J. Group Theory **14** (2011), 333–339
DOI 10.1515/JGT.2010.051

Journal of Group Theory

On Lyndon's equation in some Λ-free groups and HNN extensions

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(Communicated by A. V. Borovik)

Abstract. In this paper we study Lyndon's equation $x^p y^q z^r = 1$, with x, y, z group elements and p, q, r positive integers, in HNN extensions of free and fully residually free groups, and draw some conclusions about its behavior in Λ-free groups.

1 Introduction

The classical result of Lyndon and Schützenberger ([9]) states that any elements x, y and z of a free group F that satisfy the relation $x^p y^q = z^r$ for $p, q, r \geqslant 2$ necessarily commute. In the paper of Brady, Ciobanu, Martino and O Rourke ([1]) this result has been generalized to Λ-free groups. In particular, the following result has been obtained. Let G be a group that acts freely on a Λ-tree, where Λ is an ordered abelian group, and let x, y, z be elements in G. If $x^p y^q = z^r$ with integers $p, q, r \geqslant 4$, then x, y and z must commute. It has been unclear whether the same conclusion holds for p, q, r not all larger than 4, and in particular the proof in [1] cannot be extended to these smaller integer cases. Here we shed light on the behavior of this equation in some HNN extensions and show that for p, q, r not all larger than 4 the conclusion of [1] does not always hold (see Corollary 1). This work complements the results in [5], where Lyndon's equation is studied in various amalgams of groups.

2 Results

Theorem 1. *Let F be a finitely generated non-cyclic free group, and let u and v be non-trivial elements in F which are not proper powers. Let $G = \langle F, t \mid tut^{-1} = v \rangle$ and $r \geqslant 2$ be a given integer. Then for particular choices of u and v there exist non-commuting elements $a, b, c \in G$ such that $a^2 b^2 c^r = 1$.*

Proof. The one-relator group $H = \langle a, b, c \mid a^2 b^2 c^r = 1 \rangle$ can also be written in terms of the presentation $\langle b, c, d \mid b^{-1} d^{-1} b = c^{-r} d \rangle$. This can be seen by letting $d = ab$ and writing the relation $a^2 b^2 c^r = 1$ as $db^{-1} db^{-1} b^2 c^r = 1$, which can then be rewritten as $b^{-1} d^{-1} b = c^{-r} d$.

Thus in the HNN extension $\langle b, c, d \mid b^{-1}d^{-1}b = c^{-r}d\rangle$ of the free group generated by $\{c, d\}$, with stable letter b and associated subgroups $\langle d\rangle$ and $\langle c^{-r}d\rangle$, the equality $a^2b^2c^r = 1$, where $a = db^{-1}$, will be satisfied, but none of a, b, c will commute.

We can clearly take the HNN extension of any finitely generated non-cyclic free group F with associated cyclic subgroups of the form $\langle x\rangle$ and $\langle y^r x\rangle$, where x and y are generators of F, and an equality of the form $a^2b^2c^r = 1$ will be satisfied without any of a, b, c commuting. □

For brevity we refer the reader to [2] for a complete account of Λ-trees and the groups that act freely on them, called Λ-free groups.

Corollary 1. *There exist Λ-free groups in which*

$$a^2b^2c^r = 1$$

holds for non-commuting $a, b, c \in G$, and $r \geqslant 2$.

Proof. In [10, Theorem 3.1] it is shown that groups with a presentation of the form $\langle x, y, x_1, \dots, x_n \mid xyx^{-1}y^{\varepsilon} = w\rangle$, where w is any word in $\{x_1, \dots, x_n\}$, act freely on $(\mathbb{Z} \times \mathbb{Z})$-trees, unless $\varepsilon = 1$ and $w = 1$. The groups arising in the proof of Theorem 1, i.e. the groups $\langle b, c, d \mid b^{-1}d^{-1}b = c^{-r}d\rangle$ where $r \geqslant 2$, have this form and will therefore act on a Λ-tree. They also satisfy the equation $a^2b^2c^r = 1$, where $a = db^{-1}$, with non-commuting a, b, c. □

Before we state the next results we need to make the following observations. Let F be a free group with basis X. We remind the reader (see for example [8, Chapter I.4]) that a *Whitehead automorphism* of F is an automorphism τ of one of the following two kinds:

(1) τ permutes the elements of $X^{\pm 1} = X \cup X^{-1}$;

(2) for some fixed $a \in X^{\pm 1}$, τ carries each of the elements $x \in X^{\pm 1}$ into one of x, $a^{-1}x$, xa or $a^{-1}xa$.

We now consider the special situation with $X = \{x_1, x_2, x_3\}$ and F free on X. In F we consider the two words

(1) $w_1(x_1, x_2, x_3) = x_1^p x_2^q x_3^r$ with $p \geqslant 2$, $q \geqslant 3$, $r \geqslant 3$;

(2) $w_2(x_1, x_2, x_3) = g_1 x_3 u^{\alpha_1} x_3^{-1} g_2 x_3 u^{\alpha_2} x_3^{-1} \dots g_n x_3 u^{\alpha_n} x_3^{-1}$ with $n \geqslant 1$, $1 \neq u = u(x_1, x_2)$ and u is not conjugate to a power of x_1 or x_2, α_i non-zero integers and $g_1, \dots, g_n$ freely reduced words in $\langle x_1, x_2\rangle$, with $g_i \neq 1$ for $i = 1, \dots, n$.

Remark 1. Assume that $p, q, r \geqslant 3$. We first note that $w_1 \neq w_2$ and w_1 is *minimal* (with respect to length) in its automorphic orbit. It can be easily seen that if τ is a Whitehead automorphism of F of type (2), if we apply τ to $w_1(x_1, x_2, x_3)$, then the length strictly increases. The minimality of w_1 also shows that it cannot be a primitive element by [8, Proposition 4.17]. In fact, w_1 is a word of minimal rank, also called a *regular* word, that is, there is no Nielsen transformation from $\{x_1, x_2, x_3\}$ to a system d, f, g with $x_1^p x_2^q x_3^r \in \{d, f\}$ (see [6]).

Remark 2. We now consider $w_2(x_1, x_2, x_3)$. If w_2 is minimal, then there is no Whitehead automorphism τ such that the length strictly decreases when applying τ. Hence, if w_2 is minimal, there is no automorphism taking w_1 to w_2 by [8, Proposition 4.17], as the only automorphism taking w_1 to w_2 would be a permutation, and the form of the two words does not allow for a permutation to send w_1 to w_2. If w_2 is not minimal, then each Whitehead automorphism which decreases the length of w_2 will take w_2 to a word of the same form. To see this, notice that u contains both x_1 and x_2. If, for instance, $g_1 = g_1' x_2$, then an automorphism which replaces x_2 by $x_2 x_3 = x_2'$ gives $x_2' x_3^{-1}$ at all other places where x_2 occurred, especially inside u.

Remark 3. If $p = 2$ and $q, r \geqslant 3$, then w_1 is still minimal, and when we apply a Whitehead automorphism τ to $w_1(x_1, x_2, x_3)$ the length strictly increases, except when τ is of the form $x_1 \to x_1 x_2^{-1}$, $x_2 \to x_2$, $x_3 \to x_3$, in which case the length stays the same. If w_2 is minimal, the only automorphisms that could take w_1 to w_2 are of the form τ composed with permutations, and one can see that such automorphisms cannot take w_1 to a word of the form w_2. As in Remark 2, if w_2 is not minimal, then each Whitehead automorphism which decreases the length of w_2 will take w_2 to a word of the same form.

From the minimality of w_1 and the facts about w_2 in the above paragraphs we get the following.

Lemma 1. *Let F be free with basis $\{x_1, x_2, x_3\}$ and $w = w_1(x_1, x_2, x_3) = x_1^p x_2^q x_3^r$ with $p \geqslant 2$, $q \geqslant 3$, $r \geqslant 3$. Then there is no automorphism α of F with $\alpha(x_i) = y_i$, $i = 1, 2, 3$, such that $\alpha^{-1}(w)$ is, written in y_1, y_2, y_3, of the form*

$$g_1 y_3 u^{\alpha_1} y_3^{-1} g_2 y_3 u^{\alpha_2} y_3^{-1} \dots g_n y_3 u^{\alpha_n} y_3^{-1}$$

with $n \geqslant 1$, $1 \neq u = u(y_1, y_2)$ and u not conjugate to a power of y_1 or y_2, all α_i non-zero integers, and $g_1, \dots, g_n$ freely reduced words in $\langle y_1, y_2 \rangle$, with $g_i \neq 1$ for $i = 1, \dots, n$.

Lemma 1 states that words of the form w_1 and w_2 cannot be in the same automorphic orbit.

Theorem 2. *Let F be a finitely generated free group, u and v non-trivial elements in F that are not proper powers, and $G = \langle F, t \mid tut^{-1} = v \rangle$. Let $a, b, c \in G$ satisfy $a^p b^q c^r = 1$ with $p \geqslant 2$, $q \geqslant 3$, $r \geqslant 3$.*

(i) *If u is not conjugate to v^{-1}, then a, b, c must commute.*

(ii) *If u is conjugate to v^{-1}, then a, b, c either commute or generate the Klein bottle group $\langle x, y \mid xyx^{-1}y = 1 \rangle$.*

Proof. Let $H = \langle a, b, c \rangle$. We will consider three cases:

(1) u and $v^{\pm 1}$ are not conjugate;

(2) u and v are conjugate;

(3) u and v^{-1} are conjugate.

In case (1) assume that H is not abelian. Then H cannot be free of rank 2 or 3 because the word $a^p b^q c^r$ is regular by Remark 1. Thus, by [4, Theorem 2.2], H must be a one-relator group, that is, in our case, $H = \langle a, b, c \mid a^p b^q c^r = 1 \rangle$. Furthermore, by the proof of [4, Theorem 2.2], there is a Nielsen transformation from $\{a, b, c\}$ to a system $\{x_1, x_2, x_3\}$ for which we may assume, without loss of generality, that $x_1, x_2 \in F$, $\langle x_1, x_2 \rangle$ non-cyclic, $x_3 = t$ and H has a presentation of the form

$$H = \langle x_1, x_2, x_3 \mid w(x_1, x_2, x_3) = 1 \rangle,$$

with $w(x_1, x_2, x_3) = g_1 x_3 h^{\alpha_1} x_3^{-1} g_2 x_3 h^{\alpha_2} \dots g_n x_3 h^{\alpha_n} x_3^{-1}$, $n \geqslant 1$, $h = u^\alpha \in \langle x_1, x_2 \rangle$, $\alpha \neq 0$, $\alpha_i \neq 0$ and $g_i \in \langle x_1, x_2 \rangle$ non-trivial and freely reduced for $i = 1, \dots, n$. But this contradicts Lemma 1 if h is not conjugate to a power of x_1 or x_2. Therefore H is abelian if h is not conjugate to a power of x_1 or x_2.

Now suppose that h is conjugate to a power of x_1 or x_2. Without loss of generality we may assume that $h = x_1^\gamma$. Since $h = u^\alpha$ and u is not a proper power in F we get that $x_1 = u^\delta$ in F. With respect to the equation $a^p b^q c^r = 1$ and because $H = \langle a, b, c \rangle$ we may replace x_1 by u. Hence, let $x_1 = u$. We may also assume that x_2 is not a proper power in F. Now let both x_1 and x_2 be not a proper power in F. Using this and the cancellation arguments in [4, Theorems 1 and 2], we see that v is in $\langle x_1, x_2 \rangle$ because H is not free of rank 2 or 3. Hence, H has a presentation of the form $K = \langle x_1, x_2, t \mid t x_1 t^{-1} = v \rangle$ with $v = v(x_1, x_2)$ freely reduced in x_1 and x_2, and $t = x_3$. But in the free group on a, b, c there is no automorphism φ with $\varphi(a) = x_1$, $\varphi(b) = x_2$ and $\varphi(c) = t$ such that $\varphi^{-1}(a^p b^q c^r)$ with $2 \leqslant p$, $3 \leqslant q$, $3 \leqslant r$ is, written in x_1, x_2 and t, of the form $t x_1 t^{-1} v^{-1}$. This gives a contradiction. Hence, H is abelian in this case as well.

In case (2), we can assume without loss of generality that $u = v$. Then G is fully residually free, which implies that H is fully residually free. Thus H has the same universal theory as that of free groups. Therefore $a^p b^q c^r = 1$ with $p, q, r \geqslant 2$ implies that a, b and c commute.

In addition we remark that, for the case $u = v$, by the classification given in [7, Theorem 5], any non-abelian, non-free rank 3 subgroup K of G is a free rank one extension of centralizers of a free group of rank 2, that is, in our case: $K = \langle x_1, x_2, x_3 \mid x_3 h x_3^{-1} = h \rangle$ with $h \in \langle x_1, x_2 \rangle$; additionally, either h is regular and not a proper power in G or h is not regular, in which case K is isomorphic to $\langle x, y, \mid xy = yx \rangle \star \langle z \mid \rangle$ (the free product of a free abelian group of rank 2 and the integers).

In case (3), we can assume without loss of generality that $u = v^{-1}$, that is, $G = \langle F, t \mid t u t^{-1} = u^{-1} \rangle$. Since $t^2 u t^{-2} = u$, one can easily extend the arguments in [7, Theorem 5] (which only rely on the Nielsen cancellation method, and no residual properties, in a group with relation $u = v$, in order to obtain a classification of rank 3 subgroups), regarding non-abelian, non-free rank 3 subgroups of fully residually free groups to the case $u = v^{-1}$ and obtain that any non-abelian, non-free rank 3 subgroup K of G has a presentation $K = \langle x_1, x_2, x_3 \mid x_3 h x_3^{-1} = h^{\varepsilon} \rangle$, with $\varepsilon = \pm 1$ and $h \in \langle x_1, x_2 \rangle$; additionally, either h is regular and not a proper power in G or h is not regular, in which case K is isomorphic to $\langle x, y \mid xyx^{-1} = y^{\varepsilon} \rangle \star \langle z \mid \rangle$, where $\varepsilon = 1$ or $\varepsilon = -1$.

We assume now that $H = \langle a, b, c \rangle$ is not free, not abelian and of rank 3. Then H is isomorphic to a subgroup K of G as described above, that is, $H \cong K = \langle x_1, x_2, x_3 \mid x_3 h x_3^{-1} = h^{\varepsilon} \rangle$. If $\varepsilon = 1$, then H is fully residually free and so $a^p b^q c^r = 1$ implies that a, b and c commute.

Now let $\varepsilon = -1$. In case h is regular, and not a proper power, then, if H is non-free and not abelian, by the above arguments H has to be a one-relator group, that is, in our case, $H = \langle a, b, c \mid a^p b^q c^r = 1 \rangle$ since $a^p b^q c^r$ is regular. By the same arguments as in case (1), there must be a Nielsen transformation from $\{a, b, c\}$ to a system $\{x_1, x_2, x_3\}$ for which, without loss of generality, H has a presentation of the form $\langle x_1, x_2, x_3 \mid x_3 h x_3^{-1} h = 1 \rangle$. But $x_3 h x_3^{-1} h$ is a word of the form w_2, and so by Lemma 1 this cannot happen.

If h is not regular, then $H \cong K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle \star \langle z \mid \rangle$. But since a, b, c satisfy $a^p b^q c^r = 1$ and $a^p b^q c^r$ is regular, [6, Theorem 5.2] implies that H must in fact be a rank 2 subgroup, which is not the case.

Finally, let $H = \langle a, b, c \rangle$ be a non-abelian, non-free rank 2 subgroup. Then by [3, Theorem 1] H must be the Klein bottle group

$$V = \langle x, y \mid xyx^{-1} = y^{-1} \rangle \cong \langle u, v \mid u^2 = v^2 \rangle.$$

In V the elements u and v do not commute. Thus by taking, for instance, $a = u$, $b = u^{-1}$ and $c = v^{-1}$, since $u^{12} u^{-8} = v^4$, one gets that $a^{12} b^8 c^4 = 1$. □

The groups in Theorem 2 for which the translation lengths of u and v are equal are in fact Λ-free groups by Bass' work (see [10, Theorem 2.4.1]). We may extend Theorem 2 to the following result, after reminding the reader that a group G is called *n-free* for a positive integer n if every subgroup of G generated by n elements is free.

Theorem 3. *Let L be a non-cyclic, 2-free, fully residually free group, u and v non-trivial elements in L that are not proper powers and u is not conjugate to v^{-1}. Let $G = \langle L, t \mid tut^{-1} = v \rangle$. If $a, b, c \in G$ satisfy $a^p b^q c^r = 1$ for $p \geqslant 2$, $q \geqslant 3$, $r \geqslant 3$ then a, b, c must commute.*

Proof. We first remark that L is also 3-free by [7].

If u is conjugate to v, then we may assume that $u = v$. Then G is fully residually free and hence Theorem 3 holds.

From now on we assume that u is not conjugate to v. In the proof of Theorem 2 we used the classification of the rank 3 subgroups of G for the case that L is a non-abelian free group (see [4]). In [4], in addition to the standard Nielsen cancellation method in HNN groups, one only needs three properties of L and G respectively:

(1) the subgroups $\langle u \rangle$ and $\langle v \rangle$ are malnormal in L;

(2) L is 3-free;

(3) each two-generator subgroup of G is free.

Now let L, as in the statement of Theorem 3, be a non-cyclic, 2-free, fully residually free group. We have to show that the properties (1), (2) and (3) also hold in this more general situation.

(1) holds because L is 2-free. Let $x \in L$ be such that $xu^{\alpha}x^{-1} = u^{\beta}$ for some integers $\alpha, \beta \neq 0$. Since L is 2-free, the subgroup $\langle x, u \rangle$ of L is cyclic. Hence $x \in \langle u \rangle$.

(2) holds by the above remark that L is 3-free.

We now show that (3) also holds. In [3], in the special case that L is a non-abelian free group we have used, besides the Nielsen cancellation method in HNN groups and property (1), only the fact that L is 2-free. But this we assume anyway for L. Hence (3) also holds for the more general situation. We may now apply analogous arguments to the ones in the proof of Theorem 2. □

Corollary 2. *Let* $S = \langle a_1, b_1, \ldots, a_n, b_n \mid [a_1, b_1] \ldots [a_n, b_n] = 1 \rangle$, $n \geqslant 2$, *be an orientable surface group of genus* $\geqslant 2$ *or* $S = \langle a_1, \ldots, a_n \mid a_1^2 \ldots a_n^2 = 1 \rangle$ *a non-orientable surface group of genus* $\geqslant 4$. *Let u and v be non-trivial elements in S that are not proper powers and u is not conjugate to* v^{-1}, *and let* $G = \langle S, t \mid tut^{-1} = v \rangle$. *Then if for* $a, b, c \in G$ *and* $p \geqslant 2$, $q \geqslant 3$, $r \geqslant 3$ *the equality* $a^p b^q c^r = 1$ *holds, the elements a, b, c must commute.*

Proof. In both cases S is a non-cyclic, 2-free, fully residually free group. Hence Corollary 2 holds by Theorem 3. □

Acknowledgments. The first-named author was supported by the Marie Curie Reintegration Grant 230889 and the SNF (Switzerland) through project number 200020-121506. The third-named author gratefully acknowledges the hospitality of the mathematics department at the University of Fribourg and the support provided by the SNF (Switzerland) through project number 200020-121506.

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Received 20 April, 2010; revised 22 June, 2010

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