Geophys. J. Int. (2009) **179,** 923–928 doi: 10.1111/j.1365-246X.2009.04340.x

The construction of exact Taylor states. I: The full sphere

P. W. Livermore,[1] G. Ierley[1] and A. Jackson[2]

[1]Institute of Geophysics and Planetary Physics, Scripps Institution of Oceanography, UCSD, San Diego, CA, USA. E-mail: phil@ucsd.edu
[2]Institut für Geophysik, ETH, Zürich, Switzerland

Accepted 2009 July 17. Received 2009 June 16; in original form 2008 October 1

SUMMARY

The dynamics of the Earth's fluid core are described by the so-called magnetostrophic balance between Coriolis, pressure, buoyancy and Lorentz forces. In this regime the geomagnetic field is subject to a continuum of theoretical conditions, which together comprise Taylor's constraint, placing restrictions on its internal structure. Examples of such fields, so-called Taylor states, have proven difficult to realize except in highly restricted cases. In previous theoretical developments, we showed that it was possible to reduce this infinite class of conditions to a finite number of coupled quadratic homogeneous equations when adopting a certain regular truncated spectral expansion for the magnetic field. In this paper, we illustrate the power of these results by explicitly constructing two families of exact Taylor states in a full sphere that match the same low-degree observationally derived model of the radial field at the core–mantle boundary. We do this by prescribing a smooth purely poloidal field that fits this observational model and adding to it an expediently chosen unconstrained set of interior toroidal harmonics of azimuthal wavenumbers 0 and 1. Formulated in terms of the toroidal coefficients, the resulting system is purely linear and can be readily solved to find Taylor states. By calculating the extremal members of the two families that minimize the Ohmic dissipation, we argue on energetic ground that the toroidal field in the Earth's core is likely to be dominated by low order azimuthal modes, similar to the observed poloidal field. Finally, we comment on the extension of finding Taylor states within a general truncated spectral expansion with arbitrary poloidal and toroidal coefficients.

Key words: Numerical solutions; Electromagnetic theory; Dynamo: theories and simulations; Planetary interiors.

1 INTRODUCTION

Recent numerical simulations of the geodynamo have successfully reproduced magnetic fields with many geophysically realistic characteristics such as Earth-like intensities, dipolar dominance and even global reversals (Dormy *et al.* 2000; Kono & Roberts 2002). Yet a fundamental problem in all of these models is their inability to access the extreme parameter regime appropriate to Earth's core. Of particular note is the Ekman number E, a measure of viscosity non-dimensionalized by the Coriolis force estimated to be $O(10^{-15})$ in the core, yet even state-of-the-art simulations cannot attain values smaller than $O(10^{-7})$ due to inherent computational difficulties (Kageyama *et al.* 2008). Consequently, the force balance in the modelled Earth's core is almost certainly incorrect: not only does viscosity play far too large a role in the dynamics but, alongside other parameter restrictions, introduces artificially large inertial effects (Sreenivasan & Jones 2006).

The correct dynamical regime in the Earth's core is described by the so-called magnetostrophic balance between the Coriolis, pressure, buoyancy and Lorentz forces (Fearn 1998); notably absent are inertia and viscosity which are believed subdominant. Forty-six years ago Taylor (1963) showed that, in the more general setting of any rapidly rotating homogeneous Boussinesq fluid, a certain condition applies to the magnetic field that has subsequently been termed Taylor's constraint. He showed that the azimuthal Lorentz force must average to zero over any geostrophic surface, namely fluid cylinders concentric with the rotation axis. This can be written

$$\mathcal{T}(s) \equiv \int_{C(s)} ([\nabla \times \mathbf{B}] \times \mathbf{B})_\phi \, s \, \mathrm{d}\phi \, \mathrm{d}z = 0 \,, \tag{1}$$

where (s, ϕ, z) are cylindrical polar coordinates, $\mathbf{B}$ denotes the magnetic field and $C(s)$ any geostrophic contour. In the presence of an inner core, these cylinders partition into three distinct sets: those outside the tangent cylinder, and those above and below the inner core (Livermore *et al.* 2008). Taylor's condition is also satisfied in the wider setting of fluids that are compressible but stratified (of which the Boussinesq approximation is a special case), as described by the anelastic approximation (Smylie & Rochester 1984).

Examples of 3-D Taylor states, magnetic fields that satisfy Taylor's constraint, have been hard to find. The geomagnetic field in the core is believed to be one, although its internal structure is hidden from view since only the radial component of the field on the core–mantle boundary (CMB) is observable. Current geodynamo models are not yet in an Earth-like parameter regime and cannot produce Taylor states, although there is some evidence that, by measuring the 'Taylorization' of the magnetic field, the correct regime is being approached (Rotvig & Jones 2002; Takahashi *et al.* 2005). By imposing particular symmetry on the field, Jault & Cardin (1999) showed that a 3-D Taylor state could be constructed, although their example is arguably rather artificial. Progress is much easier when assuming axisymmetry and several examples of Taylor states have been found by solving the axisymmetric dynamo equations with small E (Soward & Jones 1983; Hollerbach & Ierley 1991). However, since these are all specific cases, it has not been possible to address the broader question of what, if anything, characterizes the internal structure of a Taylor state. The issue of uniqueness is of significant interest since we can only observe the geomagnetic field on the CMB. Imposing the divergence-free condition on the field (providing an infinite set of constraints when written as $\int_V \mathbf{B} \cdot \mathrm{d}S = 0$ over an arbitrary volume V) reduces the three unknown components of field to just two: the toroidal and poloidal scalars. If we additionally impose the infinite continuum of Taylor constraints, how constrained does the field, in particular its hidden toroidal component, now become?

Taylor's constraint has been difficult to address for two main reasons. First, it is defined in cylindrical coordinates, whereas the natural coordinate system to use in a sphere is spherical polar coordinates presenting immediate problems in converting between the two. Second, and perhaps more important, is that Taylor's constraint represents an infinite set of conditions, intractably difficult if only a finite number of degrees of freedom are available, as is typically the case in any geophysical model. In a recent study on the mathematical structure of Taylor's constraint (Livermore *et al.* 2008) we showed that, on expanding the field in a certain truncated spectral expansion, not only did this continuum collapse to a finite number of constraints but that the problem was, after all, greatly underdetermined. This admits the possibility of ubiquitous Taylor states although the question of how to find any such examples was left open, in general requiring the solution of many coupled quadratic equations. The purpose of this paper is to explore some of these issues by explicitly constructing various families of Taylor states. Before doing so however, we first discuss some wider aspects applicable to a spherical harmonic expansion with arbitrary radial representation.

Let us represent the magnetic field in a truncated set of poloidal and toroidal vector spherical harmonics,

$$\mathbf{B} = \sum_{l=1}^{L_{\max}} \sum_{m=0}^{l} \mathbf{S}_l^{m\,s/c} + \mathbf{T}_l^{m\,s/c},$$

where

$$\mathbf{S}_l^{m\,s/c} = \nabla \times \nabla \times \left[Y_l^{m\,s/c}(\theta, \phi)\, S_l^{m\,s/c}(r)\, \hat{\mathbf{r}} \right],$$

$$\mathbf{T}_l^{m\,s/c} = \nabla \times \left[Y_l^{m\,s/c}(\theta, \phi)\, T_l^{m\,s/c}(r)\, \hat{\mathbf{r}} \right],$$

in spherical polar coordinates (r, θ, ϕ) and with $\hat{\mathbf{r}}$ denoting the unit position vector. The notation $Y_l^{m\,s/c}$ represents a spherical harmonic of degree l, order m, and azimuthal dependence $\sin m\phi$ or $\cos m\phi$ as appropriate; we adopt the usual Schmidt quasi normalization as is common in geomagnetism. The quantity $\mathcal{T}(s)$ of eq. (1), being purely quadratic in the magnetic field, can now be viewed as a double sum over all contributions or 'interactions' from each possible pair of harmonics. Because of the symmetries inherent in the spherical harmonic representation, many of these interactions are zero, a fact made explicit in Livermore *et al.* (2008) by writing down an appropriate set of 'selection rules'. One way of producing a Taylor state is to construct a magnetic field using a set of harmonics that contains no mutual interactions, ensuring that $\mathcal{T}$ is identically zero. We can readily compile an illustrative list of simple Taylor states in order of increasing angular complexity.

(i) Any single spherical harmonic (of any radial dependence).

(ii) Any purely axisymmetric toroidal or poloidal field.

(iii) Any field that has a single harmonic of each wavenumber m.

(iv) Any purely poloidal or toroidal field that, for each wavenumber m, has wavenumber dependence of either $\cos m\phi$ or $\sin m\phi$ but not both.

(v) Any purely poloidal or toroidal field that, for each wavenumber m, has only two latitudinal modes, l_1 and l_2, of different parity.

(vi) Any field that is either symmetric or anti-symmetric with respect to a rotation of π radians about the x-axis.

For instance, (i) follows as no spherical harmonic can interact with itself and (v) by noting that only harmonics of the same wavenumber and equatorial symmetry interact.

More generally, we can find Taylor states even within sets of harmonics that do interact. Assuming a full sphere (with no inner core) we may write each harmonic scalar in a truncated expression of the form, for example,

$$S_l^{mc}(r) = r^{l+1}\, W(r^2) \tag{2}$$

for some polynomial W of degree $N_{\max}$ that guarantees that the magnetic field is everywhere smooth, including at the origin where there is a coordinate singularity (Livermore *et al.* 2007). In Livermore *et al.* (2008), we showed that

$$\mathcal{T}(s) = s^2 \sqrt{1 - s^2}\, Q_N(s^2) \tag{3}$$

for some polynomial Q_N, of maximum degree N related both to $N_{\max}$ and $L_{\max}$. This remarkable result has the implication that the seeming infinity of Taylor constraints is reduced to just $N + 1$, for it is readily observed that we can obtain a Taylor state by simply equating each coefficient of Q_N to zero. In fact, if we assume electrically insulating boundary conditions, the coefficients of Q_N (themselves quadratic functions of the spectral coefficients) are linearly and homogeneously related and only N conditions are required to find a Taylor state. Despite this vast simplification, we remain in the position of having to solve N coupled quadratic equations for which there is no simple algorithm. The key idea in this paper is that by further exploitation of the selection rules, we can find a set of harmonics that do not mutually interact and so their associated coefficients appear only linearly in the constraint equations assuming all other coefficients are taken to be prescribed. This system is readily solved and immediately produces an exact Taylor state. In the remainder of the paper, we will illustrate this procedure by explicitly constructing a variety of examples of exact Taylor states in a full sphere surrounded by an electrically insulating mantle. For simplicity, we will assume that the poloidal field is prescribed and that our task is to find a toroidal component that produces a Taylor state, a procedure analogous to the asymptotic analysis of Greenspan (1974). To expedite matters still further, we will look only for large-scale solutions, working within a truncation of spherical harmonic degree three. The angular structure of the poloidal field on the CMB is chosen to be that of the xCHAOS model (Olsen & Mandea 2008) at epoch 2004 whose coefficients are given in Table 1, extended inside the core by the radial profile proportional to

$$(2l + 3) r^{l+1} - (2l + 1) r^{l+3},$$

Table 1. Gauss coefficients in units of nT from the xCHAOS model of Olsen & Mandea (2008) at epoch 2004 up to degree 3.

l	m	g_l^m	h_l^m
1	0	−29566.51	
1	1	−1682.50	5100.53
2	0	−2324.61	
2	1	3052.82	−2572.00
2	2	1658.94	−503.82
3	0	1335.88	
3	1	−2302.15	−205.03
3	2	1247.91	275.08
3	3	681.37	−520.12

Note: The magnetic field outside the core in the electrically insulating mantle is represented by $\mathbf{B} = -\nabla V$ where $V = a\sum_{l=1}^{3}\sum_{m=0}^{l}(\frac{a}{r})^{l+1}[g_l^m \cos m\phi + h_l^m \sin m\phi]P_l^m(\cos\theta)$, P_l^m is a Schmidt quasi-normalized associated Legendre function and $a =$ 6371.2 km is the Earth's mean radius.

which produces the field of least Ohmic dissipation consistent with the imposed boundary conditions (Backus *et al.* 1996). Note that, had we chosen $L_{\max} = 2$, then this purely poloidal field would have trivially been a Taylor state by observation (v) above. The truncation $L_{\max} = 3$ provides the simplest field within this set that is not, by itself, a Taylor state and is therefore illustrative of a general poloidal field. In the absence of any toroidal field, it is straightforward to show that the polynomial Q_N appearing in (3) due to the poloidal field is proportional to

$$Q_S(\sigma) = 5.94 - 17.22\sigma + 10.24\sigma^2, \quad (4)$$

where $\sigma = s^2$. Let us write each toroidal scalar $T_l^m(r)$ in terms of the representation

$${}_nT_l^m(r) = r^{l+1}\,(1 - r^2)\,P_{n-1}^{(3/2,l+1/2)}(2r^2 - 1), \quad (5)$$

where $P_k^{(\alpha,\beta)}$ is a Jacobi polynomial of degree k and $n \geq 1$ (Livermore 2009). Then we may write the toroidal field in terms of the individual basis vectors ${}_n\mathbf{T}_l^{m\,s/c}$. For both toroidal and poloidal harmonics, we omit the superscript m when $m = 0$. The precise details of the radial representation do not matter here, except to note that each is of the form $r^{l+1}\,W(r^2)$ for some polynomial W of degree n, and that each toroidal scalar function vanishes at $r = 1$ ensuring that electrically insulating boundary conditions are satisfied.

2 AN AXISYMMETRIC TOROIDAL FIELD

We consider here the sequential addition of modes from the class of axisymmetric toroidal modes to the prescribed poloidal field in an attempt to find an exact Taylor state. This is perhaps the simplest such construction since, there being no interaction between any such modes [case (ii) above], the analysis is guaranteed to be linear. The interaction associated with each new mode provides an additional polynomial, linear in the toroidal coefficient, which is added to (4). This results in a Taylor integral proportional to

$$Q(\sigma) = Q_S(\sigma) + \sum_{i=1}^{N} \alpha_i\, Q_{d_i}^i(\sigma)$$

after N modes that must vanish in a Taylor state. The polynomial $Q_{d_i}^i$, associated with toroidal mode i is of degree d_i that must be determined by inspecting which poloidal harmonics it interacts with. Fig. 1 shows d_i for a variety of simple toroidal harmonics with the

${}_1\mathbf{T}_1$	$\mathbf{S}_2$	$d_i = 2$
${}_1\mathbf{T}_3$	$\mathbf{S}_2$	$d_i = 3$
${}_1\mathbf{T}_2$	$\mathbf{S}_1, \mathbf{S}_3$	
${}_2\mathbf{T}_1$	$\mathbf{S}_2$	
${}_3\mathbf{T}_1$	$\mathbf{S}_2$	$d_i = 4$
${}_2\mathbf{T}_2$	$\mathbf{S}_1, \mathbf{S}_3$	
${}_2\mathbf{T}_3$	$\mathbf{S}_2$	
${}_1\mathbf{T}_4$	$\mathbf{S}_1, \mathbf{S}_3$	
${}_1\mathbf{T}_5$	$\mathbf{S}_2$	

Figure 1. The degree d_i of the polynomial $Q_{d_i}^i$ resulting from the interactions between toroidal and poloidal modes.

interacting poloidal harmonics. It is clear that d_i is not necessarily increased by the addition of each new toroidal mode; thus a situation with more degrees of freedom than constraints is quickly reached. For example, consider adding just one mode, ${}_1\mathbf{T}_1$. Fig. 1 shows that $d_1 = 2$; thus the addition of $Q_2^1(\sigma)$ leaves $Q(\sigma)$ a polynomial of degree 2. Noting the linear degeneracy, there are 2 constraints (being the coefficients of σ^0 and σ^1) but only one degree of freedom (α_1). A solution is therefore not possible in general. By adding in ${}_1\mathbf{T}_2$, we note that although the degrees of freedom increases by one, according to Fig. 1, so does the degree of Q. Thus the problem is still overdetermined. On adding in ${}_1\mathbf{T}_3$ and ${}_2\mathbf{T}_1$, the degree of $Q(\sigma)$ remains three (providing three constraints due to the linear degeneracy) but now four degrees of freedom are available. On writing the toroidal field as $\alpha_1\,{}_1\mathbf{T}_1 + \alpha_2\,{}_1\mathbf{T}_2 + \alpha_3\,{}_1\mathbf{T}_3 + \alpha_4\,{}_2\mathbf{T}_1$, by solving for $\{\alpha_1, \alpha_2, \alpha_3\}$ in terms of α_4, we obtain a 1-D family of solutions parametrized by α_4. We note that this is made possible by the fact that Q_2^1, Q_3^2 and Q_3^3 are linearly independent and enables a solution of the linear 3×3 system.

Of the resulting infinite class of Taylor states, we now show two extremal solutions. First, the solution with the simplest radial profile is obtained by setting $\alpha_4 = 0$ and is shown in Fig. 2(a). Adopting the Ohmic dissipation as a measure of complexity, the solution that

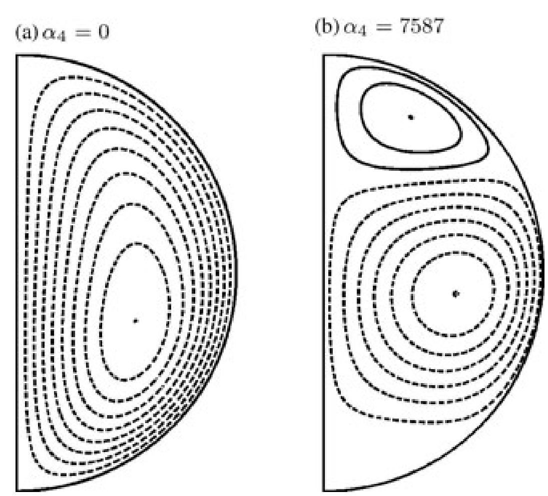

Figure 2. Contours of $\mathbf{B}_\phi$ in a meridian plane illustrating examples of axisymmetric toroidal fields within the given four harmonic set that, when added to the given poloidal field, produce an exact Taylor state. Left-hand panel shows the solution with $\alpha_4 = 0$ corresponding to harmonics with the simplest radial profile. Right-hand panel shows $\alpha_4 = 7587$(approximately) corresponding to the minimum dissipation solution. The contour levels are equally spaced, with solid denoting positive values and dashed negative.

minimizes

$$\frac{\eta}{\mu_0}\int_V |\nabla \times \mathbf{B}|^2 \, \mathrm{d}V,$$

(where $\eta = 1.6\,\mathrm{m}^2\,\mathrm{s}^{-1}$ is the magnetic diffusivity and μ_0 the permeability of free space) is $\alpha_4 = 7587$ and is shown in 2(b).

The rms strength of the prescribed poloidal field, B_S, is 0.5 mT and, although relatively weak compared to more realistic estimates, it is still of interest to compare the relative toroidal field strength in both extremal examples. Case (a) yields a toroidal field that takes rms value $B_T = B_S/17$ and in case (b) $B_T = B_S/29$. Thus, by admitting more spatial structure, we can find a Taylor state with a substantially weaker toroidal field. Case (b) yields a toroidal dissipation of 1.68×10^5 W, compared to that of the poloidal field of 8.38×10^7 W when adopting a core radius of 3485 km. Of course, these only correspond to extremal solutions; α_4 is arbitrary and toroidal field strengths exceeding any given threshold can be obtained.

Finally, we comment on the robustness of these results. First, although the toroidal field appears to be qualitatively robust under small changes in the poloidal spectrum, large changes in the spectrum could significantly alter (4) and therefore the required toroidal field for a Taylor state. Second, the poloidal profile used inside the core, being not only physically meaningful, is expedient in the sense that it the lowest degree polynomial solution that is both regular at the origin and satisfies the electrically insulating boundary conditions. Any change in this profile will introduce higher degree terms into (4), requiring more toroidal modes to produce a Taylor state. Whilst an interesting extension, such analysis introduces more complexities than is warranted in this short communication. Finally, the toroidal field of minimum energy, rather than that of minimum dissipation, has a qualitatively similar structure to that of Fig. 2(b).

3 A TOROIDAL FIELD OF NON-ZERO WAVENUMBER

We now illustrate the more general case of the addition of asymmetric toroidal modes of single azimuthal wavenumber. This increases the complexity of the analysis since in general any such set of modes will mutually interact, introducing quadratic terms in the unknown coefficients. In general, the system therefore comprises coupled quadratic equations and is great deal harder to solve. Such a case is illustrated by looking for Taylor states with 8 simple toroidal modes of wavenumber $m = 1$

$$\left\{{}_1\mathbf{T}_1^{1s}, {}_1\mathbf{T}_1^{1c}, {}_1\mathbf{T}_2^{1s}, {}_1\mathbf{T}_2^{1c}, {}_1\mathbf{T}_3^{1s}, {}_1\mathbf{T}_3^{1c}, {}_2\mathbf{T}_1^{1s}, {}_2\mathbf{T}_1^{1c}\right\}$$

chosen so that $Q(\sigma)$ is a polynomial of degree not more than four. Fig. 3 shows all the interactions with the fixed $m = 1$ poloidal field. By recalling the linear degeneracy there are four constraints, leaving a solution space that has dimension four.

The manner in we look for solutions is illustrated by a simple example. Consider solving the set of two coupled equations in three unknowns

$$xy + yz + z + 2y = 1, \qquad 2xy + z + x = 0$$

which has a one parameter family of solutions. Regarding this free parameter as x or z, we could go ahead and solve the equations for the remaining variables but in each case this requires solution of a quadratic system and, as it turns out, is needlessly complicated. A simpler approach is to regard y as the free parameter; since x and z do not explicitly couple, this results in a purely linear problem that is readily solved. It is clear that, which ever way we chose to solve the equations, the same space of solutions is obtained. This simple

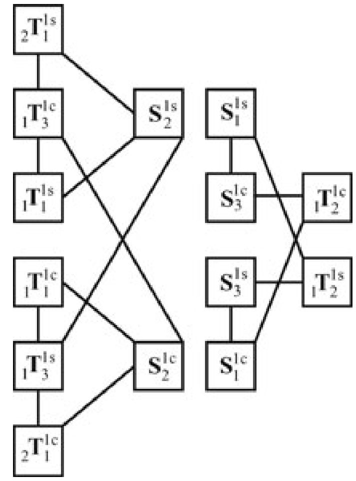

Figure 3. The interactions between the eight $m = 1$ toroidal modes and the ($m = 1$) fixed poloidal field, shown by connecting solid lines. Because each spherical harmonic has definite equatorial symmetry, the modes split into two separate groups. An expedient way of constructing a Taylor state is to choose four toroidal modes that do not interact (from either or both symmetry groups), and consider all other modes as prescribed.

argument can be extended to the search for a Taylor state. Provided we can find a subset of four harmonics that do not mutually interact, by regarding the remaining harmonic coefficients as prescribed, we are led directly to a linear problem. Of course, we could choose a different subset and solve the resulting coupled quadratic equations; however, it is clear that both solutions, differently parametrized, describe the same space.

Of the 70 ways to choose four harmonics from a set of eight, there are twenty four possible sets that do not self-interact. In fact, due to the rather simple form of poloidal field there are only 12 viable options. The remaining choices produce degenerate sets of polynomials in which at least two $Q^i_{d_i}$ are proportional, making the resulting system insoluble. One example of a viable set is $\{{}_1\mathbf{T}_1^{1s}, {}_1\mathbf{T}_2^{1s}, {}_1\mathbf{T}_2^{1c}, {}_1\mathbf{T}_3^{1s}\}$ which can be confirmed by inspecting Fig. 3 and by checking that these $Q^i_{d_i}$ are linearly independent. In following this algorithm, although we only need solve a linear system to find a Taylor state, the resulting four chosen coefficients are non-linearly dependent on the remaining unspecified coefficients and it is not therefore straightforward to find extremal examples within this set. However, by using the non-linear optimization software IPOPT available on the NEOS system[1] (Gropp & Moré 1997) we can find the extremal solution of the field of least Ohmic dissipation; contours of $\mathbf{B}_\phi$ for the non-axisymmetric toroidal field are shown in Fig. 4. This extremal toroidal field has a dissipation of 5.77×10^5 W and an rms strength $B_T = 1/17 B_S$.

4 GENERALIZATION

We now extend the ideas of the previous sections to finding a Taylor state within a more general truncated expansion by the judicious choice of a non-interacting subset of harmonics for whose coefficients we solve. Let us adopt an expansion in both toroidal and poloidal basis functions of maximum spherical harmonic degree $L_{\max}$ and radial truncation $N_{\max}$ of the form (5) or the poloidal equivalent, that satisfy electrically insulating boundary conditions (Livermore 2009). Taylor's constraint then reduces to $L_{\max} + 2N_{\max} - 1$

[1] http://neos.mcs.anl.gov/neos/

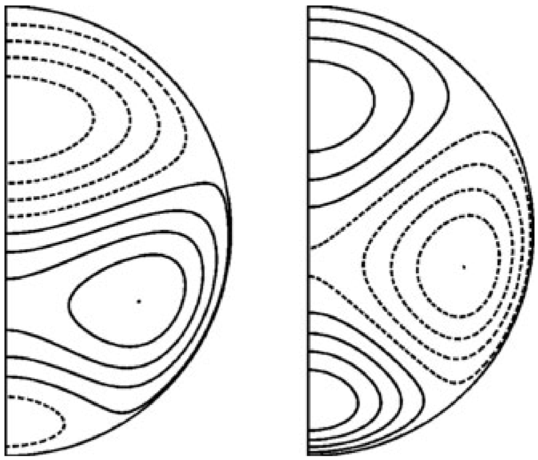

Figure 4. Contours of $\mathbf{B}_\phi$ in a meridian plane at (a) $\phi = 0$, (b) $\phi = \pi/2$ corresponding to the $m = 1$ toroidal field producing an exact Taylor state of minimal dissipation within the set of eight harmonics given. The contour levels are equally spaced, with solid lines denoting positive values and dashed negative. The rms toroidal field strength is 1/17 of that of the prescribed poloidal field.

equations in $2N_{\max}\,L_{\max}(L_{\max}+2)$ unknowns. Noting the linear degeneracy, we are required to find a subset of non-interacting harmonics that number $L_{\max} + 2N_{\max} - 2$. If we assume that $L_{\max} = N_{\max}$ then, of the set of size $O(L_{\max}^3)$, we need to find an appropriate subset of size $3L_{\max}$. This is straightforward; for instance, any subset of the following sets of size $O(L_{\max}^2)$ will suffice: (i) the axisymmetric toroidal modes or (ii) all radial modes for one harmonic per azimuthal wavenumber.

However, there are further constraints on the choice of subset: we must ensure that the polynomials $Q^i_{d_i}$ produced not only span the required space of polynomials but are linearly independent. These two conditions mean that we can invert the linear system and solve for the unknown coefficients. It is not immediately apparent that any non-interacting set produces a spanning set $\{Q^i_{d_i}\}$. For instance, it is simple to produce a set of $Q^i_{d_i}$ of degree less than that necessary. As a concrete example, consider the case $L_{\max} = N_{\max} = 4$. This produces $Q(\sigma)$ of degree 10 and has 10 associated constraints. The set

$$\{{}_4\mathbf{T}_1, {}_3\mathbf{T}_1, {}_2\mathbf{T}_1, {}_1\mathbf{T}_1, {}_4\mathbf{T}_2, {}_3\mathbf{T}_2, {}_2\mathbf{T}_2, {}_1\mathbf{T}_2, {}_1\mathbf{T}_3, {}_2\mathbf{T}_3\} \tag{6}$$

is non-interacting and of size 10, yet each $Q^i_{d_i}$ produced is of degree nine or less and there is no way of annulling the highest exponent of σ. These polynomials Q^i form a linearly dependent set and are clearly not viable. In contrast, by including some toroidal modes of higher spatial complexity, the set

$$\{{}_4\mathbf{T}_3, {}_3\mathbf{T}_3, {}_2\mathbf{T}_3, {}_1\mathbf{T}_3, {}_4\mathbf{T}_2, {}_3\mathbf{T}_2, {}_2\mathbf{T}_2, {}_1\mathbf{T}_2, {}_1\mathbf{T}_1, {}_2\mathbf{T}_1\} \tag{7}$$

produces a linearly independent set of $Q^i_{d_i}$. Although we have chosen to isolate toroidal harmonics for whose coefficients we solve, this is simply for illustrative purposes and, in the general case, any subset (either purely toroidal, purely poloidal or of mixed type) is perfectly acceptable.

Lack of linear independence can, as in the former case above, arise due to the absence of high order modes. However, rather more subtle effects can arise between the $Q^i_{d_i}$ that lead to a near singular system, particularly for large problems. This comes about due to the existence of exact Taylor states within an extended system that can be well approximated by the truncated expansions. In general, an algorithmic selection of an optimally well-conditioned subset may be required. For small systems however, this is less of a problem; for instance, the latter case above is sufficiently well conditioned (with a condition number 10^6 assuming a poloidal field ${}_1\mathbf{S}_3 + {}_1\mathbf{S}_4$) to allow an accurate linear solution.

5 DISCUSSION

In this paper, we have illustrated the ubiquity of Taylor states by explicitly constructing two families of large-scale solutions that match the same observationally derived low-degree radial field profile on the CMB. This is possible by exploiting the reduction of the continuum of Taylor constraints to a finite number within a suitably defined truncated spectral expansion. The solution of the remaining coupled quadratic equations can be facilitated by isolating a subset of harmonics that do not interact and solving the resulting linear system. Of particular note is the lack of constraint on the spectra and structure of a Taylor state, owing to the applicability of this method to such a wide class of fields with arbitrary distribution of spectral power.

Although of fundamental importance, Taylor's constraint is not, by itself, sufficient to describe a fully dynamically consistent steady state. In particular, our extremal examples will not be, in general, solutions of the steady state induction equation with a flow consistent with the magnetostrophic equations as a full analysis requires (Fearn 1998). In principle, this study could be extended by accounting for these effects, in which the second may be handled by adopting Taylor's original method for obtaining the magnetostrophic flow as a function of the magnetic field. However, such an analysis is not guaranteed to produce a solution that is stable and may not, after all, be realizable as a steady state in any time-dependent geodynamo simulation in the Taylor regime. It is far from obvious how to predict, in advance, whether the solutions would be linearly stable or not; indeed, encoding stability of the system into the optimization is probably too problematic to implement. More broadly, we speculate that many Taylor states are linearly unstable which, if true, would explain why it has been so hard to find such examples historically by seeking steady solutions directly. An alternative investigation into stability of (non-extremal) Taylor states would be to study the time-dependent magnetostrophic system in a sphere, in a similar vein to the $\alpha\omega$ dynamo in a duct geometry of Jones & Wallace (1992), who found a large range of temporal dynamics including steady states and oscillations. Perhaps one of the most promising avenues in this direction is to revisit the axisymmetric analysis of Hollerbach & Ierley (1991), using a polynomial rather than Bessel function representation in radius, in which exact Taylor states can be found (Livermore *et al.* 2008).

The dominant source of dissipation in the Earth's core is Ohmic owing to the relatively small magnitude of both thermal and viscous diffusivities. It is therefore possible that the geodynamo arranges itself in such a fashion as to be close to minimizing its Ohmic dissipation in order to be maximally efficient, and we briefly comment on how our results bear on this issue. The procedure adopted to find the extremal member of least dissipation in both families of solutions is not guaranteed to yield the global optimum over all possible Taylor states of single toroidal wavenumber that match the low-degree observational model, but merely an approximation to it. This is not only because of the severe truncation imposed on the models, but because we did not optimize over both toroidal and poloidal components simultaneously. Nevertheless, seeking a Taylor state with an $m = 1$ rather than $m = 0$ toroidal field raises our approximation to the minimum toroidal dissipation from 1.68×10^5

to 5.77×10^5 W with an associated increase of the rms toroidal field by a factor of 2. This can be readily understood by considering the electromagnetic torque balance in the models that may be written

$$[S, S] + [T, S] + [T, T] = 0$$

for poloidal–poloidal interactions, toroidal–poloidal and toroidal–toroidal, respectively. We find the toroidal field is considerably weaker than the poloidal field and so $[T, T]$ may be neglected in comparison with the other terms. Regarding $[S, S]$ as fixed, we require $[T, S]$ to be equal and opposite in sign. However, since the toroidal field consists of only one wavenumber, only this wavenumber in the poloidal field contributes to $[T, S]$. The strength of the toroidal field therefore depends on the shape of the wavenumber spectrum of the rms poloidal field, highly skewed towards the low wavenumbers. In particular, the largest contribution, that from the $m = 0$ component, is three times that from $m = 1$. Therefore, equilibrating this torque balance in the $m = 1$ wavenumber requires a toroidal field about three times greater than by the $m = 0$ wavenumber. Extending this argument to balancing the torque in higher azimuthal wavenumbers leads to very large toroidal fields that are energetically costly and are unlikely to be realized. This leads us to speculate that the toroidal field spectrum in the Earth's core must be comparable to that of the poloidal field, concentrated at small azimuthal wavenumbers.

ACKNOWLEDGMENTS

PWL was supported by NSF grant number DMS0724331 and AJ by the European commission under contract number 028670. Some of this work was carried out at the Kavli Institute of Theoretical Physics, University of California Santa Barbara, during the programme on dynamo theory supported by NSF grant number PHY05-51164.

REFERENCES

Backus, G., Parker, R. & Constable, C., 1996. *Foundations of Geomagnetism,* Cambridge Univ. Press, Cambridge, UK.

Dormy, E., Valet, J.P. & Courtillot, V., 2000. Numerical models of the geodynamo and observational constraints, *Geochem. Geophys. Geosyst.,* **1,** 1–42.

Fearn, D., 1998. Hydromagnetic flow in planetary cores, *Rep. Prog. Phys.,* **61,** 175–235.

Greenspan, H.P., 1974. On α-dynamos, *Stud. Appl. Math.,* **53**(1), 35–43.

Gropp, W. & Moré, J., 1997. Optimization environments and the NEOS server, in *Approximation Theory and Optimization,* pp. 167–182, eds Buhmann, M.D. & Iserles, A., Cambridge Univ. Press, Cambridge, UK.

Hollerbach, R. & Ierley, G., 1991. A modal α^2-dynamo in the limit of asymptotically small viscosity, *Geophys. Astrophys. Fluid Dyn.,* **56,** 133–158.

Jault, D. & Cardin, P., 1999. On dynamic geodynamo models with imposed velocity as energy source, *Phys. Earth planet. Int.,* **111,** 75–81.

Jones, C.A. & Wallace, S.G., 1992. Periodic, chaotic and steady solutions in $\alpha\omega$-dynamos, *Geophys. Astrophys. Fluid Dyn.,* **67,** 37–64.

Kageyama, A., Miyagoshi, T. & Sato, T., 2008. Formation of current coils in geodynamo simulations, *Nature,* **454**(7208), 1106–1109.

Kono, M. & Roberts, P.H., 2002. Recent geodynamo simulations and observations of the geomagnetic field *Rev. Geophys.,* **40**(4), 1013, doi:10.1029/2000RG00102.

Livermore, P., 2009. A compendium of generalized orthogonal polynomials, Tech. rep., Scripps Institute of Oceanography, Available at: http://repositories.cdlib.org/sio/techreport/98/.

Livermore, P., Jones, C.A. & Worland, S., 2007. Spectral radial basis functions for full sphere computations, *J. Comp. Phys.,* **227,** 1209–1224.

Livermore, P., Ierley, G. & Jackson, A., 2008. The structure of Taylor's constraint in three dimensions, *Proc. R. Soc. A,* **464,** 3149–3174.

Olsen, N. & Mandea, M., 2008. Rapidly changing flows in the Earth's core, *Nat. Geosci.,* **1,** 390–394.

Rotvig, J. & Jones, C.A., 2002. Rotating convection-driven dynamos at low Ekman number, *Phys. Rev. E,* **66,** 056308:1–15.

Smylie, D.E. & Rochester, M.G., 1984. The dynamics of the Earth's inner and outer cores, *Rep. Prog. Phys.,* **47,** 855–906.

Soward, A. & Jones, C.A., 1983. α^2-Dynamos and Taylor's constraint, *Geophys. Astrophys. Fluid Dyn.,* **27,** 87–122.

Sreenivasan, B. & Jones, C.A., 2006. The role of inertia in the evolution of spherical dynamos, *Geophys. J. Int.,* **164,** 467–476.

Takahashi, F., Matsushima, M. & Honkura, Y., 2005. Simulations of a quasi-Taylor state geomagnetic field including polarity reversals on the earth simulator, *Science,* **309,** 459–461.

Taylor, J.B., 1963. The magneto-hydrodynamics of a rotating fluid and the Earth's dynamo problem, *Proc. R. Soc. A,* **9,** 274–283.