APPLICATION OF RELIABILITY THEORY TO INSURANCE

E. Straub

Swiss Reinsurance Company, Zurich

1. There is a general rule applicable to all insurance and reinsurance fields according to which the level of the so-called technical minimum premium should be fixed such that a certain stability criterion is satisfied for the portfolio under consideration. The two bestknown such criteria are

(i) the probability that there is a technical loss in any of the future years should be less than a given percentage

(ii) the probability that the company gets "ruined" i.e. initial reserves plus accumulated premiums minus accumulated claims becomes negative at any time of a given period in the future should be less than a tolerated percentage.

Confining ourselves to criterion (i) in the present paper we may then say that the problem of calculating technical minimum premiums is broadly spoken equivalent with the problem of estimating loss probabilities. Since an exact calculation of such probabilities is only possible for a few very simple and therefore mostly unrealistic risk models and since e.g. Esscher's method is not always very easy to apply in practice it might be worthwhile to describe in the following an alternative approach using results and techniques from Reliability Theory in order to establish bounds for unknown loss probabilities.

It would have been impossible for me to write this paper without having had the opportunity of numerous discussions with the Reliability experts R. Barlow and F. Proschan while I was at Stanford University. In particular I was told the elegant proof of theorem 3 given below by R. Barlow recently.

2. About ten years ago a group of American statisticians started to work on a subfield of Applied Probability Theory and Statistics which is now called Reliability Theory. For a detailed introduction

to this theory we refer to the book "Mathematical Theory of Reliability" [1]. For the purpose of the present note we only need to make the following remarks:

Let us first look at a few definitions:

—The *reliability of a system* (of any kind) is usually defined as the probability that the system is able to perform its function(s) during a given time period. Using the words of the authors of the book mentioned before, Mathematical Reliability Theory is "a body of ideas, mathematical models and methods directed toward the solution of problems in predicting, estimating or optimizing the probability of survivial, mean life, or, more generally, life distribution of components or systems".

—Another key notion in Reliability Theory is the notion of *failure rate*. If the life time X of a given system is a non-negative and continous stochastic variable with distribution $V(x)$ and density $v(x)$ then the ratio

$$r(x) = \frac{v(x)}{1 - V(x)}$$

is called failure rate (function). This notation corresponds to the intuitive interpretation because $r(x)dx$ is the conditional probability that the system fails in the time interval $(x, x + dx)$ given that it was functioning up to time x. Therefore, from a mathematical point of view, the failure rate $r(x)$ is identical with the force of mortality μ_x since $\mu_x dx$ is the probability that a man of age x dies in $(x, x + dx)$.

—Finally a distribution $V(x)$ is called
IFR (i.e., with increasing failure rate)
if $r(x)$ is nondecreasing in x
and DFR (i.e., with decreasing failure rate)
if $r(x)$ is nonincreasing in x
"prominent" examples of IFR-distributions are

(i) Gamma distributions with density $v(x) = \frac{\mu}{\Gamma(\alpha)}(\mu x)^{\alpha-1}e^{-\mu x}$ where $\alpha \geqslant 1$

(ii) normal distributions

(iii) Weibull distributions with $v(x) = \mu\alpha x^{\alpha-1} e^{-\mu x^\alpha}$ where $\alpha \geqslant 1$

important DFR-distributions are

(i) Gamma distributions with $\alpha \leqslant 1$

(ii) Weibull distributions with $\alpha \leqslant 1$

(iii) Pareto distributions with $v(x) = \alpha(1 + x)^{-\alpha-1}$, $\alpha \geqslant 0$

For the exponential distribution $V(x) = 1 - e^{-\mu x}$ we get

$$r(x) = \frac{\mu e^{-\mu x}}{e^{-\mu x}} = \mu = \text{constant}$$

i.e., the exponential distribution belongs to both the IFR- and the DFR-class.

There are of course distributions belonging to neither of these two classes. From an actuarial point of view the log normal distribution is one of these "regrettable" examples.

3. After this very short excursion into Reliability Theory we return to our insurance rating problem. Following the remarks made in section 1 we are concerned with the problem of estimating loss probabilities $\overline{F}_t(x)$ of the form

$$\overline{F}_t(x) = \sum_{n=0}^{\infty} P_n(t)\, \overline{V}^{(n)}(x)$$

where $\overline{F}_t(x) = 1 - F_t(x) =$ Prob. (total of claims arising in $(0, t)$ exceeds the premium x)

$P_n(t) =$ probability that there are n claims in $(0, t)$

$V(x) =$ distribution of the individual claims amount

$\overline{V}^{(n)}(x) = 1 - V^{(n)}(x)$ where $V^{(n)}(x)$ denotes the n-th convolution of $V(x)$

(assuming mutual independence between the individual claims as well as independence of these claim amounts time from t)

Unfortunately there are only a few very special cases—and mostly unrealistic cases—where it is actually possible to calculate $F_t(x)$ in an exact manner. Besides that the classical approximation technique, namely Esscher's method, is not always easily applicable to practical situations. Therefore it might be worthwhile in the following to look into an alternative approach which leads to upper limits for $F_t(x)$ provided that

(i) $p_n(t) = \dfrac{(\lambda t)^n}{n!} e^{-\lambda t}$ Poisson-distributed number of claims

(ii) $V(x)$ is IFR
(iii) $\mu_1 = \int x dV(x)$ and $\mu_2 = \int x^2 dV(x)$ are given.

It is easy to see that it is actually possible to calculate limits of this sort if we mention the following two results from Reliability Theory:
a) Barlow and Marshall [2] have calculated (sharp) upper limits $U(y, \mu_2)$ with the property that

$$\bar{V}(y) \leqslant U(y, \mu_2), \; y \geqslant 0$$

for each IFR-distribution $V(x)$ with $\mu_1 = 1$ and given μ_2.
b) The convolution of any two IFR-distributions is again IFR [1]. (But the corresponding theorem for DFR-distributions does not hold true as can be verified e.g., by convoluting two Gamma distributions with $1/2 < \alpha \leqslant 1$)
using these results we get

$$\bar{F}_t(x) \leqslant \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \, U\left(\frac{x}{n\mu_1}, 1 + \frac{\sigma^2}{n\mu_1^2}\right)$$

where $\sigma^2 = \mu_2 - \mu_1^2$.

As a numerical illustration to this section we have calculated a few values in Appendix No. 1.

4. Sometimes we may prefer to work with a specified distribution $V(x)$ rather than just assuming that $V(x)$ belongs to a class of distributions such as the IFR class considered above. But then we run into difficulties again if the convolution powers $V^{(n)}(x)$ of $V(x)$ can not be calculated as nice and explicite mathematical expressions (e.g., in the case of the Pareto distribution). In such a situation the following result may be of some help:

Theorem 1: The inequalities

$$\Gamma_n(nR(x/n)) \leqslant (\geqslant) \; V^{(n)}(x) \leqslant (\geqslant) \; \Gamma_n(R(x)), \; n = 1, 2, 3, \ldots$$

hold true for the n-th convolution $V^{(n)}(x)$ of an IFR (DFR) distribution $V(x)$

where
$$\Gamma_n(y) = \begin{cases} 1 - e^{-y} \sum_{j=0}^{n-1} \frac{y^j}{j!} & \text{for } y \geqslant 0 \\ 0 & \text{for } y \leqslant 0 \end{cases}$$

and
$$R(x) = -\log \bar{V}(x) = \int_0^x r(\xi)\, d\xi \qquad \text{(hazard function).}$$

As a concrete example we may take the Pareto distribution $\bar{V}(x) = (1 + x)^{-\alpha}$ which is DFR. In this case we have $R(x) = \alpha \log (1 + x)$ and $nR(x/n) = n\alpha \log (1 + x/n)$

and therefore

$$\Gamma_n(\alpha \log (1 + x)) \leqslant V^{(n)}(x) \leqslant \Gamma_n(n \log (1 + x/n))$$

with regard to practical calculations we remind that

$$\Gamma_n(y) = 1 - \sum_{j=0}^{n-1} \frac{y^j}{j!} e^{-y} = \bar{P}_n(y) = 1 - \text{Poisson distribution.}$$

Since the probabilities $P_n(y)$ are tabulated, it is easily possible to calculate the above bounds even without using a computer.

In Appendix 2 we have given a few numerical values for the probability of loss $\bar{F}_1(x)$ calculated on the basis of this theorem assuming a Poisson-Pareto model i.e. $p_n(1) = \frac{\lambda^n}{n!} e^{-\lambda}$ and $\bar{V}(x) = (1 + x)^{-\alpha}$.

For the actual calculation a FORTRAN program has been used written by J. Hofmann of Swiss Reinsurance Company, Zurich.

5. The above-mentioned theorem turns out to be a special case of the following much more general inequalities.

Theorem 2:

a) If for two distributions $V(x)$ and $G(x)$ with $V(0) = G(0) = 0$ the function $R(x) = G^{-1}(V(x))$ is convex then

$$G^{(n)}(nR(x/n)) \leqslant V^{(n)}(x) \leqslant G^{(n)}(R(x)) \text{ for } n = 1, 2, 3, \ldots$$

b) If for two distributions $V(x)$ and $G(x)$ with $V(0) = G(0) = 0$ the function $R(x) = G^{-1}(V(x))$ is concave then

$$G^{(n)}(R(x)) \leqslant V^{(n)}(x) \leqslant G^{(n)}(nR(x/n)) \text{ for } n = 1, 2, 3, \ldots$$

Here $G^{-1}(y)$ stands for the inverse of the distribution $G(x)$.

$R(x)$ is usually called "generalized hazard function". This notion is used because in the special case where $G(x) = 1 - e^{-x}$

$R(x)$ is equal to the hazard function $R(x) = \int_0^x r(\xi)\, d\xi$

$r(x)$ being again the failure rate of $V(x)$.

Proof of Theorem 2:

(i) any function $R(x)$ is called convex (concave) if

$R(\alpha x + (1 - \alpha)\, y) \leqslant (\geqslant)\ \alpha R(x) + (1 - \alpha)\, R(y)$ for all $x, y \geqslant 0$ and all $\alpha \in [0, 1]$

(ii) if $R(0) = 0$ and $R(x)$ convex and if $x \geqslant y$ then $yR(x) \geqslant xR(y)$

because we can write $y = \frac{yx}{x} + \left(1 - \frac{y}{x}\right) 0$ with $\frac{y}{x} \in [0,1]$

and therefore $R(y) \leqslant \frac{y}{x} R(x) + \left(1 - \frac{y}{x}\right) R(0)$

i.e. $xR(y) \leqslant yR(x)$ qed.

(iii) if $R(x)$ is convex and $R(0) = 0$ then

$R(x + y) \geqslant R(x) + R(y)$ for $x, y \geqslant 0$ i.e. convexity implies superadditivity.

Because assuming two values x and y with $x \geqslant y$ and $R(x + y) < R(x) + R(y)$ we would have

$x = \frac{y}{x}\, y + \left(1 - \frac{y}{x}\right) (x + y)$ with $\frac{y}{x} \in [0,1]$ and

$$R(x) \leqslant \frac{y}{x} R(y) + \left(1 - \frac{y}{x}\right) R(x + y) < \frac{y}{x} R(y) + \left(1 - \frac{y}{x}\right)[R(x) + R(y)]$$

or $0 < xR(y) - yR(x)$ which contradicts (ii) qed.

(iv) We now proceed to prove by induction that

$V^{(n)}(x) \leqslant G^{(n)}(R(x))$ if $R(x) = G^{-1}(V(x))$ is convex.

The statement is true for $n = 1$ by definition because of $V^{(1)}(x) = V(x) = G(R(x)) = G^{(1)}(R(x))$.

We assume that it is also true for $n - 1$ in order to get

$$V^{(n)}(x) = \int_0^x V^{(n-1)}(x - \xi)\, dV(\xi) \leqslant \int G^{(n-1)}(R(x - \xi))\, dG(R(\xi)).$$

Using the superadditivity of $R(x)$ proved in (iii) we have

$R(x) \geqslant R(\xi) + R(x - \xi)$ or $R(x - \xi) \leqslant R(x) - R(\xi)$

i.e. $V^{(n)}(x) \leqslant \int_0^x G^{(n-1)}(R(x) - R(\xi))\, dG(R)(\xi)) = G^{(n)}(R(x))$ qed.

(v) Proof of $G^{(n)}\left(nR\left(\frac{x}{n}\right)\right) \leqslant V^{(n)}(x)$ if $R(x) = G^{-1}(V(x))$ is convex.

The statement is again true for $n = 1$ by definition of $R(x)$, we assume that it holds also true for $n - 1$ and carry out the step from $n - 1$ to n as follows

$$V^{(n)}(x) = \int_0^x V^{(n-1)}(x - \xi)\, dV(\xi) \geqslant$$

$$\geqslant \int_0^x G^{(n-1)}\left((n-1)\, R\left(\frac{x-\xi}{n-1}\right)\right) dG(R(\xi))$$

Using the convexity of $R(x)$ we may write

$$R\left(\frac{x}{n}\right) \leqslant \frac{n-1}{n} R\left(\frac{x-\xi}{n-1}\right) + \frac{1}{n} R(\xi)$$

because of $$\frac{x}{n} = \frac{n-1}{n}\,\frac{x-\xi}{n-1} + \frac{1}{n}\,\xi$$

or $$(n-1)\, R\left(\frac{x-\xi}{n-1}\right) \geqslant nR\left(\frac{x}{n}\right) - R(\xi)$$ leading to

$$V^{(n)}(x) \geqslant \int_0^x G^{(n-1)}\left[nR\left(\frac{x}{n}\right) - R(\xi)\right] dG(R(\xi))$$

$$= \int^{nR(x/n)} G^{(n)}\left(nR\left(\frac{x}{n}\right) - \eta)\, dG(\eta\right) = G^{(n)}\left(nR\left(\frac{x}{n}\right)\right)$$

if we observe that $G^{(n-1)}\left(nR\left(\frac{x}{n}\right) - \eta\right) = 0$ for $\eta > nR\left(\frac{x}{n}\right)$ qed.

(vi) Part *b* of the theorem can be proved in the same way if convexity of $R(x)$ is replaced by concavity.

6. The theorem given in this section together with a certain chain property of the Γ-family will be used later on to improve the upper and lower bounds for convolution powers of IFR and DFR distributions stated in theorem 1.

First we borrow from Reliability Theory a notion of ordering tor distribution functions which is based on the definition of the generalized hazard function [2]:

Notation:

If $R(x) = G^{-1}(V(x))$ is convex we say "$V(x)$ is convex ordered with respect to $G(x)$" and write "$V(x) \underset{c}{\leqslant} G(x)$" or "$V \underset{c}{\leqslant} G$".

Theorem 3:

a) if $G(x)$, $H(x)$ and $V(x)$ are distribution functions with $G(0) = H(0) = V(0) = 0$ and $V \underset{c}{\leqslant} H \underset{c}{\leqslant} G$ then

$$G^{(n)}\left(nR\left(\frac{x}{n}\right)\right) \leqslant H^{(n)}\left(nS\left(\frac{x}{n}\right)\right) \leqslant V^{(n)}(x) \leqslant H^{(n)}(S(x)) \leqslant G^{(n)}(R(x))$$

for $n = 1, 2, 3, \ldots$

b) if $G(x)$, $H(x)$ and $V(x)$ are distribution functions with $G(0) = H(0) = V(0) = 0$ and $G \underset{c}{\leqslant} H \underset{c}{\leqslant} V$ then

$$G^{(n)}(R(x)) \leqslant H^{(n)}(S(x)) \leqslant V^{(n)}(x) \leqslant H^{(n)}\left(nS\left(\frac{x}{n}\right)\right) \leqslant G^{(n)}\left(nR\left(\frac{x}{n}\right)\right)$$

for $n = 1, 2, 3, \ldots$

where $R(x) = G^{-1}(V(x))$ and $S(x) = H^{-1}(V(x))$

The theorem says in other words that bounds for the convolution powers of a given $V(x)$ calculated on the basis of a distrubution $G(x)$ can generally be improved if there is another distribution $H(x)$ which with respect to convex ordering lies in between of $G(x)$ and $V(x)$.

Proof of theorem 3, part b:

We write $T(x) = G^{-1}(H(x))$ and use the abbreviations $G^{-1}H(x)$ and $TS(x)$ for $G^{-1}(H(x))$ and $T(S(x))$. By definition we have $TS(x) = G^{-1}HH^{-1}V(x) = G^{-1}V(x) = R(x)$. All the three functions $R(x)$, $S(x)$ and $T(x)$ are convex by assumtion.

Applying theorem 2 to $G \leqslant H$ we get

$$H^{(n)}(S(x) \geqslant G^{(n)}(TS(x)) = G^{(n)}(R(x))$$

which proves the left hand side of part a)

Furthermore using theorem 2 again we have

$$H^{(n)}\left(nS\left(\frac{x}{n}\right)\right) \leqslant G^{(n)}\left(nT\left(\frac{nS\left(\frac{x}{n}\right)}{n}\right)\right) = G^{(n)}\left(nTS\left(\frac{x}{n}\right)\right) =$$

$$= G^{(n(}\left(nR\left(\frac{x}{n}\right)\right) \text{ qed}$$

(Part a) is proved in the same way)

Finally we would like to mention the following result proved by van Zwet ⌈3]:

The Gamma family forms a chain with respect to convex ordering i.e.

$$\Gamma_\alpha(x) \underset{c}{\leqslant} \Gamma_\beta(x) \text{ if and only if } \alpha \geqslant \beta.$$

This chain property together with theorem 3 and the fact that Gamma distributions are easy to convolute are of great practical value for the calculation of bounds for $V^{(n)}(x)$.

7. *Final Remarks*

We have tried to demonstrate in this paper how to use certain reliability techniques for the calculation of bounds for the probability of loss. The determination of such bounds is, however, by no means the only possibl‹ relationship between Reliability Theory and Insurance Risk Theory. In particular we would like to mention that there is also a useful way of getting bounds for probabilities of ruin [4]. Furthermore it seems that a great variety of statistical procedures developped in Reliability Theory could also be applied successfully to various practical insurance problems.

Appendix no. 1

Upper bounds for the probability of loss in % if the individual claim is IFR-distributed with

$$\mu_1 = \int_0^\infty x\,dV(x) = 1 \text{ and } \mu_2 = \int_3^\infty x^2 dV(x) = 1 + \sigma^2$$

and if the number of claims is Poisson distributed with parameter λ.

	σ^2	$x = 1$	$x = 2$	$x = 3$	$x = 4$						
$\lambda = 1$	0.25	42.0	26.0	12.1	4.9						
	0.50	44.9	26.9	13.5	6.8						
	0.75	40.7	23.5	14.7	9.2						
	1.00	36.9	21.1	16.5	9.7						
$\lambda = 2$	0.25	76.7	56.3	37.1	21.0	11.5	5.4				
	0.50	73.0	54.3	37.2	25.3	15.2	8.3				
	0.75	69.0	49.7	36.5	25.2	17.3	10.5				
	1.00	64.8	46.4	33.1	25.1	18.1	11.5				
$\lambda = 3$	0.25		76.9	60.7	42.3	28.1	16.7	8.7	4.5		
	0.50		74.1	58.7	42.3	32.2	20.4	12.3	7.3		
	0.75		69.9	57.0	43.7	33.4	22.5	14.9	9.6		
	1.00		67.0	52.8	42.7	33.2	23.7	16.9	11.4		
$\lambda = 4$	0.25			83.4	61.9	47.1	32.1	20.2	12.1	6.8	4.0
	0.50			74.9	63.7	50.2	36.4	24.9	16.1	10.1	6.5
	0.75			72.8	60.5	50.1	37.6	27.5	18.9	13.0	8.8
	1.00			68.8	58.6	48.7	38.0	29.0	21.1	14.8	10.4
$\lambda = 5$	0.25				76.8	64.2	49.5	35.2	24.8	15.0	9.3
	0.50				87.1	66.1	52.7	39.6	28.1	19.6	13.6
	0.75				83.9	64.8	52.9	43.6	30.9	23.4	17.0
	1.00				71.7	62.7	52.3	42.7	32.8	24.9	18.0
					$x = 4$	$x = 5$	$x = 6$	$x = 7$	$x = 8$	$x = 9$	$x = 10$

APPENDIX NO. 2

Upper (U) and lower bounds (L) for the probability of loss if the number of claims is Poisson and the individual claims amount is Pareto-distributed.

		$\alpha = 0\ 80$		$\alpha = 1\ 00$		$\alpha = 1\ 25$	
		U	L	U	L	U	L
$\lambda = 0\ 25$	$x = 1\ 0$	136	135	120	119	103	101
	$x = 2\ 0$	102	100	084	081	066	062
	$x = 4\ 0$	071	067	053	049	037	033
	$x = 8\ 0$	047	041	032	026	019	015
	$x = 16\ 0$	030	024	018	013	009	006
$\lambda = 0\ 50$	$x = 1\ 0$	256	253	230	226	200	196
	$x = 2\ 0$	199	191	167	156	134	124
	$x = 4\ 0$	144	129	112	095	081	065
	$x = 8\ 0$	099	079	070	050	045	028
	$x = 16\ 0$	066	044	042	024	024	011
$\lambda = 1\ 00$	$x = 1\ 0$	454	447	417	408	375	363
	$x = 2\ 0$	372	352	325	299	273	241
	$x = 4\ 0$	288	245	235	185	181	127
	$x = 8\ 0$	213	184	160	094	111	052
	$x = 16\ 0$	153	079	104	041	064	018
$\lambda = 2\ 00$	$x = 1\ 0$	717	705	681	665	638	615
	$x = 2\ 0$	635	597	582	530	520	452
	$x = 4\ 0$	539	446	.471	354	394	258
	$x = 8\ 0$	441	277	362	182	279	102
	$x = 16\ 0$	351	138	268	069	188	028
$\lambda = 4\ 00$	$x = 1\ 0$	930	922	914	901	893	874
	$x = 2\ 0$	891	858	863	812	824	750
	$x = 4\ 0$	836	731	789	638	727	521
	$x = 8\ 0$	767	517	698	375	611	233
	$x = 16\ 0$	687	265	598	137	491	052

REFERENCES

[1] R BARLOW, F PROSCHAN and L HUNTER, Mathematical Theory of Reliability Theory, John Wiley and Sons

[2] R BARLOW and W VAN ZWET, Asymptotic properties of isotonic estimators for the generalized failure rate, Part I, Berkeley 1969

[3] W VAN ZWET, Convex transformations of random variables, Mathematical Centre, Amsterdam 1964

[4] R MOREY, Some stochastic properties of a compourd renewal damage model, Operations Research 1967(?)