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Locking-free DGFEM for elasticity problems in polygons

THOMAS P. WIHLER†

Seminar for Applied Mathematics, ETH Zürich, CH-8092 Zürich, Switzerland

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The h-version of the discontinuous Galerkin finite element method (h-DGFEM) for nearly incompressible linear elasticity problems in polygons is analysed. It is proved that the scheme is robust (locking-free) with respect to volume locking, even in the absence of H^2-regularity of the solution. Furthermore, it is shown that an appropriate choice of the finite element meshes leads to robust and optimal algebraic convergence rates of the DGFEM even if the exact solutions do not belong to H^2.

Keywords: DGFEM; locking; elasticity problems; singular solutions; graded meshes; discontinuous Galerkin methods.

1. Introduction

In mechanical engineering, partial differential equations are often solved by low-order finite element methods (FEMs). In many applications, the convergence of these schemes may strongly depend on various problem parameters. Unfortunately, this can result in non-robustness of the convergence: i.e. the asymptotic convergence regime of the method is reached only at such high numbers of degrees of freedom that the scheme is practically not feasible. In computational mechanics, this non-robustness of the FEM is termed *locking*. An additional problem is caused by the fact that many practical examples are based on non-smooth domains, and therefore boundary singularities may arise. In this paper, however, it will be shown that locking effects may be circumvented by using a discontinuous Galerkin finite element method (DGFEM) and that singular solution behaviour can be resolved by applying an appropriate mesh refinement strategy.

There exist different kinds of locking: *shear locking* typically appears if the corresponding domains are very thin and plate and shell theories, which include shear deformation, are used. In addition, in shell theories and their finite element models, there arises *membrane locking* which is caused by the interaction between bending and membrane energies. Finally, problems dealing with nearly incompressible materials are often accompanied by the so-called *volume locking*; this type of locking is very typical for elasticity problems in biology and will be explored in this paper.

In order to overcome volume locking, a variety of approaches have been suggested. For example, low-order mixed FEMs, where an extra variable for the divergence term is introduced, yield adequate numerical results (Brezzi & Fortin, 1991). These methods are closely related to under-integration schemes. A further possibility is the use of non-conforming methods, where the global continuity of the numerical solutions is not any more enforced (see Kouhia & Stenberg, 1995, for example).

†Email: twihler@sam.math.ethz.ch

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In 1983, M. Vogelius proved the absence of volume locking for the p-version of the FEM on smooth domains (Vogelius, 1983). Moreover, in 1992 Babuška & Suri showed that, on polygonal domains, the h-FEM is locking-free on regular triangular elements with $p \geqslant 4$. In addition, they proved that, for conforming methods, locking cannot be avoided on quadrilateral meshes for any $p \geqslant 1$. Recently, Hansbo & Larson (2002) suggested the use of a discontinuous FEM (DGFEM). Assuming at least H^2 regularity, they showed that the h-version of the DGFEM does not lock for all $p \geqslant 1$.

Following the classical approach of Wheeler (1978) and Riviere & Wheeler (2000), this paper is devoted to the exploration of the non-symmetric interior penalty Galerkin (NIPG) version of the DGFEM for linear elasticity problems (with mixed boundary conditions) in convex and non-convex polygons. Based on a recent regularity result by Guo & Schwab (2000) it will be proved here that, even if the exact solutions of the elasticity problems are singular (i.e. not H^2 any more), the h-version of the NIPG is free of volume locking. Additionally, the use of so-called '$\underline{\gamma}$-graded meshes' leads this method to converge at an optimal algebraic rate (independently of the compressibility of the material). On non-graded (uniform) meshes, the DGFEM (NIPG with $p = 1$) is still free of locking. However, due to the occurrence of singularities, the algebraic convergence rates may be suboptimal.

The DGFEM above is closely related to non-conforming methods of Crouzeix–Raviart type. Brenner & Sung (1992) showed that these schemes are locking-free even for $p = 1$. However, their results are based on the assumption that the displacements are H^2 regular, and therefore the case of non-convex polygons is in general not covered by that work. Nevertheless, applying the regularity results and the mesh refinement strategies presented in this paper (Theorems 3.4 and 5.10), it may be proved that the convergence statements in Brenner & Sung (1992) are extensible to the case where the exact solutions of the elasticity problems exhibit corner singularities.

The outline of the paper is as follows. In Sections 2 and 3, the linear elasticity problem and its regularity on polygons are presented. In Section 4, the DGFEM (NIPG) is introduced. Section 5 contains the error analysis of the DGFEM and the proof of the main result (optimal, robust convergence of the NIPG). In Section 6, the theoretical results are confirmed with some numerical examples.

2. Problem Formulation

Let Ω be a polygon in $\mathbb{R}^2$. Its boundary $\Gamma := \partial\Omega$ is assumed to consist of a Dirichlet part Γ_D with $|\Gamma_\mathrm{D}| > 0$ and of a Neumann part Γ_N:

$$\overline{\Gamma} = \overline{\Gamma}_\mathrm{D} \cup \overline{\Gamma}_\mathrm{N}.$$

The linear elasticity problem reads as follows:

$$\begin{aligned} -\nabla \cdot \underline{\underline{\sigma}}(\underline{u}) &= \underline{f} && \text{in } \Omega \\ \underline{u} &= \underline{g}_\mathrm{D} && \text{on } \Gamma_\mathrm{D} \\ \underline{\underline{\sigma}}(\underline{u}) \cdot \underline{n}_\Omega &= \underline{g}_\mathrm{N} && \text{on } \Gamma_\mathrm{N}. \end{aligned} \tag{2.1}$$

Here, $\underline{u} = (u_1, u_2)$ is the *displacement* and $\underline{\underline{\sigma}} = \{\sigma_{ij}\}_{i,j=1}^2$ is the *stress tensor* for homogeneous isotropic material given by

$$\underline{\underline{\sigma}}(\underline{u}) = 2\mu\underline{\underline{\epsilon}}(\underline{u}) + \lambda\nabla \cdot \underline{u}\,\underline{\underline{1}}_{2\times 2}, \tag{2.2}$$

where $\underline{\underline{\epsilon}}(\underline{u}) = \{\epsilon_{ij}(\underline{u})\}_{i,j=1}^{2}$ with

$$\epsilon_{ij}(\underline{u}) = \tfrac{1}{2}(\partial_{x_i} u_j + \partial_{x_j} u_i) \tag{2.3}$$

the *symmetric gradient* of $\underline{u}$. Furthermore, μ and λ are the so-called *Lamé coefficients* satisfying

$$0 < \min\{\mu, \mu + \lambda\},$$

and $\underline{n}_\Omega$ is the unit outward vector of Ω on Γ.

3. Regularity

3.1 *Weighted Sobolev spaces*

The regularity of (2.1) will be measured in terms of certain weighted Sobolev spaces. In order to do so, set

$$SP(\Omega, \Gamma_{\mathrm{D}}, \Gamma_{\mathrm{N}}) := \{A_i : i = 1, 2, \ldots, M\},$$

where A_i, $i = 1, \ldots, M$, denote the 'singular points', e.g. corners and vertices of changing boundary condition type of Ω. Moreover, introduce a weight vector $\underline{\beta} = (\beta_1, \ldots, \beta_M)$ with $0 \leqslant \beta_i < 1$, and for any number $k \in \mathbb{R}$ set $\underline{\beta} + k := (\beta_1 + k, \ldots, \beta_M + k)$. Then, let $\Phi_{\underline{\beta}}$ be a *weight function* on Ω given by

$$\Phi_{\underline{\beta}}(x) = \prod_{i=1}^{M} r_i^*(x)^{\beta_i}, \qquad r_i^*(x) = |x - A_i|.$$

Furthermore, for any integers $m \geqslant l \geqslant 0$, denote by $H_{\underline{\beta}}^{m,l}(\Omega)^2$ the so-called *weighted Sobolev spaces* on Ω (Babuška & Guo, 1988, 1989; Guo & Babuška, 1993) which are understood to be the completions of $C^\infty(\overline{\Omega})^2$ with respect to the norms

$$\|\underline{u}\|^2_{H_{\underline{\beta}}^{m,l}(\Omega)} = \|\underline{u}\|^2_{H^{l-1}(\Omega)} + \sum_{k=l}^{m} \||D^k \underline{u}| \Phi_{\underline{\beta}+k-l}\|^2_{L^2(\Omega)}, \qquad l \geqslant 1,$$

$$\|\underline{u}\|^2_{H_{\underline{\beta}}^{m}(\Omega)} = \sum_{k=0}^{m} \||D^k \underline{u}| \Phi_{\underline{\beta}+k}\|^2_{L^2(\Omega)}, \qquad l = 0.$$

CONVENTION 3.1 Since the weight function $\Phi_{\underline{\beta}}$ controls the local behaviour of the solution in the vicinity of a (singular) vertex, it is obvious to work locally with the weight function $\Phi_\beta = r^\beta$ with

$$\beta := \beta_i \qquad \text{and} \qquad r(x) := |x - A_i|,$$

where A_i denotes the corresponding vertex of the polygon.

REMARK 3.2 In this paper, the spaces $H^{2,2}_{\underline{\beta}}(\Omega)^2$ will play a main role and it may be proved easily that for all $\varepsilon > 0$ and for each function $\underline{u} \in H^{2,2}_{\underline{\beta}}(\Omega)^2$, there holds $\underline{u}|_{\tilde{\Omega}_\varepsilon} \in H^2(\tilde{\Omega}_\varepsilon)^2$, where

$$\tilde{\Omega}_\varepsilon := \Omega \setminus \bigcup_{i=1}^{M} \{x \in \mathbb{R}^2 : |x - A_i| < \varepsilon\}.$$

Moreover, $H^{2,2}_{\underline{0}}(\Omega) = H^2(\Omega)$.

Finally, the spaces $H^{k-\frac{1}{2},l-\frac{1}{2}}_{\underline{\beta}}(\gamma)^2$, $l = 1, 2$, are defined as the trace spaces of $H^{k,l}_{\underline{\beta}}(\Omega)$ on $\gamma \subset \Gamma$ and

$$\|\underline{g}\|_{H^{k-\frac{1}{2},l-\frac{1}{2}}_{\underline{\beta}}(\gamma)} := \inf_{\substack{\underline{G} \in H^{k,l}_{\underline{\beta}}(\Omega)^2 \\ \underline{G}|_\gamma = \underline{g}}} \|\underline{G}\|_{H^{k,l}_{\underline{\beta}}(\Omega)}.$$

3.2 *Regularity of generalized Stokes problems*

In order to obtain a regularity result for the elasticity problem (2.1), the following *generalized Stokes problem* in the polygon Ω is considered:

$$\begin{aligned} -\nabla \cdot \underline{\underline{\sigma}}(\underline{u}, p) &= \underline{f} && \text{in } \Omega \\ -\nabla \cdot \underline{u} &= h && \text{in } \Omega \\ \underline{u} &= \underline{g}_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \\ \underline{\underline{\sigma}}(\underline{u}, p) \cdot \underline{n}_\Omega &= \underline{g}_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}. \end{aligned} \tag{3.1}$$

Here, $\underline{u}$ is the velocity field, p a Lagrange multiplier corresponding to the (hydrostatic) pressure in the incompressible limit and $\underline{\underline{\sigma}}(\underline{u}, p)$ the hydrostatic stress tensor of $\underline{u}$ defined by

$$\underline{\underline{\sigma}}(\underline{u}, p) = -p\,\underline{\underline{1}} + 2\nu\underline{\underline{\epsilon}}(\underline{u}),$$

where $\underline{\underline{\epsilon}}(\underline{u})$ is given as in (2.3) and $\nu > 0$ is the (kinematic) viscosity. If $\Gamma_{\mathrm{N}} = \emptyset$, the following compatibility condition is supposed to be fulfilled:

$$\int_\Omega h\,\mathrm{d}x + \int_{\partial\Omega} \underline{g}_{\mathrm{D}} \cdot \underline{n}_\Omega\,\mathrm{d}s = 0. \tag{3.2}$$

In Guo & Schwab (2000) the following regularity result was proved:

THEOREM 3.3 Let $k \geqslant 0$ and $|\Gamma_{\mathrm{D}}| > 0$. In addition, if $\Gamma_{\mathrm{N}} = \emptyset$, let (3.2) be satisfied. Then there exists a weight vector $\underline{\beta} = (\beta_1, \ldots, \beta_M)$ with $0 \leqslant \beta_i < 1$, $i = 1, \ldots, M$, such that for $\underline{f} \in H^{k,0}_{\underline{\beta}}(\Omega)^2$, $h \in H^{k+1,1}_{\underline{\beta}}(\Omega)$, $\underline{g}_{\mathrm{D}} \in H^{k+\frac{3}{2},\frac{3}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{D}})^2$ and $\underline{g}_{\mathrm{N}} \in H^{k+\frac{1}{2},\frac{1}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{N}})^2$ the generalized Stokes problem (3.1) admits a unique solution $(\underline{u}, p) \in H^{k+2,2}_{\underline{\beta}}(\Omega)^2 \times$

$H^{k+1,1}_{\underline{\beta}}(\Omega)$ and the a priori estimate

$$\begin{aligned}&\|\underline{u}\|_{H^{k+2,2}_{\underline{\beta}}(\Omega)} + \|p\|_{H^{k+1,1}_{\underline{\beta}}(\Omega)} \\ &\quad \leqslant C\Big(\|\underline{f}\|_{H^{k,0}_{\underline{\beta}}(\Omega)} + \|h\|_{H^{k+1,1}_{\underline{\beta}}(\Omega)} + \|\underline{g}_{\mathrm{D}}\|_{H^{k+\frac{3}{2},\frac{3}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{D}})} + \|\underline{g}_{\mathrm{N}}\|_{H^{k+\frac{1}{2},\frac{1}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{N}})}\Big)\end{aligned} \tag{3.3}$$

holds true.

3.3 *Regularity of linear elasticity problems*

A regularity result for linear elasticity problems in polygons was proved in Guo & Babuška (1993, Theorem 5.2). However, referring to the previous Theorem 3.3, a more specific statement, which clarifies the regularity of the linear elasticity problem (2.1) in dependence on the Lamé coefficient λ, may be developed.

THEOREM 3.4 Let Ω be a polygon in $\mathbb{R}^2$ and $|\Gamma_{\mathrm{D}}| > 0$. Then there exists a weight vector $\underline{\beta} = (\beta_1, \ldots, \beta_M)$ with $0 \leqslant \beta_i < 1$, $i = 1, \ldots, M$, such that for $\underline{f} \in H^{k,0}_{\underline{\beta}}(\Omega)^2$, $\underline{g}_{\mathrm{D}} \in H^{k+\frac{3}{2},\frac{3}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{D}})^2$ and $\underline{g}_{\mathrm{N}} \in H^{k+\frac{1}{2},\frac{1}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{N}})^2$ the linear elasticity problem (2.1) has a unique solution $\underline{u} \in H^{k+2,2}_{\underline{\beta}}(\Omega)^2$. In addition, there exists a constant $C > 0$ independent of λ such that the ensuing estimate holds true:

$$\begin{aligned}&\|\underline{u}\|_{H^{k+2,2}_{\underline{\beta}}(\Omega)} + |\lambda|\|\nabla\cdot\underline{u}\|_{H^{k+1,1}_{\underline{\beta}}(\Omega)} \\ &\quad \leqslant C\Big(\|\underline{f}\|_{H^{k,0}_{\underline{\beta}}(\Omega)} + \|\underline{g}_{\mathrm{D}}\|_{H^{k+\frac{3}{2},\frac{3}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{D}})} + \|\underline{g}_{\mathrm{N}}\|_{H^{k+\frac{1}{2},\frac{1}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{N}})}\Big).\end{aligned} \tag{3.4}$$

Proof. As already mentioned above, the unique solution $\underline{u}_{\mathrm{elast}}$ of the linear elasticity problem (2.1) belongs to $H^{k+2,2}_{\underline{\beta}}(\Omega)$ (see Guo & Babuška, 1993, Theorem 5.2). Therefore, the choice

$$h := -\nabla\cdot\underline{u}_{\mathrm{elast}} \in H^{k+1,1}_{\underline{\beta}}(\Omega)$$

leads to the following solution $(\underline{u}, p)$ of the generalized Stokes problem (3.1):

$$p = -\lambda\nabla\cdot\underline{u}_{\mathrm{elast}}$$

and

$$\underline{u} = \underline{u}_{\mathrm{elast}}.$$

Hence, using (3.3) implies that

$$\begin{aligned}\|\underline{u}\|_{H^{k+2,2}_{\underline{\beta}}(\Omega)} + |\lambda|\|\nabla\cdot\underline{u}\|_{H^{k+1,1}_{\underline{\beta}}(\Omega)} \leqslant C\Big(&\|\underline{f}\|_{H^{k,0}_{\underline{\beta}}(\Omega)} + \|\nabla\cdot\underline{u}\|_{H^{k+1,1}_{\underline{\beta}}(\Omega)} \\ &+\|\underline{g}_{\mathrm{D}}\|_{H^{k+\frac{3}{2},\frac{3}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{D}})} + \|\underline{g}_{\mathrm{N}}\|_{H^{k+\frac{1}{2},\frac{1}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{N}})}\Big).\end{aligned} \tag{3.5}$$

Thus, if $|\lambda| < 2C$, it follows that

$$\begin{aligned}
&\|\underline{u}\|_{H^{k+2,2}_{\underline{\beta}}(\Omega)} + |\lambda| \|\nabla \cdot \underline{u}\|_{H^{k+1,1}_{\underline{\beta}}(\Omega)} \\
&\leqslant \tilde{C} \|\underline{u}\|_{H^{k+2,2}_{\underline{\beta}}(\Omega)} \\
&\leqslant \tilde{C} \left(\|\underline{f}\|_{H^{k,0}_{\underline{\beta}}(\Omega)} + \|\underline{g}_{\mathrm{D}}\|_{H^{k+\frac{3}{2},\frac{3}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{D}})} + \|\underline{g}_{\mathrm{N}}\|_{H^{k+\frac{1}{2},\frac{1}{2}}_{\underline{\beta}}(\Gamma_{\mathrm{N}})} \right)
\end{aligned}$$

for a constant $\tilde{C}$ independent of $|\lambda| \in (0, 2C)$. In the last step, Theorem 5.2 in Guo & Babuška (1993) was applied.

Alternatively, if $|\lambda| \geqslant 2C$, the term $C \|\nabla \cdot \underline{u}\|_{H^{k+1,1}_{\underline{\beta}}(\Omega)}$ in the right-hand side of (3.5) may obviously be absorbed into the left-hand side. □

4. The DGFEM

4.1 *Finite element meshes*

Consider a *regular*† partition (FE mesh) $\mathcal{T}$ of Ω into open triangles K:

$$\mathcal{T} = \{K_i\}_i, \qquad \bigcup_{K \in \mathcal{T}} \overline{K} = \overline{\Omega}.$$

The elements $K \in \mathcal{T}$ are images of the reference triangle

$$\hat{T} := \{(\hat{x}, \hat{y}) : -1 \leqslant \hat{y} \leqslant -\hat{x}, \hat{x} \in (-1, 1)\} \tag{4.1}$$

under affine maps $\underline{F}_K$, i.e. for each $K \in \mathcal{T}$ there exists a constant matrix $\underline{\underline{A}}_K \in \mathbb{R}^{2\times 2}$ and a constant vector $\underline{b}_K \in \mathbb{R}^2$ such that with

$$\underline{F}_K(\underline{x}) = \underline{\underline{A}}_K \underline{x} + \underline{b}_K \tag{4.2}$$

there holds

$$K = \underline{F}_K(\hat{T}). \tag{4.3}$$

Moreover, for each $K \in \mathcal{T}$, introduce

$$h_K := \mathrm{diam}(K)$$

and

$$\rho_K := \sup\{\mathrm{diam}(B) : B \text{ is a ball contained in } K\}.$$

The so-called *mesh size* of $\mathcal{T}$ is given by

$$h_{\mathcal{T}} := \sup_{K \in \mathcal{T}} h_K. \tag{4.4}$$

†i.e. the intersection of any two elements is either empty, a vertex or an entire side.

Finally, in order to account for the singular behaviour of solutions near the singular points of the polygon Ω, the following set has to be defined:

$$\mathcal{K}_0 := \{K \in \mathcal{T} : \partial K \cap SP(\Omega, \Gamma_\mathrm{D}, \Gamma_\mathrm{N}) \neq \emptyset\}.$$

Henceforth, the finite element meshes are assumed to satisfy the following property:

$$h_K \leqslant C\rho_K, \qquad \forall K \in \mathcal{T}, \tag{4.5}$$

for a constant $C > 0$ independent of $K \in \mathcal{T}$.

4.2 *FE spaces*

Let $\mathcal{T}$ be a regular finite element mesh consisting of triangles $K \in \mathcal{T}$. The discontinuous finite element spaces that will be appropriate for the DGFEM are defined as follows:

$$\mathcal{S}^{1,0}(\Omega, \mathcal{T}) := \{\underline{u} \in L^2(\Omega)^2 : \underline{u}|_K \in \mathcal{P}_1(K)^2, K \in \mathcal{T}\}. \tag{4.6}$$

Here,

$$\mathcal{P}_1(K) := \{u(x, y) = ax + by + c : a, b, c \in \mathbb{R}\}$$

is the space of all linear functions on K.

4.3 *Trace operators for the DGFEM*

First of all, assume that there exists an index set $\mathcal{I} \subset \mathbb{N}$ such that the elements in the subdivision $\mathcal{T}$ are numbered in a certain way:

$$\mathcal{T} = \{K_i\}_{i \in \mathcal{I}}.$$

Furthermore, denote by $\mathcal{E}$ the set of all element edges associated with the mesh $\mathcal{T}$. Additionally, let Γ_int be the union of all edges $e \in \mathcal{E}$ not lying on $\partial\Omega$:

$$\Gamma_\mathrm{int} := \bigcup_{\substack{e \in \mathcal{E}: \\ e \cap \partial\Omega = \emptyset}} e.$$

Moreover, define

$$\Gamma_\mathrm{int,D} := \Gamma_\mathrm{int} \cup \{e \in \mathcal{E} : e \subset \Gamma_\mathrm{D}\}.$$

Obviously, for each $e \in \Gamma_\mathrm{int}$, there exist two indices i and j with $i > j$ such that K_i and K_j share the interface e:

$$e = \partial K_i \cap \partial K_j.$$

Thus, the following mapping is well-defined:

$$\begin{array}{cccc} \varphi : & \Gamma_\mathrm{int} & \longrightarrow & \mathbb{N}^2 \\ & e & \longmapsto & \begin{pmatrix} \varphi_1(e) := i \\ \varphi_2(e) := j \end{pmatrix}. \end{array}$$

If $e \in \mathcal{E} \setminus \Gamma_{\text{int}}$, i.e. if e is a boundary edge, there is a unique element $K_i \in \mathcal{T}$ such that

$$e = \partial K_i \cap \Gamma.$$

Hence, the above definition may be expanded as follows:

$$\begin{array}{rccc} \varphi : & \mathcal{E} \setminus \Gamma_{\text{int}} & \longrightarrow & \mathbb{N} \\ & e & \longmapsto & \varphi(e) := i. \end{array}$$

On $e \in \Gamma_{\text{int}}$, let $\underline{\nu}_e$ be the normal vector which points from $K_{\varphi_1(e)}$ to $K_{\varphi_2(e)}$; for boundary edges $e \subset \Gamma$, set $\underline{\nu}_e = \underline{n}_\Omega$.

Since the DGFEM is based on functions in

$$H^{1,1}(\Omega, \mathcal{T}) = \{v \in L^2(\Omega) : v|_K \in W^{1,1}(K), K \in \mathcal{T}\} \not\subset \mathcal{C}^0(\Omega),$$

the discontinuities over element boundaries have to be controlled in a certain way. Consider therefore $\underline{v} \in H^{1,1}(\Omega, \mathcal{T})^2$. Then, for $e \in \Gamma_{\text{int}}$ and $\underline{x} \in e$, introduce the following *average* at $\underline{x} \in e$:

$$\langle \underline{v} \rangle_e := \frac{\underline{v}^+ + \underline{v}^-}{2},$$

and the (numbering-dependent) *jump* at $\underline{x} \in e$,

$$[\underline{v}]_e := \underline{v}^+ - \underline{v}^-.$$

Here, $\underline{v}^+$, $\underline{v}^-$ denote the traces of $\underline{v}$ onto e taken from within the interior of the elements $K_{\varphi_1(e)}$ and $K_{\varphi_2(e)}$, respectively. For $e \subset \Gamma$, let $\langle \underline{v} \rangle_e := \underline{v}$ and $[\underline{v}]_e := \underline{v}$.

4.4 *Variational formulation*

There is a wide variety of DG methods for linear elliptic problems. Most of them are examples of the so-called *flux formulation* introduced by Cockburn & Shu (1998). In this very general formulation, the normal derivatives are replaced by *numerical fluxes*, which may also be interpreted as Lagrange multipliers. Since there are many possibilities to choose the numerical fluxes, a considerable number of different DG methods may be obtained (see Arnold *et al.*, 2001 for details). In this paper, the so-called non-symmetric interior penalty Galerkin method (NIPG) will be analysed. It was originally introduced in Wheeler (1978) and extensively studied in Arnold (1982), Rivière *et al.* (1999), Arnold *et al.* (2001), Wihler (2003) (and the references therein).

In order to define the NIPG for the linear elasticity problem (2.1), the following product operator on $L^2(K)^{2\times 2} \times L^2(K)^{2\times 2}$, $K \in \mathcal{T}$, is introduced:

$$\underline{\underline{\alpha}} : \underline{\underline{\beta}} := \sum_{i,j=1}^{2} \alpha_{ij} \beta_{ij},$$

with the induced norm

$$\|\underline{\underline{\alpha}}\|_K := \sqrt{\int_K \underline{\underline{\alpha}} : \underline{\underline{\alpha}} \, \mathrm{d}x}.$$

DEFINITION 4.1 (NIPG) For $\tau = 1$, define a bilinear form B_{DG} by

$$\begin{aligned} B_{\mathrm{DG}}(\underline{u}, \underline{v}) := & \sum_{K \in \mathcal{T}} \int_K \underline{\underline{\sigma}}(\underline{u}) : \underline{\underline{\epsilon}}(\underline{v}) \, \mathrm{d}x \\ & - \sum_{e \in \Gamma_{\mathrm{int,D}}} \int_e (\left\langle \underline{\underline{\sigma}}(\underline{u}) \cdot \underline{\nu}_e \right\rangle_e \cdot [\underline{v}]_e - \tau [\underline{u}]_e \cdot \left\langle \underline{\underline{\sigma}}(\underline{v}) \cdot \underline{\nu}_e \right\rangle_e) \, \mathrm{d}s \\ & + \mu \sum_{e \in \Gamma_{\mathrm{int,D}}} \frac{1}{|e|} \int_e [\underline{u}]_e \cdot [\underline{v}]_e \, \mathrm{d}s, \end{aligned}$$

and a corresponding linear functional L_{DG} by

$$\begin{aligned} L_{\mathrm{DG}}(\underline{v}) := & \sum_{K \in \mathcal{T}} \int_K \underline{f} \cdot \underline{v} \, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \underline{g}_{\mathrm{N}} \cdot \underline{v} \, \mathrm{d}s \\ & + \int_{\Gamma_{\mathrm{D}}} (\underline{\underline{\sigma}}(\underline{v}) \cdot \underline{n}_\Omega) \cdot \underline{g}_{\mathrm{D}} \, \mathrm{d}s + \mu \sum_{\substack{e \in \mathcal{E}: \\ e \subset \Gamma_{\mathrm{D}}}} \frac{1}{|e|} \int_e \underline{g}_{\mathrm{D}} \cdot \underline{v} \, \mathrm{d}s. \end{aligned}$$

Then, the DGFEM for the linear elasticity problem (2.1) reads as follows:

Find $\underline{u}_{\mathrm{DG}} \in \mathcal{S}^{1,0}(\Omega, \mathcal{T})$ such that

$$B_{\mathrm{DG}}(\underline{u}_{\mathrm{DG}}, \underline{v}) = L_{\mathrm{DG}}(\underline{v}) \qquad \forall \underline{v} \in \mathcal{S}^{1,0}(\Omega, \mathcal{T}). \tag{4.7}$$

REMARK 4.2 The choice $\tau = -1$ in Definition 4.1 leads to the symmetric interior penalty Galerkin method (SIPG) for the elasticity problem (2.1). However, to prove absence of volume locking for this scheme, an additional stabilization term of the form

$$\lambda \sum_{e \in \Gamma_{\mathrm{int,D}}} \frac{1}{|e|} \int_e [\underline{u} \cdot \underline{\nu}_e]_e [\underline{v} \cdot \underline{\nu}_e]_e \, \mathrm{d}s$$

must be added to the bilinear form B_{DG} (Hansbo & Larson, 2002).

PROPOSITION 4.3 (Consistency) If the exact solution $\underline{u}_{ex}$ of the linear elasticity problem (2.1) belongs to $H^{2,2}_{\underline{\beta}}(\Omega)^2$ for any weight vector $\underline{\beta} = (\beta_1, \ldots, \beta_M)$ with $\beta_i \in [0, 1)$, $i = 1, \ldots, M$, then the DGFEM (4.7) is consistent:

$$B_{\mathrm{DG}}(\underline{u}_{ex} - \underline{u}_{\mathrm{DG}}, \underline{v}) = 0 \qquad \forall \underline{v} \in \mathcal{S}^{1,0}(\Omega, \mathcal{T}). \tag{4.8}$$

Proof. See Wihler (2003). □

REMARK 4.4 Proposition 4.3 shows that, in contrast to many other non-conforming finite element methods, the consistency error of the DGFEM vanishes. This property results from the fact that the discontinuities of the DG solutions over element boundaries are handled with the aid of some extra inter-element terms in the bilinear form B_{DG}. Nevertheless, the analysis of the DGFEM is comparable to that of non-conforming, non-consistent methods, since there, similar expressions occur in the corresponding residual terms.

Finally, the following norm is associated to the DGFEM:

$$\|\underline{u}\|_{\mathrm{DG}}^2 := \sum_{K\in\mathcal{T}} \|\underline{\underline{\epsilon}}(\underline{u})\|_K^2 + \frac{\mu}{m_{\mathrm{elast}}} \sum_{e\in\Gamma_{\mathrm{int,D}}} |e|^{-1} \int_e |[\underline{u}]_e|^2 \,\mathrm{d}s, \tag{4.9}$$

where $m_{\mathrm{elast}} := 2\min\{\mu, \mu+\lambda\}$.

REMARK 4.5 The norm in (4.9) is equivalent to the element-wise H^1 norm. A corresponding result may be found in Brenner (2002), where a discrete Korn inequality was proved.

PROPOSITION 4.6 (Coercivity) The bilinear form B_{DG} is coercive on $\mathcal{S}^{1,0}(\Omega, \mathcal{T})$. More precisely,

$$B_{\mathrm{DG}}(\underline{u}, \underline{u}) \geqslant m_{\mathrm{elast}} \|\underline{u}\|_{\mathrm{DG}}^2$$

for all $\underline{u} \in \mathcal{S}^{1,0}(\Omega, \mathcal{T})$.

Proof. Set

$$\underline{\underline{\epsilon}}_0(\underline{u}) := \underline{\underline{\epsilon}}(\underline{u}) - \tfrac{1}{2}\nabla\cdot\underline{u}\,\underline{\underline{1}}_{2\times 2}.$$

Then, for $K \in \mathcal{T}$, there holds that

$$\begin{aligned}
\int_K \underline{\underline{\sigma}}(\underline{u}) : \underline{\underline{\epsilon}}(\underline{u})\,\mathrm{d}x &= 2\mu \int_K \underline{\underline{\epsilon}}(\underline{u}) : \underline{\underline{\epsilon}}(\underline{u})\,\mathrm{d}x + \lambda \int_K |\nabla\cdot\underline{u}|^2\,\mathrm{d}x \\
&= 2\mu \int_K (\underline{\underline{\epsilon}}_0(\underline{u}) + \tfrac{1}{2}\nabla\cdot\underline{u}\,\underline{\underline{1}}_{2\times 2}) : (\underline{\underline{\epsilon}}_0(\underline{u}) + \tfrac{1}{2}\nabla\cdot\underline{u}\,\underline{\underline{1}}_{2\times 2})\,\mathrm{d}x \\
&\quad + \lambda \int_K |\nabla\cdot\underline{u}|^2\,\mathrm{d}x \\
&= 2\mu \int_K \{\underline{\underline{\epsilon}}_0(\underline{u}) : \underline{\underline{\epsilon}}_0(\underline{u}) + \tfrac{1}{2}|\nabla\cdot\underline{u}|^2\}\,\mathrm{d}x + \lambda \int_K |\nabla\cdot\underline{u}|^2\,\mathrm{d}x \\
&= 2\mu \int_K \underline{\underline{\epsilon}}_0(\underline{u}) : \underline{\underline{\epsilon}}_0(\underline{u})\,\mathrm{d}x + (\mu+\lambda) \int_K |\nabla\cdot\underline{u}|^2\,\mathrm{d}x.
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\int_K \underline{\underline{\epsilon}}(\underline{u}) : \underline{\underline{\epsilon}}(\underline{u})\,\mathrm{d}x &= \int_K (\underline{\underline{\epsilon}}_0(\underline{u}) + \tfrac{1}{2}\nabla\cdot\underline{u}\,\underline{\underline{1}}_{2\times 2}) : (\underline{\underline{\epsilon}}_0(\underline{u}) + \tfrac{1}{2}\nabla\cdot\underline{u}\,\underline{\underline{1}}_{2\times 2})\,\mathrm{d}x \\
&= \int_K \{\underline{\underline{\epsilon}}_0(\underline{u}) : \underline{\underline{\epsilon}}_0(\underline{u}) + \tfrac{1}{2}|\nabla\cdot\underline{u}|^2\}\,\mathrm{d}x,
\end{aligned}$$

it follows that

$$\int_K \underline{\underline{\sigma}}(\underline{u}) : \underline{\underline{\epsilon}}(\underline{u})\,\mathrm{d}x \geqslant m_{\mathrm{elast}} \int_K \underline{\underline{\epsilon}}(\underline{u}) : \underline{\underline{\epsilon}}(\underline{u})\,\mathrm{d}x.$$

Thus,

$$\begin{aligned}
B_{\mathrm{DG}}(\underline{u}, \underline{u}) &\geqslant m_{\mathrm{elast}} \sum_{K\in\mathcal{T}} \int_K \underline{\underline{\epsilon}}(\underline{u}) : \underline{\underline{\epsilon}}(\underline{u})\,\mathrm{d}x + \mu \sum_{e\in\Gamma_{\mathrm{int,D}}} |e|^{-1} \int_e |[\underline{u}]|^2\,\mathrm{d}s \\
&\geqslant m_{\mathrm{elast}} \|\underline{u}\|_{\mathrm{DG}}^2.
\end{aligned}$$

□

Note that the coercivity constant in Proposition 4.6 is independent of λ as $\lambda \to \infty$. This remarkable property of the NIPG will be essential for the error analysis in this paper and may not simply be generalized to other DG methods.

5. Error Analysis

5.1 *The Crouzeix–Raviart interpolant*

From the analysis of other non-conforming methods (see Brenner & Scott, 2002, for example), it is well-known that the Crouzeix–Raviart element does not lock. This can be shown by introducing the so-called Crouzeix–Raviart interpolant (Crouzeix & Raviart, 1973) which provides some essential properties for the circumvention of volume locking. These properties are typically not available for continuous (low-order) elements.

Therefore, this interpolant will also be used for the error analysis of the DGFEM considered in this paper. However here, the original definition must be extended to weighted Sobolev spaces. This can be done straightforwardly.

PROPOSITION 5.1 Let $K \in \mathcal{T}$ be a triangle with vertices A_1, A_2, A_3. Then, for each $\beta \in [0, 1)$ and for $\Phi_\beta(x) = r^\beta = |x - A_1|^\beta$, there exists an interpolant

$$\pi_K : H^{2,2}_\beta(K)^2 \longrightarrow \mathcal{P}_1(K)^2$$

such that the following properties are satisfied:

(a) $\displaystyle\int_e (\underline{u} - \pi_K \underline{u})\, \mathrm{d}s = \underline{0}, \quad \forall e \in \mathcal{E}_K := \{e \in \mathcal{E} : e \subset \partial K\};$

(b) $\displaystyle\int_e (\underline{u} - \pi_K \underline{u}) \cdot \underline{n}_e\, \mathrm{d}s = 0, \quad \forall e \in \mathcal{E}_K;$

(c) $\displaystyle\int_K \nabla \cdot (\underline{u} - \pi_K \underline{u})\, \mathrm{d}x = 0.$

Here, for $e \in \mathcal{E}_K$, $\underline{n}_e$ denotes the unit outward vector of K on e.

Proof. For $\underline{u} \in H^{2,2}_\beta(K)^2$ the interpolant $\pi_K \underline{u} \in \mathcal{P}_1(K)^2$ is uniquely defined by

$$\pi_K \underline{u}(x_e^M) := \frac{1}{|e|} \int_e \underline{u}\, \mathrm{d}s, \quad \forall e \in \mathcal{E}_K,$$

where x_e^M denotes the midpoint of $e \in \mathcal{E}_K$. Then, (a) and (b) follow directly from this definition. (c) results from (b) and from Green's formula:

$$\begin{aligned}\int_K \nabla \cdot (\underline{u} - \pi_K \underline{u})\, \mathrm{d}x &= \int_{\partial K} (\underline{u} - \pi_K \underline{u}) \cdot \underline{n}_{\partial K}\, \mathrm{d}s \\ &= \sum_{e \in \mathcal{E}_K} \int_e (\underline{u} - \pi_K \underline{u}) \cdot \underline{n}_e\, \mathrm{d}s \\ &= 0.\end{aligned}$$

□

In order to study the approximation properties of π_K on $H^{2,2}_\beta(K)$, $K \in \mathcal{T}$, some new (optimal) interpolation error estimates have to be established.

PROPOSITION 5.2 For $\underline{u} \in H^{2,2}_\beta(K)^2$, $K \in \mathcal{T}$, the interpolant $\pi_K \underline{u}$ from Proposition 5.1 satisfies the following estimates:

$$\|\underline{u} - \pi_K \underline{u}\|_{L^2(K)} + h_K |\underline{u} - \pi_K \underline{u}|_{H^1(K)} \leqslant C h_K^{2-\beta} |\underline{u}|_{H^{2,2}_\beta(K)} \tag{5.1}$$

$$|\underline{u} - \pi_K \underline{u}|_{H^{2,2}_\beta(K)} \leqslant |\underline{u}|_{H^{2,2}_\beta(K)}, \tag{5.2}$$

and

$$\|\nabla \cdot (\underline{u} - \pi_K \underline{u})\|_{L^2(K)} \leqslant C h_K^{1-\beta} |\nabla \cdot \underline{u}|_{H^{1,1}_\beta(K)} \tag{5.3}$$

$$|\nabla \cdot (\underline{u} - \pi_K \underline{u})|_{H^{1,1}_\beta(K)} \leqslant |\nabla \cdot \underline{u}|_{H^{1,1}_\beta(K)}. \tag{5.4}$$

$C > 0$ is a constant independent of h_K and of $\underline{u}$.

Proof. Set $\underline{U} := \underline{u} - \pi_K \underline{u}$. Then, since $\pi_K \underline{u} \in \mathcal{P}_1(K)^2$, there holds

$$|\underline{U}|_{H^{2,2}_\beta(K)} = |\underline{u}|_{H^{2,2}_\beta(K)} \quad \text{and} \quad |\nabla \cdot \underline{U}|_{H^{1,1}_\beta(K)} = |\nabla \cdot \underline{u}|_{H^{1,1}_\beta(K)}.$$

Thus, applying Lemma A.2 to $\underline{U}$ and Lemma A.3 to $\nabla \cdot \underline{U}$, completes the proof. □

5.2 *A priori error estimates*

In a polygon Ω consider a FE mesh $\mathcal{T}$ satisfying the conditions from Section 4.1. Moreover, let $\underline{\beta} = (\beta_1, \ldots, \beta_M)$ be a weight vector and $\Phi_{\underline{\beta}}$ the corresponding weight function described in Section 3.1. Then, on $\mathcal{S}^{1,0}(\Omega, \mathcal{T})$, define an interpolant

$$\Pi_{\mathcal{T}} : H^{2,2}_{\underline{\beta}}(\Omega)^2 \longrightarrow \mathcal{S}^{1,0}(\Omega, \mathcal{T}) \tag{5.5}$$

by

$$\Pi_{\mathcal{T}}|_K \underline{u} = \pi_K \underline{u}, \quad \forall K \in \mathcal{T}, \tag{5.6}$$

where π_K, $K \in \mathcal{T}$, is the interpolant from Proposition 5.1.

Then, the DG error $\underline{e} := \underline{u}_{\text{ex}} - \underline{u}_{\text{DG}}$, where $\underline{u}_{\text{ex}}$ is the exact solution of the linear elasticity problem (2.1) and $\underline{u}_{\text{DG}}$ is the solution of the DGFEM (4.7), may be represented as follows:

$$\underline{e} = \underbrace{\underline{u}_{\text{ex}} - \Pi_{\mathcal{T}} \underline{u}_{\text{ex}}}_{=: \underline{\eta}} + \underbrace{\Pi_{\mathcal{T}} \underline{u} - \underline{u}_{\text{DG}}}_{=: \underline{\xi}}. \tag{5.7}$$

REMARK 5.3 Since $H^{2,2}_{\underline{\beta}}(\Omega)^2 \subset \mathcal{C}^0(\overline{\Omega})^2$ (Babuška *et al.*, 1979), $\underline{u}_{\text{ex}} \in H^{2,2}_{\underline{\beta}}(\Omega)$ implies that

$$\int_e [\underline{\eta}]_e \, \mathrm{d}s = 0 \tag{5.8}$$

for all edges $e \in \Gamma_{\text{int}}$.

In the following part, it will be proved that $\|\underline{\xi}\|_{\mathrm{DG}}$ is bounded in terms of $\underline{\eta}$, and thus, due to the triangle inequality, the error $\|\underline{e}\|_{\mathrm{DG}} = \|\underline{u}_{\mathrm{ex}} - \underline{u}_{\mathrm{DG}}\|_{\mathrm{DG}}$ of the DGFEM can be controlled by $\underline{\eta}$ only.

For standard (conforming) FEMs, such error estimates are usually obtained using Galerkin orthogonality (consistency) as well as the coercivity and the continuity of the corresponding bilinear forms. In the DG context however, the latter property is typically not available on continuous spaces and alternative error estimation techniques have to be applied. A possible approach is presented in the proof of the following proposition.

PROPOSITION 5.4 Let the exact solution $\underline{u}_{\mathrm{ex}}$ of the linear elasticity problem (2.1) be in $H^{2,2}_{\underline{\beta}}(\Omega)^2$, where Ω is a polygon in $\mathbb{R}^2$. Then, with $\underline{\eta}$ and $\underline{\xi}$ as in (5.7), there holds the following stability inequality for the DGFEM (4.7):

$$\begin{aligned}\|\underline{\xi}\|^2_{\mathrm{DG}} \leqslant C\Big\{\mu^2\Big[\sum_{K\in\mathcal{T}}(h_K^{-2}\|\underline{\eta}\|^2_{L^2(K)} + |\underline{\eta}|^2_{H^1(K)}) + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2|\underline{\eta}|^2_{H^2(K)}\\ + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta}|\underline{\eta}|^2_{H^{2,2}_{\underline{\beta}}(K)}\Big] + \lambda^2\Big[\sum_{K\in\mathcal{T}}\|\nabla\cdot\underline{\eta}\|^2_{L^2(K)}\\ + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2|\nabla\cdot\underline{\eta}|^2_{H^{1,1}_{\underline{\beta}}(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta}|\nabla\cdot\underline{\eta}|^2_{H^{1,1}_{\underline{\beta}}(K)}\Big]\Big\},\end{aligned} \tag{5.9}$$

where $C > 0$ is a constant independent of μ, λ and of $\{h_K : K \in \mathcal{T}\}$.

The error bound in Proposition 5.4 is explicit with respect to the Lamé coefficients μ and λ. This fact will be essential in Section 5.3, where robust (λ-independent) convergence rates for the DGFEM will be derived.

To make clear how this explicit form of the right hand-side of (5.9) is obtained, the following auxiliary result, Lemma 5.5, is inserted prior to the proof of Proposition 5.4.

LEMMA 5.5 Let $\underline{v} = \underline{v}_1 + \underline{v}_2$, where $\underline{v}_1 \in H^{2,2}_{\underline{\beta}}(\Omega)^2$ and $\underline{v}_2 \in \mathcal{S}^{1,0}(\Omega,\mathcal{T})$. Then, there holds the bound

$$\begin{aligned}\mu^2\sum_{K\in\mathcal{T}}\|\underline{\underline{\epsilon}}(\underline{v})\|^2_K + \sum_{K\in\mathcal{T}}\sum_{\substack{e\in\mathcal{E}_K\\ e\in\Gamma_{\mathrm{int,D}}}}\|\underline{\underline{\sigma}}(\underline{v})\cdot\underline{\nu}_e\|^2_{L^1(e)} + \mu^2\sum_{e\in\Gamma_{\mathrm{int,D}}}|e|^{-1}\|[\underline{v}]_e\|^2_{L^2(e)}\\ \leqslant C\Big\{\mu^2\Big[\sum_{K\in\mathcal{T}}(h_K^{-2}\|\underline{v}\|^2_{L^2(K)} + |\underline{v}|^2_{H^1(K)}) + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2|\underline{v}|^2_{H^2(K)}\\ + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta}|\underline{v}|^2_{H^{2,2}_{\underline{\beta}}(K)}\Big] + \lambda^2\Big[\sum_{K\in\mathcal{T}}\|\nabla\cdot\underline{v}\|^2_{L^2(K)}\\ + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2|\nabla\cdot\underline{v}|^2_{H^1(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta}|\nabla\cdot\underline{v}|^2_{H^{1,1}_{\underline{\beta}}(K)}\Big]\Big\}.\end{aligned}$$

Proof. Obviously,

$$\sum_{K\in\mathcal{T}}\|\underline{\underline{\epsilon}}(\underline{v})\|^2_K \leqslant C\sum_{K\in\mathcal{T}}|\underline{v}|^2_{H^1(K)}.$$

Furthermore, Lemma A.4 and the fact that $\underline{v}|_K \in H^2(K)^2$ for all $K \notin \mathcal{K}_0$ (see Remark 3.2) imply that

$$\begin{aligned}
&\sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K \\ e\in\Gamma_{\mathrm{int,D}}}} \|\underline{\underline{\sigma}}(\underline{v})\cdot\underline{\nu}_e\|^2_{L^1(e)} \\
&\leqslant C\Big[\mu^2 \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K \\ e\in\Gamma_{\mathrm{int,D}}}} \|\underline{\underline{\epsilon}}(\underline{v}\cdot\underline{\nu}_e)\|^2_{L^1(e)} + \lambda^2 \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K \\ e\in\Gamma_{\mathrm{int,D}}}} \|\nabla\cdot\underline{v}\|^2_{L^1(e)}\Big] \\
&\leqslant C\mu^2\Big[\sum_{K\in\mathcal{T}} \|\nabla\underline{v}\|^2_{L^2(K)} + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2 |\underline{v}|^2_{H^2(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta} |\underline{v}|^2_{H^{2,2}_{\underline{\beta}}(K)}\Big] \\
&\quad + C\lambda^2\Big[\sum_{K\in\mathcal{T}} \|\nabla\cdot\underline{v}\|^2_{L^2(K)} + \sum_{K\in\mathcal{T}\setminus\mathcal{K}_0} h_K^2 |\nabla\cdot\underline{v}|^2_{H^1(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta} |\nabla\cdot\underline{v}|^2_{H^{1,1}_{\underline{\beta}}(K)}\Big].
\end{aligned}$$

Additionally, by the standard trace theorem (see Schwab, 1998, Theorem A.11), there holds

$$\begin{aligned}
\sum_{e\in\Gamma_{\mathrm{int,D}}} |e|^{-1} \|[\underline{v}]_e\|^2_{L^2(e)} &\leqslant C \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K \\ e\in\Gamma_{\mathrm{int,D}}}} |e|^{-1} \|\underline{v}\|^2_{L^2(e)} \\
&\leqslant C \sum_{e\in\Gamma_{\mathrm{int,D}}} (|e|^{-2} \|\underline{v}\|^2_{L^2(K)} + |\nabla\underline{v}|^2_{L^2(K)}) \\
&\leqslant C \sum_{e\in\Gamma_{\mathrm{int,D}}} (h_K^{-2} \|\underline{v}\|^2_{L^2(K)} + |\nabla\underline{v}|^2_{L^2(K)}).
\end{aligned}$$

□

Proof of Proposition 5.4. Due to the consistency of the DGFEM (see Proposition 4.3), it holds that

$$B_{\mathrm{DG}}(\underline{\xi},\underline{\xi}) = B_{\mathrm{DG}}(\underline{e}-\underline{\eta},\underline{\xi}) = -B_{\mathrm{DG}}(\underline{\eta},\underline{\xi}).$$

Therefore, by Proposition 4.6,

$$2m_{\mathrm{elast}}\|\underline{\xi}\|^2_{\mathrm{DG}} \leqslant -B_{\mathrm{DG}}(\underline{\eta},\underline{\xi}). \tag{5.10}$$

Referring to the definition (2.2) of the stress tensor $\underline{\underline{\sigma}}$, and noting that $\left\langle\underline{\underline{\sigma}}(\underline{\xi})\cdot\underline{\nu}_e\right\rangle_e$ is constant on each edge $e \in \mathcal{E}$, leads to

$$\begin{aligned}
B_{\mathrm{DG}}(\underline{\eta}, \underline{\xi}) = & \sum_{K \in \mathcal{T}} \int_K \underline{\underline{\sigma}}(\underline{\eta}) : \underline{\underline{\epsilon}}(\underline{\xi}) \, \mathrm{d}x \\
& - \sum_{e \in \Gamma_{\mathrm{int,D}}} \int_e (\left\langle \underline{\underline{\sigma}}(\underline{\eta}) \cdot \underline{\nu}_e \right\rangle_e \cdot [\underline{\xi}]_e - [\underline{\eta}]_e \cdot \left\langle \underline{\underline{\sigma}}(\underline{\xi}) \cdot \underline{\nu}_e \right\rangle_e) \, \mathrm{d}s \\
& + \mu \sum_{e \in \Gamma_{\mathrm{int,D}}} |e|^{-1} \int_e [\underline{\eta}]_e \cdot [\underline{\xi}]_e \, \mathrm{d}s \\
= & \, 2\mu \sum_{K \in \mathcal{T}} \int_K \underline{\underline{\epsilon}}(\underline{\eta}) : \underline{\underline{\epsilon}}(\underline{\xi}) \, \mathrm{d}x + \lambda \sum_{K \in \mathcal{T}} \nabla \cdot \underline{\xi} \int_K \nabla \cdot \underline{\eta} \, \mathrm{d}x \\
& - \sum_{e \in \Gamma_{\mathrm{int,D}}} \Big(\int_e \left\langle \underline{\underline{\sigma}}(\underline{\eta}) \cdot \underline{\nu}_e \right\rangle_e \cdot [\underline{\xi}]_e \, \mathrm{d}s - \left\langle \underline{\underline{\sigma}}(\underline{\xi}) \cdot \underline{\nu}_e \right\rangle_e \cdot \int_e [\underline{\eta}]_e \, \mathrm{d}s \Big) \\
& + \mu \sum_{e \in \Gamma_{\mathrm{int,D}}} |e|^{-1} \int_e [\underline{\eta}]_e \cdot [\underline{\xi}]_e \, \mathrm{d}s.
\end{aligned}$$

Using the properties of the interpolant $\Pi_{\mathcal{T}}$ (Proposition 5.1) as well as the weak continuity (5.8) of $\underline{\eta}$ results in

$$\begin{aligned}
B_{\mathrm{DG}}(\underline{\eta}, \underline{\xi}) = & \, 2\mu \sum_{K \in \mathcal{T}} \int_K \underline{\underline{\epsilon}}(\underline{\eta}) : \underline{\underline{\epsilon}}(\underline{\xi}) \, \mathrm{d}x - \sum_{e \in \Gamma_{\mathrm{int,D}}} \int_e \left\langle \underline{\underline{\sigma}}(\underline{\eta}) \cdot \underline{\nu}_e \right\rangle_e \cdot [\underline{\xi}]_e \, \mathrm{d}s \\
& + \mu \sum_{e \in \Gamma_{\mathrm{int,D}}} |e|^{-1} \int_e [\underline{\eta}]_e \cdot [\underline{\xi}]_e \, \mathrm{d}s \\
= & \, I - II + III.
\end{aligned}$$

In the remaining part of the proof, the sums I, II and III are estimated in terms of $\underline{\eta}$ and of $\underline{\xi}$. First of all, by Hölder's inequality, there holds that

$$\begin{aligned}
|I| &= \Big| 2\mu \sum_{K \in \mathcal{T}} \int_K \underline{\underline{\epsilon}}(\underline{\eta}) : \underline{\underline{\epsilon}}(\underline{\xi}) \, \mathrm{d}x \Big| \\
&\leqslant \Big[4\mu^2 \sum_{K \in \mathcal{T}} \|\underline{\underline{\epsilon}}(\underline{\eta})\|_K^2 \Big]^{\frac{1}{2}} \Big[\sum_{K \in \mathcal{T}} \|\underline{\underline{\epsilon}}(\underline{\xi})\|_K^2 \Big]^{\frac{1}{2}}.
\end{aligned}$$

Secondly, a bound for II will be established. To do so, the sum over all edges $e \in \Gamma_{\mathrm{int,D}}$ (in II) is transformed into a sum over all elements $K \in \mathcal{T}$. Again, Hölder's inequality is used:

$$
\begin{aligned}
|II| &\leqslant \sum_{e\in\Gamma_{\mathrm{int,D}}} \int_e |\left\langle \underline{\underline{\sigma}}(\underline{\eta})\cdot \nu_e \right\rangle_e | |[\underline{\xi}]_e| \,\mathrm{d}s \\
&\leqslant \sum_{e\in\Gamma_{\mathrm{int,D}}} \|[\underline{\xi}]_e\|_{L^\infty(e)} \| \left\langle \underline{\underline{\sigma}}(\underline{\eta})\cdot \underline{\nu}_e \right\rangle_e \|_{L^1(e)} \\
&\leqslant \tfrac{1}{2} \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\Gamma_{\mathrm{int}}\\ e\subset\partial K}} \|[\underline{\xi}]_e\|_{L^\infty(e)} \|\underline{\underline{\sigma}}(\underline{\eta})\cdot \underline{\nu}_e\|_{L^1(e)} \\
&\quad + \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\Gamma_{\mathrm{D}}}} \|[\underline{\xi}]_e\|_{L^\infty(e)} \|\underline{\underline{\sigma}}(\underline{\eta})\cdot \underline{\nu}_e\|_{L^1(e)} \\
&\leqslant C \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\Gamma_{\mathrm{int,D}}}} \|[\underline{\xi}]_e\|_{L^\infty(e)} \|\underline{\underline{\sigma}}(\underline{\eta})\cdot \underline{\nu}_e\|_{L^1(e)}.
\end{aligned}
$$

Now, applying the inverse inequality from Lemma A.1 to the linear polynomial $[\underline{\xi}]_e$, yields

$$
\begin{aligned}
|II| &\leqslant C \sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\Gamma_{\mathrm{int,D}}}} |e|^{-\frac{1}{2}} \|[\underline{\xi}]_e\|_{L^2(e)} \|\underline{\underline{\sigma}}(\underline{\eta})\cdot \underline{\nu}_e\|_{L^1(e)} \\
&\leqslant C \Big[\sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\Gamma_{\mathrm{int,D}}}} |e|^{-1} \|[\underline{\xi}]_e\|^2_{L^2(e)} \Big]^{\frac{1}{2}} \Big[\sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\Gamma_{\mathrm{int,D}}}} \|\underline{\underline{\sigma}}(\underline{\eta})\cdot \underline{\nu}_e\|^2_{L^1(e)} \Big]^{\frac{1}{2}} \\
&= C \sqrt{\frac{m_{\mathrm{elast}}}{\mu}} \Big[\frac{\mu}{m_{\mathrm{elast}}} \sum_{e\in\Gamma_{\mathrm{int,D}}} |e|^{-1} \|[\underline{\xi}]_e\|^2_{L^2(e)} \Big]^{\frac{1}{2}} \\
&\quad \times \Big[\sum_{K\in\mathcal{T}} \sum_{\substack{e\in\mathcal{E}_K\\ e\in\Gamma_{\mathrm{int,D}}}} \|\underline{\underline{\sigma}}(\underline{\eta})\cdot \underline{\nu}_e\|^2_{L^1(e)} \Big]^{\frac{1}{2}}.
\end{aligned}
$$

Finally, III is estimated as follows:

$$
|III| \leqslant \sqrt{\frac{m_{\mathrm{elast}}}{\mu}} \Big[\mu^2 \sum_{e\in\Gamma_{\mathrm{int,D}}} |e|^{-1} \|[\underline{\eta}]_e\|^2_{L^2(e)} \Big]^{\frac{1}{2}} \Big[\frac{\mu}{m_{\mathrm{elast}}} \sum_{e\in\Gamma_{\mathrm{int,D}}} |e|^{-1} \|[\underline{\xi}]_e\|^2_{L^2(e)} \Big]^{\frac{1}{2}}.
$$

Summing up and using (5.10) results in

$$\begin{aligned}\|\underline{\xi}\|_{\mathrm{DG}}^2 &\leqslant \frac{1}{2m_{\mathrm{elast}}}|B_{\mathrm{DG}}(\underline{\eta},\underline{\xi})| \\ &\leqslant \frac{1}{2m_{\mathrm{elast}}}(|I|+|II|+|III|) \\ &\leqslant C\max\left\{1,\sqrt{\frac{m_{\mathrm{elast}}}{\mu}}\right\}\|\underline{\xi}\|_{\mathrm{DG}}\cdot\Big[\mu^2\sum_{K\in\mathcal{T}}\|\underline{\underline{\epsilon}}(\underline{\eta})\|_K^2 \\ &\quad+\sum_{K\in\mathcal{T}}\sum_{\substack{e\in\mathcal{E}_K\\ e\in\Gamma_{\mathrm{int,D}}}}\|\underline{\underline{\sigma}}(\underline{\eta})\cdot\underline{\nu}_e\|_{L^1(e)}^2+\mu^2\sum_{e\in\Gamma_{\mathrm{int,D}}}|e|^{-1}\|[\underline{\eta}]_e\|_{L^2(e)}^2\Big]^{\frac{1}{2}}.\end{aligned} \tag{5.11}$$

Noting that

$$\max\left\{1,\sqrt{\frac{m_{\mathrm{elast}}}{\mu}}\right\}\leqslant\sqrt{2},$$

and inserting the bound from Lemma 5.5 with $\underline{v}=\underline{\eta}$ into (5.11) completes the proof. □

A direct consequence of the above statement is the ensuing corollary.

COROLLARY 5.6 Let the assumptions of Proposition 5.4 be satisfied. Then, the following a priori error estimate holds true:

$$\begin{aligned}\|\underline{u}_{\mathrm{ex}}-\underline{u}_{\mathrm{DG}}\|_{\mathrm{DG}}^2 \leqslant CC_{\mu,\lambda}\Big\{\mu^2\Big[&\sum_{K\in\mathcal{T}}(h_K^{-2}\|\underline{\eta}\|_{L^2(K)}^2+|\underline{\eta}|_{H^1(K)}^2)+\sum_{K\in\mathcal{T}\setminus\mathcal{K}_0}h_K^2|\underline{\eta}|_{H^2(K)}^2 \\ &+\sum_{K\in\mathcal{K}_0}h_K^{2-2\beta}|\underline{\eta}|_{H_{\underline{\beta}}^{2,2}(K)}^2\Big]+\lambda^2\Big[\sum_{K\in\mathcal{T}}\|\nabla\cdot\underline{\eta}\|_{L^2(K)}^2 \\ &+\sum_{K\in\mathcal{T}\setminus\mathcal{K}_0}h_K^2|\nabla\cdot\underline{\eta}|_{H_{\underline{\beta}}^{1,1}(K)}^2+\sum_{K\in\mathcal{K}_0}h_K^{2-2\beta}|\nabla\cdot\underline{\eta}|_{H_{\underline{\beta}}^{1,1}(K)}^2\Big]\Big\}.\end{aligned}$$

Here, $\underline{u}_{\mathrm{ex}}$ is the exact solution of (2.1), $\underline{u}_{\mathrm{DG}}$ is the solution of the DGFEM (4.7) and

$$C_{\mu,\lambda}=\max\{\mu^{-2},\mu^{-1}m_{\mathrm{elast}}^{-1},1\}.$$

REMARK 5.7 A few calculations show that the constant $C_{\mu,\lambda}$ from Corollary 5.6 is independent of λ if $\lambda\geqslant 0$.

Proof of Corollary 5.6. From the error splitting (5.7) it follows that

$$\begin{aligned}\|\underline{e}\|_{\mathrm{DG}}^2 &\leqslant C(\|\underline{\eta}\|_{\mathrm{DG}}^2+\|\underline{\xi}\|_{\mathrm{DG}}^2) \\ &\leqslant C\Big[\sum_{K\in\mathcal{T}}\|\underline{\underline{\epsilon}}(\underline{\eta})\|_K^2+\frac{\mu}{m_{\mathrm{elast}}}\sum_{e\in\Gamma_{\mathrm{int,D}}}|e|^{-1}\int_e|[\underline{\eta}]|^2\,\mathrm{d}s+\|\underline{\xi}\|_{\mathrm{DG}}^2\Big] \\ &\leqslant C\max\{\mu^{-2},\mu^{-1}m_{\mathrm{elast}}^{-1}\}\Big[\mu^2\sum_{K\in\mathcal{T}}\|\underline{\underline{\epsilon}}(\underline{\eta})\|_K^2+\mu^2\sum_{e\in\Gamma_{\mathrm{int,D}}}|e|^{-1}\int_e|[\underline{\eta}]|^2\,\mathrm{d}s\Big] \\ &\quad+C\|\underline{\xi}\|_{\mathrm{DG}}^2.\end{aligned}$$

Thus, using Lemma 5.5 and inserting the bound from Proposition 5.4 completes the proof. □

5.3 *Convergence rates*

It is well-known that, if $\underline{u}_{\mathrm{ex}} \in H^2(\Omega)^2$, where $\underline{u}_{\mathrm{ex}}$ denotes the exact solution of (2.1), the standard (conforming) FEM (and also the DGFEM) converges at an optimal algebraic rate, i.e.

$$\|\underline{u}_{\mathrm{ex}} - \underline{u}_{FE}\| \leqslant CN^{-\frac{1}{2}},$$

where N is the number of degrees of freedom and $\mathcal{T}$ is a uniform mesh on Ω. Unfortunately, this result is typically not anymore true if the assumption $\underline{u}_{\mathrm{ex}} \in H^2(\Omega)^2$ is weakened, i.e. $\underline{u}_{\mathrm{ex}} \in H^{2,2}_{\underline{\beta}}(\Omega)^2$ with $\underline{\beta} \succ \underline{0}$. Moreover, for conforming FEM, C depends on λ, $C \sim \sqrt{\lambda}$ as $\lambda \to \infty$.

Although the convergence rate remains algebraic in this case, the optimal order $\mathcal{O}(N^{-\frac{1}{2}})$ is usually reduced to $\mathcal{O}(N^{-\frac{\alpha}{2}})$ with $\alpha \ll 1$. This effect is even more pronounced at higher orders of approximation.

The aim of this section is to prove that the optimal convergence rate may be preserved (independently of λ for the DGFEM) even if the exact solution is singular, i.e. $\underline{u}_{\mathrm{ex}} \notin H^2(\Omega)$. The main idea is to replace the uniform meshes by so-called '$\underline{\gamma}$-graded meshes' which are able to approximate singularities at an optimal algebraic rate.

5.3.1 $\underline{\gamma}$-*Graded Meshes.*

The $\underline{\gamma}$-graded meshes are constructed in such a way that, for all singularities $A_i \in SP(\Omega, \Gamma_{\mathrm{D}}, \overline{\Gamma_{\mathrm{N}}})$, the ratio

$$\frac{\text{element diameter}}{(\text{distance to singularity})^{\gamma_i}}$$

is kept bounded, where $\gamma_i \geqslant 0$ is an appropriate real number (grading factor) corresponding to the singular point A_i.

A more precise definition may be found in Babuška *et al.* (1979).

DEFINITION 5.8 Let $\underline{\gamma}$ be a weight vector as defined in Section 3.1 and $\Phi_{\underline{\gamma}}$ the corresponding weight function on Ω. Then, a mesh $\mathcal{T}_{\underline{\gamma}}$ on Ω is called a *$\underline{\gamma}$-graded mesh with grading vector $\underline{\gamma}$* if there exists a constant $L > 0$ such that the following properties are satisfied:

(i) if $K \in \mathcal{T}_{\underline{\gamma}} \setminus \mathcal{K}_0$ then

$$L^{-1} h_{\mathcal{T}_{\underline{\gamma}}} \Phi_{\underline{\gamma}}(x) \leqslant h_K \leqslant L h_{\mathcal{T}_{\underline{\gamma}}} \Phi_{\underline{\gamma}}(x) \qquad \forall x \in K;$$

(ii) if $K \in \mathcal{K}_0$ then

$$L^{-1} h_{\mathcal{T}_{\underline{\gamma}}} \sup_{x \in K} \Phi_{\underline{\gamma}}(x) \leqslant h_K \leqslant L h_{\mathcal{T}_{\underline{\gamma}}} \sup_{x \in K} \Phi_{\underline{\gamma}}(x).$$

Here, $h_{\mathcal{T}_{\underline{\gamma}}}$ is the mesh size (4.4) of $\mathcal{T}_{\underline{\gamma}}$.

Asymptotically, $\underline{\gamma}$-graded meshes have the same number of degrees of freedom as uniform meshes.

LEMMA 5.9 Let $\mathcal{T}_{\underline{\gamma}}$ be a $\underline{\gamma}$-graded mesh as in Definition 5.8. Then,

$$N := \dim(\mathcal{S}^{1,0}(\Omega, \mathcal{T}_{\underline{\gamma}})) \leqslant C h_{\mathcal{T}_{\underline{\gamma}}}^{-2},$$

where $C > 0$ is a constant independent of $\{h_K : K \in \mathcal{T}_{\underline{\gamma}}\}$.

Proof. See Babuška *et al.* (1979, Lemma 4.1). □

5.3.2 *Main Result.* Now, the main result of this paper is established. It is shown that the DGFEM (NIPG) converges independently of the Lamé coefficient λ, and, moreover, that the algebraic convergence rates are optimal on $\underline{\gamma}$-graded meshes.

THEOREM 5.10 (Robust Optimal Convergence) Let the assumptions of Theorem 3.4 be satisfied. Moreover, let $\mathcal{T}_{\underline{\gamma}}$ with $(1, 1, \ldots, 1) \succ \underline{\gamma} \succeq \underline{\beta}$ be a $\underline{\gamma}$-graded mesh as introduced in Definition 5.8. Then, for the h-DGFEM (4.7) the following optimal error estimate holds:

$$\|\underline{u}_{\text{ex}} - \underline{u}_{\text{DG}}\|_{\text{DG}} \leqslant C C_{\mu,\lambda} N^{-\frac{1}{2}}.$$

Here, $\underline{u}_{\text{ex}} \in H^{2,2}_{\underline{\beta}}(\Omega)^2$ is the exact solution of the linear elasticity problem (2.1), $\underline{u}_{\text{DG}}$ is the solution of the DGFEM (4.7), $N = \dim(\mathcal{S}^{1,0}(\mathcal{T}_{\underline{\gamma}}, \Omega))$, $C_{\mu,\lambda}$ is the constant from Corollary 5.6 (independent of λ as $\lambda \to \infty$) and $C > 0$ is a constant independent of N and the Lamé coefficients μ and λ.

Proof. Let $\Pi_{\mathcal{T}_{\underline{\gamma}}}$ be the global interpolant from Section 5.2, i.e.

$$\Pi_{\mathcal{T}_{\underline{\gamma}}}|_K = \pi_K, \quad K \in \mathcal{T}_{\underline{\gamma}},$$

where π_K is the interpolant from Proposition 5.1. Referring to Corollary 5.6, the following error bound for the DGFEM may be obtained:

$$\begin{aligned}
&\|\underline{u}_{\text{ex}} - \underline{u}_{\text{DG}}\|^2_{\text{DG}} \\
&\leqslant C C_{\mu,\lambda}\Big\{\mu^2\Big[\sum_{K\in\mathcal{T}_{\underline{\gamma}}}(h_K^{-2}\|\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}}\|^2_{L^2(K)} + |\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}}|^2_{H^1(K)}) \\
&\quad + \sum_{K\in\mathcal{T}_{\underline{\gamma}}\setminus\mathcal{K}_0} h_K^2 |\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}}|^2_{H^2(K)} + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta}|\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}}|^2_{H^{2,2}_{\underline{\beta}(K)}}\Big] \\
&\quad + \lambda^2\Big[\sum_{K\in\mathcal{T}_{\underline{\gamma}}}\|\nabla\cdot(\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}})\|^2_{L^2(K)} + \sum_{K\in\mathcal{T}_{\underline{\gamma}}\setminus\mathcal{K}_0} h_K^2|\nabla\cdot(\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}})|^2_{H^1(K)} \\
&\quad + \sum_{K\in\mathcal{K}_0} h_K^{2-2\beta}|\nabla\cdot(\underline{u}_{\text{ex}} - \pi_K \underline{u}_{\text{ex}})|^2_{H^{1,1}_{\underline{\beta}(K)}}\Big]\Big\}.
\end{aligned}$$

Moreover, inserting the interpolation error estimates from Proposition 5.2 into the above bound yields

$$
\begin{aligned}
&\|\underline{u}_{\mathrm{ex}}-\underline{u}_{\mathrm{DG}}\|_{\mathrm{DG}}^2\\
&\leqslant CC_{\mu,\lambda}\Big\{\mu^2\Big[\sum_{K\in\mathcal{T}_{\underline{\gamma}}\setminus\mathcal{K}_0} h_K^2|\underline{u}_{\mathrm{ex}}|^2_{H^2(K)}+\sum_{K\in\mathcal{K}_0} h_K^{2-2\beta}|\underline{u}_{\mathrm{ex}}|^2_{H^{2,2}_{\underline{\beta}}(K)}\Big]\\
&\quad+\lambda^2\Big[\sum_{K\in\mathcal{T}_{\underline{\gamma}}\setminus\mathcal{K}_0} h_K^2|\nabla\cdot\underline{u}_{\mathrm{ex}}|^2_{H^1(K)}+\sum_{K\in\mathcal{K}_0} h_K^{2-2\beta}|\nabla\cdot\underline{u}_{\mathrm{ex}}|^2_{H^{1,1}_{\underline{\beta}}(K)}\Big]\Big\}\\
&=CC_{\mu,\lambda}\Big\{\sum_{K\in\mathcal{T}_{\underline{\gamma}}\setminus\mathcal{K}_0} h_K^2(\mu^2|\underline{u}_{\mathrm{ex}}|^2_{H^2(K)}+\lambda^2|\nabla\cdot\underline{u}_{\mathrm{ex}}|^2_{H^1(K)})\\
&\quad+\sum_{K\in\mathcal{K}_0} h_K^{2-2\beta}(\mu^2|\underline{u}_{\mathrm{ex}}|^2_{H^{2,2}_{\underline{\beta}}(K)}+\lambda^2|\nabla\cdot\underline{u}_{\mathrm{ex}}|^2_{H^{1,1}_{\underline{\beta}}(K)})\Big\}.
\end{aligned}
\tag{5.12}
$$

Furthermore, from the definition of the $\underline{\gamma}$-graded meshes (Definition 5.8) it follows that

$$
\begin{aligned}
&\|\underline{u}_{\mathrm{ex}}-\underline{u}_{\mathrm{DG}}\|_{\mathrm{DG}}^2\\
&\leqslant CC_{\mu,\lambda}\Big\{h^2_{\mathcal{T}_{\underline{\gamma}}}\sum_{K\in\mathcal{T}_{\underline{\gamma}}\setminus\mathcal{K}_0}\int_K r^{2\gamma}(\mu^2|D^2\underline{u}_{\mathrm{ex}}|^2+\lambda^2|D^1(\nabla\cdot\underline{u}_{\mathrm{ex}})|^2)\,\mathrm{d}x\\
&+\sum_{K\in\mathcal{K}_0} h^{2-2\beta}_{\mathcal{T}_{\underline{\gamma}}}(\sup_{x\in K} r^{\gamma})^{2-2\beta}(\mu^2|\underline{u}_{\mathrm{ex}}|^2_{H^{2,2}_{\underline{\beta}}(K)}+\lambda^2|\nabla\cdot\underline{u}_{\mathrm{ex}}|^2_{H^{1,1}_{\underline{\beta}}(K)})\Big\}.
\end{aligned}
\tag{5.13}
$$

Clearly, for all $K\in\mathcal{K}_0$, there holds $r\leqslant h_K$. Hence,

$$
h_K\leqslant Ch_{\mathcal{T}_{\underline{\gamma}}}\sup_{x\in K} r^{\gamma}\leqslant Ch_{\mathcal{T}_{\underline{\gamma}}}h_K^{\gamma},
$$

and therefore

$$
h_K\leqslant Ch_{\mathcal{T}_{\underline{\gamma}}}^{\frac{1}{1-\gamma}}.
$$

This implies that

$$
\sup_{x\in K} r^{\gamma}\leqslant Ch_K^{\gamma}\leqslant Ch_{\mathcal{T}_{\underline{\gamma}}}^{\frac{\gamma}{1-\gamma}}\leqslant Ch_{\mathcal{T}_{\underline{\gamma}}}^{\frac{\beta}{1-\beta}}.
$$

Thus, (5.13) transforms to

$$
\begin{aligned}
&\|\underline{u}_{\mathrm{ex}}-\underline{u}_{\mathrm{DG}}\|_{\mathrm{DG}}^2\\
&\leqslant CC_{\mu,\lambda}h^2_{\mathcal{T}_{\underline{\gamma}}}\Big\{\sum_{K\in\mathcal{T}_{\underline{\gamma}}\setminus\mathcal{K}_0}\int_K r^{2\gamma}(\mu^2|D^2\underline{u}_{\mathrm{ex}}|^2+\lambda^2|D^1(\nabla\cdot\underline{u}_{\mathrm{ex}})|^2)\,\mathrm{d}x\\
&\quad+\sum_{K\in\mathcal{K}_0}(\mu^2|\underline{u}_{\mathrm{ex}}|^2_{H^{2,2}_{\underline{\beta}}(K)}+\lambda^2|\nabla\cdot\underline{u}_{\mathrm{ex}}|^2_{H^{1,1}_{\underline{\beta}}(K)})\Big\},
\end{aligned}
$$

and from the definition of the weight function $\Phi_{\underline{\beta}}$ (Section 3.1), it follows that

$$
\begin{aligned}
&\|\underline{u}_{\text{ex}} - \underline{u}_{\text{DG}}\|_{\text{DG}}^2 \\
&\leqslant CC_{\mu,\lambda} h_{\mathcal{T}_{\underline{\gamma}}}^2 \Big\{ \sum_{K \in \mathcal{T}_{\underline{\gamma}} \setminus \mathcal{K}_0} \int_K \Phi_{\underline{\beta}}^2 (\mu^2 |D^2 \underline{u}_{\text{ex}}|^2 + \lambda^2 |D^1(\nabla \cdot \underline{u}_{\text{ex}})|^2)\, \mathrm{d}x \\
&\quad + \sum_{K \in \mathcal{K}_0} (\mu^2 |\underline{u}_{\text{ex}}|^2_{H^{2,2}_{\underline{\beta}}(K)} + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|^2_{H^{1,1}_{\underline{\beta}}(K)}) \Big\} \\
&\leqslant CC_{\mu,\lambda} h_{\mathcal{T}_{\underline{\gamma}}}^2 \Big\{ \int_\Omega \Phi_{\underline{\beta}}^2 (\mu^2 |D^2 \underline{u}_{\text{ex}}|^2 + \lambda^2 |D^1(\nabla \cdot \underline{u}_{\text{ex}})|^2)\, \mathrm{d}x \\
&\quad + \sum_{K \in \mathcal{K}_0} (\mu^2 |\underline{u}_{\text{ex}}|^2_{H^{2,2}_{\underline{\beta}}(K)} + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|^2_{H^{1,1}_{\underline{\beta}}(K)}) \Big\} \\
&\leqslant CC_{\mu,\lambda} h_{\mathcal{T}_{\underline{\gamma}}}^2 (\mu^2 |\underline{u}_{\text{ex}}|^2_{H^{2,2}_{\underline{\beta}}(\Omega)} + \lambda^2 |\nabla \cdot \underline{u}_{\text{ex}}|^2_{H^{1,1}_{\underline{\beta}}(\Omega)}).
\end{aligned}
$$

Finally, by Lemma 5.9, i.e.

$$h_{\mathcal{T}_{\underline{\gamma}}} \leqslant CN^{-\frac{1}{2}},$$

and with the aid of the regularity result, Theorem 3.4, the proof is complete. □

REMARK 5.11 On uniform meshes $\mathcal{T}_{\underline{0}}$, it holds that

$$h_{\mathcal{T}_{\underline{0}}} \sim h_K \sim \frac{1}{\sqrt{N}} \qquad \forall K \in \mathcal{T}_{\underline{0}}.$$

Therefore, (5.12) directly implies that, even if $\underline{\gamma} = \underline{0}$, the DGFEM still converges independently of λ. However, due to the appearance of the term $h_K^{2-2\beta}$, the rate of convergence is no longer optimal for $\underline{\beta} \succ \underline{0}$.

6. Numerical results

The aim of this section is to confirm the previous theoretical results with some practical examples. More precisely, it will be shown that, even if the exact solutions of the corresponding problems are singular, the convergence rate of the DGFEM remains of order $\mathcal{O}(N^{-\frac{1}{2}})$ on $\underline{\gamma}$-graded meshes, as expected. Moreover, the robustness of the method against volume locking will be illustrated.

6.1 *L-shaped domain*

6.1.1 *Model problem.* Let Ω be the polygonal domain with vertices

$$A_1 = (0,0),\ A_2 = (-1,-1),\ A_3 = (1,-1),\ A_4 = (1,1),\ A_5 = (-1,1).$$

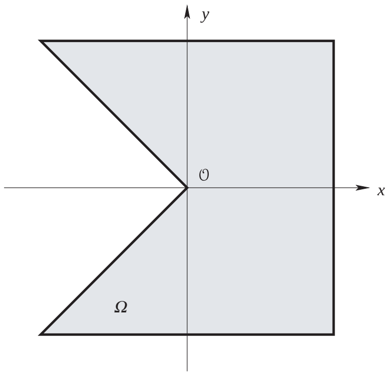

FIG. 1. Polygonal domain Ω.

Note that the origin $\mathcal{O} = (0, 0)$ is a re-entrant corner of Ω (see Fig. 1). Then, consider the following model problem:

$$\begin{aligned} -\nabla \cdot \underline{\underline{\sigma}}(\underline{u}) &= \underline{0} \quad \text{in} \quad \Omega \\ \underline{u} &= \underline{g}_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}} = \partial\Omega. \end{aligned} \tag{6.1}$$

Here, $\underline{g}_{\mathrm{D}} := \underline{u}_{\mathrm{ex}}|_{\Gamma_{\mathrm{D}}}$, where $\underline{u}_{\mathrm{ex}}$ is the exact solution of (6.1) given by its polar coordinates

$$u_r(r, \theta) = \frac{1}{2\mu} r^{\alpha}(-(\alpha + 1)\cos((\alpha + 1)\theta) + (C_2 - (\alpha + 1))C_1 \cos((\alpha - 1)\theta))$$

$$u_{\theta}(r, \theta) = \frac{1}{2\mu} r^{\alpha}((\alpha + 1)\sin((\alpha + 1)\theta) + (C_2 + \alpha - 1)C_1 \sin((\alpha - 1)\theta)),$$

where $\alpha \approx 0{\cdot}544\,484$ is the solution of the equation

$$\alpha \sin(2\omega) + \sin(2\omega\alpha) = 0$$

with $\omega = \frac{3\pi}{4}$, and

$$C_1 = -\frac{\cos((\alpha + 1)\omega)}{\cos((\alpha - 1)\omega)}, \qquad C_2 = \frac{2(\lambda + 2\mu)}{\lambda + \mu}.$$

6.1.2 *Robust optimal convergence rates on $\underline{\gamma}$-graded meshes.* A few calculations show that the exact solution $\underline{u}_{\mathrm{ex}}$ of the model problem (6.1) belongs to $H^{2,2}_{\underline{\beta}}(\Omega)^2$ with $\underline{\beta} = (\beta_1, 0, 0, 0, 0)$ for all $1 > \beta_1 > 1 - \alpha \approx 0{\cdot}455\,516$. Thus, in order to obtain the optimal convergence rate, a $\underline{\gamma}$-graded mesh with refinement towards the origin must be used for the numerical simulations.

Figure 4 shows the errors of the DGFEM for $\lambda \in \{1, 100, 500, 1000, 5000\}$ ($\mu = 1$) in the energy norm

$$\|\underline{u}\|^2_{\mathrm{DG}} = \sum_{K \in \mathcal{T}} \|\underline{\underline{\epsilon}}(\underline{u})\|^2_K + \frac{1}{m_{\mathrm{elast}}} \sum_{e \in \Gamma_{\mathrm{int,D}}} |e|^{-1} \int_e |[\underline{u}]_e|^2 \, \mathrm{d}s$$

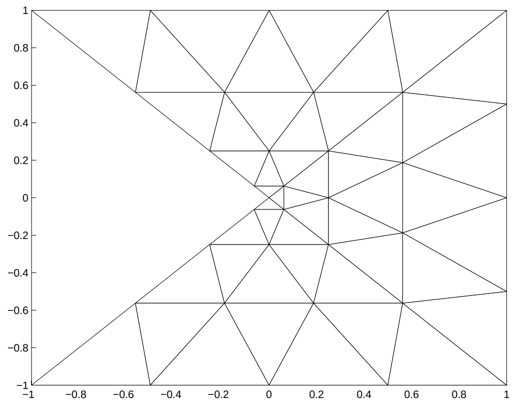

FIG. 2. $\underline{\gamma}$-graded mesh with refinement towards the origin ($\underline{\gamma} = (\frac{1}{2}, 0, 0, 0, 0)$).

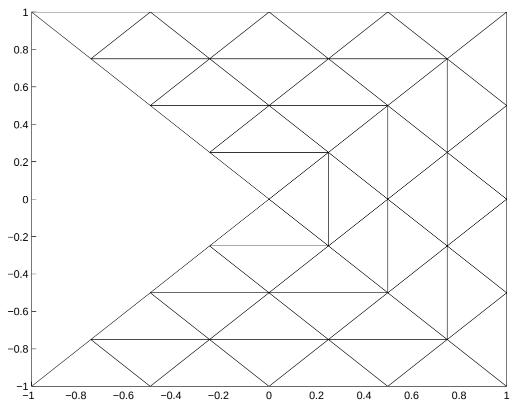

FIG. 3. Uniform mesh (i.e. $\underline{\gamma}$-graded mesh with $\underline{\gamma} = (0, 0, 0, 0, 0)$).

on a $\underline{\gamma}$-graded mesh with grading vector $\underline{\gamma} = (\frac{1}{2}, 0, 0, 0, 0)$ (see Fig. 2). Obviously, the convergence rate of the DGFEM is already almost optimal for approximately 5000 degrees of freedom (∼800 elements). Moreover, the expected robustness of the DGFEM with respect to the Lamé coefficient λ is clearly visible.

In Fig. 5 the energy error of the DGFEM on a uniform mesh (i.e. $\underline{\gamma} = (0, 0, 0, 0, 0)$) is presented. Although the DGFEM still converges robustly, the optimal convergence rate is no longer achieved (see Remark 5.11) and the use of $\underline{\gamma}$-graded meshes is found to be justified.

In addition, the L^2 errors for the computations above are shown in Figs 6 and 7. Again, the performance of the DGFEM on a uniform mesh is notably worse. However, the convergence rate of the L^2 error seems to be twice as high as of the energy error.

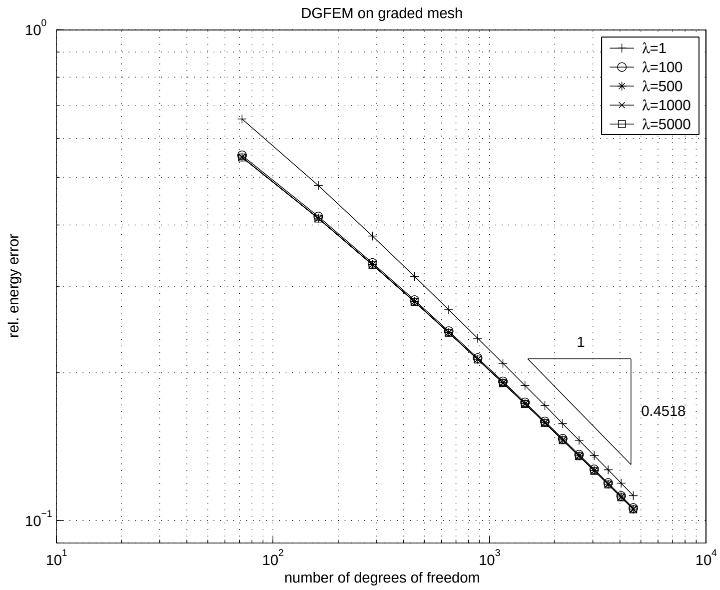

FIG. 4. Performance of the DGFEM on the L-shaped domain with $\underline{\gamma} = (\frac{1}{2}, 0, 0, 0, 0)$ ($\underline{\gamma}$-graded mesh).

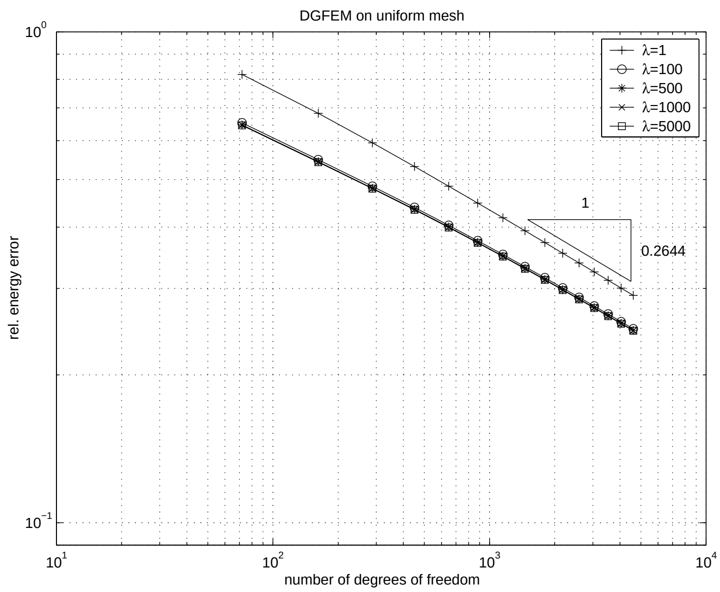

FIG. 5. Performance of the DGFEM on the L-shaped domain with $\underline{\gamma} = \underline{0}$ (uniform mesh).

6.1.3 *Volume locking.* Figures 8 and 9 show that the standard (i.e. conforming) FEM does not converge independently of λ. Although the asymptotic rate of convergence is optimal on $\underline{\gamma}$-graded meshes, the onset of the errors' decay is remarkably retarded for

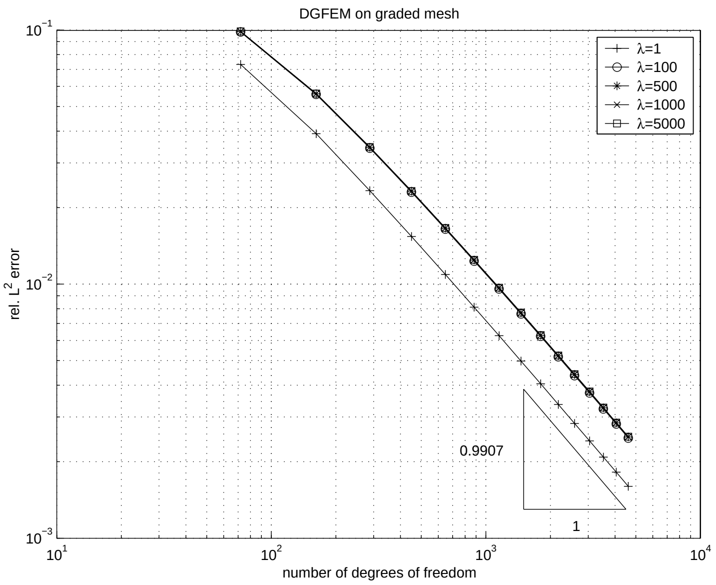

FIG. 6. Performance of the DGFEM on the L-shaped domain with $\underline{\gamma} = (\frac{1}{2}, 0, 0, 0, 0)$ ($\underline{\gamma}$-graded mesh).

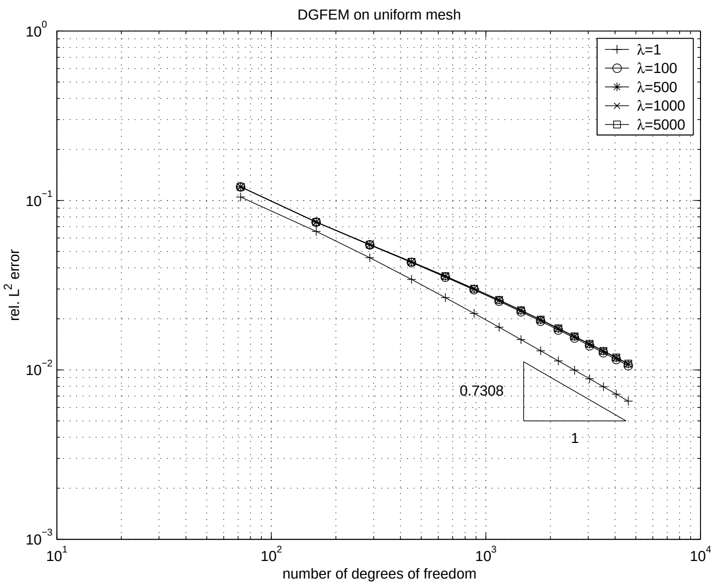

FIG. 7. Performance of the DGFEM on the L-shaped domain with $\underline{\gamma} = \underline{0}$ (uniform mesh).

$\lambda \to \infty$. This non-robustness of the convergence rate with respect to λ is widely known as 'volume locking' which, in contrast to the DGFEM, seems to be unavoidable for low-order standard h-FEMs in the primal variables. The initial ascent of the energy norm for large λ

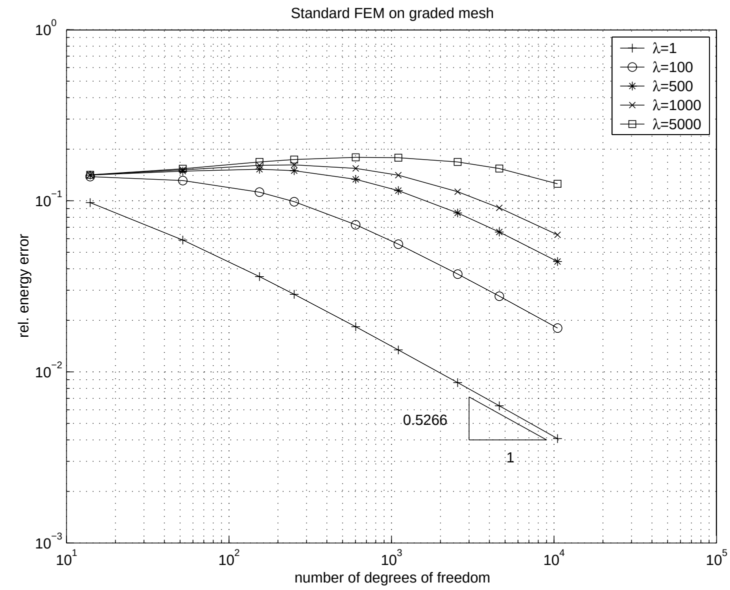

FIG. 8. Performance of the conforming FEM on the L-shaped domain with $\underline{\gamma} = \underline{0}$ (uniform mesh).

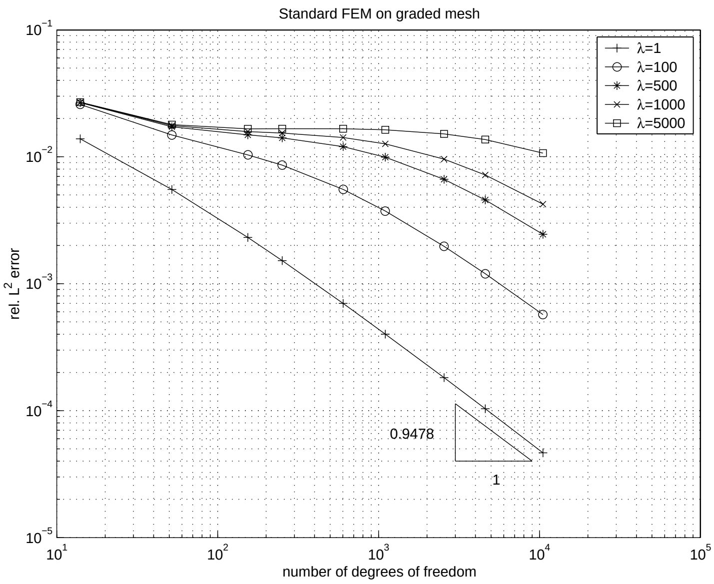

FIG. 9. Performance of the conforming FEM on the L-shaped domain with $\underline{\gamma} = (\frac{1}{2}, 0, 0, 0, 0)$ ($\underline{\gamma}$-graded mesh).

results from the fact that the finite element spaces are not nested due to the structure of the $\underline{\gamma}$-graded meshes.

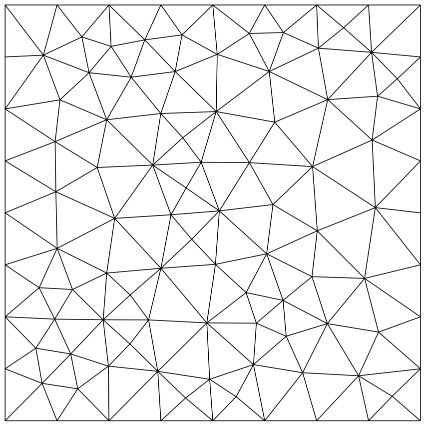

FIG. 10. Computational mesh.

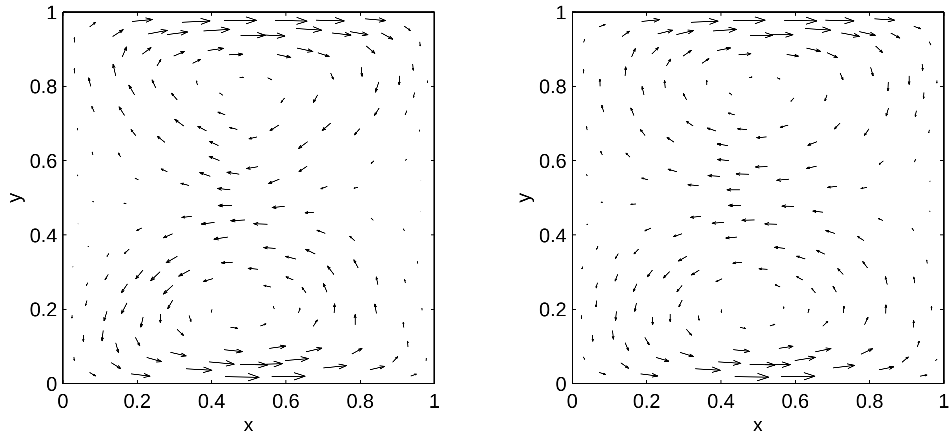

FIG. 11. Standard FEM/DGFEM for $\lambda = 100$.

6.2 *An example on the unit square*

Consider the following problem on $\Omega = (0,1)^2$:

$$\begin{aligned} -\nabla \cdot \underline{\underline{\sigma}}(\underline{u}) &= \underline{0} \quad \text{in} \quad \Omega \\ \underline{u} &= \binom{g_D^{(1)}}{0} \quad \text{on} \quad \Gamma_D = \partial\Omega \end{aligned} \tag{6.2}$$

with

$$g_D^{(1)}(x,y) = \begin{cases} 1 - 4(x - \frac{1}{2})^2 & \text{if } (x,y) \in (0,1) \times \{1\} \\ 0 & \text{else.} \end{cases}$$

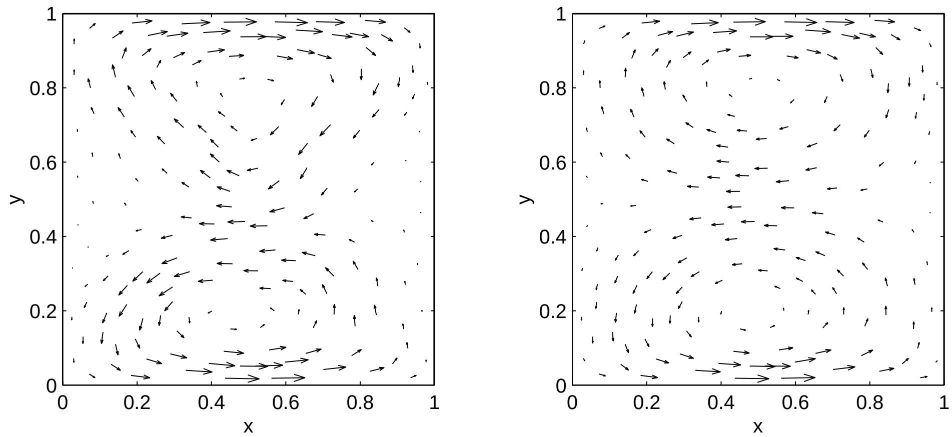

FIG. 12. Standard FEM/DGFEM for $\lambda = 500$.

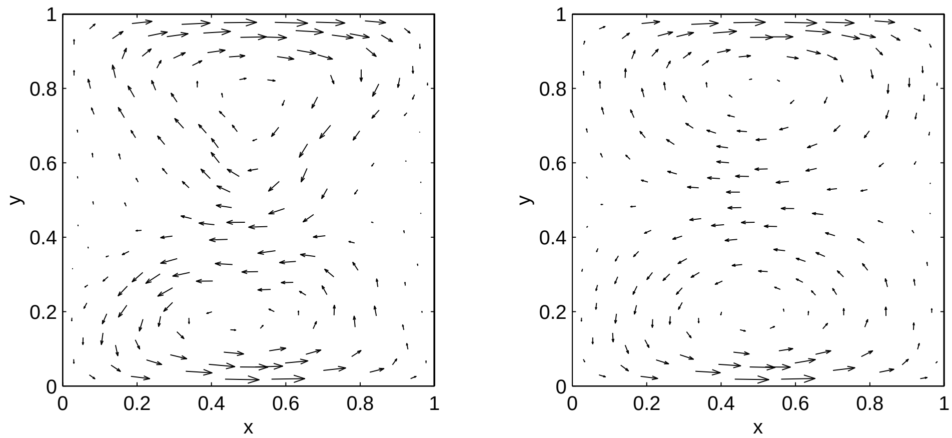

FIG. 13. Standard FEM/DGFEM for $\lambda = 1000$.

Due to Theorem 3.4, the exact solution of this problem belongs to $H^2(\Omega)^2$. Therefore, referring to the analysis above, no mesh refinement is required for the DGFEM to converge optimally. The computational (uniform) mesh is shown in Fig. 10. Additionally, the results for different choices of λ are presented (Figs 11–14). In contrast to the DGFEM, the standard FEM shows clear evidence of locking.

Acknowledgement

The author is indebted to C. Schwab (ETH, Zurich) for useful discussions and suggestions.

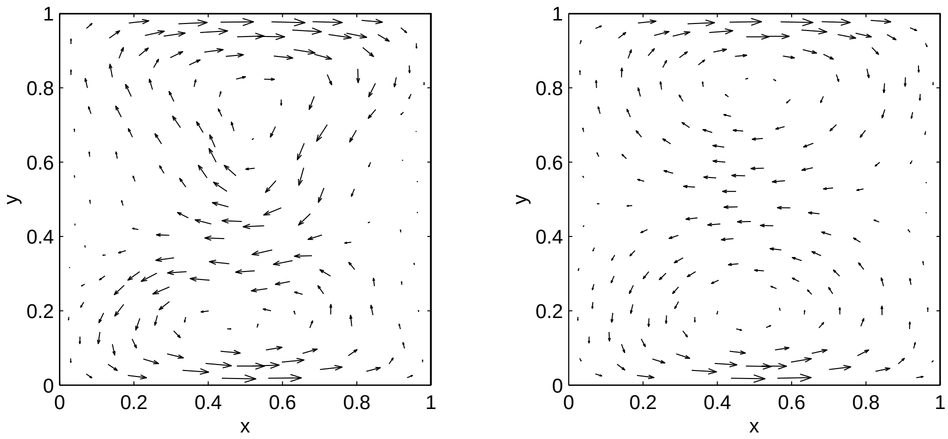

FIG. 14. Standard FEM/DGFEM for $\lambda = 5000$.

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Appendix A

LEMMA A.1 Let $I = [a, b]$, $a < b$ be an interval in $\mathbb{R}$ and $h_I = b - a$. Then, for every $u \in \mathcal{P}_1(I)$ it holds that

$$\|u\|_{L^\infty(I)} \leqslant 4\sqrt{2} h_I^{-\frac{1}{2}} \|u\|_{L^2(I)}.$$

Proof. See Quarteroni (1984). □

The proofs of the following lemmas may be found in Wihler (2003).

LEMMA A.2 Let $K \subset \mathbb{R}^2$ be a triangle with vertices A_1, A_2, A_3. Then, for each $\underline{u} \in H_\beta^{2,2}(K)^2$, where $\beta \in [0, 1)$ and $\Phi_\beta(x) = r^\beta = |x - A_1|^\beta$, there holds

$$\|\underline{u}\|^2_{H_\beta^{2,2}(K)} \leqslant C\Big(|\underline{u}|^2_{H_\beta^{2,2}(K)} + \sum_{e \in \mathcal{E}_K} \Big| \int_e \underline{u}\, \mathrm{d}s \Big|^2\Big).$$

Here, $C > 0$ is a constant (independent of $\underline{u}$) and $\mathcal{E}_K = \{e_1, e_2, e_3\}$ is the set of all edges of K.

LEMMA A.3 Let the assumptions of Lemma A.2 be satisfied. In addition, let

$$\int_K u\, \mathrm{d}x = 0.$$

Then, there holds

$$\|u\|_{L^2(K)} \leqslant C|u|_{H^{1,1}_\beta(K)},$$

where $C > 0$ is a constant independent of u.

LEMMA A.4 Let the assumptions of Lemma A.2 be satisfied. Then, the following inequalities hold true:

(a) $|\underline{u}|_{L^1(\partial K)} \leqslant C(\|\underline{u}\|_{L^2(K)} + h_K^{1-\beta}|\underline{u}|_{H^{1,1}_\beta(K)})$;

(b) $|\nabla\underline{u}|_{L^1(\partial K)} \leqslant C(|\underline{u}|_{H^1(K)} + h_K^{1-\beta}|\underline{u}|_{H^{2,2}_\beta(K)})$.