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On the positive, "radial" solutions of a semilinear elliptic equation in $\mathbb{H}^N$

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Abstract. We discuss various kinds of existence and non existence results for positive solutions of Emden–Fowler type equations in the hyperbolic space. The main tools are perturbation analysis, variational methods, Pohozeav type identities and reduction to Matukuma equations.

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1 Introduction

In this paper we consider the equation

$$\Delta_{\mathbb{H}^N} u + \lambda u + u^p = 0, \quad u > 0, \tag{1.1}$$

where $\Delta_{\mathbb{H}^N}$ is the Laplace–Beltrami operator on the hyperbolic space $\mathbb{H}^N$, $N \geq 3$, λ is a real parameter and $p > 1$.

The corresponding equation in the Euclidean space arises in geometry and physics and has led to interesting mathematical studies. It is called *scalar field equation* if $\lambda < 0$, the *Emden–Fowler* equation if $\lambda = 0$ and the *conformal scalar curvature* equation if $p = \frac{N+2}{N-2}$.

We are interested in solutions which depend only on the hyperbolic distance from a fixed center. In order to express (1.1) for such "radial" solutions, we recall that the hyperbolic space $\mathbb{H}^N$ is the set of points of the hyperboloid

$$\mathcal{H} := \{(x_1, x_2, \dots, x_{N+1}) : x_{N+1}^2 - (x_1^2 + x_2^2 + \cdots + x_N^2) = 1,\ x_{N+1} > 1\}$$

in $\mathbb{R}^{N+1}$ endowed with the Lorentz metric

$$d_H(x, y) = \operatorname{arccosh}(-x_1 y_1 - \cdots - x_N y_N + x_{N+1} y_{N+1}).$$

Note that the distance from an arbitrary point to the *origin* $e_{N+1} := (0, 0, \dots, 1)$ is $d(e_{N+1}, y) = \operatorname{arccosh}(y_{N+1})$.

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For the analysis it is more convenient to use the ball model. It is obtained by a stereographic projection from $\mathcal{H}$ onto $\mathbb{R}^N$. A point $x \in \mathcal{H}$ is mapped to the point $z \in \mathbb{R}^N$ which is obtained by intersecting the line joining x and $-e_{N+1}$ with $\{x \in \mathbb{R}^{N+1} : x_{N+1} = 0\}$. Then $\mathbb{H}^N$ is given by the unit ball $B_1 \subset \mathbb{R}^N$ with the Riemannian metric

$$ds^2 = \frac{4}{(1-|z|^2)^2}|dz|^2, \quad z \in B_1.$$

In these coordinates the hyperbolic distance from z to the origin becomes

$$d_H(z,0) = 2\,\mathrm{arctanh}(|z|).$$

In polar coordinates we have $z = \rho\theta$, where $|z| = \rho$ and θ is a point on the unit sphere $\mathbb{S}^{N-1}$. The change of variable $\rho = \tanh(t/2)$ leads to

$$ds^2 = dt^2 + \sinh^2(t)|d\theta|^2.$$

Consequently

$$\Delta_{\mathbb{H}^N} = \sinh^{-(N-1)}(t)\frac{\partial}{\partial t}\left(\sinh^{N-1}(t)\frac{\partial}{\partial t}\right) + \sinh^{-2}(t)\Delta_S,$$

where Δ_S is the spherical Laplacian and $t = d_H(0,z)$ is the hyperbolic distance. This reduction is well known, cf. e.g. [9, Section 3.9]. The “radial” solutions of (1.1) satisfy the ordinary differential equation

$$u''(t) + (N-1)\coth(t)u'(t) + \lambda u(t) + u^p(t) = 0 \quad \text{in } \mathbb{R}^+, \quad u > 0. \tag{1.2}$$

Kumaresan and Prajapat [13] observed that the moving plane method of Gidas, Ni and Nirenberg [7, 8] extends to $\mathbb{H}^N$. Thus the radial solutions play an important role.

The goal of this paper is to present a general picture of the set of positive, radial solutions. Particular results have been obtained by Stapelkamp [18, 19], Mancini and Sandeep [15] and Bonforte, Gazzola, Grillo and Vazquez [4]. We also mention the paper [1] where more general solutions of (1.1) are considered.

In [15] and [4] it was observed that for any solution u the energy functional

$$\mathcal{E}(t) := \frac{u'^2(t)}{2} + \lambda\frac{u^2(t)}{2} + \frac{u^{p+1}(t)}{p+1}$$

is monotonically decreasing since $\mathcal{E}'(t) = -(N-1)\coth(t)u'^2(t) < 0$. This implies that u is bounded for $t > 0$. Notice that u can be singular at the origin. Denote by $J = (d_0, d_1)$ the maximal interval of existence of a positive solution u. Hence if $0 < d_0 < d_1 < \infty$, then u vanishes at its endpoints and yields a solution in an annulus. This class will be denoted by $S(d_0, d_1)$. If $J = (0, d_1)$ where $d_1 < \infty$, then u vanishes at d_1. The class of these solutions defined in a (possibly punctured) ball will be denoted by $B(d_1)$. Similarly if $J = (d_1, \infty)$ where $0 < d_1$, then u vanishes at d_1. These solutions defined in outer balls belong to the class $B^c(d_1)$. All other solutions exist for all $t > 0$ and form the class $E(0, \infty)$.

This paper is organized as follows. In Section 2 we discuss the local behavior of the solutions at the origin and at infinity. The main tool is perturbation analysis ([3,10]). This method provides also the existence of local solutions. We then study their global behavior. The first approach carried out in Section 3 is by combining the local results of Section 2 with variational methods proposed in [15] and nonexistence results derived by means of Pohozaev type identities. The second approach in Section 4 consists in transforming (1.2) into a Matukuma equation and applying the criteria derived by Yanagida and Yotsutani [20, 22].

It should be pointed out that the local structure is almost completely understood whereas many questions concerning the global behavior and uniqueness are still open.

2 Classification of the positive radial solutions

2.1 Asymptotic behavior as $t \to \infty$

2.1.1 General remarks

Throughout this section we shall assume that $u(t)$ exists for large t. Then either $u(t) \in B^c(d_1)$ or $u(t) \in E(0, \infty)$. Because $\mathcal{E}(t)$ is decreasing and bounded from below, $u(t)$ converges to a constant solution as $t \to \infty$. Hence as $t \to \infty$ we have

$$u(t) \to \begin{cases} 0 & \text{if } \lambda \geq 0, \\ 0 \text{ or } \Lambda := (-\lambda)^{1/(p-1)} & \text{if } \lambda < 0. \end{cases}$$

For the next considerations it will be useful to transform (1.2) into a first order system. Set

$$U := \begin{pmatrix} u \\ u' \end{pmatrix}, \quad A(t) := \begin{pmatrix} 0 & 1 \\ -\lambda & -(N-1)\coth t \end{pmatrix}, \quad \mathcal{F}(U) := \begin{pmatrix} 0 \\ -|u|^{p-1}u \end{pmatrix}.$$

In this notation (1.2) reads as

$$U' = A(t)U + \mathcal{F}(U). \tag{2.1}$$

By the variation of constants the system (2.1) can be written in the form

$$U(t) = y(t) + \int_{t_0}^{t} e^{A(t-s)} \mathcal{F}(U)(s)\, ds, \tag{2.2}$$

where $y(t)$ is a solution of the linear system $y' = Ay$.

The results on the asymptotic behavior of the solutions as $t \to \infty$ are based on well-known stability analysis for perturbed linear systems, cf. [3] and [10, Chapter VIII and X]. Let us now recall the principal results.

Let $\|A\| = \sum_{i,j=1}^{N} |a_{ij}|$ be the matrix norm. Assume that there exists $t_0 > 0$ such that $A(t) = A_0 + B(t)$ where A_0 is a constant matrix and $B(t)$ has the property $\int_{t_0}^{\infty} \|B(s)\|\, ds < \infty$. Under these assumptions the behavior of the perturbed nonlinear system (2.1) is very similar to the behavior of the linear system $Y' = A_0 Y$.

Let ω_1 and ω_2 be the eigenvalues of A_0 and φ_1 and φ_2 be the corresponding eigenfunctions. Then the following lemma holds true.

Lemma 2.1. *Let $U(t)$ be a solution of (2.1) such that $U(t) \to 0$ as $t \to \infty$.*

(i) *If $\omega_k = \alpha \pm i\beta$, $\alpha < 0$ and $\beta \neq 0$, then there exist constants c_1, c_2 such that*

$$\begin{aligned} U(t) = c_1 e^{\alpha t}[\cos\beta t\, \mathrm{Re}\{\varphi_1\} - \sin\beta t\, \mathrm{Im}\{\varphi_1\} + o(1)] \\ + c_2 e^{\alpha t}[\sin\beta t\, \mathrm{Re}\{\varphi_2\} + \cos\beta t\, \mathrm{Im}\{\varphi_2\} + o(1)] \end{aligned}$$

as $t \to \infty$. Conversely for given c_1, c_2 such a solution exists for large t.

(ii) *If $\omega_1 < \omega_2 < 0$, then there exist constants c_1, c_2 such that*

$$U(t) = c_1 e^{\omega_1 t}(1 + o(1))\varphi_1 \quad \text{or} \quad U(t) = c_2 e^{\omega_2 t}(1 + o(1))\varphi_2 \qquad \text{as } t \to \infty.$$

Moreover, such solutions exist for large t.

(iii) *If $\omega_1 < 0 \le \omega_2$, then there exists for large t a one-parameter family of solutions to (2.1) such that $U(t) \to 0$ as $t \to \infty$. In addition,*

$$U(t) = c e^{\omega_1 t}(1 + o(1))\varphi_1 \quad \text{if } t \to \infty.$$

(iv) *If $\omega_1 = \omega_2 < 0$ and $\varphi_1 = \mathrm{const.}\times\varphi_2$, then either $U(t) = c_1 e^{\omega_1 t}(1+o(1))\varphi_1$ or $U(t) = c_2 e^{\omega_1 t} t(1 + o(1))\varphi_1$. Moreover, such solutions exist for large t.*

(v) *If $\omega_2 > 0$, then $U = 0$ is unstable.*[1]

[1] The case $\omega_2 = 0$ is more involved and no general statements are possible.

2.1.2 The case $u(t) \to 0$ as $t \to \infty$

In this case we set

$$A_0 := \begin{pmatrix} 0 & 1 \\ -\lambda & -(N-1) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ 0 & (N-1)(1-\coth t) \end{pmatrix}.$$

The eigenvalues of A_0 are

$$\omega_1 = -\sqrt{\lambda_0^2 - \lambda} - \lambda_0, \quad \text{where } \lambda_0 := \frac{N-1}{2}, \tag{2.3}$$

$$\omega_2 = \sqrt{\lambda_0^2 - \lambda} - \lambda_0.$$

From Lemma 2.1 it follows immediately that no positive solution tending to zero exists if $\lambda > \lambda_0^2$.

Definition 2.2. Let u be a positive solution of (1.2) tending to zero a infinity. It is said that u *decays rapidly at infinity* if $e^{\lambda_0 t} u(t) \to u_\infty < \infty$ as $t \to \infty$. However, if $\lim_{t\to\infty} e^{\lambda_0} u = \infty$, then we say that u *decays slowly at infinity*.

Lemma 2.1 applied to (1.2) yields

Lemma 2.3. (i) *Let $0 < \lambda < \lambda_0^2$. If u is a solution in $E(0,\infty)$ or in $B^c(d_1)$, then two possibilities can occur if $t \to \infty$:*

$$u(t)e^{(\lambda_0 + \sqrt{\lambda_0^2 - \lambda})t} \to u_\infty \quad \textit{(rapidly decaying solution),}$$

$$u(t)e^{(\lambda_0 - \sqrt{\lambda_0^2 - \lambda})t} \to \tilde{u}_\infty \quad \textit{(slowly decaying solution).}$$

Moreover for fixed $t_0 > 0$ and sufficiently small $|u(t_0)|^2 + |u'(t_0)|^2$ there exists a one-parameter family of rapidly decaying and a two-parameter family of slowly decaying solutions of (1.2).

(ii) *Assume $\lambda < 0$. Every solution $u \in E(0,\infty)$ or $u \in B^c(d_1)$ tending to zero satisfies*

$$u(t)e^{(\lambda_0 + \sqrt{\lambda_0^2 - \lambda})t} \to u_\infty \quad \textit{as } t \to \infty \quad \textit{(rapidly decaying solution).}$$

In addition for fixed t_0 and sufficiently small $|u(t_0)|^2 + |u'(t_0)|^2$ there exists a one-parameter family of rapidly decaying solutions.

(iii) *Let $\lambda = \lambda_0^2$. Then as $t \to \infty$*

$$u(t)e^{\lambda_0 t} \to u_\infty \quad \textit{(rapidly decaying solution),}$$

$$u(t)e^{\lambda_0 t} t^{-1} \to \tilde{u}_\infty \quad \textit{(slowly decaying solution).}$$

Conversely such solutions exist for large t.

Let us now discuss the case $\lambda = 0$ which requires an additional argument because $\omega_2 = 0$ (cf. the footnote to Lemma 2.1 (v)). It has already been studied in [4]. We give here a different proof.

Suppose that $u(t)$ exists and tends to zero for $t \to \infty$. It is not difficult to see that all solutions tending to zero are either monotone decreasing if they belong to $E(0, \infty)$ or they have at most one local maximum if they are in $B^c(d_1)$. In fact, this follows immediately from (1.2) in the case $\lambda \geq 0$. If $\lambda < 0$, we need in addition the monotonicity of $\mathcal{E}(t)$. Hence there exists $t_0 > 0$ such that $u' \neq 0$ for $t \geq t_0$. Consider the function $w := \frac{u'}{u}$. For large t it is negative and satisfies the Riccati type equation

$$w' + w^2 + (N-1)(1+\delta(t))w + u^{p-1} = 0, \tag{2.4}$$

where $\delta(t) := \coth(t) - 1 \to 0$ as $t \to \infty$.

Proposition 2.4. *The solutions of* (2.4) *satisfy either*

$$\lim_{t\to\infty} w(t) = 0 \quad \textit{or} \quad \lim_{t\to\infty} w(t) = -(N-1).$$

Proof. It is easy to see that w is bounded from above. We claim that w is also bounded from below. Suppose the contrary. Then

$$w' = -w^2(1+o(1)) \quad \text{implies} \quad w(t) = \frac{1}{(t-t_0)(1+o(1)) + w^{-1}(t_0)}.$$

Since $w(t_0)$ is negative for large t_0, it follows that w blows up for finite t, in contradiction to our assumption. Hence $\lim_{t\to\infty} w'(t) = 0$ implies that we have $w \to 0$ or $w \to -(N-1)$ as $t \to \infty$. □

This proposition leads to

Lemma 2.5. *Assume $\lambda = 0$. If $u \in E(0, \infty)$ or in $B^c(d_1)$, then one of the two possibilities occur as $t \to \infty$:*

$$u(t)e^{(N-1)t} \to u_\infty \qquad \textit{(rapidly decaying solution),}$$

$$u(t)t^{\frac{1}{p-1}} \to c(N,p) := \left(\frac{N-1}{p-1}\right)^{\frac{1}{p-1}} \qquad \textit{(slowly decaying solution).}$$

Moreover, there exist locally a one-parameter family of rapidly decaying solutions and a two-parameter family of slowly decaying solutions.

Proof. The first case occurs if in Proposition 2.4 we have $w \to -(N-1)$. Then $u(t)e^{(N-1)t} \to u_\infty$ as $t \to \infty$ and u is a rapidly decaying solution. The existence of such local solutions follows from Lemma 2.1 (ii).

If $w \to 0$, we deduce from

$$\frac{u''}{u'} + N - 1 + \delta(t) + \frac{u^p}{u'} = 0$$

and from Bernoulli–L'Hospital's rule that $0 = \lim_{t\to\infty} \frac{u'}{u} = \lim_{t\to\infty} \frac{u''}{u'}$ that

$$\lim_{t\to\infty} \frac{u^p}{u'} = -(N-1).$$

This implies that

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} u(t) = c(N, p).$$

It remains to prove the existence of such a solution. Set

$$\mathcal{G}(t) := w^2(t) + (N-1)\coth(t)w(t) + u^{p-1}(t).$$

Choose $u(t_0)$ and $u'(t_0)$ such that $\mathcal{G}(t_0) < 0$ and $w(t_0) > 1 - N$. Then by equation (2.4), $w' > 0$ near t_0. Observe that $w(t)$ increases until $w'(\tau) = 0$ or equivalently $\mathcal{G}(\tau) = 0$. This is impossible because $w(t) > 1 - N$. Consequently $w(t)$ increases and tends to zero as $t \to \infty$. This completes the proof. □

2.1.3 The case $\lambda < 0$ and $u(t) \to \Lambda$ as $t \to \infty$.

The goal of this section is to determine the decay rate of u near Λ. The arguments will be exactly the same as for Lemma 2.3.

Replace u in (1.2) by $\Lambda + v$. Then v solves for large t the linearized equation

$$v'' + (N-1)\coth(t)v' - \lambda(p-1)v + O(v^2) = 0. \tag{2.5}$$

Exactly the same arguments as in Section 2.1.2 apply. The only differences are the matrix A_0 which has to be replaced by

$$\tilde{A}_0 := \begin{pmatrix} 0 & 1 \\ \lambda(p-1) & -(N-1) \end{pmatrix},$$

and the inhomogeneous term $\mathcal{F}(V)$, $V = (v, v')$ which has to be changed accordingly. The eigenvalues of $\tilde{A}_0$ are

$$\beta_\pm = \pm\sqrt{\lambda_0^2 + \lambda(p-1)} - \lambda_0.$$

This implies that either

$$e^{(\lambda_0+\sqrt{\lambda_0^2+\lambda(p-1)})t}v(t) \to v_\infty \quad \text{as } t \to \infty, \tag{2.6}$$

or

$$e^{(\lambda_0-\sqrt{\lambda_0^2+\lambda(p-1)})t}v(t) \to \tilde{v}_\infty \quad \text{as } t \to \infty. \tag{2.7}$$

In accordance with the solutions u tending to 0 we say that u tends *rapidly* to Λ in the first case of (2.6) and it decays *slowly* to Λ in the second case.

Lemma 2.6. *Suppose that u is a solution of* (1.2) *which exists for $t > t_0$ and tends to Λ as $t \to \infty$.*

(i) *If $-\lambda_0^2 < (p-1)\lambda < 0$, then either*

$$e^{-t\beta_-}(u(t)-\Lambda) \to v_\infty \quad \textit{or} \quad e^{-t\beta_+}(u(t)-\Lambda) \to \tilde{v}_\infty \quad \textit{as } t \to \infty.$$

Moreover for (u, u') close to $(\Lambda, 0)$ there exists a one-parameter family of rapidly decaying local solutions and a two-parameter family of slowly decaying solutions.

(ii) *Assume $\lambda(p-1) < -\lambda_0^2$. Then u oscillates around λ and tends eventually to Λ. Moreover for (u, u') close to $(\Lambda, 0)$ there exists locally a two-parameter family of solutions of this type.*

(iii) *Let $-\lambda(p-1) = \lambda_0^2$. Then*

$$(u(t)-\Lambda)e^{\lambda_0 t} \to v_\infty \quad \textit{or} \quad (u(t)-\Lambda)e^{\lambda_0 t}t^{-1} \to \tilde{v}_\infty \quad \textit{as } t \to \infty.$$

Conversely such solutions exist for large t.

2.2 Behavior at $t = 0$

Assume that u exists at $t = 0$. It belongs therefore either to $B(d_1)$ or to $E(0, \infty)$. For small t we can write (1.2) as

$$u'' + \frac{N-1}{t}(1 + a(t))u' + \lambda u + u^p = 0,$$

where $a(t) = t \coth t - 1 = O(t^2)$. Proceeding as in [2] we shall first perform the *Emden–Fowler transformation*

$$x = (2-N)\log(t), \quad v = t^{\frac{2}{p-1}}u, \quad \sigma := \frac{2}{(p-1)(N-2)}.$$

Then, setting $v' := dv/dx$ we have

$$v'' - (1-2\sigma)v' - \sigma(1-\sigma)v + O(e^{-\frac{2x}{N-2}})(v + v') + v^p(N-2)^{-2} = 0. \tag{2.8}$$

We are interested in the behavior of $v(x)$ as $x \to \infty$. According to the results in [2] which are based on the analysis of perturbed linear systems [10] considered in the previous sections, it follows that v is bounded and converges either to $v_0 := 0$ or, in the case $\sigma < 1$, to $v_1 := \{\sigma(1-\sigma)(N-2)^2\}^{\frac{1}{p-1}}$.

If $v \to 0$ at $x \to \infty$, then the corresponding linear system is

$$Y' = \begin{pmatrix} 0 & 1 \\ \sigma(1-\sigma) & 1-2\sigma \end{pmatrix} Y.$$

The eigenvalues of the matrix are $-\sigma$ and $1-\sigma$. Hence for all positive σ there is a family of solutions behaving like

$$v(x) = e^{-\sigma x}(c + o(1))$$

(equivalently $u(t) = u_0(1 + o(t))$ as $t \to 0$).

If $\sigma > 1$, there is an additional family of solutions behaving like

$$v(x) = e^{(1-\sigma)x}(c + o(1))$$

(equivalently $u(t) = t^{-(N-2)}(c + o(t))$ as $t \to 0$).

Lemma 2.1 does not apply if $\sigma = 1$ because $\omega_2 = 0$. The arguments of Theorem 5.1 (iii) in [2] show that if a solution u exists which is singular at the origin, then

$$\begin{aligned} &\lim_{t\to 0} t^{N-2} u(t) = 0, \\ &\limsup_{t\to 0} t^{\beta} u(t) = \infty \qquad \text{for all } 0 < \beta < N-2. \end{aligned} \tag{2.9}$$

Let us now discuss the case when $v \to v_1$ and consequently $\sigma < 1$. To this end, set $v(x) = v_1 + \eta$ and observe that for small η

$$\eta'' - (1-2\sigma)\eta' + \sigma(1-\sigma)(p-1)\eta + O(\eta^2) = 0.$$

The linear equation has solutions of the form

$$\begin{aligned} &\eta_1 = c_1 e^{(\gamma_1+\gamma_2)x} \quad \text{and} \quad \eta_2 = c_2 e^{(-\gamma_1+\gamma_2)x}, \\ &\gamma_2 = \frac{1-2\sigma}{2} \quad \text{and} \quad \gamma_1 = \sqrt{\gamma_2^2 - \sigma(1-\sigma)(p-1)}. \end{aligned}$$

Notice that these solutions tend to zero at $x = \infty$ only if $\sigma > 1/2$.

In conclusion we have the following lemma.

Lemma 2.7. (i) *If $t \to 0$, then either u is regular and behaves like $u(t) \to u_0$ and $u'(t) \to 0$, or u is singular and behaves like*

$$u(t) = \begin{cases} t^{-(N-2)}(c + o(t)) & \text{if } p < \frac{N}{N-2}, \\ t^{-\frac{2}{p-1}}(1 + o(1)) & \text{if } \frac{N}{N-2} < p. \end{cases}$$

Furthermore there exists for all $p > 1$ a one-parameter family of regular solutions. In the cases listed above there is a two-parameter family of singular solutions.

(ii) *If $p = N/(N-2)$, then the singular solutions satisfy* (2.9)

(iii) *If $p > \frac{N+2}{N-2}$, no solutions exist which are singular at $t = 0$.*

Remark 2.8. From the monotonicity of $\mathcal{E}(t)$ it follows that if $u(t) \to \Lambda$ as $t \to 0$, then $u(t) \equiv \Lambda$.

If $\sigma = 1/2$, then the linear system has a center in v_1. A more subtle analysis is required to determine the behavior of η for the nonlinear equation.

3 Global behavior

In this section we study the different classes of solutions. For the sake of completeness we shall also list some known results.

Write E_{rr} for the set of solutions in $E(0,\infty)$ which are regular at zero and rapidly decreasing at infinity and E_{ss} for the set solutions in $E(0,\infty)$ which are singular at zero and slowly decaying at infinity. Likewise we define E_{rs}, E_{sr}, B_r, B_s, B_r^c and B_s^c.

3.1 The case $S(d_0, d_1)$, $0 < d_0 < d_1$

By classical arguments the variational problem

$$\mathcal{J}(v) = \int_{d_0}^{d_1} (v'^2 - \lambda v^2) \sinh^{N-1} t \, dt \to \min,$$

where $v \in \mathcal{K}$ and

$$\mathcal{K} := \left\{ v \in C^1(d_0, d_1) : v(d_0) = v(d_1) = 0, \int_{d_0}^{d_1} |v|^{p+1} \sinh^{N-1} t \, dt = 1 \right\},$$

has a positive solution for every $p > 1$ provided $\lambda < \lambda_S(d_0, d_1)$ where $\lambda_S(d_0, d_1)$ is the Dirichlet eigenvalue of the radial part of $\Delta_{\mathbb{H}^N}$ in (d_0, d_1).

3.2 Pohozaev type identity: Integral form

An important tool for proving the nonexistence of solutions is the Pohozaev identity. We present a version which has been derived in [19] for the study of the Brezis–Nirenberg problem in $\mathbb{H}^N$ and also in [15]. Since we use here different coordinates, we shall state it for the sake of completeness.

In a first step we transform (1.2) into an equation without first order derivatives. For this purpose set

$$u(t) = \sinh^{-\frac{N-1}{2}}(t)v(t) = \sinh^{-\lambda_0}(t)v(t).$$

Then $v(t)$ solves

$$v'' - a(t)v + b(t)v^p = 0, \tag{3.1}$$

where

$$a(t) = \lambda_0 - \lambda + \lambda_0 \frac{N-3}{2}\coth^2(t) \quad \text{and} \quad b(t) = \sinh^{-\lambda_0(p-1)}(t).$$

If we multiply (3.1) with $v'g$ and integrate, we obtain

$$\begin{aligned}\frac{1}{2}\int_0^T g'v'^2\,dt = \frac{v'^2 g}{2}\Big|_0^T - \frac{agv^2}{2}\Big|_0^T + \frac{bgv^{p+1}}{p+1}\Big|_0^T \\ + \frac{1}{2}\int_0^T (ag)'v^2\,dt - \frac{1}{p+1}\int_0^T (bg)'v^{p+1}\,dt.\end{aligned} \tag{3.2}$$

Multiplication of (3.1) with $g'v$ and integration yields

$$\begin{aligned}\frac{1}{2}\int_0^T g'v'^2\,dt = \frac{1}{2}g'vv'\Big|_0^T - \frac{1}{4}v^2g''\Big|_0^T \\ + \int_0^T \left[\frac{g'''}{4} - \frac{ag'}{2}\right]v^2\,dt + \int_0^T \frac{g'b}{2}v^{p+1}\,dt.\end{aligned} \tag{3.3}$$

Suppose that

$$v(0) = v(T) = 0, \quad |v'(T)| < \infty \quad \text{and} \quad \lim_{t\to 0} v(t)v'(t) = 0. \tag{3.4}$$

Then (3.2) and (3.3) lead to the following *Pohozaev* type identity:

$$\frac{v'^2 g}{2}\Big|_0^T + \int_0^T \left[\frac{a'g}{2} + ag' - \frac{g'''}{4}\right]v^2\,dt = \int_0^T \left[\frac{(bg)'}{p+1} + \frac{g'b}{2}\right]v^{p+1}\,dt. \tag{3.5}$$

3.3 $B(T)$

Let $\lambda_B(T)$ be the first Dirichlet eigenvalue of $\Delta_{\mathbb{H}^N}$ in the geodesic ball B_T. Observe that $\lambda_B(T) > \lambda_0^2$ and that for $N = 3$ we have $\lambda_B(T) = 1 + (\frac{\pi}{T})^2$.

The variational method described in Section 3.1 for subcritical exponents applies also in this case. Mancini and Sandeep [15] established the uniqueness. More precisely

- if $1 < p < \frac{N+2}{N-2}$ and $\lambda < \lambda_B(T)$, then there exists a unique, positive solution of (1.2) in $B(0,T)$ which is regular at the origin.

S. Stapelkamp [19] (cf. also [18]) has studied the case of the critical exponent $p = \frac{N+2}{N-2}$ and she has obtained the following result:

- If
$$\lambda_B(T) > \lambda > \lambda^* := \begin{cases} \frac{N(N-2)}{4} & \text{if } N > 3, \\ 1 + \left(\frac{\pi}{2T}\right)^2 & \text{if } N = 3, \end{cases}$$
then there exists a unique solution in $B(0,T)$ which is regular at the origin.
- If $\lambda \leq \lambda^*$ or $\lambda \geq \lambda_B(T)$, no solution exists in $B(0,T)$ which is regular at the origin.

She has established the existence by means of the method of concentration compactness and the uniqueness by an argument of Kwong and Li [14]. The nonexistence was shown by means of (3.5).

Next we extend this nonexistence result.

Lemma 3.1. (i) *Assume*
$$\lambda \leq \begin{cases} \frac{N(N-2)}{4} & \textit{if } N > 3, \\ 1 + \left(\frac{\pi}{2T}\right)^2 & \textit{if } N = 3. \end{cases}$$
If $p \geq \frac{N+2}{N-2}$, then $B_r = \emptyset$.

(ii) *If $p > \frac{N+2}{N-2}$, then for any λ, $B_s = \emptyset$.*

Proof. (i) If $u \in B(0,T)$ is regular at the origin, then the properties (3.4) are satisfied. Set $g = \sinh t$. Then the left-hand side of (3.5) becomes
$$\frac{v'^2(T)g(T)}{2} + \int_0^T \left[\frac{N(N-2)}{4} - \lambda\right] (\cosh t) v^2 \, dt.$$
For $\lambda \leq \frac{N(N-2)}{4}$ and $v \neq 0$ this expression is positive. The right-hand side of (3.5) however is positive if and only if $p < \frac{N+2}{N-2}$. If $N = 3$, we obtain a sharper result by choosing $g = \sin(\omega t)$, $\omega = \frac{\pi}{2T}$. Then the left-hand side of (3.5) is positive if

$\lambda \leq 1 + (\frac{\pi}{2T})^2$. The right-hand side is

$$\int_0^T b\omega \cos \omega t \left[\frac{1}{2} + \frac{1}{p+1} - \frac{(p-1)\tan \omega t \coth t}{(p+1)\omega}\right] v^{p+1}\, dt.$$

Since $\tan \omega t \coth t/\omega \geq 1$, the integral above is negative if $p \geq 5$.

(ii) The second assertion follows from Lemma 2.7. □

Remark 3.2. (i) In general it is not clear if for $1 < p \leq \frac{N+2}{N-2}$ and $\lambda < \lambda_B(T)$ there exist solutions in B_s.

(ii) From the maximum principle it follows that for any p no positive solutions exist in B_r if $\lambda > \lambda_B(T)$.

(iii) There is an interval $(\lambda^*, \lambda_B(T))$ which is not covered by the nonexistence result of Lemma 3.1 above. Stapelkamp [19] has shown that in the critical case $p = \frac{N+2}{N-2}$, $B(T)$ has a regular solution in this interval. We conjecture that this is also true for p close to $\frac{N+2}{N-2}$.

3.4 $E(0, \infty)$

Notice that λ_0^2 is the lowest point in the L^2-spectrum of $\Delta_{\mathbb{H}^N}$. It follows therefore from the maximum principle that $E(0, \infty)$ does not contain a solution which is regular at the origin if $\lambda > \lambda_0^2$. Mancini and Sandeep [15] proved that there exists a unique, rapidly decreasing solution which is regular at zero, in the following cases:

- $1 < p < \frac{N+2}{N-2}$ and $\lambda \leq \lambda_0^2$,
- $N \geq 4$, $p = \frac{N+2}{N-2}$ and $\frac{N(N-2)}{4} < \lambda \leq \lambda_0^2$.

The existence was established by means of variational methods and the uniqueness followed from an argument of Kwong and Li [14].

Mancini and Sandeep [15] observed that (3.5) implies the nonexistence of solutions in $E(0, \infty)$ which are regular at zero and rapidly decreasing at infinity in the following cases:

- $N \geq 3$, $p \geq \frac{N+2}{N-2}$ and $\lambda \leq \frac{N(N-2)}{4}$,
- $N = 3$, $p \geq 5$ and $\lambda \leq 1$.

Lemma 3.3. (i) *Assume* $1 < p < \frac{N+2}{N-2}$. *Then at least one of the classes* $B(t)$ *or* $E(0, \infty)$ *contains a solution which is singular at the origin.*

(ii) *If* $\lambda < 0$, *then for any* $p > 1$, $E(0, \infty)$ *contains solutions which are regular at zero and converge to* Λ *as* $t \to \infty$.

(iii) *If* $p \geq \frac{N+2}{N-2}$ *and* $\lambda \leq \frac{N(N-2)}{4}$, *then* E_{rs} *contains a continuum of solutions.*

Proof. The first assertion follows from Lemma 2.7 and the second is a consequence of the monotonicity of $\mathcal{E}(t)$. In fact, if $u(0)$ is so small that $\mathcal{E}(0) < 0$, then $\mathcal{E}(t)$ stays negative and converges eventually to its minimum $\mathcal{E}(\Lambda)$. The third assertion is a consequence of Lemma 2.7 which guarantees the existence of a regular local solution at the origin. By Lemma 3.1 (i) this solution cannot vanish and belongs therefore to $E(0,\infty)$. In view of Mancini and Sandeep's result, $E_{\mathrm{rr}} = \emptyset$. □

3.5 $B^c(T)$

The previous considerations lead to the following

Lemma 3.4. *If we have $p > \frac{N+2}{N-2}$ and $\lambda \le \frac{N(N-2)}{4}$ if $N > 3$ or $\lambda \le 1$ if $N = 3$, then $B^c(d_1)$ contains a rapidly decreasing solution for some d_1.*

Proof. By Lemma 2.3 there exists locally a one-parameter family of rapidly decreasing solutions. By Mancini and Sandeep's nonexistence result this solutions are not in $E_{\mathrm{rr}}(0,\infty)$ and by Lemma 2.7 (iii) and Remark 2.1 this solution cannot belong to $E_{\mathrm{sr}}(0,\infty)$. Hence it vanishes at some d_1. The case $N = 3$ is treated in Theorem 4.4. □

4 Global results for $N = 3$

4.1 Main results

The aim of this section is to transform (1.2) into a Matukuma equation and to use the existence and uniqueness results by Yanagida and Yotsutani [20].

Throughout this section we shall assume that $N = 3$. The arguments used here apply also to higher dimensions, but the discussion is much more involved and difficult to carry out.

Observe that for $N = 3$ we have $\lambda_0 = 1$. According to Lemma 2.3 a rapidly decaying solution behaves like $e^{-(1+\sqrt{1-\lambda})t}$ and a slowly decreasing solutions like $e^{-(1-\sqrt{1-\lambda})t}$.

The main results of this section are stated in the next theorems. In order to express our first theorem, we introduce the following notation: $u(t;\alpha)$ is the unique (local) solution of (1.2) such that $u(0;\alpha) = \alpha > 0$ and $u'(0;\alpha) = 0$.

Theorem 4.1. *If $1 < p < 5$ and if $\lambda \le 1$, then there exists a unique positive rapidly decaying solution to* (1.2). *More precisely, there exists an $\alpha^* > 0$ such that for all $\alpha \in (0,\alpha^*)$*

(i) $u(t;\alpha)$ *converges slowly to* 0 *as* $t \to \infty$ *if* $\lambda \ge 0$,

(ii) $u(t;\alpha) \to \Lambda$ *as* $t \to \infty$ *if* $\lambda < 0$.

In addition $u(t;\alpha^*)$ decays rapidly to 0 as $t \to \infty$ and $u(t;\alpha)$ has a finite zero if $\alpha > \alpha^*$.

Theorem 4.1 is a sightly more precise version of Theorem 1.3 in [15] whereas the next results are new to our knowledge.

Theorem 4.2. *If $1 < p < 5$ and if $\lambda \le 1$, then there exists a continuum of positive solutions in $E(0,\infty)$ which decay rapidly to zero as $t = \infty$ and which are singular at $t = 0$. Also, there exists a continuum of solutions in $B^c(d_1)$ for some d_1, which decay rapidly at $t = \infty$.*

Remark 4.3. We can describe the structure of solutions, shooting from infinity. Let

$$\beta^* := \lim_{t\to\infty} e^{(1+\sqrt{1-\lambda})t} u(t;\alpha^*),$$

where α^* is defined in Theorem 4.1. Then

(i) any solution u to (1.2) with $\lim_{t\to\infty} e^{(1+\sqrt{1-\lambda})t} u = \beta \in (0,\beta^*)$ is singular at $t = 0$.

(ii) any solution u to (1.2) with $\lim_{t\to\infty} e^{(1+\sqrt{1-\lambda})t} u = \beta > \beta^*$ must have finite zero.

(iii) the solution u to (1.2) with $\lim_{t\to\infty} e^{(1+\sqrt{1-\lambda})t} u = \beta^*$ is nothing but the unique solution $u(t;\alpha^*)$ in Theorem 4.1.

We note that Chern, Z.-H. Chen, J.-H Chen and Tang [5] investigated the structure of positive singular solutions of $\Delta u - u + u^p = 0$ in the Euclidean whole space case.

In accordance with Lemma 3.3 we have

Theorem 4.4. *If $p \ge 5$ and if $\lambda \le 1$, then any solution of (1.2) which decays rapidly at $t = \infty$ vanishes at some $d_0 > 0$.*

This result corresponds to the nonexistence result in Theorem 3.2 by Ni and Serrin [16] for the equation $\Delta u + f(u) = 0$ in the Euclidean space.

Theorem 4.5. *Suppose that $p \ge 5$.*

(i) *If $\lambda < 0$, then for any positive α, $u(t;\alpha)$ belongs to $E(0,\infty)$ and converges to Λ.*

(ii) *If $0 \le \lambda \le 1$, then for any positive α, $u(t;\alpha)$ belongs to $E(0,\infty)$ and converges slowly to 0.*

4.2 Transformation to a Matukuma type equation

Let $\Phi(t)$ be a solution to the linear problem[2]

$$\frac{1}{\sinh^2 t}\{(\sinh^2 t)\Phi'\}' + \lambda\Phi = 0. \tag{4.1}$$

Assume in the sequel that $\lambda \le 1$. For simplicity we shall set

$$\mu = \sqrt{1-\lambda}.$$

The solutions which are regular at the origin are multiples of

$$\Phi(t) = \begin{cases} \frac{\sinh \mu t}{\sinh t} & \text{if } \mu > 0 \ (\lambda < 1), \\ \frac{t}{\sinh t} & \text{if } \mu = 0 \ (\lambda = 1). \end{cases}$$

Substituting $u(t) = v(t)\Phi(t)$ into (1.2), we get[3]

$$v'' + 2\left(\coth t + \frac{\Phi'}{\Phi}\right)v' + v^p\Phi^{p-1} = \frac{1}{g(t)}\{g(t)v'\}' + v^p\Phi^{p-1} = 0, \tag{4.2}$$

where $g(t) = \sinh^2 t\,\Phi^2(t)$. We now introduce the new variable (see e.g. [21])

$$\frac{1}{\tau} = \int_t^\infty \frac{1}{g(s)}\,ds.$$

Hence

$$\tau^{-1} = \begin{cases} \frac{1}{\mu}(\coth \mu t - 1) & \text{if } \lambda < 1, \\ \frac{1}{t} & \text{if } \lambda = 1. \end{cases}$$

Note that $\tau = (\mu e^{2\mu t} - 1)/2$ if $\lambda < 1$.

The function $w(\tau) = v(t(\tau))$ satisfies the 3-dimensional Matukuma equation

$$\frac{1}{\tau^2}(\tau^2 w')' + Q(\tau)w^p = 0 \quad \text{in } (0,\infty), \tag{4.3}$$

where

$$Q(\tau) = \frac{g^2\Phi^{p-1}}{\tau^4} = \begin{cases} \mu^{-4}\sinh^4(\mu t)\Phi(t)^{p-1}(\coth \mu t - 1)^4 & \text{if } \mu > 0, \\ \Phi(t)^{p-1} & \text{if } \mu = 0. \end{cases}$$

Observe that the same classification holds for positive solutions $w(\tau)$ as for $u(t)$. If u decays rapidly to zero at $t = \infty$, then by the Lemmas 2.3 and 2.5, we have $\lim_{\tau\to\infty} \tau w(\tau) = u_\infty$. If u is a slowly decaying solution or if u tends to Λ as t tends to infinity, then $\lim_{\tau\to\infty} \tau w(\tau) = \infty$. If u is regular at $t = 0$, then $w(0) > 0$ and $w'(0) = 0$, and finally if u is singular at zero, the same is true for w and is classified according to Lemma 2.7.

[2] The argument in this subsection is also valid for $N \ge 4$.

[3] This process is called Doob's h-transform, see p. 252 of [9].

4.3 Auxiliary tools for the study of Matukuma equations

The basic tools used in this chapter to study (4.3) hold under the assumptions

$$\textbf{(Q)} \qquad \begin{cases} Q \in C^1((0,\infty)) \cap C([0,\infty)), & Q > 0 \text{ in } (0,\infty), \\ \tau Q \in L^1([0,1]), & \tau^{2-p} Q \in L^1(1,\infty). \end{cases}$$

The third hypothesis guarantees the existence of a local solution which is regular at the origin. By the classical results of the oscillation theory, if $\tau^{2-p} Q \notin L^1(1,\infty)$, then any solution must have a finite zero. Thus the last condition is necessary for the existence of a positive solution for large τ.

The expression Q in (4.3) satisfies (**Q**). For a positive solution of (4.3) we have (cf. Lemma 2.1 (c) in [22])

Lemma 4.6. *The function $\tau w(\tau)$ is concave. Hence for a positive solution defined in $(0,\infty)$, τw is increasing.*

Next we introduce a function used by Ding and Ni [6] (originally an integral form) to classify positive solutions. For a positive solution w to (4.3), set

$$P(\tau; w) = \frac{1}{2}\tau^2 w'(\tau w' + w) + \frac{1}{p+1}\tau^3 Q(\tau) w^{p+1}.$$

In the sequel we set

$$\theta := \frac{p-5}{2} \quad \text{and} \quad Q_*(\tau) := \tau^{-\theta} Q(\tau)$$

Direct calculations yields

$$\frac{dP}{d\tau} = \frac{1}{p+1}\tau^{3+\theta} Q_*' w^{p+1}. \tag{4.4}$$

Hence P is monotone increasing. Kawano, Yanagida and Yotsutani [12] described the asymptotic behavior for large τ of the solutions of (4.3) by means of $P(\tau; w)$.

Proposition 4.7. *Suppose that Q_* is monotone near $\tau = \infty$. Then the following statements hold:*

(i) $\lim_{\tau\to\infty} P(\tau; w) < 0$ *if and only if w is a slowly decaying solution.*

(ii) $\lim_{\tau\to\infty} P(\tau; w) = 0$ *if and only if w is a rapidly decaying solution.*

(iii) $\lim_{\tau\to\infty} P(\tau; w) > 0$ *if and only if w vanishes at a finite point.*

If $Q_*(\tau)$ is monotone on the whole positive axis, we have the following propositions which are found in [12].

Proposition 4.8. *If* $Q_*' < 0$ *on* $(0, \infty)$*, then any solution of* (4.3) *which is regular at zero decays slowly.*

Proof. First note that $P(0, w) = 0$ for any $w(0) > 0$. By (4.4), we see that

$$P(\tau; w) = \frac{1}{p+1} \int_0^\tau s^{3+\theta} Q_*'(s) w_+^{p+1}\, ds \le 0, \not\equiv 0.$$

The assertion now follows from Proposition 4.7 (i). □

A similar argument yields

Proposition 4.9. *If* $Q_*' > 0$ *on* $(0, \infty)$*, then any solution of* (4.3) *which is regular at zero has a finite zero.*

We now study the case where Q_* is not monotone everywhere. We will provide a criterion for the uniqueness of rapidly decaying solutions belonging to $E(0, \infty)$.

In order to state the result, we need the following two functions:

$$G(\tau) := \frac{1}{p+1} \tau^3 Q(\tau) - \frac{1}{2} \int_0^\tau s^2 Q(s)\, ds,$$
$$H(\tau) := \frac{1}{p+1} \tau^{2-p} Q(\tau) - \frac{1}{2} \int_0^\tau s^{1-p} Q(s)\, ds.$$

Straightforward calculations yield

$$G'(\tau) = \tau^{p+1} H'(\tau) = \frac{1}{p+1} \tau^{(p+1)/2} Q_*'(\tau)$$

and

$$\frac{d}{d\tau} P(\tau; w) = G'(\tau) w^{p+1} = H'(\tau)(\tau w)^{p+1}. \tag{4.5}$$

Integrating (4.5) and keeping in mind that $P(0, w) = 0$, we find

$$P(\tau; w) = G(\tau) w^p - (p+1) \int_0^\tau G(s) w^p w'\, ds \tag{4.6}$$

and

$$P(\tau; w) = H(\tau)(\tau w)^{p+1} - (p+1) \int_0^\tau H(s)(sw)^p (sw)'\, ds. \tag{4.7}$$

In the sequel we assume

$$(Q_1) \qquad \begin{cases} G > 0 \text{ in } (0, \tau_G), & G < 0 \text{ in } (\tau_G, \infty), \\ H < 0 \text{ in } (0, \tau_H), & H > 0 \text{ in } (\tau_H, \infty). \end{cases}$$

Thus, we assume that G and H has only one zero. The following result is essentially Theorem 1 in [20].

Proposition 4.10. *If there exists $\tau_* > 0$ such that*

$$Q_*'(\tau) > 0, \quad \tau \in (0, \tau_*), \quad Q_*'(\tau) < 0, \quad \tau > \tau_*$$

and if the properties (Q_1) *hold, then there exists a unique positive rapidly decaying solution to* (4.3). *More precisely, there exists $\gamma_* > 0$ such that $w(\tau;\gamma)$ is positive and* $\lim_{\tau\to\infty} \tau w(\tau;\gamma) = \infty$ *as $\tau \to \infty$ for $\gamma \in (0, \gamma_*)$, $w(\tau;\gamma_*)$ is positive and decays rapidly, and $w(\tau;\gamma)$ has a finite zero for $\gamma > \gamma_*$.*

Remark 4.11. Proposition 4.10 holds in fact under the weaker assumption

$$0 < \tau_H \le \tau_G < \infty,$$

where τ_H and τ_G are the largest positive zero of H and the smallest positive zero of G, respectively. This is the exact assumption in Theorem 1 in [20].

4.4 Proofs of the Theorems 4.1–4.5

First we want to analyze $Q_*(\tau)$ in order to apply Propositions 4.8, 4.9 and 4.10.

If $\mu > 0$, then

$$Q_*(\tau) = (2\mu)^{-(p+3)/2} \frac{(1 - e^{-2\mu t})^{(p+3)/2}}{\sinh^{p-1} t}. \tag{4.1}$$

Since $d\tau/dt > 0$ and since we are interested in the slope of Q_*, it suffices to examine the derivative of

$$S(t) := \frac{(1 - e^{-2\mu t})^{(p+3)/2}}{\sinh^{p-1} t}$$

as a function of t. We have

$$S'(t) = \frac{(1 - e^{-2\mu t})^{(p+1)/2}}{\sinh^p t} \left\{\mu(p+3)e^{-2\mu t} \sinh t - (p-1)(1-e^{-2\mu t}) \cosh t\right\}.$$

Set

$$T(t) := \mu(p+3)\frac{e^{-2\mu t}}{1 - e^{-2\mu t}} \sinh t - (p-1)\cosh t$$

so that

$$\mu(p+3)e^{-2\mu t} \sinh t - (p-1)(1 - e^{-2\mu t}) \cosh t = T(t)(1 - e^{-2\mu t}).$$

Since

$$\frac{e^{-2\mu t}}{1 - e^{-2\mu t}} = \frac{1}{e^{2\mu t} - 1},$$

the essential part in order to determine the sign of $S'(t)$ is

$$X(t) := \frac{T(t)}{(e^{2\mu t} - 1)\cosh t} = \mu(p+3)\tanh t - (p-1)(e^{2\mu t} - 1).$$

For $t \geq 0$ the graph of $\tanh t$ is monotone increasing and concave, while that of $e^{2\mu t} - 1$ is monotone increasing and convex. Thus, if there exists $t_0 > 0$ such that $X(t_0) = 0$, then t_0 is a unique solution of $X(t) = 0$. Near $t = 0$, $\tanh t \approx t$ while $e^{2\mu t} - 1 \approx 2\mu t$. Hence, if

$$\mu(p+3) - 2\mu(p-1) > 0,$$

then $X(t) = 0$ has a unique solution for $t > 0$. This condition is satisfied for all $p < 5$. If $p \geq 5$, then $X(t) \leq 0, \not\equiv 0$.

If $\mu = 0$, then

$$Q_*(\tau) = \frac{t^{(p+3)/2}}{\sinh^{p-1} t} \quad \text{and} \quad Q_*'(\tau) = \frac{t^{(p+1)/2}}{\sinh^p t}\left\{\frac{p+3}{2}\sinh t - (p-1)t\cosh t\right\}.$$

Again, we see that the shape of the graph of Q_* is the same as for $\mu > 0$.

The proof of Theorem 4.5 is now immediate. It follows from the previous observations, Proposition 4.8 and the Lemmas 2.3 and 3.3.

Proof of Theorem 4.1. We apply Proposition 4.10; we have only to check the values of

$$\begin{aligned}
\lim_{\tau\to\infty} G(\tau) &= \int_0^\infty \frac{d}{d\tau}G(\tau)\,d\tau = \frac{1}{p+1}\int_0^\infty \tau^{(p+1)/2}\frac{d}{d\tau}Q_*(\tau)\,d\tau,\\
\lim_{\tau\to 0} H(\tau) &= \int_0^\infty \frac{d}{d\tau}H(\tau)\,d\tau = \frac{1}{p+1}\int_0^\infty \tau^{-(p+1)/2}\frac{d}{d\tau}Q_*(\tau)\,d\tau
\end{aligned}$$

for $1 < p < 5$. Since

$$\tau = \frac{\mu}{\coth \mu t - 1} = \frac{\mu \sinh \mu t}{\cosh \mu t - \sinh \mu t} = \frac{\mu e^{2\mu t} - 1}{2} = \frac{\mu}{2}(e^{2\mu t} - 1),$$

we see that $\tau \sim t$ and

$$\frac{dQ_*}{d\tau} = (2\mu)^{-(p+3)/2}\frac{(1 - e^{-2\mu t})^{(p+3)/2}}{\sinh^p t}T(t) \sim t^{-(p-3)/2}$$

near $t = 0$. Also, $\tau \sim e^{2\mu t}$ and $dQ_*/d\tau \sim e^{-(p-1)t}$ near $t = \infty$. Hence, we get

$$dG/d\tau \in L^1([0,1]) \quad \text{and} \quad dH/d\tau \in L^1([1,\infty))$$

and we have only to check the signs of $\lim_{\tau\to\infty} G(\tau)$ and $\lim_{\tau\to 0} H(\tau)$ to ensure that Proposition 4.10 applies. In the following, note that $G(\tau)$, $H(\tau)$ and $Q_*(\tau)$ are indeed functions of t although we use these expressions.

By change of variables, we have

$$\int_0^\infty \frac{d}{d\tau} G(\tau)\, d\tau = \int_0^\infty \frac{d}{dt} G(\tau)\, dt$$

and

$$\int_0^\infty \frac{d}{d\tau} H(\tau)\, d\tau = \int_0^\infty \frac{d}{dt} H(\tau)\, dt.$$

Again first, we consider the case $\mu > 0$. Near $t = \infty$, we see that

$$\frac{d}{dt} G(\tau) \sim e^{\{(p+1)\mu-(p-1)\}t}$$

if $(p+1)\mu - (p-1) \geq 0$. Then $dG/dt \notin L^1([1,\infty))$ and G must have a finite zero.

If $(p+1)\mu-(p-1) < 0$, then integration by parts yields, in view of $d\tau/dt > 0$ and $Q_* > 0$,

$$\begin{aligned}\int_0^\infty \frac{d}{dt} G(\tau)\, dt &= \left[\frac{1}{p+1}\tau^{(p+1)/2} Q_*(\tau)\right]_{t=0}^{t=\infty} - \frac{1}{2}\int_0^\infty \tau^{(p-1)/2} Q_*(\tau) \frac{d\tau}{dt}\, dt \\ &= -\frac{1}{2}\int_0^\infty \tau^{(p-1)/2} Q_*(\tau)\frac{d\tau}{dt}\, dt < 0.\end{aligned}$$

Here we used the facts that

$$\tau^{(p+1)/2} Q_*(\tau) \sim e^{(p+1)\mu t-(p-1)t} \to 0 \quad \text{as } t \to \infty$$

and that

$$\tau^{(p+1)/2} Q_*(\tau)|_{t=0} = 0.$$

Thus, in any case G has a finite zero.

For H, since $dQ_*/dt \sim t^{-(p-3)/2}$ near $t = 0$ we always have for all $\mu > 0$, $dH/dt \notin L^1([0,1])$. Hence, H also has a finite zero. In case of $\mu > 0$, all the conditions of Proposition 4.10 are satisfies and the conclusion follows.

If $\mu = 0$, we have $\tau = t$ and therefore

$$\frac{d}{d\tau} G = t^{(p+1)/2} \frac{d}{dt} Q_*(\tau) \sim t^{p+2} e^{-(p-1)t}$$

near $t = \infty$ and the integration by parts shows us

$$\lim_{\tau\to\infty} G(\tau) < 0.$$

Similarly, we have

$$\frac{d}{d\tau}H = t^{-(p+1)/2}\frac{d}{dt}Q_*(\tau) \sim t^{-(p-1)t}.$$

If $p \in [2,5)$, we see that $dH/d\tau \notin L^1([0,1])$. If $p \in (1,2)$, then integration by parts again yields

$$\begin{aligned}\int_0^\infty \frac{d}{dt}H(\tau)\,dt &= \left[\frac{1}{p+1}\tau^{-(p+1)/2}Q_*(\tau)\right]_{t=0}^{t=\infty} \\ &\qquad - \frac{1}{2}\int_0^\infty \tau^{-(p-1)/2}Q_*(\tau)\frac{d\tau}{dt}\,dt \\ &= -\frac{1}{2}\int_0^\infty \tau^{-(p-1)/2}Q_*(\tau)\,dt < 0.\end{aligned}$$

Here also note that $\tau^{-(p+1)/2}Q_*(\tau) \sim t/\sinh^{p-1} t$ near $t = 0$ and $t = \infty$ and that the value converges to 0 as $t \to 0$ or $t \to \infty$ if $1 < p < 2$. Thus, H has a finite zero near $t = 0$. Hence, all the conditions of Proposition 4.10 are satisfied if $1 < p < 5$ and if $\mu \geq 0$. Thus, we have proved Theorem 4.1. □

To prove Theorems 4.2 and 4.4, we first reduce our problem to (4.3) and then use the Kelvin transform. Let $\sigma = 1/t$ and $W(\sigma) = \tau w(\tau)$. Then we see that

$$\frac{1}{\tau^2}(\tau^2 w')' = \sigma^3(\sigma^2 W')',$$

and (4.3) is reduced to

$$\frac{1}{\sigma^2}(\sigma^2 W')' + \sigma^{p-5}Q\left(\frac{1}{\sigma}\right)W^p = 0. \tag{4.2}$$

Then we need to consider the behavior of

$$\tilde{Q}_*(\sigma) := \sigma^{-(p-5)/2}\left\{\sigma^{p-5}Q\left(\frac{1}{\sigma}\right)\right\} = \tau^{-(p-5)/2}Q(\tau).$$

More precisely, we have to investigate the sign of

$$\frac{d}{d\sigma}\tilde{Q}_*(\sigma) = \frac{d}{d\tau}\left(\tau^{-(p-5)/2}Q(\tau)\right)\frac{d\tau}{d\sigma}. \tag{4.3}$$

Proof of Theorem 4.2. If $1 < p < 5$, then as in the proof of Theorem 4.1, we see that $\tilde{Q}_*(\sigma)$ has the properties as $Q_*(\tau)$ has. Thus the conclusion comes from Proposition 4.10 and the structure of solutions which decay rapidly at $t = \infty$ is the same as in Theorem 4.1. □

Proof of Theorem 4.4. If $p \geq 5$, then $\tilde{Q}_*$ becomes monotone increasing in σ by equation (4.3) and $d\tau/d\sigma = -\sigma^{-2}$. Thus, we can apply Proposition 4.9 to show Theorem 4.4. □

5 Concluding remarks and open problems

(1) The method presented here can be extended to more general problems, for instance:

- $\Delta_{\mathbb{H}^N} u + K(\cosh(x_N))u^p = 0$, for particular functions K,
- boundary value problems in balls with Robin boundary conditions

$$\Delta_{\mathbb{H}^N} u + \lambda u + u^p = 0 \text{ in } B, \quad u > 0 \text{ in } B, \quad u + \kappa \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B,$$

  where B is the geodesic unit ball in $\mathbb{H}^N$ and ν is the unit outer normal, as considered by Kabeya, Yanagida and Yotsutani [11] in the Euclidean space,
- to other semilinear quasilinear equations which can be reduced to an ordinary differential equations.

(2) Except for $B_r(d_1)$ and $E_{\mathrm{rr}}(0,\infty)$ the question of uniqueness is still open. We expect that there is at most one solution in $S(d_0, d_1)$ for fixed $0 < d_0 < d_1 < \infty$, and in $B_r^c(d_1)$ for fixed d_1.

This conjecture is supported by the fact that in contrast to the singular solutions the regular and rapidly decreasing solutions form only a one-parameter family. For singular solutions no uniqueness is to be expected.

(3) Since there are variational solutions in $E_{\mathrm{rr}}(0,\infty)$ for $p < \frac{N+2}{N-2}$ and $\lambda < \lambda_0^2$, it is reasonable that there are also variational solutions in $B_r^c(d_1)$ for any $d_1 > 0$.

Bibliography

[1] C. Bandle, A. Brillard and M. Flucher, Green's function, harmonic transplantation and best Sobolev constant in spaces of constant curvature, *Trans. Amer. Math. Soc.* **350** (1998), 1103–1128.

[2] C. Bandle and M. Marcus, The positive radial solutions of a class of semilinear elliptic equations, *J. Reine Angew. Math.* **401** (1989), 25–59.

[3] R. Bellman, *Stability Theory of Differential Equations*, Dover, New York, 1969.

[4] M. Bonforte, F. Gazzola, G. Grillo and J. L. Vazquez, Classification of radial solutions to the Emden–Fowler equation on the hyperbolic space, submitted.

[5] J.-L. Chern, Z.-Y. Chen, J-H. Chen and Y.-L. Tang, On the classification of standing wave solutions for the Schrödinger equation, *Comm. Partial Differential Equations* **35** (2010), 275–301; Erratum, *ibid.*, 1920–1921.

[6] W.-Y. Ding and W.-M. Ni, On the elliptic equation $\Delta u + K(|x|)u^p = 0$ and related topics, *Duke Math. J.* **52** (1985), 485–506.

[7] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.

[8] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Adv. Math. Suppl. Stud.* **7A** (1981), 369–402.

[9] A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, American Mathematical Society, Providence, 2009.

[10] P. Hartman, *Ordinary Differential Equations*, Second edition, Birkhäuser-Verlag, Basel, 1982.

[11] Y. Kabeya, E. Yanagida and S. Yotsutani, Global structure of solutions for equations of Brezis–Nirenberg type on the unit ball, *Proc. Roy. Soc. Edinburgh Sect. A* **131** (2001), 647–665.

[12] N. Kawano, E. Yanagida and S. Yotsutani, Structure theorems for positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbf{R}^n$, *Funkcial. Ekvac.* **36** (1993), 557–579.

[13] S. Kumaresan and J. Prajapat, Analogue of Gidas–Ni–Nirenberg result in hyperbolic space and sphere, *Rend. Istit. Mat. Univ. Trieste* **30** (1998), 107–112.

[14] M.-K. Kwong and Y. Li, Uniqueness of radial solutions of semilinear elliptic equations, *Trans. Amer. Math. Soc.* **333** (1992), 339–363.

[15] G. Mancini and K. Sandeep, On a semilinear elliptic equation in $\mathbb{H}^n$, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **7** (2008), 635–671.

[16] W.-M. Ni and J. Serrin, Nonexistence theorems for singular solutions of quasilinear partial differential equations, *Comm. Pure Appl. Math.* **39** (1986), 379–399.

[17] W.-M. Ni and S. Yotsutani, Semilinear elliptic equations of Matukuma-type and related topics, *Japan J. Appl. Math.* **5** (1988), 1–32.

[18] S. Stapelkamp, The Brezis-Nirenberg problem on $\mathbb{H}^n$: Existence and uniqueness of solutions, in: *Elliptic and Parabolic Problems* (Rolduc and Gaeta 2001), World Scientific, Singapore (2002), 283–290.

[19] S. Stapelkamp, *Das Brezis-Nirenberg Problem im* $\mathbb{H}^n$, Dissertation, Universität Basel, 2003.

[20] E. Yanagida and S. Yotsutani, Classifications of the structure of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^n$, *Arch. Ration. Mech. Anal.* **124** (1993), 239–259.

[21] E. Yanagida and S. Yotsutani, Pohozaev identity and its applications, *RIMS Kokyuroku* **834** (1993), 80–90.

[22] E. Yanagida and S. Yotsutani, Existence of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^n$, *J. Differential Equations* **115** (1995), 477–502.

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