W^{1,\infty}(0, 1)$ then

$$\tilde{f}(x, u, u') \equiv g(u) + h(u') + \frac{d}{dx}\left[\frac{\sqrt{-h_0 g_0}}{2} \tan\left(\sqrt{\left(\frac{-g_0}{h_0}\right)}(x - \tfrac{1}{2})\right) u^2\right] \text{ almost everywhere in } (0, 1),$$

and therefore

$$I(u) \equiv \int_0^1 [g(u(x)) + h(u'(x))] \, dx = \int_0^1 \tilde{f}(x, u(x), u'(x)) \, dx,$$

for every $u \in W_0^{1,\infty}(0, 1)$ if $\pi^2 h_0 + g_0 > 0$ and only in $\mathscr{D}(0, 1)$ if $\pi^2 h_0 + g_0 = 0$. □

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