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Energy, angular momentum and wave action associated with density waves in a rotating magnetized gas disc

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ABSTRACT

Both fast and slow magnetohydrodynamic (MHD) density waves propagating in a thin rotating magnetized gas disc are investigated. In the tight-winding or WKBJ regime, the radial variation of MHD density-wave amplitude during wave propagation is governed by the conservation of wave action surface density $\mathcal{N}$ which travels at a relevant radial group speed C_g. The wave energy surface density $\mathcal{E}$ and the wave angular momentum surface density $\mathcal{J}$ are related to $\mathcal{N}$ by $\mathcal{E} = \omega\mathcal{N}$ and $\mathcal{J} = m\mathcal{N}$ respectively, where ω is the angular frequency in an inertial frame of reference and the integer m, proportional to the azimuthal wavenumber, corresponds to the number of spiral arms. Consequently, both wave energy and angular momentum are conserved for spiral MHD density waves. For both fast and slow MHD density waves, net wave energy and angular momentum are carried outward or inward for trailing or leading spirals, respectively. The wave angular momentum flux contains separate contributions from gravity torque, advective transport and magnetic torque. While the gravity torque plays an important role, the latter two can be of comparable magnitudes to the former. Similar to the role of gravity torque, the part of MHD wave angular momentum flux by magnetic torque (in the case of either fast or slow MHD density waves) propagates outward or inward for trailing or leading spirals, respectively. From the perspective of global energetics in a magnetized gas sheet in rotation, trailing spiral structures of MHD density waves are preferred over leading ones. With proper qualifications, the generation and maintenance as well as transport properties of MHD density waves in magnetized spiral galaxies are discussed.

Key words: gravitation – MHD – waves – ISM: general – galaxies: magnetic fields – galaxies: spiral.

1 INTRODUCTION

Large-scale optical spiral structures of disc galaxies are generally understood, as widely accepted, as spiral density-wave patterns (Lin & Shu 1964, 1966; Goldreich & Lynden-Bell 1965; Toomre 1964, 1969, 1981; Binney & Tremaine 1987; Lin 1987; Bertin & Lin 1996). High-resolution radio-continuum observations of synchrotron emissions from nearby spiral galaxies (Sofue, Fujimoto & Wielebinski 1986; Beck & Hoernes 1996; Beck et al. 1996; Zweibel & Heiles 1997) have prompted a series of theoretical studies on large-scale magnetohydrodynamic (MHD) density waves (Fan & Lou 1996, 1997; Lou & Fan 1997, 1998a, in preparation), with magnetic fields playing important dynamic roles in a thin rotating thermal gas disc.[1] MHD density waves in the context of spiral galaxies have been studied decades ago by Lynden-Bell (1966) and Roberts & Yuan (1970). Substantial progress has also been made recently to incorporate large-scale effects of cosmic ray gas in the MHD-density-wave scenario (Lou & Fan 1999, in preparation) by extending the basic formalism first explored by Parker (1965).

It has been shown, from the WKBJ dispersion relation, that a rotating thermal gas disc embedded with an azimuthal magnetic field can support both large-scale fast and slow MHD density-wave modes (Fan & Lou 1996). The fast MHD density-wave mode can appear globally in the disc region of almost rigid rotation, as well as in the disc region with an almost flat rotation curve. In contrast, the slow MHD density-wave mode can manifest

[1] Within a composite disc system, large-scale spiral density-wave structures in the stellar disc and the magnetized thermal gas disc are coupled mainly through the mutual gravitational interaction (Lou & Fan 1997, 1998b).

over a relatively large radial range only in the almost rigidly rotating disc portion (Lou & Fan 1998a, in preparation). It has been demonstrated that while fast MHD density waves have the enhancement of azimuthal magnetic field perturbations well correlated with that of surface mass density perturbations, there is a significant phase shift (i.e., a phase difference $\gtrsim\pi/2$) between the two perturbation enhancements associated with slow MHD density waves. Based on this, we proposed that the observed 'anticorrelation' between optical spiral arms and the polarized radio-continuum emission spiral arms in the nearby galaxy NGC 6946 (Beck & Hoernes 1996; Beck et al. 1996) can be actually caused by the dominant presence of slow MHD density waves (Fan & Lou 1996; Lou & Fan 1998a), because optical and polarized radio-continuum emission arms occupy the inner disc region[2] where the rotation curve is gradually rising, i.e., the inner disc rotation is almost rigid (cf. Kormendy & Norman 1979; Carignan et al. 1990). We also identified the overall in-phase correlation of large-scale optical and radio-continuum spiral structures of the 'Whirlpool galaxy' M51 (NGC 5194) as a salient manifestation of fast MHD density waves (Roberts & Yuan 1970; Mathewson, van der Kruit & Brouw 1972; Neininger 1992; Neininger & Horellou 1996; Neininger 1998, private communications). Another important nearby galaxy showing features of fast MHD density waves is the Andromeda nebula (also referred to as M31 or NGC 224; Beck, Berkhuijsen & Wielebinski 1980; Koper 1993, and extensive references therein; Beck, Berkhuijsen & Hoernes 1998; Hoernes, Berkhuijsen & Xu 1998; Lou & Fan 1998a, 1999). Recent polarized radio-continuum observations of synchrotron emissions at 6.0, 3.5 and 13 cm wavelengths from the grand-design spiral galaxy NGC 2997 (Han et al. 1999 preprint) in the southern sky appear to reveal another example of large-scale fast MHD density waves in a disc galaxy; in particular, the isolated 'magnetic arm' (in the sense without accompanying optical features; cf. Block et al. 1994a,b) in the southeast quadrant is predicted to be associated with a neutral hydrogen arm (Lou, Han & Fan 1999) which can be verified by forthcoming 21 cm HI observations. From the perspective of swing amplification (Goldreich & Lynden-Bell 1965; Toomre 1981), it has also been shown (Fan & Lou 1997) that fast MHD density waves tend to grow in the disc portion with a strong differential rotation, while the growth of slow MHD density waves is preferentially favoured in the disc portion with a weak differential rotation. We note that large-scale galactic structures of M51, M31 and NGC 2997 indeed occupy disc portions with more or less flat rotation curves.

It is an important first step forward to derive the WKBJ dispersion relations of fast and slow MHD density waves (Fan & Lou 1996; Lou & Fan 1998a) in a magnetized gas disc. A more complete analysis of the MHD-density-wave scenario requires a comprehensive consideration for several closely relevant aspects. Among others, observations indicate that most spiral galaxies have trailing spiral arms (e.g. Pasha 1985), i.e., the extension direction of spiral arms is opposite to the sense of galactic disc rotation. In order to understand this statisitical prevalence of trailing spirals, one must go beyond the local dispersion relation, because both trailing and leading density waves are allowed on an equal footing, as implied by the WKBJ dispersion relation. In fact, the earlier antispiral theorem states that if a steady-state solution of a time-reversible set of equations has the form of a trailing spiral, there must be also an identical solution in the form of a leading spiral (Lynden-Bell & Ostriker 1967). It is possible to get around this antispiral theorem. For example, spirals may arise from unstable normal modes; in that case, the system is not really in a steady state. Alternatively, the spiral formation can be influenced by dissipative processes that are not time-reversible (Binney & Tremaine 1987; Bertin & Lin 1996). In the framework of density-wave theory (without magnetic fields), Lynden-Bell & Kalnajs (1972) have shown that trailing spiral waves transport energy and angular momentum radially outward by the dominant gravity torque, and thus lower the rotational kinetic energy of a stellar disc. The disc rotational energy thus extracted is subsequently converted into the kinetic energy of random motions of stars as a result of the resonant interaction between density waves and stars occurred at the Lindblad resonances; eventually, the system reaches an energetically more stable configuration. This scenario offers a natural interpretation for the prevailing appearance of trailing spiral structures in disc galaxies. In essence, this interpretation involves the energy transfer from systematic disc rotation to random stellar motions, which is in fact a time-irreversible process.

Along a separate line of attack, the search for globally unstable normal modes in a thin rotating disc has been systematically carried out (Lau, Lin & Mark 1976; Lau & Bertin 1978; Bertin et al. 1989a,b; Bertin & Lin 1996) in the so-called modal formalism for self-excited density waves (without magnetic fields). The existence of unstable normal modes for given rotation curves (Kent 1986, 1987) can be physically understood as follows. A short trailing wave packet propagates inwards inside corotation (where the pattern speed $\Omega_p \equiv \omega/m$ is locally equal to the disc angular rotation speed Ω). Under appropriate conditions, there exists a so-called Q-barrier which shields the inner Lindblad resonance from exposure to incoming waves and reflects the incoming short trailing wave packet back in the form of a long trailing wave packet. As the long trailing wave packet propagates outward towards corotation, the wave packet is partially transmitted across the corotation region as short trailing waves, and the rest is reflected back to become short trailing waves. As the wave action inside and outside corotation is respectively negative and positive, the reflected short trailing waves are amplified as a result of the wave action conservation (cf. Bertin & Lin 1996). As this basic process repeats itself, an unstable (in the sense that the amplitude of the short trailing wave inside corotation increases after each cycle) normal density-wave mode appears. A necessary condition for amplifying a normal mode is stipulated by the radiation condition that no waves should propagate towards corotation from outside. Such a radiation condition can only be implemented by trailing waves, and this provides an alternative explanation for the prevalence of trailing spiral structures in disc galaxies. It is noted that the argument for trailing waves in the modal approach by Bertin & Lin (1996) and the argument given by Lynden-Bell & Kalnajs (1972) rely on the same physical mechanism, namely the outward transport of angular momentum (or energy) associated with *trailing* density waves.

With the foregoing information in mind, there are several purposes to carry out the analysis presented in this paper. First, in the context of a magnetized gas disc in rotation, we would like systematically to extend density-wave results regarding the

[2] A recent wavelet analysis of Fricke et al. (1998 preprint) indicates that magnetic arms of NGC 6946 might extend far into the outer disc portion with strong differential rotation. It would be difficult to accommodate their results in the current framework.

conservation and transport of energy, angular momentum, and wave action to the corresponding fast and slow MHD-density-wave results. While conceptually clear, these generalizations are themselves non-trivial and reveal specific physical properties of MHD density waves valid in a rotating magnetized gas disc. Applications of these theoretical results include, but are not limited to, magnetized spiral galaxies. Secondly, in the context of magnetized spiral galaxies, it would be more realistic and sensible to consider the above theoretical problem for large-scale MHD density waves in a thin *composite* disc system consisting of a stellar disc and a magnetized gas disc coupled through the mutual gravitational interaction. By simple estimates, one can readily show (Lou & Fan 1998b) that gravity torques associated with stellar and gas surface mass density *perturbations* can be comparable in magnitudes, even though the background stellar disc is more massive than the background gas disc. Generally speaking, gravity torque and advective transport effects (with the former more effective) can be comparable in orders of magnitude associated with stellar and gas discs respectively. As the magnetic torque is of the same order of magnitude as the gravity torque and advective transport in a magnetized gas disc, the present MHD analysis indeed carries a considerable practical significance. To avoid an overkill (in terms of mathematical analysis) and to grasp the essential ideas at the present stage, it is fairly sufficient to apply qualitatively the results obtained in a magnetized gas disc alone *together* with the well-known density-wave results in a stellar disc (Lynden-Bell & Kalnajs 1972; Goldreich & Tremaine 1979) for a more satisfactory and consistent interpretation of trailing spiral arms observed in various electromagnetic wavebands. Thirdly, analogous to hydrodynamic density waves, it is important to discuss the conceptual framework for the excitation and maintenance of MHD density waves in a rotating magnetized gas disc (Fan & Lou 1997). Finally, the theoretical results derived here with adaptations can be applied to magnetized gas accretion discs in various astrophysical contexts. In particular, trailing spiral MHD density waves (both fast and slow) in the presence of resonances can be effective means of removing disc angular momentum.

The main theme of this paper is to analyse the propagation of angular momentum, energy and wave action associated with both fast and slow MHD density waves. We present, in Section 2, the formulation of MHD density waves. In Section 3 the WKBJ analysis of MHD density waves is described. The expressions for angular momentum and energy fluxes of fast and slow MHD density waves are derived in Section 4 in the WKBJ regime. In Section 5 we calculate the wave angular momentum density, the energy density and the wave action density. In Section 6 we discuss the statistical prevalence of magnetized spiral galaxies. Summary and discussion are contained in Section 7. Finally, some mathematical details are presented in Appendix A for the convenience of reference.

2 THE BASIC FORMALISM

For large-scale spiral MHD density waves, our formalism starts from a rotating magnetized gas sheet embedded with an azimuthal magnetic field (Fan & Lou 1996). For reasons similar to those stated in Lou & Fan (1998a), the investigation here does not include the more massive stellar disc component in order to show clearly the main features of fast and slow MHD density waves without unnecessary mathematical complications. A great deal of new information can be gleaned from this simple model, and it is possible to incorporate the gravitational influence of density waves in the stellar disc component (Lou & Fan 1997, 1998b, in preparation). In such a composite system consisting of stellar and magnetized gaseous discs, the basic features of fast and slow MHD density waves persist in the gaseous disc, with density enhancements in the stellar and gaseous discs tracking each other *within* the Lindblad resonances (Lou & Fan 1997). In terms of wave angular momentum transport, we will provide estimates for stellar and gas discs presently in Section 6.

The basic MHD equations governing a rotating magnetized gas sheet are the mass conservation equation, the momentum equation, the magnetic induction equation, the divergence-free condition for the magnetic field $\boldsymbol{B}$, and Poisson's equation for the negative gravitational potential Φ. These equations read

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{V}) = 0, \tag{2.1}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = -\frac{\nabla P}{\rho} + \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi\rho} + \nabla\Phi, \tag{2.2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{V}\times\boldsymbol{B}), \tag{2.3}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{2.4}$$

$$\nabla^2\Phi = -4\pi G(\rho + \rho_{\mathrm{R}}), \tag{2.5}$$

where ρ is the thermal gas mass density, $\boldsymbol{V}$ is the gas flow velocity, P is the thermal gas pressure, G is the universal gravitational constant, and ρ_{R} is the remaining mass density (i.e., that of halo stars and dark matter together in a typical galactic system) which contributes only to the background disc equilibrium but does not directly take part in MHD-density-wave perturbations.

We shall consider MHD-density-wave perturbations in such a thin magnetized gas disc using the cylindrical coordinate system (r, θ, z). The disc rotation is described by $\boldsymbol{V}_0 = (0, V_0, 0)$, where $V_0 = \Omega r$ is the linear rotation speed, and Ω is the angular rotation speed. A background azimuthal magnetic field is described by $\boldsymbol{B}_0 = (0, B_0, 0)$, where $B_0 = F_0/r$, and F_0 is a constant. For simplicity, we assume a polytropic relation between P and ρ, namely $P = K\rho^\gamma$ with constant proportion coefficient K and polytropic index γ. The pressure force term $\nabla P/\rho$ on the right-hand side of equation (2.2) can be written as ∇H, with $H \equiv \gamma P/[(\gamma-1)\rho]$ being the enthalpy (cf. Binney & Tremaine 1987). We consider MHD perturbations tangential to the thin disc; components of velocity and magnetic field perturbations perpendicular to the disc can be set to zero consistently. MHD-density-wave equations corresponding to equations (2.1)–(2.5) are then:

$$\frac{\partial \mu}{\partial t} + \frac{1}{r}\frac{\partial(\mu_0 r v_r)}{\partial r} + \frac{\mu_0}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{V_0}{r}\frac{\partial \mu}{\partial \theta} = 0, \tag{2.6}$$

$$\frac{\partial v_r}{\partial t} - \frac{2V_0 v_\theta}{r} + \frac{V_0}{r}\frac{\partial v_r}{\partial \theta} = -\frac{\partial h}{\partial r} - \frac{B_0 \tau}{4\pi\mu_0 r}\left[\frac{\partial(r b_\theta)}{\partial r} - \frac{\partial b_r}{\partial \theta}\right] + \frac{\partial \phi}{\partial r}, \tag{2.7}$$

$$\frac{\partial v_\theta}{\partial t} + \frac{V_0}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r}\frac{\mathrm{d}(rV_0)}{\mathrm{d}r} v_r = -\frac{1}{r}\frac{\partial h}{\partial \theta} + \frac{1}{\mathrm{r}}\frac{\partial \phi}{\partial \theta}, \tag{2.8}$$

$$\frac{\partial b_r}{\partial t}=-\frac{1}{r}\frac{\partial(V_0 b_r)}{\partial\theta}+\frac{1}{r}\frac{\partial(B_0 v_r)}{\partial\theta}, \tag{2.9}$$

$$\frac{\partial b_\theta}{\partial t}=\frac{\partial(V_0 b_r)}{\partial r}-\frac{\partial(B_0 v_r)}{\partial r}, \tag{2.10}$$

$$\frac{1}{r}\frac{\partial(r b_r)}{\partial r}+\frac{1}{r}\frac{\partial b_\theta}{\partial\theta}=0, \tag{2.11}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=-4\pi G\mu(r,\theta)\delta(z), \tag{2.12}$$

where μ_0 is the background gas surface mass density, μ is the perturbation in μ_0, τ is the thickness of the gas disc, v_r and v_θ are the radial and azimuthal velocity perturbations, b_r and b_θ are the radial and azimuthal magnetic field perturbations, $h=C_{\rm S}^2\mu/\mu_0$ is the enthalpy perturbation, where $C_{\rm S}^2\equiv\gamma K\mu_0^{\gamma-1}$ defines the thermal sound speed $C_{\rm S}$, ϕ is the perturbation in Φ due to the mass density perturbation $\mu(r,\theta)\delta(z)$, and $\delta(z)$ is the Dirac delta function in z.

Since the background gas disc is axisymmetric and steady, we consider an MHD-density-wave mode for which all perturbation variables carry the same $\exp(\mathrm{i}\omega t-\mathrm{i}m\theta)$ dependence, where ω is the angular frequency in an inertial frame of reference, and the integer m, which relates to the azimuthal wavenumber m/r, specifies the number of spiral arms. It follows from equations (2.6)–(2.12) that

$$\mathrm{i}(\omega-m\Omega)\mu+\frac{1}{r}\frac{\partial(r\mu_0 v_r)}{\partial r}-\frac{\mathrm{i}m\mu_0}{r}v_\theta=0, \tag{2.13}$$

$$\mathrm{i}(\omega-m\Omega)v_r-2\Omega v_\theta=-\frac{B_0\tau}{4\pi\mu_0 r}\left[\frac{\partial(r b_\theta)}{\partial r}+\mathrm{i}m b_r\right]+\frac{\partial(\phi-h)}{\partial r}, \tag{2.14}$$

$$\mathrm{i}(\omega-m\Omega)v_\theta+\frac{1}{r}\frac{\mathrm{d}(rV_0)}{\mathrm{d}r}v_r=-\frac{\mathrm{i}m}{r}(\phi-h), \tag{2.15}$$

$$\mathrm{i}(\omega-m\Omega)b_r=-\frac{\mathrm{i}m}{r}B_0 v_r, \tag{2.16}$$

$$\mathrm{i}\omega b_\theta=\frac{\partial(V_0 b_r)}{\partial r}-\frac{\partial(B_0 v_r)}{\partial r}, \tag{2.17}$$

$$b_\theta=-\frac{\mathrm{i}}{m}\frac{\partial(r b_r)}{\partial r}, \tag{2.18}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)-\frac{m^2}{r^2}\phi+\frac{\partial^2\phi}{\partial z^2}=-4\pi G\mu(r,\theta)\delta(z). \tag{2.19}$$

We derive the WKBJ results for fast and slow MHD density waves from equations (2.13)–(2.19) in the following sections.

3 TIGHT-WINDING SPIRAL MHD DENSITY WAVES

As an example for the adopted convention of notations, the solution for the surface mass density perturbation μ from equations (2.13)–(2.19) can be written in the form

$$\mu=\tilde{\mu}(r)\exp\left[\mathrm{i}\int^r k(s)\mathrm{d}s\right]\exp(\mathrm{i}\omega t-\mathrm{i}m\theta), \tag{3.1}$$

where $k(r)$ is the local radial wavenumber, and $\tilde{\mu}(r)$ is the complex amplitude of $\mu(r)$. The tight-winding or WKBJ approximation assumes that both the perturbation magnitude $\tilde{\mu}(r)$ and the radial wavenumber $k(r)$ vary slowly on radial spatial scales much larger than k^{-1} and $kr\gg m$. The local solution to Poisson's equation (2.19) is approximately given by (Shu 1970; Goldreich & Tremaine 1979; Lin & Lau 1979)

$$\frac{1}{r^{1/2}}\frac{\partial(r^{1/2}\phi)}{\partial r}\Big|_{z=0}\approx 2\pi G\,\mathrm{i}\,\mathrm{sgn}(k)\mu, \tag{3.2}$$

where the fractional error is in the order of $O(kr)^{-2}$. By the sign convention here, $k>0$ and $k<0$ correspond to leading and trailing spirals respectively. With this local approximation and to the leading order of large kr, the dispersion relations of fast and slow MHD density waves can be derived from equations (2.13)–(2.18) (Fan & Lou 1996). For fast MHD density waves in a disc with either weak or strong differential rotation, the dispersion relation is given by

$$(\omega-m\Omega)^2\approx\kappa^2+k^2\left(C_{\rm S}^2+C_{\rm A}^2-\frac{2\pi G\mu_0}{|k|}\right), \tag{3.3}$$

where $C_{\rm S}$ is the thermal sound speed, $C_{\rm A}\equiv[B_0^2\tau/(4\pi\mu_0)]^{1/2}$ is the Alfvén wave speed, and $\kappa\equiv[(2\Omega/r)\mathrm{d}(rV_0)/\mathrm{d}r]^{1/2}$ is the epicyclic frequency. In the disc portion with an almost rigid rotation, that is $|\mathrm{d}\Omega/\mathrm{d}r|\ll\Omega/r$, slow MHD density waves can appear in an extended radial range. The dispersion relation for slow MHD density waves is given by

$$(\omega-m\Omega)^2\approx\frac{k^2C_{\rm A}^2(C_{\rm S}^2-2\pi G\mu_0/|k|)m^2/r^2}{\kappa^2+k^2(C_{\rm A}^2+C_{\rm S}^2-2\pi G\mu_0/|k|)}. \tag{3.4}$$

Properties of fast and slow MHD density waves have been extensively discussed by Lou & Fan (1998a, in preparation). We note again that for fast MHD density waves, perturbation enhancements of surface mass density μ and azimuthal magnetic field b_θ are roughly in phase, whereas the two perturbation enhancements are significantly phase-shifted for slow MHD density waves. For the convenience of reference, the corresponding expressions for the radial group speed $C_{\rm g}=-\partial\omega/\partial k$ are given below. For fast MHD density waves,

$$C_{\rm g}^{\rm F}=-\frac{k\left(C_{\rm S}^2+C_{\rm A}^2-\pi G\mu_0/|k|\right)}{(\omega-m\Omega)}, \tag{3.5}$$

where the superscript F over $C_{\rm g}$ indicates the association with fast MHD density waves, and for slow MHD density waves,

$$C_{\rm g}^{\rm S}=-\frac{kC_{\rm A}^2m^2/r^2\left[\kappa^2\left(C_{\rm S}^2-\pi G\mu_0/|k|\right)+C_{\rm A}^2\pi G\mu_0|k|\right]}{\tilde{\omega}\left[\kappa^2+k^2\left(C_{\rm A}^2+\Delta\right)\right]^2}, \tag{3.6}$$

where $\tilde{\omega}\equiv\omega-m\Omega$, $\Delta\equiv C_{\rm S}^2-2\pi G\mu_0/|k|$, and the superscript S over $C_{\rm g}$ indicates the association with slow MHD density waves.

In the magnitude order next to the leading order of large kr, equations (2.13)–(2.18) and (3.2) contain information for the variation of perturbation magnitudes. The relevant derivations are tedious but straightforward. We simply state the results here. For fast MHD density waves, one has

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r\mu_0 k}{\tilde{\omega}^2}\left(C_\mathrm{S}^2+C_\mathrm{A}^2-\frac{\pi G\mu_0}{|k|}\right)|\tilde{v}_r|^2\right]=0, \tag{3.7}$$

given a relatively general disc rotation curve, whereas for slow MHD density waves, one has

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{\left[kr\mu_0\left(\frac{\Delta}{\tilde{\omega}^2-m^2\Delta/r^2}+\frac{C_\mathrm{A}^2}{\tilde{\omega}^2}\right)+\frac{\mathrm{sgn}(k)r}{4\pi G}\left(\frac{2\pi G}{|k|}\right)^2\frac{\left(m^2\kappa^4\mu_0^2\right)/(4\Omega^2r^2)+\tilde{\omega}^2k^2\mu_0^2}{[\tilde{\omega}^2-(m^2/r^2)\Delta]^2}\right]|\tilde{v}_r|^2\right\}=0 \tag{3.8}$$

in the disc portion with an almost rigid rotation. As will be apparent in the next section, equations (3.7) and (3.8) actually represent the flux conservation of angular momentum, energy and wave action associated with fast and slow MHD density waves respectively.

4 ANGULAR MOMENTUM AND ENERGY FLUXES

From the mass conservation equation (2.1), the momentum equation (2.2), the magnetic induction equation (2.3) and Poisson's equation (2.5), the general law of angular momentum conservation is given by

$$\frac{\partial(\rho\boldsymbol{r}\times\boldsymbol{V})}{\partial t}+\nabla\cdot\left[\rho\boldsymbol{V}(\boldsymbol{r}\times\boldsymbol{V})+P(\boldsymbol{r}\times\hat{\boldsymbol{I}})+\frac{|\boldsymbol{B}|^2(\boldsymbol{r}\times\hat{\boldsymbol{I}})}{8\pi}-\frac{\boldsymbol{B}(\boldsymbol{r}\times\boldsymbol{B})}{4\pi}+\frac{\nabla\Phi(\boldsymbol{r}\times\nabla\Phi)}{4\pi G}-\frac{|\nabla\Phi|^2(\boldsymbol{r}\times\hat{\boldsymbol{I}})}{8\pi G}\right]=0, \tag{4.1}$$

while the general law of energy conservation is given by

$$\frac{\partial}{\partial t}\left(\frac{\rho|\boldsymbol{V}|^2}{2}-\frac{|\nabla\Phi|^2}{8\pi G}+\frac{|\boldsymbol{B}|^2}{8\pi}+\frac{P}{\gamma-1}\right)+\nabla\cdot\left[\frac{\rho|\boldsymbol{V}|^2}{2}\boldsymbol{V}+\frac{\gamma P\boldsymbol{V}}{(\gamma-1)}-\rho\Phi\boldsymbol{V}+\frac{\Phi}{4\pi G}\nabla\frac{\partial\Phi}{\partial t}+\frac{\boldsymbol{B}\times(\boldsymbol{V}\times\boldsymbol{B})}{4\pi}\right]=0, \tag{4.2}$$

where $\hat{\boldsymbol{I}}$ is the unit tensor $\{\delta_{ij}\}$. From these two conservation equations (4.1) and (4.2), one can readily identify the angular momentum density and flux density, and the energy density and flux density associated with MHD density waves in the current context.

4.1 Angular momentum fluxes of fast and slow MHD density waves

It follows from equation (4.1) that the angular momentum flux across the surface of an infinite cylinder of radius r is given by

$$\begin{aligned}\boldsymbol{F}_\mathrm{J}=&\int\mathrm{d}\boldsymbol{S}\cdot\left[\rho\boldsymbol{V}(\boldsymbol{r}\times\boldsymbol{V})+P(\boldsymbol{r}\times\hat{\boldsymbol{I}})+\frac{|\boldsymbol{B}|^2(\boldsymbol{r}\times\hat{\boldsymbol{I}})}{8\pi}-\frac{\boldsymbol{B}(\boldsymbol{r}\times\boldsymbol{B})}{4\pi}+\frac{\nabla\Phi(\boldsymbol{r}\times\nabla\Phi)}{4\pi G}-\frac{|\nabla\Phi|^2(\boldsymbol{r}\times\hat{\boldsymbol{I}})}{8\pi G}\right]\\=&\int_{-\infty}^{+\infty}\mathrm{d}zr\int_0^{2\pi}\mathrm{d}\theta\hat{\boldsymbol{r}}\cdot\left[\rho\boldsymbol{V}(\boldsymbol{r}\times\boldsymbol{V})-\frac{\boldsymbol{B}(\boldsymbol{r}\times\boldsymbol{B})}{4\pi}+\frac{\nabla\Phi(\boldsymbol{r}\times\nabla\Phi)}{4\pi G}\right],\end{aligned} \tag{4.3}$$

where the three terms in the last equality correspond to the angular momentum transfer by three distinctly different physical mechanisms, namely, the advective transport, the magnetic torque and the gravity torque. To the first order of perturbations, one would have $\boldsymbol{F}_\mathrm{J}=0$ due to the azimuthal periodicity contained in the exp $(-\mathrm{i}m\theta)$ dependence of all relevant perturbation variables. To the second order of perturbations, the z-component of $\boldsymbol{F}_\mathrm{J}$ (now denoted by $\mathcal{F}_\mathrm{J}$) associated with MHD density waves is

$$\begin{aligned}\mathcal{F}_\mathrm{J}=&\int_{-\infty}^{+\infty}\mathrm{d}zr\int_0^{2\pi}\mathrm{d}\theta\Big[\rho_0v_r(rv_\theta)+\rho v_r(rV_0)+\rho_0v_r^{(2)}(rV_0)\\&-\frac{b_r(rb_\theta)}{4\pi}-\frac{b_r^{(2)}(rB_0)}{4\pi}+\frac{1}{4\pi G}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta}+\frac{1}{4\pi G}\frac{\partial\Phi_0}{\partial r}\frac{\partial\phi^{(2)}}{\partial\theta}\Big],\end{aligned} \tag{4.4a}$$

where the superscript $^{(2)}$ indicates a second-order perturbation quantity, and we have made use of the facts that $\boldsymbol{V}_0=V_0\hat{\theta},\boldsymbol{B}_0=B_0\hat{\theta}$ and the background disc axisymmetry. Clearly, one has $\int_0^{2\pi}\mathrm{d}\theta[\partial\phi^{(2)}/\partial\theta]=0$, and also $\int_{-\infty}^{+\infty}\mathrm{d}zr\int_0^{2\pi}\mathrm{d}\theta b_r^{(2)}=0$ as a direct result of $\nabla\cdot\boldsymbol{b}^{(2)}=0$. Since $\partial[\int\mathrm{d}x^3\rho^{(2)}]/\partial t=0$, the mass conservation equation (2.1) implies that $\int_{-\infty}^{+\infty}\mathrm{d}zr\int_0^{2\pi}\mathrm{d}\theta[\rho v_r+\rho_0v_r^{(2)}]=0$. Equation (4.4a) can then be simplified as

$$\mathcal{F}_\mathrm{J}=\int_{-\infty}^{+\infty}\mathrm{d}zr\int_0^{2\pi}\mathrm{d}\theta\left[\rho_0v_r(rv_\theta)-\frac{b_r(rb_\theta)}{4\pi}+\frac{1}{4\pi G}\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta}\right]. \tag{4.4b}$$

One can now specifically calculate the three terms of $\mathcal{F}_\mathrm{J}$ separately. The angular momentum flux across a straight cylinder of radius r due to the gravity torque is defined by

$$\mathcal{F}_\mathrm{J}^\mathrm{G}=\frac{1}{4\pi G}\int_0^{2\pi}\mathrm{d}\theta\int_{-\infty}^{\infty}\mathrm{d}zr\left[\frac{\partial\Re(\phi)}{\partial\theta}\right]\left[\frac{\partial\Re(\phi)}{\partial r}\right], \tag{4.5}$$

where the operator $\Re(\cdots)$ takes the real part of the argument. In the WKBJ approximation, we have

$$\begin{aligned}\Re[\phi(r,\theta,z)]=&\Re[\tilde{\phi}(r)]\exp(-|k(r)z|)\cos\left(\int^r k(s)\mathrm{d}s-m\theta+\omega t\right)\\&-\Im[\tilde{\phi}(r)]\exp(-|k(r)z|)\sin\left(\int^r k(s)\mathrm{d}s-m\theta+\omega t\right),\end{aligned}$$

where the operator $\Im(\cdots)$ takes the imaginary part of the argument. Therefore

$$\mathcal{F}_\mathrm{J}^\mathrm{G}=-\mathrm{sgn}(k)\frac{mr|\tilde{\phi}(r)|^2}{4G} \tag{4.6}$$

by averaging equation (4.5) over time t (cf. Goldreich &

Tremaine 1979). The angular momentum flux $\mathcal{F}_J^G$ due to the gravity torque is outward and inward for trailing and leading spirals respectively. The advective transport of angular momentum flux $\mathcal{F}_J^A$ is defined by

$$\mathcal{F}_J^A = r^2\mu_0\int_0^{2\pi} d\theta \Re(v_r)\Re(v_\theta). \quad (4.7)$$

From wave equations (2.13)–(2.18), one can express v_θ in terms of v_r, namely

$$v_\theta = \frac{i\tilde{\omega}\kappa^2 v_r/(2\Omega) + (im/r)\Delta[d\ln(r\mu_0)/dr]v_r + (im/r)\Delta dv_r/dr}{\tilde{\omega}^2 - (m^2/r^2)\Delta}. \quad (4.8)$$

Substituting equation (4.8) into equation (4.7), taking time averages and keeping leading-order terms for large kr, we derive respectively for fast MHD density waves

$$\mathcal{F}_J^A = -\frac{\pi mkr\Delta\mu_0}{\tilde{\omega}^2}|\tilde{v}_r|^2, \quad (4.9)$$

and for slow MHD density waves

$$\mathcal{F}_J^A = -\frac{\pi mkr\Delta\mu_0}{\tilde{\omega}^2 - (m^2/r^2)\Delta}|\tilde{v}_r|^2. \quad (4.10)$$

For fast MHD density waves *within* the Lindblad resonances (i.e., $\tilde{\omega}^2 < \kappa^2$), dispersion relation (3.3) implies $\Delta < 0$. Thus the advective angular momentum flux $\mathcal{F}_J^A$ as given by (4.9) is inward and outward for trailing and leading spirals respectively. For *neutral* slow MHD density waves (i.e., $\Delta>0$), dispersion relation (3.4) implies $\tilde{\omega}^2 - (m^2/r^2)\Delta < 0$. Thus the advective angular momentum flux $\mathcal{F}_J^A$ as given by (4.10) is inward and outward for trailing and leading spirals respectively.

The angular momentum flux due to the magnetic torque is defined by

$$\mathcal{F}_J^B = -\frac{r^2}{4\pi}\int_0^{2\pi} d\theta\int_{-\infty}^{\infty} dz\Re(b_r)\Re(b_\theta). \quad (4.11a)$$

Using the divergence-free condition (2.18) for the magnetic field perturbation $\boldsymbol{b}$

$$b_\theta = -\frac{i}{m}\frac{\partial(rb_r)}{\partial r},$$

one obtains from equation (4.11a)

$$\mathcal{F}_J^B = -\frac{r^2\tau}{4}\frac{kr}{m}|\tilde{b}_r|^2, \quad (4.11b)$$

where τ is the disc thickness and a time average has been taken. From equation (2.16), the radial components of magnetic field and velocity perturbations b_r and v_r are related to each other by

$$b_r = -\frac{mB_0}{\tilde{\omega}r}v_r.$$

It then follows from equation (4.11b) that

$$\mathcal{F}_J^B = -\frac{mkr}{4\tilde{\omega}^2}B_0^2\tau|\tilde{v}_r|^2 = -\frac{\pi mkr\mu_0}{\tilde{\omega}^2}C_A^2|\tilde{v}_r|^2. \quad (4.12)$$

For both fast and slow MHD density waves, the angular momentum flux $\mathcal{F}_J^B$ due to the magnetic torque is outward and inward for trailing and leading spirals respectively. In short, for cases of interest, the angular momentum fluxes due to gravity and magnetic torques are always in the same sense, which is opposite to the sense of angular momentum flux by advective transport; this is true for both fast and slow MHD density waves.

The net angular momentum flux $\mathcal{F}_J$ associated with a fast MHD density wave is given by the sum of corresponding $\mathcal{F}_J^G$, $\mathcal{F}_J^A$ and $\mathcal{F}_J^B$, namely

$$\mathcal{F}_J = \mathcal{F}_J^G + \mathcal{F}_J^A + \mathcal{F}_J^B = -\left[\mathrm{sgn}(k)\frac{mr|\tilde{\phi}(r)|^2}{4G} + \frac{\pi mkr\Delta\mu_0}{\tilde{\omega}^2}|\tilde{v}_r|^2 + \frac{\pi mkr\mu_0}{\tilde{\omega}^2}C_A^2|\tilde{v}_r|^2\right].$$

For fast MHD density waves,

$$\tilde{\phi}(r) \approx -\frac{\mathrm{sgn}(k)2\pi G\mu_0}{\tilde{\omega}}\tilde{v}_r$$

and the net angular momentum flux can be written as

$$\mathcal{F}_J = -\frac{\pi mkr\mu_0}{\tilde{\omega}^2}\left(C_S^2 + C_A^2 - \frac{\pi G\mu_0}{|k|}\right)|\tilde{v}_r|^2. \quad (4.13)$$

For *neutral* fast MHD density waves within the entire radial wavenumber k range, dispersion relation (3.3) requires that the magnetic Q parameter $Q_M \equiv (C_S^2 + C_A^2)^{1/2}\kappa/(\pi G\mu_0) > 1$ (Lou & Fan 1998a). Sufficiently away from corotation, dispersion relation (3.3) allows a short-wave branch and a long-wave branch (Lou & Fan 1998a). For short- and long-wave branches, $C_A^2 + \Delta$ is positive and negative respectively. By the adopted sign conventions here, trailing and leading spirals correspond to $k < 0$ and $k > 0$ respectively. Therefore the net angular momentum flux $\mathcal{F}_J$ associated with fast MHD density waves as given by (4.13) is outward and inward for trailing and leading short-wave spirals respectively, while it is inward and outward for trailing and leading long-wave spirals respectively.[3]

In parallel, the net angular momentum flux associated with a slow MHD density wave is given by

$$\mathcal{F}_J = \mathcal{F}_J^G + \mathcal{F}_J^A + \mathcal{F}_J^B = -\left[\mathrm{sgn}\,(k)\frac{mr|\tilde{\phi}(r)|^2}{4G} + \frac{\pi mkr\Delta\mu_0}{\tilde{\omega}^2 - (m^2/r^2)\Delta}|\tilde{v}_r|^2 + \frac{\pi mkr\mu_0}{\tilde{\omega}^2}C_A^2|\tilde{v}_r|^2\right].$$

As $\tilde{\phi}$ and $\tilde{v}_r$ are related by

$$\tilde{\phi} = \frac{2\pi G\mu_0}{|k|}\frac{(im\kappa^2)/(2\Omega r) - \tilde{\omega}k}{\tilde{\omega}^2 - (m^2/r^2)\Delta}\tilde{v}_r$$

for a slow MHD density wave, it follows that

$$|\tilde{\phi}|^2 = \left(\frac{2\pi G\mu_0}{|k|}\right)^2\frac{(m^2\kappa^4)/(4\Omega^2r^2) + \tilde{\omega}^2k^2}{[\tilde{\omega}^2 - (m^2/r^2)\Delta]^2}|\tilde{v}_r|^2.$$

The net angular momentum flux associated with a slow MHD

[3] Note that the short-wave results are naturally consistent with the WKBJ approximation, whereas the long-wave results are heuristic and require further analysis which is beyond the scope of this paper.

density wave can thus be written as

$$\mathcal{F}_{\rm J} = -\left\{\pi mkr\mu_0\left[\frac{\Delta}{\tilde{\omega}^2-(m^2/r^2)\Delta}+\frac{C_{\rm A}^2}{\tilde{\omega}^2}\right]\right.$$
$$\left.+{\rm sgn}(k)\frac{mr}{4G}\left(\frac{2\pi G\mu_0}{|k|}\right)^2\frac{(m^2\kappa^4)/(4\Omega^2r^2)+\tilde{\omega}^2k^2}{[\tilde{\omega}^2-(m^2/r^2)\Delta]^2}\right\}|\tilde{v}_r|^2. \tag{4.14}$$

For neutral slow MHD density waves with $\Delta > 0$ and $\tilde{\omega}^2 - (m^2/r)^2\Delta < 0$, one can readily show that

$$\frac{\Delta}{\tilde{\omega}^2-(m^2/r^2)\Delta}+\frac{C_{\rm A}^2}{\tilde{\omega}^2} > 0$$

by using dispersion relation (3.4). Therefore the net angular momemtum flux $\mathcal{F}_{\rm J}$ associated with slow MHD density waves is outward and inward for trailing and leading spirals respectively.

4.2 Energy fluxes of fast and slow MHD density waves

Along a separate line, the energy flux $\mathcal{F}_{\rm E}$ is naturally identified with

$$\mathcal{F}_{\rm E} = \int_{-\infty}^{+\infty}{\rm d}zr\int_0^{2\pi}{\rm d}\theta\hat{\boldsymbol{r}}\cdot\left[\frac{\rho|\boldsymbol{V}|^2}{2}\boldsymbol{V}+\frac{\gamma P\boldsymbol{V}}{(\gamma-1)}-\rho\Phi\boldsymbol{V}\right.$$
$$\left.+\frac{\Phi}{4\pi G}\nabla\frac{\partial\Phi}{\partial t}+\frac{\boldsymbol{B}\times(\boldsymbol{V}\times\boldsymbol{B})}{4\pi}\right] \tag{4.15}$$

according to the energy conservation equation (4.2). To the second order of perturbations, $\mathcal{F}_{\rm E}$ can be written as

$$\mathcal{F}_{\rm E} = \int_{-\infty}^{+\infty}{\rm d}zr\int_0^{2\pi}{\rm d}\theta\left[\frac{V_0^2\rho v_r}{2}+\frac{V_0^2\rho_0v_r^{(2)}}{2}+\rho_0V_0v_\theta v_r+pv_r\right.$$
$$+\frac{pv_r}{(\gamma-1)}+\frac{\gamma p_0v_r^{(2)}}{(\gamma-1)}-\rho_0\phi v_r-\Phi_0\rho v_r-\Phi_0\rho_0v_r^{(2)}$$
$$+\frac{\phi}{4\pi G}\frac{\partial^2\phi}{\partial r\partial t}+\frac{\Phi_0}{4\pi G}\frac{\partial^2\phi^{(2)}}{\partial r\partial t}$$
$$\left.+\frac{B_0b_\theta v_r-V_0b_\theta b_r+B_0(b_\theta v_r+B_0v_r^{(2)}-v_\theta b_r-V_0b_r^{(2)})}{4\pi}\right].$$

It can be shown from $\partial[\int {\rm d}x^3\rho^{(2)}]/\partial t = 0$ and the mass conservation equation (2.1) that

$$\int_{-\infty}^{+\infty}{\rm d}zr\int_{-0}^{2\pi}{\rm d}\theta\left[\frac{V_0^2\rho v_r}{2}+\frac{V_0^2\rho_0v_r^{(2)}}{2}\right] = 0$$

and

$$\int_{-\infty}^{+\infty}{\rm d}zr\int_0^{2\pi}{\rm d}\theta\left[-\Phi_0\rho v_r-\Phi_0\rho_0v_r^{(2)}\right] = 0.$$

Using $p_0 = K\rho_0^\gamma$, $p = \gamma K\rho_0^{\gamma-1}\rho$ and the mass conservation equation (2.1), we have

$$\int_{-\infty}^{+\infty}{\rm d}zr\int_0^{2\pi}{\rm d}\theta\left[\frac{pv_r}{(\gamma-1)}+\frac{\gamma p_0v_r^{(2)}}{(\gamma-1)}\right] = 0.$$

The Poisson equation (2.5) and the mass conservation equation (2.1) together imply that

$$\int_{-\infty}^{+\infty}{\rm d}zr\int_0^{2\pi}{\rm d}\theta\frac{\Phi_0}{4\pi G}\frac{\partial^2\phi^{(2)}}{\partial r\partial t} = 0.$$

Finally, we obtain

$$\int_{-\infty}^{+\infty}{\rm d}zr\int_0^{2\pi}{\rm d}\theta B_0\left[b_\theta v_r+B_0v_r^{(2)}-v_\theta b_r-V_0b_r^{(2)}\right] = 0$$

from the induction equation (2.3) and $\partial[\int {\rm d}x^3\boldsymbol{b}^{(2)}]/\partial t = 0$. With these results, the second-order MHD-density-wave energy flux is given by

$$\mathcal{F}_{\rm E} = \int_{-\infty}^{+\infty}{\rm d}zr\int_0^{2\pi}{\rm d}\theta\left(\rho_0V_0v_\theta v_r+pv_r-\rho_0\phi v_r+\frac{\phi}{4\pi G}\frac{\partial^2\phi}{\partial r\partial t}\right.$$
$$\left.+\frac{B_0b_\theta v_r-V_0b_\theta b_r}{4\pi}\right). \tag{4.16}$$

Using the WKBJ approximation, it is then straightforward to show (see Appendix A) that

$$\mathcal{F}_{\rm E} = \frac{\omega}{m}\mathcal{F}_{\rm J} \tag{4.17}$$

which is valid for *both* fast and slow MHD density waves.

On the basis of the above analysis (see equations 4.13 and 4.14), it is obvious that amplitude equations (3.7) and (3.8) represent the flux conservations of both angular momentum and energy associated with fast and slow MHD density waves respectively. In the tight-winding or WKBJ approximation, both fast and slow MHD density waves therefore do not exchange net wave angular momentum and wave energy with the background disc in the region away from corotation and Lindblad resonances. At these resonances, the above analysis breaks down, and stars and gas particles may absorb or emit angular momentum and energy from or to MHD density waves (Lynden-Bell & Kalnajs 1972; Mark 1974).

5 ANGULAR MOMENTUM, ENERGY AND ACTION DENSITIES OF MHD DENSITY WAVES

Let us take a tight-winding spiral MHD density wave with an angular frequency $\omega_{\rm R}$ and a radial wavenumber $k(r)$ that satisfy the dispersion relation of either fast or slow MHD density waves (see equations 3.3 and 3.4). Suppose that such a wave is excited by some external masses which were slowly turned on at $t = -\infty$ (cf. Mark 1974). Since density waves exert gravitational forces on external masses (external electric current is absent), one can determine the MHD density wave energy and angular momentum by calculating the work and torque due to the back reaction of the external masses on MHD density waves.

A perturbation component of the external mass density variation with given ω and m can be written in the form

$$\sigma_{\rm e}(r,\theta,t)\delta(z) = S_{\rm e}(r)\exp{\rm i}(\omega t-{\rm i}m\theta)\delta(z), \tag{5.1}$$

where $\sigma_{\rm e}$ is the external surface mass density, $\delta(z)$ is the Dirac delta function in z, $S_{\rm e}$ is the magnitude of $\sigma_{\rm e}$, and $\omega = \omega_{\rm R} + {\rm i}\omega_{\rm I}$ with $\omega_{\rm I} < 0$ and $|\omega_{\rm I}| \ll |\omega_{\rm R}|$. For given ω and m, the corresponding Poisson equation is given by

$$\nabla^2\phi = -4\pi G(\sigma_{\rm e}+\mu)\delta(z), \tag{5.2}$$

where μ is the gas surface mass density perturbation, and ϕ is the negative gravitational potential perturbation produced by σ_e and μ together. In the WKBJ approximation (Shu 1970), the solution to Poisson's equation (5.2) at $z=0$ is

$$\sigma_e + \mu = \frac{|k|}{2\pi G}\phi \qquad \text{or} \qquad \sigma_e = -\mu + \frac{|k|}{2\pi G}\phi. \tag{5.3}$$

From the MHD-density-wave perturbation equations in Sections 2 and 3, one generally derives a relation

$$\mu = F(\omega, k, m)\phi,$$

where the specific functional forms of F for fast (denoted by $F^{\rm F}$) and slow (denoted by $F^{\rm S}$) MHD density waves are, of course, different. From the relevant dispersion relation $D^{\rm F(S)}(\omega_{\rm R}, k, m) = F^{\rm F(S)}(\omega_{\rm R}, k, m) - |k|/(2\pi G) \cong 0$, one has approximately

$$\sigma_e \approx -{\rm i}\omega_{\rm I}\frac{\partial D(\omega_{\rm R}, k, m)}{\partial\omega}\phi. \tag{5.4}$$

The gravitational torque density associated with MHD density waves exerting on external masses is

$$\sigma_e\delta(z)\boldsymbol{r}\times\nabla\phi = \sigma_e\delta(z)\frac{\partial\phi}{\partial\theta}\hat{z};$$

the net change in the wave angular momentum from $t=-\infty$ to t can thus be written as

$$J = -\int_{-\infty}^{t}{\rm d}t\int_0^{2\pi}\int_0^{\infty}\sigma_e\frac{\partial\phi}{\partial\theta}\bigg|_{z=0} r{\rm d}r{\rm d}\theta. \tag{5.5a}$$

In the limit of $\omega_{\rm I}\rightarrow 0$, one can readily show that

$$J = \int_0^{2\pi}\int_0^{\infty}\frac{m}{4}\frac{\partial D}{\partial\omega}|\phi|^2_{z=0} r{\rm d}r{\rm d}\theta. \tag{5.5b}$$

Up to the divergence of an arbitrary vector function (cf. Toomre 1969; Kalnajs 1971; Mark 1974), the surface density of wave angular momentum $\mathcal{J}$ can be identified from equation (5.5b) as

$$\mathcal{J} = \frac{m}{4}\frac{\partial D}{\partial\omega}|\phi|^2_{z=0}. \tag{5.6}$$

Similarly, the net change in the wave energy from $t=-\infty$ to t can be written as

$$E = -\int_{-\infty}^{t}{\rm d}t\int_{-\infty}^{\infty}{\rm d}z\int_0^{2\pi}{\rm d}\theta\int_0^{\infty}r{\rm d}r\boldsymbol{j}_e\cdot\nabla\phi, \tag{5.7a}$$

where $\boldsymbol{j}_e$ is the external mass flux density which is related to the external mass density ρ_e through the mass conservation equation $\partial\rho_e/\partial t + \nabla\cdot\boldsymbol{j}_e = 0$. Once again, up to the divergence of an arbitrary vector function and in the limit of $\omega_{\rm I}\rightarrow 0$, the surface density of wave energy is identified from equation (5.7a) as

$$\mathcal{E} = \frac{\omega}{m}\mathcal{J} = \frac{\omega}{4}\frac{\partial D}{\partial\omega}|\phi|^2_{z=0}. \tag{5.7b}$$

For fast MHD density waves, we have

$$F^{\rm F} = \frac{-k^2\mu_0}{\tilde{\omega}^2 - \kappa^2 - k^2(C_{\rm A}^2 + C_{\rm S}^2)}. \tag{5.8}$$

It follows that

$$\frac{\partial D^{\rm F}}{\partial\omega} = \frac{2k^2\mu_0\tilde{\omega}}{[\tilde{\omega}^2 - \kappa^2 - k^2(C_{\rm A}^2 + C_{\rm S}^2)]^2} = \frac{\tilde{\omega}}{2\pi^2G^2\mu_0}, \tag{5.9}$$

where dispersion relation (3.3) of fast MHD density waves is used. Substitutions of equation (5.9) into equations (5.6) and (5.7b) give explicit expressions of the angular momentum surface density $\mathcal{J}^{\rm F}$ and the energy surface density $\mathcal{E}^{\rm F}$ respectively for fast MHD density waves, namely

$$\mathcal{J}^{\rm F} = \frac{m\tilde{\omega}}{8\pi^2G^2\mu_0}|\tilde{\phi}|^2_{z=0}$$

and

$$\mathcal{E}^{\rm F} = \frac{\omega\tilde{\omega}}{8\pi^2G^2\mu_0}|\tilde{\phi}|^2_{z=0},$$

where the superscript $^{\rm F}$ indicates the association with fast MHD density waves. By invoking the following relation between ϕ and v_r,

$$|\tilde{\phi}|^2_{z=0} \approx \frac{4\pi^2G^2\mu_0^2}{\tilde{\omega}^2}|\tilde{v}_r|^2,$$

we derive

$$\mathcal{J}^{\rm F} = \frac{m\mu_0}{2\tilde{\omega}}|\tilde{v}_r|^2 \tag{5.10}$$

and

$$\mathcal{E}^{\rm F} = \frac{\omega\mu_0}{2\tilde{\omega}}|\tilde{v}_r|^2. \tag{5.11}$$

By expressions (5.10) and (5.11), it is clear that inside corotation (i.e., $\tilde{\omega}<0$), both $\mathcal{J}$ and $\mathcal{E}$ are negative, whereas outside corotation (i.e., $\tilde{\omega}>0$), both $\mathcal{J}^{\rm F}$ and $\mathcal{E}^{\rm F}$ are positive; this conclusion holds true for either trailing or leading spiral MHD density waves. Based on equations (3.5), (4.13), (4.17), (5.10) and (5.11), it can be further shown that the angular momentum flux and energy flux calculated in previous sections can be consistently written, respectively, as

$$\mathcal{F}_{\rm J}^{\rm F} = 2\pi r\mathcal{J}^{\rm F}C_{\rm g}^{\rm F} \tag{5.12}$$

and

$$\mathcal{F}_E^{\rm F} = 2\pi r\mathcal{E}^{\rm F}C_{\rm g}^{\rm F}, \tag{5.13}$$

where $C_{\rm g}^{\rm F} = -\partial\omega/\partial k$ is the radial group speed of fast MHD waves given by equation (3.5). For the short-wave branch[4] of trailing fast MHD density waves, $C_{\rm g}^{\rm F}<0$ sufficiently inside corotation and $C_{\rm g}^{\rm F}>0$ sufficiently outside corotation. *Therefore short trailing spiral fast MHD density waves transport angular momentum and energy outward sufficiently away from corotation.* In contrast, angular momentum and energy are transported radially inward by short leading fast MHD density waves sufficiently away from corotation. For the long-wave branch sufficiently away from corotation (Lou & Fan 1998a), trailing and leading fast MHD density waves transport both angular momentum and energy inward and outward respectively.

In parallel to what has been done for fast MHD density waves, we now calculate the angular momentum and energy associated with slow MHD density waves in the disc portion with an almost rigid rotation. One now has

$$F^{\rm S} = \frac{\tilde{\omega}^2k^2\mu_0 - k^2(m^2/r^2)C_{\rm A}^2\mu_0}{\tilde{\omega}^2(\kappa^2 + k^2C_{\rm A}^2 + k^2C_{\rm S}^2) - k^2(m^2/r^2)C_{\rm A}^2C_{\rm S}^2} \tag{5.14}$$

[4] As in the hydrodynamic case (cf. Binney & Tremaine 1987), dispersion relation (3.3) of fast MHD density waves contains a short-wave branch as well as a long-wave branch (Fan & Lou 1996; Lou & Fan 1998a).

(cf. Lou & Fan 1998a), where the superscript $^{\rm S}$ of F indicates the association with slow MHD density waves. It follows that

$$\frac{\partial D^{\rm S}}{\partial\omega}=\frac{2\tilde{\omega}k^2(m^2/r^2)C_{\rm A}^2\left(\kappa^2+k^2C_{\rm A}^2\right)\mu_0}{\left[\tilde{\omega}^2\left(\kappa^2+k^2C_{\rm A}^2+k^2C_{\rm S}^2\right)-k^2(m^2/r^2)C_{\rm A}^2C_{\rm S}^2\right]^2}. \qquad (5.15)$$

After tedious but straightforward calculations and rearrangement, we obtain the angular momentum surface density $\mathcal{J}^{\rm S}$ of slow MHD density waves

$$\mathcal{J}^{\rm S}=\left(\frac{m}{2\tilde{\omega}}\right)\left(\frac{\mu_0k^2}{4\pi^2G^2\mu_0^2}\right)\left[\frac{\Delta\left(\kappa^2+k^2C_{\rm A}^2+k^2C_S^2\right)}{\kappa^2+k^2C_{\rm A}^2}\right]|\tilde{\phi}|^2_{z=0}.$$

Using the relation

$$|\tilde{\phi}|^2_{z=0}=\left(\frac{4\pi^2G^2\mu_0^2}{k^2}\right)\left\{\frac{m^2\kappa^4/(4\Omega^2r^2)+\tilde{\omega}^2k^2}{[\tilde{\omega}^2-(m^2/r^2)\Delta]^2}\right\}|\tilde{v}_r|^2$$

(see steps leading to equation 4.14 in Section 4) and dispersion relation (3.4) of slow MHD density waves with $\kappa^2\approx 4\Omega^2$, we derive after rearrangement

$$\mathcal{J}^{\rm S}=\left(\frac{m\mu_0}{2\tilde{\omega}}\right)\left[\frac{\left(\kappa^2+k^2C_{\rm A}^2+k^2\Delta\right)^2}{(m^2/r^2)\Delta(\kappa^2+k^2\Delta)}\right]|\tilde{v}_r|^2 \qquad (5.16)$$

for the surface density of wave angular momentum. Correspondingly, the surface density of slow MHD wave energy is given by

$$\mathcal{E}^{\rm S}=\left(\frac{\omega\mu_0}{2\tilde{\omega}}\right)\left[\frac{\left(\kappa^2+k^2C_{\rm A}^2+k^2\Delta\right)^2}{(m^2/r^2)\Delta(\kappa^2+k^2\Delta)}\right]|\tilde{v}_r|^2. \qquad (5.17)$$

Similar to fast MHD density waves, the surface density of wave angular momentum $\mathcal{J}^{\rm S}$ and the surface density of wave energy $\mathcal{E}^{\rm S}$ are both negative inside corotation, and are both positive outside corotation. By equation (4.14), the net angular momentum flux transported by slow MHD density waves is

$$\mathcal{F}_{\rm J}^S=-\left\{\pi mkr\mu_0\left[\frac{\Delta}{\tilde{\omega}^2-(m^2/r^2)\Delta}+\frac{C_{\rm A}^2}{\tilde{\omega}^2}\right]\right.$$
$$\left.+\mathrm{sgn}(k)\frac{mr}{4G}\left(\frac{2\pi G\mu_0}{|k|}\right)^2\frac{m^2\kappa^4/(4\Omega^2r^2)+\tilde{\omega}^2k^2]}{[\tilde{\omega}^2-(m^2/r^2)\Delta]^2}\right\}|\tilde{v}_r|^2.$$

According to equations (3.6), (4.14), (4.17), (5.16) and (5.17), one can consistently show that

$$\mathcal{F}_{\rm J}^{\rm S}=2\pi r\mathcal{J}^{\rm S}C_{\rm g}^{\rm S} \qquad (5.18)$$

and

$$\mathcal{F}_{\rm E}^{\rm S}=2\pi r\mathcal{E}^{\rm S}C_{\rm g}^{\rm S}, \qquad (5.19)$$

where $C_{\rm g}^{\rm S}$ is the radial group speed of slow MHD density waves given by equation (3.6). It was shown earlier (Lou & Fan 1998a) that there is only one wave branch for slow MHD density waves. From equations (5.18) and (5.19), it is seen again that *trailing/leading slow MHD density waves transport angular momentum and energy outward/inward both inside and outside corotation.*

From the two pairs of equations (5.10)–(5.11) and (5.16)–(5.17), one can also derive the wave action surface density (Toomre 1969; Shu 1970) for fast (denoted by $\mathcal{N}^{\rm F}$) and slow (denoted by $\mathcal{N}^{\rm S}$) MHD density waves respectively, namely

$$\mathcal{N}^{\rm F}=\frac{\mu_0}{2\tilde{\omega}}|\tilde{v}_r|^2 \qquad (5.20)$$

and

$$\mathcal{N}^{\rm S}=\left(\frac{\mu_0}{2\tilde{\omega}}\right)\left[\frac{\left(\kappa^2+k^2C_{\rm A}^2+k^2\Delta\right)^2}{(m^2/r^2)\Delta(\kappa^2+k^2\Delta)}\right]|\tilde{v}_r|^2. \qquad (5.21)$$

At this point, it is apparent that equations (3.7) and (3.8) also represent the conservations of wave action associated with fast and slow MHD density waves respectively.

6 TRAILING MAGNETIZED SPIRAL GALAXIES

In a simple sketch, a typical system hosting a spiral galaxy contains a massive spherical halo (dark matter included), a luminous stellar disc, a magnetized gas disc and an oblate spheroid of cosmic ray gas. Over more than three decades, much has been known about large-scale density waves (Lin & Shu 1964, 1966; Toomre 1964, 1969; Goldreich & Lynden-Bell 1965; Athanasoula 1984; Binney & Tremaine 1987; Lin 1987; Bertin & Lin 1996) in a stellar disc (using either a distribution function approach or a fluid description) through observational, theoretical and numerical investigations. The importance of understanding large-scale density-wave perturbations in a composite system consisting of rotating stellar and gas discs has also gained a considerable appreciation in various galactic contexts over the past decade or so (Jog & Solomon 1984; Bertin & Romeo 1988; Elmegreen 1995; Jog 1996; Lou & Fan 1998b). One significant revelation is that a composite disc system can be unstable as a result of mutual gravitational coupling, even though the two discs themselves are separately stable; this can happen even when the two disc masses are significantly different. In the galactic context, a direct implication is that the gas disc, typically an order-of-magnitude less massive than the stellar disc, can play a non-trivial role in the dynamics of a composite disc system through the gravitational interaction. In addition, various subprocesses (e.g., interstellar shocks, star formation, etc.) take place in the magnetized interstellar medium to affect the overall appearance and evolution of a spiral galaxy.

In a recent paper (Lou & Fan 1998b) on density waves in a composite disc system, we considered an example in which a gas disc surface mass density μ_0 was taken roughly to be a tenth of a stellar disc surface mass density μ_0^S. Within the Lindblad resonances, the dispersion relation of density waves in a composite disc system can be written in a form similar to the dispersion relation in a stellar disc alone, such that all effects due to the presence of a gas disc can be relegated to a factor f – referred to as the effective fractional increment factor of the surface mass density of the stellar disc. For sensible galactic parameters, f can be 20 to 30 per cent, which is ~2 to 3 times the surface mass density ratio μ_0/μ_0^S. For different sets of parameter estimates under various situations, the value of f can be even larger. This is one perspective of assessing the significance of a gas disc.

There is yet another effective way to estimate the importance of a gas disc for density waves in a composite disc system. According to equation (4.6), the angular momentum flux $\mathcal{F}_{\rm J}^{\rm G}$

due to gravity torque bears the form of

$$\mathcal{F}_{\rm J}^{\rm G} = -{\rm sgn}(k)\frac{mr|\phi(r)|^2}{4G}.$$

For density waves in a composite disc system, ϕ here should be the *net* negative gravitational potential perturbation which contains contributions from $\phi^{\rm S}$ of stellar perturbation origin and ϕ^g of gas perturbation origin respectively. Since $\phi^{\rm S} = 2\pi G\mu^{\rm S}/|k|$ and $\phi^g = 2\pi G\mu^g/|k|$ are proportional to the stellar and gas surface mass density perturbations, $\mu^{\rm S}$ and μ^g respectively, the magnitude ratio of $\mu^{\rm S}$ and μ^g associated with density waves in a composite disc system provides a direct measure for the corresponding angular momentum fluxes due to gravity torques of stellar and gas disc origins. In the tight-winding regime, one can readily show that $\mu^{\rm S}$ and μ^g are related by

$$[\kappa^2 - (\omega - m\Omega)^2 + D_{\rm S}^2k_{\rm T}^2 - 2\pi G\mu_0^{\rm S}|k_{\rm T}|]\mu^{\rm S} = 2\pi G\mu_0^S|k_{\rm T}|\mu^g,$$

where $|k_{\rm T}| \equiv (k^2 + m^2/r^2)^{1/2}$ is the total wavenumber, and $D_{\rm S}$ is the stellar velocity dispersion. For density waves in a composite disc system *within* the Lindblad resonances, the factor in front of $\mu^{\rm S}$ on the left-hand side is positive (see Lou & Fan 1998b for details). Given sensible parameter estimates, there are realistic situations in which $\mu^{\rm S}$ and μ^g can be comparable in magnitudes, even though μ_0^S is typically an order of magnitude larger than μ_0^g. As long as $\mu^{\rm S}/\mu_0^{\rm S}$ remains sufficiently small to insure a small μ^g/μ_0^g, the linear approximation can be consistently valid.[5] In such cases, stellar and gas contributions to the angular momentum flux transport by gravity torque are actually comparable. Note that arguments leading to this conclusion are valid whether the gas disc is magnetized or not. For large-scale density waves in either a stellar disc or a gas disc *alone*, angular momentum fluxes by gravity torque and by advection are comparable in order of magnitudes with the former being more dominant than the latter. The basic conclusion derived from the above analysis is that in terms of the angular momentum flux associated with density waves, gas disc plays a significant role and should be included for quantitative galactic applications.

In the preceding sections of this paper, we have specifically studied the angular momentum flux transport by gravity torque, advection and magnetic torque in a rotating magnetized gas disc alone. The physical reason for including a magnetic field is clear: in typical spiral galaxies, magnetic and thermal gas energy densities are comparable. Magnetic field plays an important dynamic role which leads to the possible existence of both fast and slow MHD density waves (Fan & Lou 1996; Lou & Fan 1998a, in preparation). Furthermore, the angular momentum flux by magnetic torque can be comparable to the angular momentum fluxes by gravity torque and advection in magnitude; their relative magnitude ratios are $C_{\rm A}^2$, $\pi G\mu_0/|k|$ and $C_S^2 - 2\pi G\mu_0/|k|$ for fast MHD density waves (see equation 4.13), and are

$$\frac{C_{\rm A}^2}{\tilde\omega^2},\quad \frac{\pi G\mu_0}{|k|}\frac{(m^2\kappa^4)/(4\Omega^2r^2) + \tilde\omega^2k^2}{k^2[\tilde\omega^2 - (m^2/r^2)\Delta]^2}\quad \text{and}\quad \frac{\Delta}{\tilde\omega^2 - (m^2/r^2)\Delta}$$

for slow MHD density waves (see equation 4.14), with $\tilde\omega^2$ defined by the dispersion relation (3.4). One important point noted earlier is that angular momentum fluxes by gravity and magnetic torques are always in the same sense, i.e., outward and inward for trailing and leading spirals respectively. Also note the distinction that

$$C_{\rm A}^2 + C_S^2 - 2\pi G\mu_0/|k| < 0$$

for short fast MHD density waves within the Lindblad resonances and sufficiently away from corotation (i.e., magnetic torque is weaker than advective transport), whereas

$$\frac{C_{\rm A}^2}{\tilde\omega^2} + \frac{\Delta}{\tilde\omega^2 - (m^2/r^2)\Delta} > 0$$

for neutral slow MHD density waves (i.e., magnetic torque is stronger than advective transport).

It was known decades ago (Lynden-Bell & Kalnajs 1972) that the gravity torque associated with trailing spiral density waves transports most angular momentum and energy fluxes outward in a stellar disc (in the absence of a magnetized gas disc), while the advective transport (or lorry transport in their terminology) plays a lesser role. For a more realistic spiral galaxy consisting of a stellar disc and a magnetized gas disc, our analysis and estimates here (see also Lou & Fan 1998b) reveal a significant contribution from MHD density waves in a magnetized gas disc to the transport process of angular momentum flux, especially in view of two possible types (i.e., fast and slow) of MHD density waves (Fan & Lou 1996). Whereas it remains true that the outward angular momentum flux by gravity torque (now containing comparable stellar and gas contributions) is most significant in trailing spirals, one should not underestimate the significant outward angular momentum flux by magnetic torque associated with either fast or slow MHD density waves. For *quantitative* galactic applications, it is also necessary to take into account the advective transport of angular momentum flux of both stellar and gas disc origins. For both fast and slow MHD density waves, the net angular momentum and energy fluxes are outward for trailing spirals. It is more consistent and also satisfactory to conclude that the observed statistical prevalence of trailing spiral galaxies can now be established for a more realistic galactic model with a rotating magnetized gas disc included.

7 SUMMARY AND DISCUSSION

The main result of this analysis is that in the WKBJ regime, both trailing fast and slow MHD density waves transport net angular momentum, energy and action radially outward in a magnetized rotating gas disc. From the theoretical perspective, this result is satisfying (and is also anticipated to some extent), because previous theoretical analyses (Shu 1970; Lynden-Bell & Kalnajs 1972; Mark 1974) have already shown that in a thin rotating stellar disc, angular momentum, energy and action are transported radially outward for trailing spiral density waves. In a real disc galaxy, large-scale density-wave perturbations in the stellar disc and in the magnetized gas disc are coupled through the mutual gravitational interaction (Lou & Fan 1997, 1998b); in the WKBJ approximation, one can derive an MHD dispersion relation in a composite disc system which generalizes the previous dispersion relation in the absence of magnetic field (Lin & Shu 1968; Kato 1972; Jog & Solomon 1984; Bertin & Romeo 1988; Elmegreen 1995; Jog 1996). In the context of MHD density waves in such a composite system of stellar and gas discs, our present analysis together with those of Lynden-Bell & Kalnajs (1972) and Mark

[5] Even if μ^g/μ_0^g falls within a weakly or moderately non-linear regime, wave angular momentum fluxes by gravity torques in stellar and gas discs should remain roughly comparable on an intuitive basis.

(1974) strongly hint at a more general conclusion that trailing fast and slow MHD density waves transport angular momentum and energy outward in a composite disc system. Therefore a composite disc system can reach a lower energy state with the appearance of trailing spiral MHD density waves, and our result consistently strengthens such an interpretation for the prevalence of trailing spiral structures in disc galaxies seen in optical and radio bands as well as other electromagnetic wavelengths (Lynden-Bell & Kalnajs 1972; Pasha 1985).

Another important point is that under the WKBJ approximation and in the disc region away from corotation and Lindblad resonances, the angular momentum and energy fluxes of both fast and slow MHD density waves are conserved. There is thus no net angular momentum and energy exchanges between the rotating disc and MHD density waves, except at these relevant resonances. Without magnetic fields, Lynden-Bell & Kalnajs (1972; see also Mark 1974) have shown that in a fairly broad context, stars emit angular momentum to density waves at the inner Lindblad resonance, and absorb angular momentum from density waves at corotation and outer Lindblad resonances. Thus the emission and absorption are arranged in an orderly manner to reduce the angular momentum at the inner disc portion and to increase the angular momentum at the outer disc portion such that the systematic disc rotation energy can be reduced. While the emission and absorption properties do not depend on whether the density wave is trailing or leading, only the trailing density wave carries angular momentum outward such that the communication between emitter and absorbers can be consistently established. At the inner Lindblad resonance of a stellar disc, stars emit an amount of angular momentum ΔJ corresponding to an amount of energy $\Delta E = \Omega_{\rm p}\Delta J$, where $\Omega_{\rm p} \equiv \omega/m$ is the density-wave pattern speed. The disc rotation energy of stars with an angular momentum $J_0 - \Delta J$ is $\Omega(J_0 - \Delta J) = E_0 - \Omega\Delta J$, where J_0 and E_0 are the angular momentum and energy of stars before emission, and Ω is the disc angular rotation speed at the inner Lindblad resonance. Since $\Omega > \Omega_{\rm p}$ at the inner Lindblad resonance, the amount of energy transformed from disc rotation of stars to random motions of stars after the emission is $(\Omega - \Omega_{\rm p})\Delta J$. Similarly, the amount of energy converted to random motions of stars at the outer Lindblad resonance after absorbing an amount of angular momentum ΔJ is $(\Omega_{\rm p} - \Omega)\Delta J$, with $\Omega_{\rm p} > \Omega$ at the outer Lindblad resonance. At corotation, there is no net energy exchange between the systematic disc rotation and random motions of stars.

For MHD density waves in a composite system of a stellar disc and a magnetized gas disc (Lou & Fan 1997), the above process of angular momentum transfer between stars and large-scale MHD density waves should also apply, because stars interact with both fast and slow MHD density waves mainly through the mutual gravitational coupling (Lou & Fan 1998b); the calculations and conclusions of Lynden-Bell & Kalnajs (1972) regarding the transfer of energy and angular momentum should remain applicable to density-wave perturbations in the stellar disc even in the presence of perturbations in a magnetized gas disc. The physical properties of fast MHD density waves are similar to hydrodynamic density waves (in a composite disc system; see Jog & Solomon 1984) in many ways, except that the sound speed $C_{\rm S}$ is replaced by the magneto-sonic speed $(C_S^2 + C_{\rm A}^2)^{1/2}$ to accommodate the magnetic pressure effect in the thermal gas disc. The perturbation enhancements of surface mass densities in the stellar and gas discs track each other *within* the Lindblad resonances. The principal radial range for the manifestation of fast MHD density waves remains between the inner and outer Lindblad resonances with an evanescent region characterized by a width depending on the magnetic Q_M parameter around corotation.[6] Therefore, for fast MHD density waves, stars also emit angular momentum and energy to waves at the inner Lindblad resonance and absorb angular momentum and energy from waves at the corotation and outer Lindblad resonances. Consequently, the disc rotation energy can be converted to the energy of random stellar motions at the Lindblad resonances through interactions between stars and fast MHD density waves. Slow MHD density waves tend to appear within and around corotation, with a radial extension of their structural patterns depending on the disc rotation curve: the weaker the differential rotation, the larger the the radial extension. Slow MHD density waves cannot extend to the Lindblad resonances (Lou & Fan 1998a); this situation excludes possible interactions between slow MHD density waves and stars at the Lindblad resonances. Nevertheless, it appears plausible that the angular momentum and energy fluxes, transported outward by trailing slow MHD density waves, are eventually consumed by dissipative processes in interstellar medium outside corotation; the net effect is that the disc rotation energy is systematically transferred to the thermal energy of interstellar gas.

We now consider the propagation of density waves and the interaction between density waves and stars in the context of density-wave generation. It has been shown that inside corotation, the wave angular momentum $\mathcal{J}$ and energy $\mathcal{E}$ are negative, while outside corotation, the wave angular momentum $\mathcal{J}$ and energy $\mathcal{E}$ are positive; this conclusion is true for density waves in a disc without magnetic field (Lynden-Bell & Kalnajs 1972; Mark 1974) as well as for fast and slow MHD density waves in a magnetized gas disc, as demonstrated in this paper (see the two pairs of equations 5.10 and 5.11, and 5.16 and 5.17). Therefore, as stars emit angular momentum and energy to waves at the inner Lindblad resonance, density waves within corotation are damped, because the angular momentum and energy of density waves are negative inside corotation. Meanwhile, as stars absorb angular momentum and energy from waves at the outer Lindblad resonance, density waves outside corotation are also damped, because the angular momentum and energy of density waves are positive outside corotation. Density waves can be generated at corotation. Modal analyses (Bertin et al. 1989a,b; Bertin & Lin 1996) indicate that under proper conditions, a disc can support globally unstable (i.e., growing) normal modes of density waves. Several ingredients are necessary for such globally growing normal modes. First, there must be an effective Q barrier inside corotation but outside the inner Lindblad resonance to reflect incoming density waves backwards; the main reason to shield the inner Lindblad resonance is to avoid wave damping there. Secondly, the effective Q value near corotation needs to be larger than but close to unity, such that waves can partially transmit through corotation; the partially reflected waves at corotation are then amplified as a result of wave action conservation. Thirdly, an appropriate radiation condition outside corotation is crucial. This

[6] Note that both hydrodynamic density waves in a gas disc and fast MHD density waves in a magnetized gas disc allow their short-wave branch to pass smoothly through the Lindblad resonances. For a stellar component described by a stellar distribution function, however, the short-wave branch terminates at the Lindblad resonances (cf. Binney & Tremaine 1987).

radiation condition involves two key aspects. In the absence of external energy sources outside corotation, no waves should come from outside towards corotation. Furthermore, outgoing waves are absorbed either at the outer Lindblad resonance through interactions with stars or by dissipations in interstellar medium; the net effect is to avoid reflected waves coming back towards corotation. Only trailing density waves can fulfil the above radiation condition and, consequently, be amplified (Bertin & Lin 1996). The modal analysis contains essentially all aspects discussed above, namely, the propagation of waves and the interaction between waves and stars at the resonances.

As fast MHD density waves are similar to hydrodynamic density waves in many ways, we expect that the self-excitation mechanism of fast MHD density waves should be more or less the same as that discussed in Bertin et al. (1989a,b; Bertin & Lin 1996). In contrast, the basic properties of slow MHD density waves are distinctly unique by the lack of a hydrodynamic counterpart. The self-excitation mechanism of slow MHD density waves should be explored further by using a modal approach. We call attention here to several aspects of slow MHD density waves that are closely relevant to a modal analysis. For disc galaxies whose spiral structures are confined in the inner disc portion with a gradually rising rotation curve, the inner Lindblad resonance usually disappears (cf. Kormendy & Norman 1979). Slow MHD density waves can manifest over a sufficiently large radial range only in the inner galactic disc portion where the rotation curve is gradually rising and the disc differential rotation is sufficiently weak (Fan & Lou 1996; Lou & Fan 1998a). In the absence of the inner Lindblad resonance, slow MHD density waves may well extend to the disc centre where an incoming trailing slow MHD density wave can be reflected back in the form of an outgoing leading wave towards corotation. In this case, the galactic centre acts as a reflection point which is essential for the existence of growing normal modes. For a modal analysis of generating slow MHD density waves, one should therefore be mindful about the conditions around the galactic centre. In the absence of an evanescent region around corotation, slow MHD density waves can be transmitted through corotation. Trailing slow MHD density waves, carrying positive angular momentum $\mathcal{J}$ and energy $\mathcal{E}$ (cf. equations 5.16 and 5.17), propagate outward outside corotation, and can be damped through dissipative processes by interstellar gas in the outer disc. In such a manner, slow MHD density waves can be excited and sustained in the inner disc portion.

Goldreich & Tremaine (1979) studied the excitation of density waves in a fluid disc at corotation and Lindblad resonances by a periodically varying external potential. Such an external potential can be due to a central bar or, in rare cases, to a satellite galaxy (e.g., the 'Whirlpool galaxy' M51). In addition to a non-wave part, density waves can be continuously generated and sustained in the disc by a rigidly rotating external potential characterized by a fairly large radial spatial scale. It is of considerable interest to generalize their study for exciting MHD density waves in a magnetized gas disc. An important goal of such a pursuit is to clearly identify distinct physical conditions by which either fast or slow MHD density waves tend to manifest more prominently.

The time-scale of setting up normal spiral density-wave modes can be estimated from the wave group speed (cf. Section 3) and the size of a galaxy (Bertin & Lin 1996). For a slower group speed and a larger galaxy size, normal modes of density waves may not be established during the lifetime of a galaxy. For a slow MHD density wave (Fan & Lou 1996; Lou & Fan 1998a), the larger the radial wavenumber k, the tighter the spiral winding, and the smaller the radial group speed (see equation 3.6). Therefore a galactic disc might not be able to excite long-lasting normal modes of slow MHD density waves which are extremely tightly wound. Tight-winding slow MHD density waves may be excited, for example, by swing amplifications of a shearing wave packet (Fan & Lou 1997). On the other hand, for relatively open slow MHD density waves, their radial group speed may be sufficiently large to establish long-lasting normal modes. From these considerations, a galactic disc with a nearly rigid rotation and with a relatively low surface gas mass density μ_0 or relatively high thermal sound speed C_S (such that relatively open slow MHD density waves can exist) is likely to sustain normal modes of slow MHD density waves. Since fast and slow MHD density waves are governed by a single set of equations (cf. equations 2.13–2.19), it is essential to identify and distinguish the physical conditions for which fast or slow MHD density waves can manifest more dominantly. In general terms, physical features of fast and slow MHD density waves can coexist or mingle in magnetized spiral galaxies (Fan & Lou 1997).

The angular momentum and energy transports by spiral density waves also contribute to secular evolution of spiral galaxies (e.g. Lynden-Bell & Kalnajs 1972; Bertin & Lin 1996; Zhang 1996). Using typical parameters for spiral galaxies, Bertin (1983) estimated that the time-scale of galactic evolution caused by the appearance of spiral density waves is $\sim$200 times the rotation period just beyond corotation; this estimated time-scale is longer than the Hubble time. Gnedin, Goodman & Frei (1995) measured gravity torques along spiral arms of the galaxy M100 (NGC 4321) and concluded that the secular evolution time-scale is $\sim$5–10 Gyr, about an order of magnitude shorter than that estimated by Bertin (1983). Courteau, de Jong & Broeils (1996) showed evidence for secular evolution by observing an ensemble of late-type spiral galaxies. All these estimates did not include the breaking effect of galactic magnetic torque. On the basis of our analysis, magnetic torques associated with trailing MHD density waves always transport angular momentum outward and can have comparable effects as gravity torques do. Thus the outward transport of angular momentum by density waves in a composite system of stars and a magnetized gas disc becomes more efficient than the angular momentum transport by gravity torque and advection in a single stellar disc alone. Consequently, the secular evolution of magnetized spiral galaxies should be enhanced, especially in view of the fact that the global star formation rate can be significantly influenced by the presence of magnetic fields.

Finally, we note briefly that the problem of angular momentum redistribution in a rotating disc is extremely important in various astrophysical contexts, ranging from planetary rings, protostar formation to active galactic nuclei on grandiose scales. It has been shown that spiral waves generated in a proto-stellar disc can effectively remove angular momentum and can therefore affect the star formation process (e.g. Shu et al. 1990; Pickett, Durisen & Davis 1996, and references therein). In a magnetized accretion disc, in addition to the angular momentum transport by the Balbus–Hawley type of instabilities (Balbus & Hawley 1991, 1998; Hawley 1995), spiral magnetosonic fast waves in a disc (e.g. Tagger et al. 1990; Tagger, Pellat & Coroniti 1992) can also be effective in redistributing angular momentum. The problem of examining the role of spiral MHD density waves in removing angular momentum from a magnetized accretion disc is thus an astrophysical application of considerable interest.

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APPENDIX A

Derivations of equation (4.17) are summarized here. In terms of real parts of perturbations, equation (4.16) can be written explicitly as

$$\begin{aligned}\mathcal{F}_{\rm E} = &\int_0^{2\pi} r\mathrm{d}\theta\big[\mu_0 V_0 \Re(v_\theta)\Re(v_r) + C_S^2\Re(\mu)\Re(v_r)\\ &-\mu_0\Re(\phi)\Re(v_r)\big] + \int_{-\infty}^{+\infty}\mathrm{d}z\int_0^{2\pi} r\mathrm{d}\theta\frac{\Re(\phi)}{4\pi G}\frac{\partial^2\Re(\phi)}{\partial r\partial t}\\ &+\int_{-\infty}^{+\infty}\mathrm{d}z\int_0^{2\pi} r\mathrm{d}\theta\frac{B_0\Re(b_\theta)\Re(v_r) - V_0\Re(b_\theta)\Re(b_r)}{4\pi},\end{aligned}\qquad (A1)$$

where the right-hand side has been grouped into three integrals, and $\Re(\cdots)$ takes the real part of the argument. By equation (2.15), one has

$$\begin{aligned}C_S^2\Re(\mu) - \mu_0\Re(\phi) = &\frac{\omega r}{m}\mu_0\Re(v_\theta) - V_0\mu_0\Re(v_\theta)\\ &+\frac{\mu_0}{m}\frac{\mathrm{d}(rV_0)}{\mathrm{d}r}\Im(v_r),\end{aligned}\qquad (A2)$$

where $\Im(\cdots)$ takes the imaginary part of the argument. Since $\int_0^{2\pi}\mathrm{d}\theta\Im(v_r)\Re(v_r) = 0$ as a result of azimuthal periodicity, the first part of integral (A1) reduces to

$$\int_0^{2\pi} r\mathrm{d}\theta\frac{\omega r}{m}\mu_0\Re(v_\theta)\Re(v_r). \qquad (A3)$$

Under the WKBJ approximation, one has

$$\frac{\Re(\phi)}{4\pi G}\frac{\partial^2\Re(\phi)}{\partial r\partial t} = -\omega k\frac{\Re(\phi)\Re(\phi)}{4\pi G}. \qquad (A4)$$

By taking time average, the second part of integral ($A1$) is

$$-\omega\,\mathrm{sgn}(k) r\frac{|\tilde{\phi}(r)|^2}{4G} = \frac{\omega}{m}\mathcal{F}_{\mathrm{J}}^{\mathrm{G}} \tag{A5}$$

according to equation (4.6).

We now consider the third part of integral ($A1$). By equation (2.16), one has

$$B_0\Re(v_r) = \frac{r}{m}(m\Omega - \omega)\Re(b_r). \tag{A6}$$

It then follows that

$$\frac{\Re(b_\theta)}{4\pi}[B_0\Re(v_r) - V_0\Re(b_r)] = -\frac{\omega r}{m}\frac{\Re(b_r)\Re(b_\theta)}{4\pi}. \tag{A7}$$

Integral (A1) can now be written as

$$\mathcal{F}_{\mathrm{E}} = \frac{\omega}{m}\left[\int_0^{2\pi} r\mathrm{d}\theta\mu_0\Re(v_r)r\Re(v_\theta) + \mathcal{F}_{\mathrm{J}}^{\mathrm{G}} - \int_{-\infty}^{+\infty}\mathrm{d}z\int_0^{2\pi} r\mathrm{d}\theta\frac{\Re(b_r)r\Re(b_\theta)}{4\pi}\right]. \tag{A8}$$

In reference to equation (4.4b), we immediately arrive at result (4.17), namely $\mathcal{F}_{\mathrm{E}} = (\omega/m)\mathcal{F}_{\mathrm{J}}$.

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