BIFURCATION IN $L^p(\mathbb{R}^N)$ FOR A SEMILINEAR ELLIPTIC EQUATION

C. A. STUART

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1. *Introduction*

The main results of this paper concern the non-linear eigenvalue problem:

$$\Delta u(x) + r(x)f(u(x)) + \lambda u(x) = 0 \quad \text{for } x \in \mathbb{R}^N, \tag{1.1}$$

$$\lim_{|x| \to \infty} u(x) = 0, \tag{1.2}$$

where $r\colon \mathbb{R}^N \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ are given functions satisfying the hypotheses (C1) and (C2) of § 4. In particular, $f \in C^1(\mathbb{R})$ and $f(0) = f'(0) = 0$. Thus $u \equiv 0$ is a solution of (1.1), (1.2) for all $\lambda \in \mathbb{R}$ and the spectrum of the linearisation about this solution is the interval $[0, \infty)$. We are interested in the existence of solutions (λ, u) with $\lambda < 0$ and $u \not\equiv 0$ and, above all, in the behaviour of such solutions as λ approaches 0. For $1 \leqslant p \leqslant \infty$, there is L^p-bifurcation at 0 if there exists a sequence $\{(\lambda_n, u_n)\}$ of solutions of (1.1), (1.2) such that

$$\lambda_n \to 0 \quad \text{and} \quad |u_n|_p \to 0 \quad \text{as } n \to \infty, \tag{1.3}$$

where $|\cdot|_p$ denotes the usual norm on $L^P(\mathbb{R}^N)$. The notation for this is $0 \in B(p)$. On the other hand, if (1.3) is replaced by

$$\lambda_n \to 0 \quad \text{and} \quad |u_n|_p \to \infty \quad \text{as } n \to \infty, \tag{1.4}$$

then there is asymptotic L^p-bifurcation and we write $0 \in B_\infty(p)$. The results in this paper give conditions on r, f, and p which distinguish the following situations:

(a) $0 \notin B(p)$, Theorems 4.8 and 5.7;
(b) $0 \in B(p)$, Theorems 5.7 and 5.9;
(c) $0 \in B_\infty(p)$, Theorem 5.7.

The earlier work on these questions can be summarised as follows. The first papers [**14**, **25–28**] deal only with L^2-bifurcation and apply to cases where either

(i) $$\lim_{|x| \to \infty} r(x) = 0 \tag{1.5}$$

or

(ii) $N \geqslant 2$ and r is radially symmetric. (1.6)

Subsequently [**29**, **33**, **34**], results were obtained for L^p-bifurcation with $p \neq 2$, but only in cases where either

(iii) $N = 1$ and r is even and decreasing (1.7)

or

(iv) $N \geqslant 2$ and r is radially symmetric and radially decreasing with

$$\lim_{|x| \to \infty} r(x) > 0. \tag{1.8}$$

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Furthermore, most of the results deal only with the case where $f(u)=|u|^{\sigma}u$ for some $\sigma>0$. In this paper, we deal with $N\geqslant 1$ without requiring any symmetry or monotonicity of r, and f need not be homogeneous. The results apply to cases where $r(\infty)=\lim_{|x|\mapsto\infty} r(x)$ exists, but we can handle both $r(\infty)=0$ and $r(\infty)>0$. A first step in this direction was taken in [**30**, **31**] where the case $N=1$ with $0<A\leqslant r(x)\leqslant B$ for $x\in\mathbb{R}$ is discussed in considerable detail.

The method that we have used is based on the variational structure of (1.1) and deals naturally with weak solutions of (1.1), (1.2) in the Sobolev space $H^1(\mathbb{R}^N)$. In § 4, we show that under our hypotheses the notions of weak and classical solution are equivalent. The point here is not so much to show that every weak solution of (1.1) is indeed a classical solution, for this follows from standard regularity theory, but rather to prove that the classical boundary condition (1.2) is equivalent to the requirement that $u\in H^1(\mathbb{R}^N)$. In particular, we can conclude that all solutions of (1.1), (1.2) (weak or classical) with $\lambda<0$ decay exponentially to zero as $|x|\rightarrow\infty$.

From the results of § 4, it follows that for each $\lambda<0$, the problem (1.1), (1.2) is equivalent to a problem of the form,

$$\nabla J_\lambda(u)=0 \quad \text{for } u\in H, \tag{1.9}$$

where H is a real Hilbert space and $J_\lambda\in C^1(H,\mathbb{R})$. Thus for $\lambda<0$, we seek non-zero stationary points of J_λ in a case where

$$\inf\{J_\lambda(u)\colon\ u\in H\}=-\infty \quad\text{and}\quad \sup\{J_\lambda(u)\colon\ u\in H\}=+\infty.$$

One way of attacking this kind of problem goes back to work by Nehari [**20**, **6**, **13**] and amounts to finding critical points of the restriction of J_λ to the following set of natural constraints:

$$V_\lambda=\{u\in H\colon\ u\neq 0 \text{ and } \langle\nabla J_\lambda(u),u\rangle=0\},$$

where $\langle\cdot,\cdot\rangle$ is the scalar product of H. If we set

$$m_\lambda=\inf\{J_\lambda(u)\colon\ u\in V_\lambda\},$$

it is enough to show that J_λ attains its minimum m_λ on V_λ. Sections 2 and 3 deal with this problem in the general Hilbert space context. In § 2 we give the basic hypotheses about J_λ that are used to show that V_λ is a nice set and that $m_\lambda>0$. However, under these conditions, m_λ need not be attained (see Lemma 5.3) and the main point of § 3 is to provide a set of conditions on J_λ which ensure that there exists $u_\lambda\in V_\lambda$ such that $J_\lambda(u_\lambda)=m_\lambda$. Roughly speaking, Theorem 3.6 asserts that if m_λ can be increased by a weakly continuous perturbation of J_λ then m_λ is attained by J_λ. Of course the problem (1.9) may have solutions on V_λ even when J_λ does not attain its minimum on V_λ. It is just that the variational characterisation of such solutions is more complicated. In some cases it is possible to circumvent this by using a non-variational approach based on a rescaling of the variables and an application of the implicit function theorem [**32**, **18**].

Finally in § 5, the general results of § 3 are applied to the problem (1.1), (1.2) giving conditions on r and f which imply that for each $\lambda<0$ there exists $u_\lambda\in V_\lambda$ such that $J_\lambda(u_\lambda)=m_\lambda$. Furthermore, (λ,u_λ) satisfies (1.1), (1.2) and $u_\lambda(x)>0$ on $\mathbb{R}^N$. Estimates for m_λ in terms of $|\lambda|$ are calculated and then used to estimate $|u_\lambda|_p$ in terms of $|\lambda|$ for $p\in[2,2N/(N-2)]$. This leads to conditions for L^p-bifurcation.

Some *a priori* lower bounds on the L^p-norms of solutions, proved in § 4, show that these conditions are sharp and we can deduce criteria for asymptotic L^p-bifurcation from them. Finally, the results are extended to the full range $1 \leq p \leq \infty$.

Although the main issue here is L^p-bifurcation for (1.1), (1.2), a preliminary step involves proving the existence of solutions for $\lambda < 0$. In cases (1.5) and (1.6) this is relatively straightforward **[2, 3, 23, 24, 27, 28]**, but when $r(\infty) > 0$ and r has no symmetry the problem is more difficult. The most general approach so far is due to P. L. Lions **[17]** using a method that he has called 'compactness-concentration'. Ding and Ni **[10]** have also obtained results of this kind by considering the Dirichlet problem on balls of increasing radii. In both cases an essential feature is the comparison between an infimum for the given problem and one associated with the problem where r is replaced by $r(\infty)$. The Theorem 3.6 is a simple general result of this kind. The same sort of comparison has also been used for other problems involving a lack of compactness, for example, by Brézis and Nirenberg **[4, 5]** in dealing with critical Sobolev exponents.

The restrictions on the form of f imposed by the condition (C2) ensure that the set V_λ is radially diffeomorphic to the unit sphere in $H^1(\mathbb{R}^N)$ and allow us to measure the effect of rescaling elements of $H^1(\mathbb{R}^N)$ so as to lie on V_λ. In this way we are able to give a relatively elementary solution of the problem. It might be possible to obtain results on L^p-bifurcation under less stringent conditions on f by using some more powerful variational techniques such as the 'mountain-pass' results. This leads one to enquire whether the hypotheses of Theorem 3.6 ensure that the restriction of J_λ to V_λ satisfies the Palais–Smale condition for values of J_λ in the interval $(-\infty, m(\Psi))$. A result of this kind would also offer the possibility of establishing the bifurcation of several branches of solutions using Lyusternik–Schnirelmann theory. At present the only result concerning the bifurcation of multiple solutions for (1.1), (1.2) is due to Ruppen [21] and deals with L^2-bifurcation under the restrictions (1.5) or (1.6).

2. *Basic hypotheses for the variational method*

We begin by setting out a list of conditions to be used in what follows.

Throughout this section H denotes a real Hilbert space with scalar product $\langle \cdot , \cdot \rangle$ and norm $\|\cdot\|$.

(H1) $\Phi \in C^1(H, \mathbb{R})$ and Φ': $H \to H^*$ is a bounded mapping.

The gradient of Φ is denoted by $\nabla\Phi$. Thus

$$\langle \nabla\Phi(u), v \rangle = \Phi'(u)v \quad \text{for all } u, v \in H.$$

For the discussion of the relevant properties of Φ it is convenient to introduce two auxiliary functions:

$$\phi(u) = \langle \nabla\Phi(u), u \rangle = \Phi'(u)u \tag{2.1}$$

and

$$\tilde{\phi}(u) = \phi(u) - 2\Phi(u) \quad \text{for } u \in H. \tag{2.2}$$

(H2) $\phi \in C^1(H, \mathbb{R})$ and ϕ': $H \to H^*$ is a bounded mapping. Furthermore, there exist constants $K>0$ and $\gamma \geqslant \alpha > 2$ such that, for all $u \in H\setminus\{0\}$,

$$\phi(u) = \Phi'(u)u \geqslant \alpha\Phi(u) > 0, \tag{2.3}$$

$$\tilde{\phi}'(u)u \geqslant \alpha\tilde{\phi}(u), \tag{2.4}$$

and

$$\tilde{\phi}(u) \leqslant K\{\|u\|^{\alpha} + \|u\|^{\gamma}\}. \tag{2.5}$$

The hypotheses (H1) and (H2) fix the structure of the non-linearities we shall discuss. The next two conditions describe their behaviour with respect to weak convergence.

(H3) $\tilde{\phi}$: $H \to \mathbb{R}$ is weakly sequentially lower semi-continuous.

(H4) $\nabla\Phi(u_k) \rightharpoonup \nabla\Phi(u)$ weakly in H whenever $u_k \rightharpoonup u$ weakly in H.

For future reference, we note some elementary consequences of the hypotheses (H1) and (H2).

(i) For $u \in H\setminus\{0\}$,

$$\tilde{\phi}(u) = \phi(u) - 2\Phi(u) \geqslant (\alpha-2)\Phi(u) > 0, \tag{2.6}$$

$$0 < \Phi(u) \leqslant \frac{K}{(\alpha-2)}\{\|u\|^{\alpha} + \|u\|^{\gamma}\}, \tag{2.7}$$

$$0 < \phi(u) = \tilde{\phi}(u) + 2\Phi(u) \leqslant \frac{K\alpha}{(\alpha-2)}\{\|u\|^{\alpha} + \|u\|^{\gamma}\}, \tag{2.8}$$

and

$$\begin{aligned} \phi'(u)u &= \tilde{\phi}'(u)u + 2\Phi'(u)u \\ &\geqslant \alpha\tilde{\phi}(u) + 2\phi(u) \quad \text{(by (2.4))} \\ &\geqslant \alpha\tilde{\phi}(u) + 2\alpha\Phi(u) \quad \text{(by (2.3))} \\ &= \alpha\phi(u). \end{aligned} \tag{2.9}$$

(ii) Setting $h(t) = \Phi(tu)t^{-\alpha}$ for $t>0$ and $u \in H\setminus\{0\}$, we have

$$h'(t) = \frac{\phi(tu) - \alpha\Phi(tu)}{t^{\alpha+1}} \geqslant 0$$

and so

$$\Phi(tu) \leqslant \Phi(u)t^{\alpha} \quad \text{for } 0<t\leqslant 1, \tag{2.10}$$

$$\Phi(tu) \geqslant \Phi(u)t^{\alpha} \quad \text{for } t \geqslant 1. \tag{2.11}$$

In particular,

$$\Phi(0) = \nabla\Phi(0) = 0. \tag{2.12}$$

(iii) Setting $k(t) = \phi(tu)t^{-\alpha}$ for $t>0$ and $u \in H\setminus\{0\}$, we have

$$k'(t) = \frac{\phi'(tu)tu - \alpha\phi(tu)}{t^{\alpha+1}} \geqslant 0 \quad \text{by (2.9)}.$$

Hence $t \to \phi(tu)t^{-2}$ is strictly increasing on $(0,\infty)$ with

$$\lim_{t\to 0}\phi(tu)t^{-2} = 0 \quad \text{and} \quad \lim_{t\to\infty}\phi(tu)t^{-2} = +\infty \tag{2.13}$$

for $u \in H\setminus\{0\}$.

Using (2.4) instead of (2.9), we can obtain similar conclusions when ϕ is replaced by $\tilde{\phi}$.

3. *Existence by constrained minimisation*

In this section we always suppose that the hypotheses (H1) and (H2) hold and we discuss the existence of solutions of the equation:

$$u = \nabla\Phi(u) \quad \text{for } u \in H\setminus\{0\}. \tag{3.1}$$

For the variational approach that we shall use, the following two functionals play a fundamental role. Let

$$J(u) = \|u\|^2 - 2\Phi(u) \tag{3.2}$$

and

$$g(u) = \|u\|^2 - \phi(u) \quad \text{for } u \in H.$$

Clearly $J, g \in C^1(H, \mathbb{R})$ and (3.1) is equivalent to

$$J'(u) = 0 \quad \text{with } u \in H\setminus\{0\}. \tag{3.3}$$

On the other hand, $g(u) = 0$ for every solution of (3.1). It is well known that one way of overcoming the difficulties connected with the indefiniteness of J on H, is to consider J subject to the natural constraint $g(u) = 0$. With this in mind we set

$$V = \{u \in H\setminus\{0\}: \ g(u) = 0\} \tag{3.4}$$

and

$$m = \inf\{J(u): \ u \in V\}. \tag{3.5}$$

Noting that

$$J(u) = g(u) + \tilde{\phi}(u) \quad \text{for all } u \in H \tag{3.6}$$

and

$$J'(u)u = 2g(u) = g'(u)u + \tilde{\phi}'(u)u, \tag{3.7}$$

we see how the auxiliary functions ϕ and $\tilde{\phi}$ associated with Φ arise in this approach to the equation (3.1). In particular, for all $u \in V$ we find that

$$J(u) = \tilde{\phi}(u), \quad g'(u)u = -\tilde{\phi}'(u)u, \tag{3.8}$$

and

$$m = \inf\{\tilde{\phi}(u): \ u \in V\}.$$

In general, V is not a bounded subset of H, but it does have many useful properties. In particular, if (H1) and (H2) hold, it follows from (3.8) that $g'(u)u < 0$ for all $u \in V$, and consequently V is a C^1-manifold of co-dimension 1.

LEMMA 3.1. *Let the conditions* (H1) *and* (H2) *hold.*
(i) *There exists $\delta > 0$ such that $\phi(u) = \|u\|^2 \geq \delta$ for all $u \in V$.*
(ii) *We have $m > 0$ and $0 < \delta \leq \|u\|^2 \leq (\alpha/(\alpha-2))J(u)$ for all $u \in V$.*
(iii) *If $u \in V$ and $J(u) = m$, then u satisfies* (3.1) *and*

$$0 < \delta \leq \|u\|^2 \leq \alpha m/(\alpha - 2).$$

Proof. (i) For $u \in V$,

$$\|u\|^2 = \phi(u) \leq \frac{\alpha K}{(\alpha-2)}\{\|u\|^\alpha + \|u\|^\gamma\} \quad \text{by (2.8).}$$

Since $\gamma \geq \alpha > 2$, it follows that there exists $\delta > 0$ such that $\|u\|^2 \geq \delta$ for all $u \in V$.

(ii) For $u \in H\backslash\{0\}$,

$$\begin{aligned}\phi(u) &= \tilde{\phi}(u) + 2\Phi(u) \\ &\leq \tilde{\phi}(u) + (2/(\alpha-2))\tilde{\phi}(u) \qquad \text{(by (2.6))} \\ &= (\alpha/(\alpha-2))\tilde{\phi}(u). \end{aligned} \tag{3.9}$$

Hence, for $u \in V$,

$$J(u) = \tilde{\phi}(u) \geq \left(1 - \frac{2}{\alpha}\right)\phi(u) = \left(1 - \frac{2}{\alpha}\right)\|u\|^2 \geq \left(1 - \frac{2}{\alpha}\right)\delta.$$

Thus, $m \geq (1 - 2/\alpha)\delta > 0$ and $\delta \leq \|u\|^2 \leq (\alpha/(\alpha-2))J(u)$ for $u \in V$.

(iii) For $u \in V$,

$$\begin{aligned} g'(u)u &= -\tilde{\phi}'(u)u \quad \text{(by (2.7))} \\ &\leq -\alpha\tilde{\phi}(u) < 0 \quad \text{(by (2.4) and (2.6))}. \end{aligned}$$

If $u \in V$ and $J(u) = m$, it follows that there exists $\xi \in \mathbb{R}$ such that $J'(u)v = \xi g'(u)v$ for all $v \in H$. Putting $v = u$, we see that $J'(u)u = \xi g'(u)u$ and so $\xi = 0$ since $g'(u)u < 0$ and $J'(u)u = 0$. Thus, $J'(u)v = 0$ for all $v \in H$ and this means that u satisfies (3.1).

Remark. The previous result shows that the manifold V is bounded away from the origin in H and that any minimising sequence for J on V is bounded. Now we prove that V is diffeomorphic to the unit sphere in H. In general, V is not a bounded subset of H.

Lemma 3.2. *Let the conditions* (H1) *and* (H2) *be satisfied. Then there exists a unique function* s: $H\backslash\{0\} \to (0, \infty)$ *such that*

if $u \in H\backslash\{0\}$ *and* $t > 0$ *then* $tu \in V$ *if and only if* $t = s(u)$.

Furthermore, $s \in C^1(H\backslash\{0\}, \mathbb{R})$ *and for all* $u \in H\backslash\{0\}$,

$$\|\nabla s(u)\| \leq \frac{\{2\,\|s(u)u\| + \|\nabla\phi(s(u)u)\|\}s(u)^2}{(\alpha-2)\phi(s(u)u)}.$$

Proof. Let ψ: $(0, \infty) \times H \to \mathbb{R}$ be defined by

$$\psi(t, u) = \|u\|^2 - \phi(tu)t^{-2} \quad \text{for } t > 0 \text{ and } u \in H.$$

Then $\psi \in C^1((0, \infty) \times H, \mathbb{R})$,

$$\psi_t(t, u) = -\frac{[\phi'(tu)tu - 2\phi(tu)]}{t^3} \leq -\frac{(\alpha-2)\phi(tu)}{t^3}$$

by (2.9), and

$$\psi_u(t, u)v = 2\langle u, v\rangle - \frac{\langle\nabla\phi(tu), v\rangle}{t}. \tag{3.10}$$

Furthermore, for $u \in H\backslash\{0\}$ and $t > 0$, $tu \in V$ if and only if $\psi(t, u) = 0$.

By remark (iii) in § 2, for each $u \in H\backslash\{0\}$, there exists a unique value $t = s(u)$

such that $\psi(s(u), u) = 0$. Furthermore,

$$\psi_t(s(u), u) \leqslant -\frac{(\alpha-2)\phi(s(u)u)}{s(u)^3} < 0,$$

and so, by the implicit function theorem, $s \in C^1(H\setminus\{0\}, \mathbb{R})$.

Also $\psi_t(s(u), u)s'(u)v + \psi_u(s(u), u)v = 0$ for all $v \in H$ and hence,

$$|s'(u)v| \leqslant \frac{|\psi_u(s(u), u)v|\, s(u)^3}{(\alpha-2)\phi(s(u)u)}.$$

But $|\psi_u(s(u), u)v| \leqslant 2\,\|u\| + s(u)^{-1}\,\|\nabla\phi(s(u)u)\|$ by (3.10) for $\|v\| \leqslant 1$. Hence

$$\|\nabla s(u)\| \leqslant \frac{\{2\,\|s(u)u\| + \|\nabla\phi(s(u)u)\|\}s(u)^2}{(\alpha-2)\phi(s(u)u)} \quad \text{for } u \in H\setminus\{0\}.$$

We can now begin to discuss the existence of solutions of equation (3.1).

THEOREM 3.3. *Let the conditions* (H1) *to* (H3) *be satisfied. Then the following conditions are equivalent:*

(i) *there exists $u \in V$ such that $J(u) = m$;*

(ii) *there exists $\{u_n\} \subset V$ such that $J(u_n) \to m$, $u_n \rightharpoonup u$ weakly in H, and $\phi(u_n) \to \phi(u)$.*

Furthermore, if $u \in V$ and $J(u) = m$ then

$$0 < \|u\|^2 \leqslant \alpha m/(\alpha-2). \tag{3.11}$$

In particular, (i) *and* (ii) *are true if ϕ is weakly sequentially continuous on H.*

Proof. That (i)$\Rightarrow$(ii) is trivial since we need only set $u_n = u$ for all $n \in \mathbb{N}$. For the converse we suppose that (ii) is satisfied. Then

$$\phi(u) = \lim \phi(u_n) \geqslant \delta > 0,$$

by Lemma 3.1, and so $u \neq 0$. Also

$$\|u\|^2 \leqslant \liminf \|u_n\|^2 = \liminf \phi(u_n) = \phi(u)$$

and so, in the notation of Lemma 3.2, $s(u) \leqslant 1$. But

$$0 < m \leqslant J(s(u)u) = \tilde{\phi}(s(u)u) \leqslant \tilde{\phi}(u)$$

by remark (iii) of § 2. By (H3),

$$\tilde{\phi}(u) \leqslant \liminf \tilde{\phi}(u_n) = \liminf J(u_n) = m$$

since $u_n \in V$. Thus $J(s(u)u) = m$ and $\tilde{\phi}(s(u)u) = \tilde{\phi}(u)$. Again by the remark (iii) of § 2, this implies that $s(u) = 1$, $u \in V$, and $J(u) = m$. The inequality (3.11) follows from Lemma 3.1.

REMARK. By definition, there exists $\{v_n\} \subset V$ such that $J(v_n) \to m$ and, by Lemma 3.1, such a sequence is bounded. Hence, by passing to a subsequence, we can suppose that $v_n \rightharpoonup v$ weakly in H. Thus the condition (ii) in Theorem 3.3 amounts to requiring that we can construct a minimising sequence such that $\phi(u_n) \to \phi(u)$. The construction of such a sequence can be arranged:

(a) trivially if ϕ is weakly sequentially continuous,

(b) by symmetrisation under certain circumstances such as those in [**29**, **7**].

In general, however, under the hypotheses (H1) to (H4), the equation (3.1) may have no solution other than $u = 0$. (See [**31**, Lemma 2.2] for an example.) In seeking conditions which imply that (ii) does hold the following result is useful.

LEMMA 3.4. *Let* (H1) *and* (H2) *hold. Then there exists* $\{u_n\} \subset V$ *such that* $J(u_n) \to m$, $u_n \rightharpoonup u$ *weakly in* H, *and* $\|\nabla J(u_n)\| \to 0$.

Proof. Since $g \in C^1(H, \mathbb{R})$, V is a closed subset of H and as such is a complete metric space. Then $J \in C(V, \mathbb{R})$ and $J(u) \geq m > 0$ for all $u \in V$. Hence by Ekeland's ε-variational principle [**11**], there exists $\{u_n\} \subset V$ such that

$$J(u_n) \leq m + n^{-1}$$

and

$$J(w) \geq J(u_n) - n^{-1} \|w - u_n\| \quad \text{for all } w \in V.$$

Then by Lemma 3.1 and the conditions (H1) and (H2), there exists N such that

$$\|u_n\| \leq N \quad \text{and} \quad \|\nabla\phi(u_n)\| \leq N \quad \text{for all } n \in \mathbb{N}.$$

Now, for all $v \in H \setminus \{0\}$,

$$\begin{aligned} J(v) - J(u_n) &= J(v) - J(s(v)v) + J(s(v)v) - J(u_n) \\ &\geq J(v) - J(s(v)v) - n^{-1} \|s(v)v - u_n\| \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} |s(v) - 1| &= |s(v) - s(u_n)| \\ &\leq \|v - u_n\| \sup\{\|\nabla s(z)\| : \|z - u_n\| < \sqrt{(\tfrac{1}{2}\delta)}\} \end{aligned}$$

for $\|v - u_n\| < \sqrt{(\tfrac{1}{2}\delta)}$, where δ is given by Lemma 3.1(i).

From Lemma 3.2 we conclude that

$$|s(v) - 1| \leq \|v - u_n\| \frac{2\{2\|u_n\| + \|\nabla\phi(u_n)\|\}}{(\alpha - 2)\phi(u_n)}$$

for $\|v - u_n\| \leq \delta_n < \min\{1, \sqrt{(\tfrac{1}{2}\delta)}\}$. Thus for $\|v - u_n\| \leq \delta_n$,

$$|s(v) - 1| \leq \frac{\|v - u_n\| \, 6N}{(\alpha - 2)\delta^2} \quad \text{(by Lemma 3.1)}.$$

Setting $D = 6N/(\alpha - 2)\delta^2$, we have that for $\|v - u_n\| \leq \delta_n$,

$$\begin{aligned} \|s(v)v - u_n\| &\leq \|s(v)v - v\| + \|v - u_n\| \\ &\leq |s(v) - 1| \, \|v\| + \|v - u_n\| \\ &\leq \|v - u_n\| \{D \|v\| + 1\} \\ &\leq \|v - u_n\| \{DN + D\delta_n + 1\} \\ &= E \|v - u_n\|, \end{aligned} \tag{3.13}$$

where $E = D(N + 1) + 1$. Also

$$|J(v) - J(s(v)v)| \leq |s(v) - 1| \, |J'(\theta(v)v)v|,$$

where $\theta(v)$ lies between 1 and $s(v)$. But $J'(u_n)u_n = 0$ by (3.7) since $u_n \in V$, and so we can choose $\delta_n > 0$ so small that

$$|J'(\theta(v)v)v| < n^{-1} \quad \text{for } \|v - u_n\| \leq \delta_n.$$

Hence for $\|v-u_n\| \leqslant \delta_n$ we have

$$|J(v)-J(s(v)v)| \leqslant \frac{D\,\|v-u_n\|}{n}, \tag{3.14}$$

and so, by (3.12) to (3.14),

$$J(v)-J(u_n) \geqslant -\frac{D\,\|v-u_n\|}{n}-\frac{E\,\|v-u_n\|}{n}=-\frac{(D+E)\,\|v-u_n\|}{n},$$

where D and E are independent of n. In particular, for $\|z\|=1$ and $0<t<\delta_n$,

$$\frac{J(u_n+tz)-J(u_n)}{t} \geqslant -\frac{(D+E)}{n},$$

and so $J'(u_n)z \geqslant -(D+E)/n$. Replacing z by $-z$, we obtain

$$|J'(u_n)z| \leqslant (D+E)/n \quad \text{for all } \|z\|=1$$

and so

$$\|\nabla J(u_n)\| \leqslant (D+E)/n \quad \text{for all } n \in \mathbb{N}.$$

REMARK. If J and g are both elements of $C^2(H, \mathbb{R})$, the existence of a sequence having the properties of Lemma 3.4 can be established by considering the behaviour as $t \to \infty$ of the differential equation

$$\dot{u}(t) = -\nabla J(u) + \langle \nabla J(u), \nabla g(u)\rangle \frac{u}{\langle \nabla g(u), u\rangle},$$
$$u(0) = w_n \in V \quad \text{where } J(w_n) \to m,$$

as in § 4.4 of [**8**].

Having established these preliminary results, we can now turn to the main theorems concerning (3.1).

THEOREM 3.5. *Let the hypotheses* (H1) *to* (H4) *be satisfied and let* $\{u_n\} \subset V$ *be a sequence having the properties established in Lemma* 3.4. *Then either*

(i) $u=0$ *or*

(ii) $u \in V$, $J(u)=m$, *and* $0<\|u\|^2 \leqslant \alpha m/(\alpha-2)$.

In particular, if $\phi(u_n) \to \phi(u)$ *or* $\tilde{\phi}(u_n) \to \tilde{\phi}(u)$, *then* (ii) *is satisfied.*

Proof. For $v \in H$,

$$\begin{aligned}\langle u, v\rangle &= \lim_{n\to\infty} \langle u_n, v\rangle \\ &= \lim_{n\to\infty} \{\tfrac{1}{2}\langle \nabla J(u_n), v\rangle + \langle \nabla\Phi(u_n), v\rangle\} \\ &= \lim_{n\to\infty} \langle \nabla\Phi(u_n), v\rangle \quad \text{(by Lemma 3.4)} \\ &= \langle \nabla\Phi(u), v\rangle \quad \text{(by (H4))}.\end{aligned}$$

Hence $u=\nabla\Phi(u)$. If $u \neq 0$, then $u \in V$ and

$$J(u)=\tilde{\phi}(u) \leqslant \liminf_{n\to\infty} \tilde{\phi}(u_n) = \liminf_{n\to\infty} J(u_n) = m$$

since $u_n \in V$. Thus $J(u) = m$ if $u \neq 0$ and by Lemma 3.3, $0 < \|u\|^2 \leq \alpha m/(\alpha - 2)$. If

$$\tilde{\phi}(u) = \lim_{n\to\infty} \tilde{\phi}(u_n) = \lim_{n\to\infty} J(u_n),$$

then $\tilde{\phi}(u) = m \neq 0$ and so $u \neq 0$. If $\phi(u) = \lim_{n\to\infty} \phi(u_n)$, then $\phi(u) \geq \delta > 0$ and so $u \neq 0$.

THEOREM 3.6. *Let Φ satisfy the conditions* (H1) *to* (H4) *and let Ψ satisfy the conditions* (H1) *and* (H2). *Suppose that $\Phi - \Psi$ and $\phi - \psi$ are weakly sequentially continuous at* 0. *Then if $m(\Phi) < m(\Psi)$, there exists $u \in V(\Phi)$ such that $J_\Phi(u) = m(\Phi)$. Furthermore, u satisfies* (3.1) *and*

$$0 < \|u\|^2 \leq \frac{\alpha m(\Phi)}{(\alpha - 2)} < \frac{\alpha m(\Psi)}{(\alpha - 2)}.$$

REMARKS. In the above statement it is understood that

$$\begin{aligned}
J_\Phi(u) &= \|u\|^2 - 2\Phi(u), & J_\Psi(u) &= \|u\|^2 - 2\Psi(u),\\
\phi(u) &= \langle \nabla\Phi(u), u\rangle, & \psi(u) &= \langle \nabla\Psi(u), u\rangle,\\
g_\Phi(u) &= \|u\|^2 - \phi(u), & g_\Psi(u) &= \|u\|^2 - \psi(u),\\
V(\Phi) &= \{u \in H\backslash\{0\}:\ g_\Phi(u) = 0\}, & V(\Psi) &= \{u \in H\backslash\{0\}:\ g_\Psi(u) = 0\},\\
m(\Phi) &= \inf\{J_\Phi(u):\ u \in V(\Phi)\}, & m(\Psi) &= \inf\{J_\Psi(u):\ u \in V(\Psi)\}.
\end{aligned}$$

Proof. Let $\{u_n\} \subset V(\Phi)$ be a sequence having the properties established in Lemma 3.4. Suppose that $u = 0$. Since Ψ satisfies the conditions (H1) and (H2), it follows from Lemma 3.2 (applied to Ψ) that there exists $s \in C^1(H\backslash\{0\}, \mathbb{R})$ such that $s(u_n)u_n \in V(\Psi)$. Setting $s_n = s(u_n)$, we note that it is enough to prove that if $u_n \rightharpoonup 0$ then

(a) $$\lim_{n\to\infty} [\tilde{\psi}(s_n u_n) - \tilde{\psi}(u_n)] = 0$$

and

(b) $$\lim_{n\to\infty} [\tilde{\psi}(u_n) - \tilde{\phi}(u_n)] = 0,$$

where $\tilde{\phi}(u) = \phi(u) - 2\Phi(u)$ and $\tilde{\psi}(u) = \psi(u) - 2\Psi(u)$. Indeed

$$m(\Psi) \leq J_\Psi(s_n u_n) = \tilde{\psi}(s_n u_n) = \tilde{\psi}(s_n u_n) - \tilde{\psi}(u_n) + \tilde{\psi}(u_n) - \tilde{\phi}(u_n) + \tilde{\phi}(u_n)$$

and

$$\lim_{n\to\infty} \tilde{\phi}(u_n) = \lim_{n\to\infty} J_\Phi(u_n) = m(\Phi).$$

Hence, if (a) and (b) are satisfied, then $m(\Psi) \leq m(\Phi)$ and we have a contradiction. This means that $u \neq 0$ and then the result follows by Theorem 3.5.

To prove (a) we shall show that $s_n \to 1$ and we begin by noting that

$$|\tilde{\psi}(s_n u_n) - \tilde{\psi}(u_n)| \leq |s_n - 1|\, |\tilde{\psi}'(\theta_n u_n) u_n|,$$

where θ_n lies between 1 and s_n. But $\|u_n\|^2 = \phi(u_n)$ and $\|u_n\|^2 = \psi(s_n u_n)/s_n^2$. If

$s_n \geq 1$, then by remark (iii) of § 2 (applied to ψ), $\psi(s_n u_n) \geq s_n^\alpha \psi(u_n)$ and so

$$1 \leq s_n^{\alpha-2} \leq \frac{\|u_n\|^2}{\psi(u_n)} = \frac{\phi(u_n)}{\psi(u_n)} = 1 + \frac{\phi(u_n) - \psi(u_n)}{\phi(u_n) + [\psi(u_n) - \phi(u_n)]}.$$

If $s_n \leq 1$, then by remark (iii) of § 2 (applied to ψ), $\psi(s_n u_n) \leq s_n^\alpha \psi(u_n)$ and so

$$1 \geq s_n^{\alpha-2} \geq \frac{\|u_n\|^2}{\psi(u_n)} = 1 + \frac{\phi(u_n) - \psi(u_n)}{\phi(u_n) + [\psi(u_n) - \phi(u_n)]}.$$

By Lemma 3.1(i), there exists $\delta > 0$ such that $\phi(u_n) \geq \delta$ for all $n \in \mathbb{N}$. Hence to prove (a) we need only show that $\lim_{n\to\infty} [\phi(u_n) - \psi(u_n)] = 0$ when $u_n \rightharpoonup 0$ (since we are supposing that $u = 0$), and this is just the weak sequential continuity of $\phi - \psi$ at 0 since $\phi(0) = \psi(0) = 0$.

Finally, we note that (b) follows from the weak sequential continuity of $\tilde{\psi} - \tilde{\phi}$ at 0, since $\tilde{\psi} - \tilde{\phi} = \psi - \phi - 2(\Psi - \Phi)$. This completes the proof of the theorem.

The previous result can be reformulated as a kind of perturbation theorem for the existence of solutions of (3.1).

COROLLARY 3.7. *Let Φ and Ψ both satisfy the conditions* (H1) *to* (H4). *Suppose, in addition, that*

(a) *there exists $v \in V(\Psi)$ such that $J_\Psi(v) = m(\Psi)$ and $\phi(tv) > \psi(tv)$ for all $t > 0$,*

(b) *$\Phi - \Psi$ and $\phi - \psi$ are weakly sequentially continuous.*

Then $m(\Phi) < m(\Psi)$ and there exists $u \in V(\Phi)$ such that $J_\Phi(u) = m(\Phi)$. Furthermore, u satisfies (3.1) *and*

$$0 < \|u\|^2 \leq \frac{\alpha m(\Phi)}{(\alpha - 2)} < \frac{\alpha m(\Psi)}{(\alpha - 2)}.$$

Proof. By the definition of ϕ and ψ,

$$\frac{d}{dt}\{\Phi(tv) - \Psi(tv)\} = \frac{\phi(tv) - \psi(tv)}{t} > 0, \quad \text{for all } t > 0,$$

and so $\Phi(tv) > \Psi(tv)$ for $t > 0$ since $\Phi(0) = \Psi(0) = 0$. For $v \in V(\Psi)$, set $r(t) = J_\Psi(tv)$ for $t \geq 0$. By the definition of $V(\Psi)$, $tv \in V(\Psi)$ if and only if $r'(t) = 0$ and $t > 0$. Hence by Lemma 3.2 (applied to Ψ), $t = 1$ is the only stationary point of r on $(0, \infty)$. Furthermore, $r(0) = 0 < r(1) = m(\Psi)$ and, by (2.11), $\lim_{t\to\infty} r(t) = -\infty$. Hence

$$J_\Psi(v) = r(1) > r(t) = J_\Psi(tv) \quad \text{for all } t \in (0, \infty)\setminus\{1\}.$$

On the other hand, by Lemma 3.2, there exists $s(v) > 0$ such that $s(v)v \in V(\Phi)$. But $\|v\|^2 = \psi(v) < \phi(v)$ and so by remark (iii) of § 2, $s(v) < 1$. Hence

$$m(\Psi) = J_\Psi(v) > J_\Psi(s(v)v) > J_\Phi(s(v)v) \geq m(\Phi).$$

The result now follows from Theorem 3.6.

4. *Properties of solutions of* (1.1)

The general theory of § 3 applies to the eigenvalue problem (1.1), (1.2) provided that the given functions r and f satisfy the following conditions.

(C1) $r: \mathbb{R}^N \to \mathbb{R}$ is Hölder continuous, $\lim_{|x|\to\infty} r(x) = r(\infty)$ exists, and there exists $t \geq 0$ such that

$$0 < A \leq r(x)(1+|x|)^t \leq B \quad \text{for all } x \in \mathbb{R}^N. \tag{4.1}$$

REMARKS. 1. Clearly $r(\infty) = 0$ if $t > 0$, and $r(\infty) > 0$ if $t = 0$.

2. The Hölder continuity of r is required in order to allow us to discuss classical solutions of (1.1). If one is content to treat only weak solutions, this condition can be dropped.

(C2) $f(s) = \sum_{i=1}^n a_i |s|^{\sigma_i} s$ for $s \in \mathbb{R}$ where $a_i > 0$ and $0 < \sigma_1 < \ldots < \sigma_n < 4/(N-2)$. (If $N \leq 2$ we simply require $\sigma_n < \infty$.)

It follows that $f \in C^1(\mathbb{R})$ and that $f(0) = f'(0) = 0$.

As explained in the introduction, we are interested in solutions of (1.1), (1.2) with $\lambda < 0$ and $u \not\equiv 0$. The set of classical solutions is

$$\mathscr{S} = \{(\lambda, u) \in \mathbb{R} \times C^2(\mathbb{R}^N): \lambda < 0, u \not\equiv 0, \text{ and (1.1), (1.2) are satisfied}\},$$

but the variational approach leads naturally to the set of weak solutions,

$$S = \Big\{(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^N): \lambda < 0, u \not\equiv 0, \text{ and } \int_{\mathbb{R}^N} [\nabla u \cdot \nabla v - rf(u)v - \lambda uv]\, dx = 0 \text{ for all } v \in C_0^\infty(\mathbb{R}^N)\Big\}.$$

In this section we establish various properties of such solutions and we show, in particular, that $\mathscr{S} = S$. In the next section we apply the results of § 4 to prove that $s \neq \varnothing$ and to discuss L^p-bifurcation.

For $u: \mathbb{R}^N \to \mathbb{R}$, let

$$F(u)(x) = r(x)f(u(x)) \tag{4.2}$$

and

$$\Phi(u) = \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{r(x)a_i}{(\sigma_i + 2)} |u(x)|^{\sigma_i+2}\, dx. \tag{4.3}$$

LEMMA 4.1. *Let* (C1) *and* (C2) *hold. Then*

(i) $F: H^1(\mathbb{R}^N) \to [H^1(\mathbb{R}^N)]^*$ *boundedly and continuously,*

(ii) $\Phi \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ *and* $\Phi'(u)v = \int_{\mathbb{R}^N} F(u)v\, dx$ *for all* $u, v \in H^1(\mathbb{R}^N)$,

(iii) $\Phi: H^1(\mathbb{R}^N) \to \mathbb{R}$ *is weakly sequentially continuous if and only if* $t > 0$.

Proof. This follows from Lemmas 5.1 and 5.2 of **[27]**.

It is convenient to use the following notation. For $\lambda < 0$,

$$\|u\|_\lambda = \{|\nabla u|_2^2 - \lambda\, |u|_2^2\}^{\frac{1}{2}} \tag{4.4}$$

defines a norm on $H^1(\mathbb{R}^N)$ which is equivalent to the usual one. (Here and elsewhere $|u|_p$ denotes the usual $L^p(\mathbb{R}^N)$ norm of u.) Let H denote $H^1(\mathbb{R}^N)$. Let

$$J_\lambda(u) = \|u\|_\lambda^2 - 2\Phi(u), \tag{4.5}$$

where Φ is defined by (4.3).

LEMMA 4.2. *Let the conditions* (C1) *and* (C2) *be satisfied and consider* $\lambda < 0$. *Then,*

(i) $J_\lambda \in C^1(H, \mathbb{R})$,

(ii) $(\lambda, u) \in S$ *if and only if* $u \in H\setminus\{0\}$ *and* $J'_\lambda(u)v = 0$ *for all* $v \in H$.

Proof. (i) This follows from Lemma 4.1 and we have

$$J'_\lambda(u)v = 2\int_{\mathbb{R}^N} (\nabla u \,.\, \nabla u - \lambda uv - rf(u)v)\,dx \quad \text{for } u, v \in H.$$

(ii) This follows from (i) since $rf(u) = F(u) \in H^*$ and $C_0^\infty(\mathbb{R}^N)$ is dense in H.

The remainder of this section is devoted to establishing the regularity of weak solutions of (1.1), the exponential decay of classical solutions of (1.1), (1.2),and finally the equivalence of weak and classical solutions. We begin this programme by quoting two fundamental results. The first deals with the regularity of the Laplacian on $\mathbb{R}^N$ and the second establishes the exponential decay of eigenvalues of the Schrödinger equation.

We recall that if $u \in L^p(\mathbb{R}^N)$ for some $1 \leqslant p \leqslant \infty$, then u defines a tempered distribution.

PROPOSITION 4.3. *Let h be a tempered distribution on $\mathbb{R}^N$ and consider the equation*

$$\Delta u + \lambda u + h = 0 \tag{4.6}$$

in the sense of distributions for $\lambda < 0$.

(i) *There is a unique tempered distribution satisfying* (4.6).

(ii) *If* $h \in L^p(\mathbb{R}^N)$ *for some* $1 \leqslant p \leqslant \infty$, *then the solution* $u \in L^p(\mathbb{R}^N)$ *and* $|\lambda|\,|u|_p \leqslant |h|_p$. *Furthermore, if* $1 < p < \infty$, *then* $u \in W^{2,p}(\mathbb{R}^N)$ *and*

$$\|u\|_{W^{2,p}(\mathbb{R}^N)} \leqslant C(\lambda, p)\,|h|_p.$$

(iii) *If* $h \geqslant 0$ $(h \not\equiv 0)$ *in the sense of distributions, then*

$$u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \quad \text{for } 1 < p < N/(N-1) \quad \text{and} \quad \inf_{|x|\leqslant R} \operatorname{ess} u(x) > 0 \text{ for all } R > 0.$$

Proof. This is part of Proposition 27 in [**9**, Chapter II, °8], combined with the Calderon–Zygmund estimate [**22**, Chapter III, Proposition 3],

$$\left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|_p \leqslant A_p\,|\Delta u|_p \quad \text{for all } u \in W^{2,p}(\mathbb{R}^N).$$

PROPOSITION 4.4. *Let* $q \in C(\mathbb{R}^N)$ *be such that* $\lim_{|x|\mapsto\infty} q(x) = 0$. *If* $u \in C^2(\mathbb{R}^N)$ *is a solution of*

$$\Delta u(x) + [\lambda + q(x)]u(x) = 0 \quad \text{for } x \in \mathbb{R}^N,$$

$$\lim_{|x|\mapsto\infty} u(x) = 0,$$

with $\lambda < 0$, *then* $\lim_{|x|\mapsto\infty} u(x)e^{\alpha|x|} = 0$ *for all* $\alpha \in (0, \surd|\lambda|)$.

Proof. Fix $\alpha \in (0, \sqrt{|\lambda|})$ and set $\delta = |\lambda| - \alpha^2$. Since $\lim_{|x|\to\infty} q(x) = 0$, there exists $R > 0$ such that $q(x) \leqslant \delta$ for all $|x| \geqslant R$. Let

$$w(x) = Me^{-\alpha(|x|-R)} \quad \text{for } x \neq 0$$

where $M = \max\{|u(x)|: \ |x| = R\}$, and for $L > R$, let

$$\Omega(L) = \{x \in \mathbb{R}^N: \ R < |x| < L \text{ and } u(x) > w(x)\}.$$

Then $\Omega(L)$ is open and, for $x \in \Omega(L)$,

$$\begin{aligned}\Delta(w-u)(x) &= \left[\alpha^2 - \frac{\alpha(N-1)}{|x|}\right] w(x) + [\lambda + q(x)]u(x) \\ &\leqslant \alpha^2 w(x) + [-|\lambda| + \delta]u(x) \quad (\text{since } u(x) > 0 \text{ on } \Omega(L)) \\ &= \alpha^2(w(x) - u(x)) \\ &< 0.\end{aligned}$$

By the maximum principle, for $x \in \Omega(L)$,

$$\begin{aligned} w(x) - u(x) &\geqslant \min\{(w-u)(x): \ x \in \partial\Omega(L)\} \\ &\geqslant \min\left\{0, \min_{|x|=L} (w-u)(x)\right\}. \end{aligned}$$

Since $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} w(x) = 0$, by letting $L \to \infty$, we find that

$$w(x) - u(x) \geqslant 0$$

for all $|x| \geqslant R$. Applying the same result to $-u$, we obtain that $|u(x)| \leqslant w(x)$ for all $|x| \geqslant R$ and the result follows.

LEMMA 4.5. *Let the conditions* (C1) *and* (C2) *hold and suppose that* $(\lambda, u) \in S$. *Then*

$$u \in C^2(\mathbb{R}^N) \cap W^{2,p}(\mathbb{R}^N) \quad \textit{for all } p \in [2, \infty)$$

and, in particular,

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} |\nabla u(x)| = 0.$$

Proof. For $1 \leqslant i \leqslant n$, set $h_i = ra_i |u|^{\sigma_i} u$. Since $u \in H = H^1(\mathbb{R}^N)$, $u \in L^p = L^p(\mathbb{R}^N)$ for $2 \leqslant p < 2N/(N-2)$ and so $h_i \in L^p$ for $\tau \leqslant p < \tau_i$ where $\tau = \max\{1, 2/(\sigma_1 + 1)\}$ and $\tau_i = 2N/(N-2)(\sigma_i + 1)$. Recall that $\tau_i = +\infty$ if $N \leqslant 2$.

Let u_i be the unique tempered distribution satisfying

$$\Delta u + \lambda u + h_i = 0 \quad \text{on } \mathbb{R}^N.$$

Then $u_i \in W^{2,p}(\mathbb{R}^N)$ for $\tau < p < \tau_i$. Furthermore, $F(u) = \sum_{i=1}^n h_i$ and $u \in H^1(\mathbb{R}^N)$ satisfies

$$\Delta u + \lambda u + F(u) = 0 \quad \text{on } \mathbb{R}^N,$$

in the sense of distributions. Hence by the uniqueness in Proposition 4.3, we have $u = \sum_{i=1}^n u_i$. Thus $u \in W^{2,p}(\mathbb{R}^N)$ for $\tau < p < \tau_n$, and by the Sobolev embeddings **[1, 15]**, $u \in L^q(\mathbb{R}^N)$ where

$$\tau < q < N\tau_n/(N - 2\tau_n) \quad \text{if } \tau_n < \tfrac{1}{2}N$$

and

$$\tau < q < \infty \qquad \text{if } \tau_n \geqslant \tfrac{1}{2}N.$$

The condition $\tau_n \geq \frac{1}{2}N$ is equivalent to $\sigma_n \leq (6-N)/(N-2)$, and in this case we have $u \in L^q$ for $2 \leq q < \infty$ and consequently $h_i \in L^p$ for $2 \leq p < \infty$. Hence when $\tau_n \geq \frac{1}{2}N$, we have $u_i \in W^{2,p}$ for $2 \leq p < \infty$ and the result is established.

For the case where $\tau_n < \frac{1}{2}N$ we first observe that $N\tau_n/(N-2\tau_n) > 2N/(N-2)$ since $\sigma_n < 4/(N-2)$. Thus when $\tau_n < \frac{1}{2}N$ we can assume that there exists $T_k > 2N/(N-2)$ such that $u \in L^q$ for $2 \leq q < T_k$. But then $h_i \in L^p$ for $\tau \leq p < T_k/(\sigma_i+1)$ and so $u_i \in W^{2,p}$ for $\tau < p < T_k/(\sigma_i+1)$. The Sobolev embeddings now yield $u_i \in L^q$ where

$$\tau < q < NT_k/(N(\sigma_i+1)-2T_k) \quad \text{if } T_k < \tfrac{1}{2}N(\sigma_i+1)$$

and

$$\tau < q < \infty \quad \text{if } T_k \geq \tfrac{1}{2}N(\sigma_i+1).$$

Thus $u \in L^q$ where

$$\tau < q < NT_k/(N(\sigma_n+1)-2T_k) \quad \text{if } T_k < \tfrac{1}{2}N(\sigma_n+1)$$

and

$$\tau < q < \infty \quad \text{if } T_k \geq \tfrac{1}{2}N(\sigma_n+1).$$

If $T_k < \frac{1}{2}N(\sigma_n+1)$, we set

$$T_{k+1} = \frac{NT_k}{N(\sigma_n+1)-2T_k},$$

and observe that $T_{k+1} > T_k$ since $T_k > 2N/(N-2)$. In this way we construct an increasing sequence T_k such that

$$\frac{T_{k+1}}{T_k} = \frac{1}{(\sigma_n+1)-2T_k/N}.$$

If $T_k < \frac{1}{2}N(\sigma_n+1)$ for all $k \in \mathbb{N}$, we would have

$$\lim_{k\to\infty} T_k = L \quad \text{and} \quad 1 = \frac{1}{(\sigma_n+1)-2L/N}.$$

Hence $L = \frac{1}{2}N\sigma_n$, implying that

$$T_k < \tfrac{1}{2}N\sigma_n < \frac{N}{2}\cdot\frac{4}{(N-2)} = \frac{2N}{N-2},$$

which is false. Thus there exists $k \in \mathbb{N}$ such that $T_k \geq \frac{1}{2}N(\sigma_n+1)$, and the proof is now completed as in the case where $\tau_n \geq \frac{1}{2}N$.

We have shown that $u \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [2, \infty)$. The remaining statements in Lemma 4.5 are consequences of this and the Hölder continuity of r and f.

LEMMA 4.6. *Let the conditions* (C1) *and* (C2) *be satisfied and consider* $(\lambda, u) \in \mathcal{S}$. *Then*

$$\lim_{|x|\to\infty} e^{\alpha|x|}u(x) = 0 \quad \textit{for all } \alpha < \sqrt{|\lambda|}.$$

Proof. Set

$$q(x) = r(x)\sum_{i=1}^{n} a_i\,|u(x)|^{\sigma_i}.$$

Clearly $q \in C(\mathbb{R}^N)$, $q(x) \geqslant 0$ for all $x \in \mathbb{R}^N$, and $\lim_{|x| \to \infty} q(x) = 0$. The result now follows from Proposition 4.4.

We end this section by summarizing the properties of solutions of (4.1).

THEOREM 4.7. *Let the conditions* (C1) *and* (C2) *be satisfied. Then*
(i) $\mathcal{S} = S$,
(ii) *if* $(\lambda, u) \in \mathcal{S}$, *then*

$$u \in L^p(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \cap W^{2,q}(\mathbb{R}^N) \quad \text{for } 1 \leqslant p \leqslant \infty \text{ and } 1 < q < \infty.$$

Furthermore,

$$\lim_{|x| \to \infty} e^{\alpha|x|} u(x) = 0 \quad \text{for all } \alpha < \surd|\lambda|$$

and

$$\lim_{|x| \to \infty} |\nabla u(x)| = 0.$$

Proof. (i) By Lemma 4.5, we already have $S \subset \mathcal{S}$. Supposing that $(\lambda, u) \in \mathcal{S}$ we conclude from Lemma 4.6 that $u \in L^p(\mathbb{R}^N)$ for all $1 \leqslant p \leqslant \infty$, and so $F(u) \in L^q(\mathbb{R}^N)$ for all $1 \leqslant q \leqslant \infty$. But then Proposition 4.3 with $h = F(u)$ implies that $u \in W^{2,q}(\mathbb{R}^N)$ for $1 < q < \infty$. In particular, $(\lambda, u) \in S$ and so $\mathcal{S} \subset S$.

(ii) This follows from the inclusions noted in part (i) and from Lemmas 4.5 and 4.6.

THEOREM 4.8. *Let the conditions* (C1) *and* (C2) *be satisfied. There exists a constant* $C(p) > 0$ *such that*

$$|u|_p \geqslant C(p) \quad \text{for all } (\lambda, u) \in S$$

under the following conditions:

(i) $N \geqslant 3$ *and*

$$\begin{aligned} &\max\{1, \tfrac{1}{2}N\sigma_n\} \leqslant p < N\sigma_1/(2-t) && \text{if } 0 < t \leqslant 2, \\ &\max\{1, \tfrac{1}{2}N\sigma_n\} \leqslant p \leqslant \infty && \text{if } t > 2, \\ &p = \tfrac{1}{2}N\sigma_1 && \text{if } t = 0,\ n = 1, \text{ and } N\sigma_1 \geqslant 2; \end{aligned}$$

(ii) $N = 2$ *and the same restrictions as in* (i) *except that we require* $p < \infty$ *in addition when* $t > 2$;
(iii) $N = 1$ *and the same restrictions as in* (i) *except that we require* $p \leqslant \sigma_1$ *in addition when* $t > 1$.

Proof. (i) For $N \geqslant 3$, there exists $A(N) > 0$ such that

$$|u|_{2N/(N-2)}^2 \leqslant A(N)\, |\nabla u|_2^2 \quad \text{for all } u \in H = H^1(\mathbb{R}^N). \tag{4.7}$$

Since $(\lambda, u) \in S$,

$$|\nabla u|_2^2 - \lambda\, |u|_2^2 - \int_{\mathbb{R}^N} rf(u)u\, dx = 0 \tag{4.8}$$

with $\lambda < 0$, and so

$$|u|_{2N/(N-2)}^2 \leqslant A(N) \sum_{i=1}^n a_i \int_{\mathbb{R}^N} r\, |u|^{\sigma_i + 2}\, dx.$$

Also $|r|_s < \infty$ if $ts > N$, and $|r|_\infty < \infty$ for $t \geq 0$. Using Hölder's inequality we obtain

$$\int_{\mathbb{R}^N} r\,|u|^{\sigma_i+2}\,dx \leq \left(\int_{\mathbb{R}^N} r^{\frac{1}{2}N}\,|u|^{\frac{1}{2}\sigma_i N}\,dx\right)^{2/N}\left(\int_{\mathbb{R}^N} |u|^{2N/(N-2)}\,dx\right)^{(N-2)/N}$$

$$\leq \left(\int_{\mathbb{R}^N} r^{\frac{1}{2}Ns_i}\,dx\right)^{2/Ns_i}\left(\int_{\mathbb{R}^N} |u|^{\frac{1}{2}\sigma_i N t_i}\,dx\right)^{2/Nt_i} |u|^2_{2N/(N-2)},$$

provided that $s_i \geq 1$ and $1/s_i + 1/t_i = 1$. Let $0 < t \leq 2$. Choosing $t_i = 2p/\sigma_i N$ we have $t_i \geq 1$ when $p \geq \frac{1}{2}\sigma_n N$, and $\frac{1}{2}Ns_i t > N$ when $p < \sigma_1 N/(2-t)$ since

$$\frac{1}{s_i} = 1 - \frac{1}{t_i} = 1 - \frac{\sigma_i N}{2p}.$$

Thus

$$1 \leq A(N) \sum_{i=1}^{n} a_i\,|r|_{\frac{1}{2}Ns_i}\,|u|_p^{\sigma_i} \quad \text{and} \quad |r|_{\frac{1}{2}Ns_i} < \infty,$$

for $1 \leq i \leq n$. This proves the result when $N \geq 3$ and $0 < t \leq 2$. For $N \geq 3$ and $t > 2$ we note that the above choice of t_i is acceptable for $\max\{1, \frac{1}{2}N\sigma_n\} \leq p \leq \infty$. For $N \geq 3$ and $t = 0$ we use $t_1 = 1$ and $|r|_\infty < \infty$ to obtain the result when $n = 1$.

(ii) For the case where $N = 2$, we replace (4.7) by the inequality

$$|u|_q \leq C(q, r)\,|\nabla u|_2^\alpha\,|u|_r^{1-\alpha} \quad \text{for } 1 \leq r \leq q, \tag{4.9}$$

where $\alpha = 1 - r/q$. This is a special case of Theorem 7.1 of [**15**, p. 34]. Now

$$|\nabla u|_2^2 \leq \int_{\mathbb{R}^N} rf(u)u\,dx \quad (\text{for } (\lambda, u) \in S, \text{ by (4.8)})$$

$$\leq \sum_{i=1}^{n} a_i\,|r|_{s_i}\,|u|^{\sigma_i+2}_{(\sigma_i+2)t_i}$$

provided that $s_i \geq 1$ and $1/s_i + 1/t_i = 1$. By (4.9),

$$|u|^{\sigma_i+2}_{(\sigma_i+2)t_i} \leq C\,|\nabla u|_2^{\alpha_i(\sigma_i+2)}\,|u|_{r_i}^{(1-\alpha_i)(\sigma_i+2)},$$

where $\alpha_i = 1 - (r_i/(\sigma_i+2)t_i)$. Choosing $t_i = p/\sigma_i$ and $r_i = p$ we have $t_i \geq 1$ and $\alpha_i = 2/(\sigma_i + 2)$. Furthermore, $(\sigma_i + 2)t_i > p = r_i \geq 1$. Hence

$$|u|^{\sigma_i+2}_{(\sigma_i+2)t_i} \leq C\,|\nabla u|_2^2\,|u|_p^{\sigma_i}$$

and so

$$1 \leq C \sum_{i=1}^{n} a_i\,|r|_{s_i}\,|u|_p^{\sigma_i}$$

where $1/s_i + 1/t_i = 1$. Noting that $s_i t > 2$, since $p < 2\sigma_i/(2-t)$ when $0 < t \leq 2$, we see that this proves the result for $0 < t \leq 2$ and $N = 2$.

The same choice of t_i and r_i yields the result when $t > 2$. If $t = 0$ and $n = 1$, we use $t_1 = 1$.

For the case where $N = 1$, we replace (4.9) by

$$|u|_q \leq C(q, r)\,|u'|_2^\alpha\,|u|_r^{1-\alpha} \quad \text{for } 1 \leq r \leq q, \tag{4.10}$$

where $\alpha = 1 - r(q+2)/q(r+2)$. The proof is similar to the case where $N = 2$. By (4.10),

$$|u|^{\sigma+2}_{(\sigma_i+2)t_i} \leq C\,|u'|_2^{\alpha_i(\sigma_i+2)}\,|u|_{r_i}^{(1-\alpha_i)(\sigma_i+2)}$$

where

$$\alpha_i = 1 - \frac{r_i[(\sigma_i+2)t_i+2]}{(\sigma_i+2)t_i(r_i+2)}.$$

We choose $r_i = p$ and t_i so that $\alpha_i(\sigma_i+2)=2$. Thus $t_i = p/(\sigma_i - p)$ and we have the further restrictions that $t_i \geq 1$ and $s_i t > 1$ where $1/s_i + 1/t_i = 1$. These requirements lead to the restrictions $\frac{1}{2}\sigma_i \leq p < \sigma_i$ and $p < \sigma_i/(2-t)$. When $t > 1$ and $p = \sigma_1$, we can take $t_i = \infty$ and $s_i = 1$.

5. *Bifurcation theory*

In this section we apply the result of § 3 to the problem (1.1), (1.2). For this we maintain the notation introduced in § 4. In particular, for $\lambda < 0$,

$$\|u\|_\lambda = \{|\nabla u|_2^2 - \lambda\, |u|_2^2\}^{\frac{1}{2}} \quad \text{and} \quad H = H^1(\mathbb{R}^N).$$

Also, F, Φ, and J_λ are defined by (4.2), (4.3), and (4.5) respectively. By Lemma 4.1, Φ satisfies the condition (H1) of § 2 and we define ϕ and $\tilde{\phi}$ by (2.1) and (2.2).

LEMMA 5.1. *Let the conditions* (C1) *and* (C2) *be satisfied. Then* Φ *satisfies the conditions* (H1) *to* (H4) *with* $\alpha = \sigma_1 + 2$ *and* $\gamma = \sigma_n + 2$. *Furthermore,* Φ, ϕ, *and* $\tilde{\phi}$ *are weakly sequentially continuous on* H *if and only if* $t > 0$.

Proof. By Lemma 4.1, Φ satisfies (H1) and

$$\phi(u) = \Phi'(u)u = \sum_{i=1}^n a_i \int_{\mathbb{R}^N} r\, |u|^{\sigma_i+2}\, dx, \tag{5.1}$$

$$\tilde{\phi}(u) = \phi(u) - 2\Phi(u) = \sum_{i=1}^n \frac{a_i\sigma_i}{\sigma_i+2} \int_{\mathbb{R}^N} r\, |u|^{\sigma_i+2}\, dx. \tag{5.2}$$

As in Lemma 4.1, it follows that ϕ, $\tilde{\phi} \in C^1(H, \mathbb{R})$, that

$$\phi'(u)v = \sum_{i=1}^n a_i(\sigma_i+2) \int_{\mathbb{R}^N} r\, |u|^{\sigma_i}\, uv\, dx, \tag{5.3}$$

$$\tilde{\phi}'(u)v = \sum_{i=1}^n a_i\sigma_i \int_{\mathbb{R}^N} r\, |u|^{\sigma_i}\, uv\, dx, \tag{5.4}$$

that ϕ' and $\tilde{\phi}'$ map H continuously and boundedly in H^*, and that ϕ and $\tilde{\phi}$ are weakly sequentially continuous if and only if $t > 0$. Furthermore,

$$\phi(u) \geq (\sigma_1+2)\Phi(u) > 0 \quad \text{if } u \neq 0,$$

$$\tilde{\phi}'(u)u \geq (\sigma_1+2)\tilde{\phi}(u),$$

and

$$\tilde{\phi}(u) \leq K\{|u|_{\sigma_1+2}^{\sigma_1+2} + |u|_{\sigma_n+2}^{\sigma_n+2}\} \leq K(\lambda)\{\|u\|_\lambda^{\sigma_1+2} + \|u\|_\lambda^{\sigma_n+2}\},$$

by the Sobolev embedding theorem. Thus (H2) is satisfied with $\alpha = \sigma_1 + 2$ and $\gamma = \sigma_n + 2$.

For (H3) we note that $s \to |s|^{\sigma_i+2}$ is a convex function and so for all $u, v \in H$,

$$\tilde{\phi}(v) \geq \tilde{\phi}(u) + \tilde{\phi}'(u)(v-u).$$

Thus if $u_k \rightharpoonup u$ weakly in H,

$$\tilde{\phi}'(u)(u_k - u) \to 0 \quad \text{and} \quad \liminf_{k\to\infty} \tilde{\phi}(u_k) \geq \tilde{\phi}(u).$$

Thus $\tilde{\phi}$ is weakly sequentially lower semi-continuous on H. For (H4), we suppose that $u_k \rightharpoonup u$ weakly in H and, for $v \in C_0^\infty(\mathbb{R}^N)$, we consider

$$\begin{aligned}\langle \nabla\Phi(u_k) - \nabla\Phi(u), v\rangle &= \Phi'(u_k)v - \Phi'(u)v \\ &= \int_{\mathbb{R}^N} r\{f(u_k) - f(u)\}v\,dx \\ &= \sum_{i=1}^{n} a_i \int_B r\{|u_k|^{\sigma_i} u_k - |u|^{\sigma_i} u\}v\,dx,\end{aligned}$$

where $B \subset \mathbb{R}^N$ is a ball such that $\operatorname{supp} v \subset B$.

As in Lemma 5.2 of [**27**], using the compactness of the embedding

$$H^1(B) \subset L^p(B) \quad \text{for } 1 \leq p < 2N/(N-2),$$

and the fact that $u \to |u|^{\sigma_i} u$ maps $L^{\sigma_i+2}(B)$ continuous into $L^{(\sigma_i+2)/(\sigma_i+1)}(B)$, we deduce, from the estimate

$$\left|\int_B r\{|u_k|^{\sigma_i} u_k - |u|^{\sigma_i} u\}v\,dx\right| \leq K\big||u_k|^{\sigma_i} u_k - |u|^{\sigma_i} u\big|_{(\sigma_i+2)/(\sigma_i+1)} |v|_{\sigma_i+2},$$

that

$$\lim_{k\to\infty} \langle \nabla\Phi(u_k) - \nabla\Phi(u), v\rangle = 0 \quad \text{for all } v \in C_0^\infty(\mathbb{R}^N).$$

But by Lemma 4.1, $\nabla\Phi$: $H \to H$ is a bounded map. It follows that

$$\lim_{k\to\infty} \langle \nabla\Phi(u_k) - \nabla\Phi(u), v\rangle = 0 \quad \text{for all } v \in H,$$

and so the condition (H4) is satisfied.

Having verified the basic hypotheses for the variational method, we can now apply the main results of § 3 to the problem (4.1). We begin by estimating the infimum of J on V.

With $H = H^1(\mathbb{R}^N)$, we set

$$V_\lambda = \{u \in H\setminus\{0\}:\ \|u\|_\lambda^2 = \phi(u)\}$$

and

$$m_\lambda = \inf\{J_\lambda(u):\ u \in V_\lambda\} = \inf\{\tilde{\phi}(u):\ u \in V_\lambda\}.$$

LEMMA 5.2. *Let the conditions* (C1) *and* (C2) *be satisfied and let* $\lambda < 0$.

(a) *For* $N \geq 1$ *and* $0 \leq t < 2$, *there exists* $\hat{\lambda} < 0$ *such that*

$$m_\lambda \leq E\,|\lambda|^{(\sigma_1(2-N)+2(2-t))/2\sigma_1} \quad \textit{for } \hat{\lambda} < \lambda < 0,$$

where E *is a positive constant depending on* N *and* t *but not on* λ.

(b) *For* $N = 1$ *and* $t \geq 0$, *there exists* $\hat{\lambda} < 0$ *such that*

$$m_\lambda \leq E\,|\lambda|^{(\sigma_1+2)/2\sigma_1} \quad \textit{for } \hat{\lambda} < \lambda < 0,$$

where E *is a positive constant.*

Proof. For $k>0$ we set $-k^2=\lambda$ and $v_k(x)=e^{-k|x|}$. Then $v_k\in H$,

$$|v_k|_2^2=\int_{\mathbb{R}^N} v_k(x)^2\,dx = C_N k^{-N},$$

where

$$C_N=\int_{\mathbb{R}^N} e^{-2|y|}\,dy,$$

$$|\nabla v_k(x)|^2=k^2 v_k(x)^2 \quad \text{for all } x\neq 0,$$

$$|\nabla v_k|_2^2 = C_N k^{2-N} \quad\text{and}\quad \|v_k\|_\lambda^2 = 2C_N k^{2-N}.$$

Also, for $1\leqslant i\leqslant n$,

$$\phi(v_k)=\sum_{i=1}^n a_i\int_{\mathbb{R}^N} r\,|v_k|^{\sigma_i+2}\,dx=\sum_{i=1}^n \phi_i(v_k)$$

where

$$\phi_i(v_k)=k^{-N}a_i\int_{\mathbb{R}^N} r\left(\frac{y}{k}\right)e^{-(\sigma_i+2)|y|}\,dy,$$

and

$$\int_{\mathbb{R}^N} r\left(\frac{y}{k}\right)e^{-(\sigma_i+2)|y|}\,dy \geqslant A\int_{|y|\geqslant 1}\left(1+\frac{|y|}{k}\right)^{-t} e^{-(\sigma_i+2)|y|}\,dy$$

$$\geqslant D_i(t)k^t \quad \text{for } 0<k\leqslant 1,$$

where

$$D_i(t)=2^{-t}A\int_{|y|\geqslant 1}|y|^{-t}e^{-(\sigma_i+2)|y|}\,dy.$$

Thus

$$\phi(v_k)\geqslant B_i(t)k^{t-N} \quad\text{where } 0<k\leqslant 1 \text{ and } B_i(t)=a_iD_i(t). \tag{5.5}$$

Now $s(v_k)v_k\in V_\lambda$ for $s(v_k)=s_k>0$ provided that

$$\|v_k\|_\lambda^2 = s_k^{-2}\phi(s_kv_k). \tag{5.6}$$

Since $\phi(s_kv_k)\geqslant s_k^{\sigma_i+2}\phi_i(v_k)$, we have that

$$2C_Nk^{2-N}\geqslant s_k^{\sigma_i}k^{t-N}B_i(t) \quad \text{for } 1\leqslant i\leqslant n \text{ and } 0<k\leqslant 1,$$

and so

$$s_k\leqslant\left\{\frac{2C_Nk^{2-t}}{B_i(t)}\right\}^{1/\sigma_i}. \tag{5.7}$$

Consequently, for $0\leqslant t<2$, there exists $k_t\in(0,1]$ such that $s_k<1$ for $0<k\leqslant k_t$. Returning to (5.6), we obtain $\|v_k\|_\lambda^2\geqslant s_k^{\sigma_1}\phi_1(v_k)$. But

$$m_\lambda\leqslant\tilde\phi(s_kv_k)=\sum_{i=1}^n\tilde\phi_i(s_kv_k),$$

where

$$\tilde\phi_i(s_kv_k)=\frac{a_i\sigma_i}{(\sigma_i+2)}\int_{\mathbb{R}^N} r s_k^{\sigma_i+2}v_k^{\sigma_i+2}\,dx$$

$$\leqslant s_k^{\sigma_1+2}a_i\int_{\mathbb{R}^N} r v_k^{\sigma_1+2}\,dx$$

$$=s_k^{\sigma_1+2}\frac{a_i}{a_1}\phi_1(v_k),$$

since $0 < s_k \leqslant 1$ and $0 < v_k < 1$. Thus

$$\begin{aligned} m_\lambda &\leqslant \left(\sum_{i=1}^n a_i\right) a_1^{-1} s_k^{\sigma_1+2} \phi_1(v_k) \\ &\leqslant \left(\sum_{i=1}^n a_i\right) a_1^{-1} \|v_k\|_\lambda^{2(\sigma_1+2)/\sigma_1} \phi_1(v_k)^{-2/\sigma_1} \\ &\leqslant E\,|\lambda|^{(2-N)(\sigma_1+2)/2\sigma_1}\,|\lambda|^{-(t-N)/\sigma_1} \\ &= E\,|\lambda|^{(\sigma_1(2-N)+2(2-t))/2\sigma_1}, \end{aligned}$$

where $0 \leqslant t < 2$ and $0 < |\lambda|^{\frac{1}{2}} < k_t$. The constant E depends on N and t.

For $N = 1$, it is possible to replace (5.5) by an alternative estimate which leads to a useful result even for $t \geqslant 2$. In fact, for $1 \leqslant i \leqslant n$,

$$\begin{aligned} \phi_i(v_k) &= a_i \int_{\mathbb{R}} r(x) e^{-(\sigma_i+2)k|x|}\,dx \\ &\geqslant a_i e^{-(\sigma_i+2)k\delta} \int_{|x|\leqslant\delta} r(x)\,dx \quad (\text{for all } \delta > 0) \\ &\geqslant \tfrac{1}{2} a_i e^{-(\sigma_i+2)k\delta} \min\left\{\int_{-\infty}^{\infty} r(x)\,dx, 1\right\} \quad (\text{for } \delta > \delta_0) \\ &\geqslant \tfrac{1}{4} a_i \min\left\{\int_{-\infty}^{\infty} r(x)\,dx,\ 1\right\} \\ &= K_i \quad (\text{for } 0 < k < k_0). \end{aligned} \tag{5.8}$$

From (5.6) we obtain, for $0 < k < k_0$,

$$2C_1 k \geqslant s_k^{\sigma_1} \phi_1(v_k) \geqslant s_k^{\sigma_1} K_1,$$

and so there exists $\hat{k} > 0$ such that $s_k < 1$ for $0 < k < \hat{k}$. Hence

$$m_\lambda \leqslant \left(\sum_{i=1}^n a_i\right) a_1^{-1} s_k^{\sigma_1+2} \phi_1(v_k)$$

as before, and

$$m_\lambda \leqslant \left(\sum_{i=1}^n a_i\right) a_1^{-1} \|v_k\|_\lambda^{2(\sigma_1+2)/\sigma_1} \phi_1(v_k)^{-2/\sigma_1} \leqslant E\,|\lambda|^{(\sigma_1+2)/2\sigma_1} \quad \text{by (5.8).}$$

Next we observe that the conditions (C1) and (C2) do not imply that J_λ attains its minimum on V_λ for $\lambda < 0$.

LEMMA 5.3. *Let the conditions* (C1) *and* (C2) *be satisfied and suppose, in addition, that* $n = 1$ *and* $0 < r(x) < r(\infty)$ *for* $x \in \mathbb{R}^N$. *Then, for* $\lambda < 0$, $J_\lambda(u) > m_\lambda$ *for all* $u \in V_\lambda$.

Proof. Suppose that there exists $u \in V_\lambda$ such that $J_\lambda(u) = m_\lambda$. Since $|u| \in V_\lambda$ and $J_\lambda(|u|) = J_\lambda(u)$, we can suppose that $u(x) \geqslant 0$ for all $x \in \mathbb{R}^N$. But then by Lemma 4.2, $(\lambda, u) \in S = \mathcal{S}$ and by Proposition 4.3(iii), $u(x) > 0$ on $\mathbb{R}^N$.

For $T \in \mathbb{R}$ and $e_1 = (1, 0, \ldots, 0)$, let $u_T(x) = u(x - Te_1)$. Then $\|u_T\|_\lambda = \|u\|_\lambda$ for all $T \in \mathbb{R}$ and

$$\phi(u_T) = \int_{\mathbb{R}^N} a_1 r(x)\,|u(x - Te_1)|^{\sigma_1+2}\,dx = \int_{\mathbb{R}^N} a_1 r(x + Te_1)\,|u(x)|^{\sigma_1+2}\,dx.$$

By dominated convergence,

$$\lim_{T\to\infty} \phi(u_T) = \int_{\mathbb{R}^N} a_1 r(\infty)\,|u(x)|^{\sigma_1+2}\,dx > \phi(u),$$

since $r(x) < r(\infty)$ and $u(x) > 0$ for all $x \in \mathbb{R}^N$.

Hence there exists $T > 0$ such that $\phi(u_T) > \phi(u)$ and setting $s(T) = s(u_T)$, we see that

$$s(T)u_T \in V_\lambda \quad \text{with } s(T) < 1.$$

Then

$$\begin{aligned} m_\lambda &\leq J_\lambda(s(T)u_T) \\ &= \bar{\phi}(s(T)u_T) \\ &= \left(\frac{\sigma_1}{\sigma_1+2}\right)\phi(s_T u_T) \quad (\text{since } n = 1) \\ &= \left(\frac{\sigma_1}{\sigma_1+2}\right) \|s_T u_T\|_\lambda^2 \\ &< \left(\frac{\sigma_1}{\sigma_1+2}\right) \|u\|_\lambda^2 \quad (\text{since } s_T < 1 \text{ and } \|u_T\|_\lambda = \|u\|_\lambda \neq 0) \\ &= J_\lambda(u) = m_\lambda. \end{aligned}$$

From this contradiction we must conclude that

$$J_\lambda(u) > m_\lambda \quad \text{for all } u \in V_\lambda.$$

In establishing the existence of solutions of (1.1), (1.2), it is convenient to distinguish three different situations. First of all, the case where $t > 0$ in (C1) is easiest since Φ is weakly sequentially continuous. Next the case where r is constant is treated by constructing a special minimising sequence by symmetrisation. Finally, cases where $t = 0$ and r is not necessarily radially symmetric are treated by comparison with the case where r is constant.

THEOREM 5.4. *Let the conditions* (C1) *and* (C2) *be satisfied with* $t > 0$. *Then for each* $\lambda < 0$, *there exists* $u_\lambda \in V_\lambda$ *such that* $J_\lambda(u_\lambda) = m_\lambda$ *and*

$$0 < \|u_\lambda\|_\lambda \leq \left\{\frac{(\sigma_1+2)m_\lambda}{\sigma_1}\right\}^{\frac{1}{2}}.$$

Furthermore, $(\lambda, u_\lambda) \in S = \mathcal{S}$ *and* $u_\lambda(x) > 0$ *for all* $x \in \mathbb{R}^N$.

Proof. The existence of u_λ follows immediately from Theorem 3.3 and Lemma 5.1. Furthermore, since $\|u\|_\lambda$, $\Phi(u)$, and $\phi(u)$ remain unchanged when u is replaced by $|u|$, we can suppose that the sequence $\{u_n\}$ in Theorem 3.3(ii) is such that $u_n \geq 0$ on $\mathbb{R}^N$. But then $u \geq 0$ on $\mathbb{R}^N$ and, applying Proposition 4.3(iii) with $h = r\,|u|^\sigma u$, we can conclude that $u_\lambda(x) > 0$ for all $x \in \mathbb{R}^N$.

THEOREM 5.5. *Let the conditions* (C1) *and* (C2) *be satisfied with* r *constant on* $\mathbb{R}^N$. *Then for each* $\lambda < 0$, *there exists* $u_\lambda \in V_\lambda$ *such that* $J_\lambda(u_\lambda) = m_\lambda$ *and*

$$0 < \|u_\lambda\|_\lambda \leq \left\{\frac{(\sigma_1+2)m_\lambda}{\sigma_1}\right\}^{\frac{1}{2}}.$$

Furthermore, $(\lambda, u_\lambda) \in S = \mathcal{S}$ *and* u_λ *is positive, radially symmetric and radially decreasing on* $\mathbb{R}^N$.

Proof. Let $\{w_n\} \subset V_\lambda$ be a sequence such that $\lim_{n\to\infty} J_\lambda(w_n) = m_\lambda$. As in Theorem 5.3, we can suppose that $w_n \geqslant 0$ on $\mathbb{R}^N$ and by Lemma 3.1(ii), $\|w_n\|_\lambda \leqslant C$ for all $n \in \mathbb{N}$. Let w_n^* be the Schwarz symmetrisation of w_n **[16, 19]**. Then $\|w_n^*\|_\lambda \leqslant \|w_n\|_\lambda$, $\phi(w_n^*) = \phi(w_n)$, $\Phi(w_n^*) = \Phi(w_n)$, and $\tilde{\phi}(w_n^*) = \tilde{\phi}(w_n)$.

Let $u_n = s(w_n^*)w_n^*$ where s: $H\backslash\{0\} \to (0, \infty)$ is defined in Lemma 3.2. Since $w_n \in V_\lambda$, we have $\|w_n\|_\lambda^2 = \phi(w_n)$ and so $\|w_n^*\|_\lambda^2 \leqslant \phi(w_n^*)$. Thus by remark (iii) of § 2, $s(w_n^*) \leqslant 1$ and

$$\begin{aligned} J_\lambda(u_n) &= \tilde{\phi}(u_n) \quad (\text{since } u_n \in V_\lambda) \\ &\leqslant \tilde{\phi}(w_n^*) \quad (\text{since } 0 < s(w_n^*) \leqslant 1) \\ &= \tilde{\phi}(w_n) \\ &= J_\lambda(w_n) \quad (\text{since } w_n \in V_\lambda). \end{aligned}$$

Hence $\{u_n\} \subset V_\lambda$ is a minimising sequence for J_λ and we can suppose that $u_n \rightharpoonup u$ weakly in H and $\|u_n\|_\lambda \leqslant C$ for all $n \in \mathbb{N}$. Since $u_n = u_n^*$, it follows that $u_n(x) = v_n(|x|)$, v_n is decreasing, and

$$C^2 \geqslant |\lambda|\, |u_n|_2^2 \geqslant |\lambda| \int_0^z \omega_N r^{N-1} v_n(r)^2\, dr \geqslant \frac{|\lambda|\, \omega_N z^N}{N} v_n(z)^2 \quad \text{for all } z > 0,$$

where ω_N is the area of the unit sphere in $\mathbb{R}^N$.

Using this estimate, it follows as in Lemma 6.3 of **[27]** that $\phi(u_n) \to \phi(u)$. Hence by Theorem 3.3, $u \in V_\lambda$ and $J_\lambda(u) = m_\lambda$. Furthermore, since $u_n = u_n^*$, it follows that $u = u^*$. The positivity of u on $\mathbb{R}^N$ is established using Proposition 4.3(iii) as in Theorem 5.4.

THEOREM 5.6. *Let the conditions* (C1) *and* (C2) *hold with* $t = 0$ *and suppose that there exists* $x_0 \in \mathbb{R}^N$ *such that*

$$\begin{aligned} &\int_{|x-x_0|=c} r(x)\, dx \gneqq r(\infty) \int_{|x-x_0|=c} dx \quad \text{for all } c > 0 \text{ if } N \geqslant 2, \\ &r(x_0 - c) + r(x_0 + c) \gneqq 2r(\infty) \qquad \text{for all } c > 0 \text{ if } N = 1. \end{aligned} \tag{5.9}$$

Then for each $\lambda < 0$, *there exists* $u_\lambda \in V_\lambda$ *such that* $J_\lambda(u_\lambda) = m_\lambda$ *and*

$$0 < \|u_\lambda\|_\lambda \leqslant \left\{ \frac{(\sigma_1 + 2)m_\lambda}{\sigma_1} \right\}^{\frac{1}{2}}.$$

Furthermore, $(\lambda, u_\lambda) \in S = \mathcal{S}$ *and* $u_\lambda(x) > 0$ *for all* $x \in \mathbb{R}^N$.

Proof. By translation we can suppose that $x_0 = 0$. Setting

$$\Psi(u) = r(+\infty) \int_{\mathbb{R}^N} \sum_{i=1}^n \frac{a_i}{(\sigma_i + 2)} |u(x)|^{\sigma_i + 2}\, dx$$

and

$$\Phi(u) = \int_{\mathbb{R}^N} r(x) \sum_{i=1}^n \frac{a_i}{(\sigma_i + 2)} |u(x)|^{\sigma_i + 2}\, dx,$$

we apply Corollary 3.7. By Theorem 5.4, there exists $v_\lambda \in V_\lambda(\Psi)$ such that $J_\Psi(v_\lambda) = m_\lambda(\Psi)$ and $v_\lambda(x) = w_\lambda(|x|) > 0$ for $x \in \mathbb{R}^N$. Then, for $t > 0$,

$$\begin{aligned}\phi(tv_\lambda) &= \sum_{i=1}^n a_i t^{\sigma_i+2} \int_{\mathbb{R}^N} r(x)\,|v_\lambda(x)|^{\sigma_i+2}\,dx \\ &= \sum_{i=1}^n a_i t^{\sigma_i+2} \int_0^\infty w_\lambda(c)^{\sigma_i+2} \int_{|y|=c} r(y)\,dy\,dc \\ &> \sum_{i=1}^n a_i t^{\sigma_i+2} \int_0^\infty w_\lambda(c)^{\sigma_i+2} r(\infty) \int_{|y|=c} dy\,dc \\ &= \psi(tv_\lambda).\end{aligned}$$

Thus the hypothesis (a) of Corollary 3.7 is satisfied and the hypothesis (b) follows from Lemma 5.2 of [**27**].

THEOREM 5.7. *Let the conditions* (C1) *and* (C2) *be satisfied and if $t = 0$, suppose either that r is constant or that r satisfies* (5.9). *For $\lambda < 0$, let $(\lambda, u_\lambda) \in S = \mathcal{S}$ be the solution given by Theorems* 5.4 *to* 5.6.

(a) *For $N \geqslant 1$ and $0 \leqslant t < 2$ there exist $\hat{\lambda} < 0$ and $E > 0$ such that $|u_\lambda|_q \leqslant E\,|\lambda|^\gamma$ for $\hat{\lambda} < \lambda < 0$, where*

$$\gamma = \frac{(2-t) - N\sigma_1/q}{2\sigma_1} \quad \textit{and} \quad \begin{cases} q \in [2, 2N/(N-2)] & \textit{when } N \geqslant 3, \\ q \in [2, \infty) & \textit{when } N \leqslant 2. \end{cases}$$

(b) *For $N = 1$ and $t \geqslant 0$ there exist $\hat{\lambda} < 0$ and $E > 0$ such that $|u_\lambda|_q \leqslant E\,|\lambda|^\gamma$ for $\hat{\lambda} < \lambda < 0$, where*

$$\gamma = \frac{1 - \sigma_1/q}{2\sigma_1} \quad \textit{and} \quad q \in [2, \infty).$$

(c) *We have $0 \in B(p)$ under the following conditions:*

$$\begin{aligned} &N \geqslant 3 \quad \textit{and} \quad p \in [2, 2N/(N-2)] && \textit{with } N\sigma_1 < p(2-t); \\ &N = 2 \quad \textit{and} \quad p \geqslant 2 && \textit{with } 2\sigma_1 < p(2-t); \\ &N = 1 \quad \textit{and} \quad p \geqslant 2 && \textit{with } \sigma_1 < \max\{p(2-t), p\}. \end{aligned}$$

We have that $0 \notin B(p)$ under the conditions given by Theorem 4.8.
We have that $0 \in B_\infty(p)$ under the following conditions: either

$$N \geqslant 2 \quad \textit{and} \quad 1 \leqslant p < N\sigma_1/(2-t) < 2N/(N-2),$$

where

$$\begin{aligned} &(2-t)\sigma_n < 2\sigma_1 && \textit{if } 0 < t < 2, \\ &n = 1 && \textit{if } t = 0; \end{aligned}$$

or

$$N = 1 \quad \textit{and} \quad 1 \leqslant p < \begin{cases} \sigma_1/(2-t) & \textit{for } 0 \leqslant t \leqslant 1, \\ \sigma_1 & \textit{for } t > 1, \end{cases}$$

where

$$\begin{aligned} &(2-t)\sigma_n < 2\sigma_1 && \textit{if } 0 < t \leqslant 1, \\ &\sigma_n < 2\sigma_1 && \textit{if } t > 1, \\ &n = 1 && \textit{if } t = 0. \end{aligned}$$

REMARKS. 1. Comparing the conditions for $0 \in B(p)$ with those for $0 \notin B(p)$, we see that the restrictions,

$$N\sigma_1 < p(2-t) \qquad \text{for } N \geq 2,$$
$$\sigma_1 < \max\{p(2-t), p\} \quad \text{for } N = 1,$$

required for L^p-bifurcation are sharp. The other restrictions can be removed by additional work as in Theorem 5.9.

2. The fact that asymptotic L^p-bifurcation can sometimes be established as a direct consequence of L^p-bifurcation (for a different value of p) was first pointed out to me by Professor J. F. Toland. His argument was based on the Cwikel–Lieb–Rosenbljum bound for eigenvalues of the Schrödinger operator (hence $N \geq 3$), whereas here we use the lower bound for the L^p-norm of a solution established for $N \geq 1$ in Theorem 4.8 by a direct application of Sobolev inequalities.

Proof. (a) For $N \geq 3$,

$$|u|_{2N/(N-2)} \leq C\,|\nabla u|_2 \quad \text{for all } u \in H^1(\mathbb{R}^N). \tag{5.10}$$

For $\lambda < 0$,

$$\|u_\lambda\|_\lambda^2 = |\nabla u_\lambda|_2^2 - \lambda\,|u_\lambda|_2^2 \leq ((\sigma_1+2)/\sigma_1)m_\lambda,$$

and so

$$|\nabla u_\lambda|_2 \leq C\,|\lambda|^\delta \tag{5.11}$$

and

$$|u_\lambda|_2 \leq |\lambda|^{\delta-\frac{1}{2}} \quad \text{for } \hat{\lambda} < \lambda < 0, \tag{5.12}$$

where $\delta = (\sigma_1(2-N) + 2(2-t))/4\sigma_1$ by Lemma 5.2(a). By Hölder's inequality, for $2 \leq q \leq 2N/(N-2)$,

$$|u_\lambda|_q \leq |u_\lambda|_2^\theta\,|u_\lambda|_{2N/(N-2)}^{1-\theta} \leq C\,|\lambda|^{\delta-\frac{1}{2}\theta},$$

where $q^{-1} = \frac{1}{2}\theta + (1-\theta)(N-2)/2N$. Since $\delta - \frac{1}{2}\theta = (q(2-t) - N\sigma_1)/2\sigma_1 q$, this proves the result for $N \geq 3$.

For $N = 1, 2$, we replace (5.10) by (4.10) and (4.9) respectively and then use (5.11, 12) to obtain the desired result.

(b) This follows as for part (a) except that we use Lemma 5.2(b) instead of 5.2(a).

(c) The bifurcation condition is found by ensuring that the exponent γ in parts (a) and (b) is positive. The conditions prohibiting bifurcation are immediate consequences of Theorem 4.8.

Bifurcation from infinity is established by using the interpolation inequality

$$|u_\lambda|_r \leq |u_\lambda|_q^\theta\,|u_\lambda|_p^{1-\theta} \quad \text{where } 1 \leq p \leq r \leq q \text{ and } \frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}.$$

Under the conditions in the statement of the theorem, we can choose $r < q$ with r satisfying the conditions of Theorem 4.8 and q the conditions for bifurcation. It follows that

$$\lim_{\lambda \to 0-} |u_\lambda|_p = +\infty \quad \text{for } 1 \leq p < r.$$

Finally, we show how 'boot-strapping' techniques can be used to extend the conclusion of Theorem 5.7 from the range $q \in [2, 2N/(N-2)]$ to the full interval $1 \leq q \leq \infty$.

LEMMA 5.8. *Let the conditions* (C1) *and* (C2) *be satisfied and consider* $(\lambda, u) \in S = \mathcal{S}$ *with* $u(x) > 0$ *on* $\mathbb{R}^N$. *Then for* $q \in [1, \infty)$,

$$|\lambda|\,|u|_q^q \leq \sum_{i=1}^n a_i \int_{\mathbb{R}^N} r u^{\sigma_i + q}\, dx,$$

with equality for $q = 1$.

Proof. For $R > 0$, let $B(R) = \{x \in \mathbb{R}^N\colon\ |x| < R\}$. Then

$$-\int_{B(R)} (\Delta u) u^{q-1}\, dx = \int_{B(R)} \lambda u^q + \sum_{i=1}^n a_i r u^{\sigma_i + q}\, dx$$

since $(\lambda, u) \in S$, and

$$-\int_{B(R)} (\Delta u) u^{q-1}\, dx = -\int_{\partial B(R)} \frac{\partial u}{\partial n} u^{q-1}\, dx + \int_{B(R)} (q-1) u^{q-2} |\nabla u|^2\, dx.$$

If $q > 1$, it follows from Theorem 4.7 that

$$\lim_{R\to\infty} \int_{\partial B(R)} \frac{\partial u}{\partial n} u^{q-1}\, dx = 0,$$

and so

$$\int_{\mathbb{R}^N} \lambda u^q + \sum_{i=1}^n a_i r u^{\sigma_i + q}\, dx \geq 0.$$

If $q = 1$, we note that

$$\begin{aligned}\int_{\partial B(R)} \frac{\partial u}{\partial n}\, dx &= -\int_{|x|>R} \Delta u\, dx \\ &= \int_{|x|>R} \lambda u + \sum_{i=1}^n a_i r u^{\sigma_i + 1}\, dx \\ &\to 0 \quad \text{as } R \to \infty,\end{aligned}$$

by the exponential decay of u established in Theorem 4.7. Hence

$$\int_{\mathbb{R}^N} \Delta u(x)\, dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \lambda u + \sum_{i=1}^n a_i r u^{\sigma_i + 1}\, dx = 0.$$

THEOREM 5.9. *Let the conditions* (C1) *and* (C2) *be satisfied and if* $t = 0$, *suppose either that* r *is constant or that* r *satisfies* (5.9). *For* $\lambda < 0$, *let* $(\lambda, u_\lambda) \in S = \mathcal{S}$ *be the solution given by Theorems* 5.4 *to* 5.6.

(a) *Suppose that* $0 \leq t < 2$ *and, when* $N \geq 3$, *suppose also that* $(N-2)\sigma_n < 2(2-t)$. *Then for* $\varepsilon > 0$ *there exist* $\hat{\lambda} \in (-1, 0)$ *and* $E > 0$ *such that*

$$|u_\lambda|_q \leq E\, |\lambda|^{\gamma - \varepsilon} \quad \text{for } \hat{\lambda} < \lambda < 0,$$

where $1 \leq q \leq \infty$ *and*

$$\gamma = \begin{cases} (q(2-t) - N\sigma_1)/2q\sigma_1 & \text{for } 1 \leq q < \infty, \\ (2-t)/2\sigma_1 & \text{for } q = \infty. \end{cases}$$

(b) *Suppose that* $N=1$ *and that* $t \geq 0$. *Then for* $\varepsilon > 0$ *there exist* $\hat{\lambda} \in (-1, 0)$ *and* $E > 0$ *such that*

$$|u_\lambda|_q \leq E\,|\lambda|^{\gamma-\varepsilon} \quad \text{for } \hat{\lambda} < \lambda < 0,$$

where $1 \leq q \leq \infty$ *and*

$$\gamma = \begin{cases} (q-\sigma_1)/2q\sigma_1 & \text{if } 1 \leq q < \infty, \\ 1/2\sigma_1 & \text{if } q = \infty. \end{cases}$$

(c) *We have that* $0 \in B(p)$ *under the following conditions: either*

$$N \geq 2, \quad 0 \leq t < 2, \quad N\sigma_1 < p(2-t),$$

and, if $N \geq 3$ *and* $p \notin [2, 2N/(N-2)]$,

$$(N-2)\sigma_n < 2(2-t);$$

or

$$N=1, \quad t \geq 0, \quad \text{and} \quad \sigma_1 < \max\{p(2-t), p\}.$$

Proof. We accomplish the extension to $1 \leq q \leq \infty$ in three steps:
(i) for $N \geq 3$ we extend to $(2N/(N-2), \infty)$;
(ii) for $N \geq 1$ we extend to $[1, 2)$;
(iii) for $N \geq 1$ we extend to $q = \infty$.

(i) As in the proof of Theorem 4.5, for $(\lambda, u) \in S = \mathcal{S}$, we can write $u = \sum_{i=1}^n u_i$ where u_i satisfies

$$-\Delta u_i(x) - \lambda u_i(x) = h_i(x) \quad \text{and} \quad h_i(x) = r(x) a_i\,|u(x)|^{\sigma_i} u(x).$$

Setting $\lambda = -k^2$ with $k > 0$, $v_i(x) = u_i(x/k)$, and $v = \sum_{i=1}^n v_i$, we find that

$$\begin{aligned} -\Delta v_i(x) &= -k^{-2}\,\Delta u_i(x/k) \\ &= k^{-2}\{h_i(x/k) - k^2 u_i(x/k)\} \\ &= -v_i(x) + k^{-2} h_i(x/k). \end{aligned}$$

Thus

$$\|v_i\|_{W^{2,p}} \leq C\,\|k^{-2} h_i(x/k)\|_{L^p} \quad \text{for } 1 < p < \infty,$$

by Proposition 4.3. By the Sobolev embedding, it follows that, for $1 < p < \infty$,

$$|v_i|_q \leq C\,|k^{-2} h_i(x/k)|_p \tag{5.13}$$

where $p \leq q \leq Np/(N-2p)$ if $p < \frac{1}{2}N$ and $p \leq q < \infty$ if $p \geq \frac{1}{2}N$. The inequality (5.13) can be written as

$$|u_i|_q \leq Ck^{-2+N(1/p-1/q)}\,|h_i|_p \tag{5.14}$$

and

$$\begin{aligned} |h_i|_p^p &= a_i^p \int_{\mathbb{R}^N} r^p\,|u(x)|^{(\sigma_i+1)p}\,dx \\ &\leq a_i^p \left(\int_{\mathbb{R}^N} r^{ps_i}\,dx \right)^{1/s_i} \left(\int_{\mathbb{R}^N} |u(x)|^{(\sigma_i+1)pt_i}\,dx \right)^{1/t_i}, \end{aligned}$$

where $s_i > 1$ and $1/s_i + 1/t_i = 1$. If $t > 0$, we choose s_i such that $ps_i t > N$. If $t = 0$, we choose $s_i = +\infty$. Then

$$|u_i|_q \leq Ck^{-2+N(1/p-1/q)}\,|u|_{p(\sigma_i+1)t_i}^{\sigma_i+1}, \tag{5.15}$$

provided that $s_i \geq 1$, $1/s_i + 1/t_i = 1$, and $ps_i t > N$ ($s_i = \infty$, $t_i = 1$, if $t = 0$).

Suppose that the desired estimate is true for $q \in [2, T]$. By Theorem 5.7 we can suppose that $T \geq 2N/(N-2)$. For $\varepsilon > 0$ we set

$$p_i = \left\{ \frac{t}{N(1+\varepsilon)} + \frac{(\sigma_i + 1)}{T} \right\}^{-1} \quad \text{and} \quad s_i = \frac{N(1+\varepsilon)}{tp_i}$$

for $1 \leq i \leq n$ ($s_i = +\infty$ if $t = 0$). Then $p_i s_i t > N$, when $t > 0$, $s_i > 1$, and furthermore $p_i > 1$ because

$$\frac{t}{N(1+\varepsilon)} + \frac{\sigma_i + 1}{T} \leq \frac{t}{N} + \frac{(\sigma_i + 1)(N-2)}{2N} \leq \frac{t}{N} + \frac{2(2-t) + N - 2}{2N} = \frac{N+2}{2N} < 1,$$

for $N \geq 3$ since $\sigma_n \leq 2(2-t)/(N-2)$. Thus

$$|u_i|_q \leq Ck^{-2+N(1/p_i - 1/q)} |u|^{\sigma_i + 1}_{p_i(\sigma_i+1)t_i},$$

by (5.15), where

$$p_i \leq q \begin{cases} \leq Np_i/(N-2p_i) & \text{if } p_i < \tfrac{1}{2}N, \\ < \infty & \text{if } p_i \geq \tfrac{1}{2}N. \end{cases}$$

Now the choice of s_i and p_i ensures that $p_i(\sigma_i + 1)t_i = T$, and so, since the desired inequality is true for $|\cdot|_T$, we obtain

$$|u_i|_q \leq Ck^{-2+N(1/p_i - 1/q) + 2(\gamma_T - \varepsilon)(\sigma_i + 1)}, \tag{5.16}$$

where $\gamma_T = (T(2-t) - N\sigma_1)/2\sigma_1 T$, for

$$p_i \leq q \begin{cases} \leq Np_i/(N-2p_i) & \text{if } p_i < \tfrac{1}{2}N, \\ < \infty & \text{if } p_i \geq \tfrac{1}{2}N. \end{cases}$$

Using the formulae for p_i and γ_T, we see that the exponent of k in (5.16) simplifies to

$$\frac{q(2-t) - N\sigma_1}{\sigma_1 q} + r,$$

where

$$\begin{aligned} r &= -2 + \frac{2\sigma_i}{\sigma_1} + \frac{t}{1+\varepsilon} - \frac{t\sigma_i}{\sigma_1} - 2\varepsilon(\sigma_i + 1) \\ &= (2-t)\left(\frac{\sigma_i}{\sigma_1} - 1\right) - \frac{\varepsilon t}{1+\varepsilon} - 2\varepsilon(\sigma_i + 1) \\ &\geq -2\varepsilon(\sigma_i + 2). \end{aligned}$$

If $p_n \geq \frac{1}{2}N$, this establishes the desired inequality for $q \in [2, \infty)$. If $p_n < \frac{1}{2}N$, the desired inequality has been established for $2 \leq q \leq Np_n/(N-2p_n)$ and we note that $Np_n/(N-2p_n) > T$ since

$$\sigma_n < \frac{2(2-t)}{N-2} \quad \text{and} \quad T \geq \frac{2N}{N-2}.$$

Setting $T_1 = Tp_n/(N-2p_n)$ and proceeding by induction as in the proof of Theorem 4.5, we construct an increasing sequence of intervals $[2, T_k]$ on which the desired inequality is satisfied. Setting

$$p_n(k) = \left\{ \frac{t}{N(1+\varepsilon)} + \frac{(\sigma_n + 1)}{T_k} \right\}^{-1},$$

we have an increasing sequence $\{p_n(k)\}$. If $p_n(k) \geqslant \frac{1}{2}N$ for some k, the result is established for $q \in [2, \infty)$. If we suppose that $p_n(k) < \frac{1}{2}N$ for all $k \in \mathbb{N}$ and set $l = \lim_{k\to\infty} T_k$, we find that

$$l = \left(\frac{t}{N(1+\varepsilon)} + \frac{(\sigma_n + 1)}{l} - \frac{2}{N}\right)^{-1}$$

since

$$T_{k+1} = \frac{Np_n(k)}{N - 2p_n(k)} = \left(\frac{1}{p_n(k)} - \frac{2}{N}\right)^{-1}.$$

This implies that

$$\sigma_n = l\left[\frac{2}{N} - \frac{t}{N(1+\varepsilon)}\right] > \frac{2N}{(N-2)}\frac{(2-t)}{N} = \frac{2(2-t)}{N-2}$$

since $l > 2N/(N-2)$. Since we have assumed that $\sigma_n < 2(2-t)/(N-2)$, we can conclude that $p_n(k) \geqslant \frac{1}{2}N$ after a finite number of steps. This completes the proof of Step (i).

(ii) For Step (ii), we consider first the case covered by part (a) of Theorem 5.7. Since $u = u_\lambda > 0$, we have by Lemma 5.8 that

$$|\lambda|\, |u|_q^q \leqslant \sum_{i=1}^n a_i \int_{\mathbb{R}^N} r u^{\sigma_i + q}\, dx \leqslant \sum_{i=1}^n a_i\, |r|_\alpha\, |u|_{(\sigma_i+q)\beta}^{\sigma_i+q}$$

for $\alpha \geqslant 1$ and $1/\alpha + 1/\beta = 1$. Furthermore, $|r|_\alpha < \infty$ if $\alpha t > N$ (for $t = 0$ we set $\alpha = +\infty$). Thus for $\alpha \geqslant 1$ with $\alpha t > N$ by Step (i), we obtain

$$|u|_q \leqslant C\,|\lambda|^{-1/q}\left(\sum_{i=1}^n |u|_{(\sigma_i+q)\beta}^{\sigma_i+q}\right)^{1/q} \leqslant C\,|\lambda|^{-1/q}\left(\sum_{i=1}^n |\lambda|^{(\gamma_i - \varepsilon)(\sigma_i+q)}\right)^{1/q}, \tag{5.17}$$

where

$$\gamma_i = \frac{(\sigma_i + q)\beta(2-t) - N\sigma_1}{2\sigma_1(\sigma_i + q)\beta}$$

provided $\sigma_1 + q \geqslant 2$. But if $|\lambda| < 1$,

$$(\gamma_i - \varepsilon)(\sigma_i + q) \geqslant \frac{(\sigma_1 + q)\beta(2-t) - N\sigma_1}{2\sigma_1\beta} - \varepsilon(\sigma_n + q)$$

and

$$|u|_q \leqslant C\,|\lambda|^{-(1/q) + ((\sigma_1+q)\beta(2-t) - N\sigma_1)/2\sigma_1\beta q - (\varepsilon/q)(\sigma_n+q)}. \tag{5.18}$$

Setting $\alpha = (N/t)(1+\varepsilon)$ and simplifying the exponent of $|\lambda|$ in (5.18), we find that

$$|u|_a \leqslant C\,|\lambda|^{(1/\sigma_1) - (t/2\sigma_1) - (N/2q) - (\varepsilon/q)(\sigma_n + 1 + q)},$$

and the desired inequality is established for $q \geqslant \max\{1, 2 - \sigma_1\}$. This argument can be repeated inductively to prove the desired inequality for $q \geqslant \max\{1, 2 - k\sigma_1\}$ for all $k \in \mathbb{N}$. This proves the assertion (a).

For part (b), since $N = 1$, we need only extend the range of q in Theorem 5.7(b) to $[1, 2)$. Furthermore, if $0 \leqslant t \leqslant 1$, the conclusion follows from part (a). Supposing that $t \geqslant 1$ we obtain (5.17) for $\sigma_1 + q \geqslant 2$, as in Step (ii) of part (a) except that now

$$\gamma_i = \frac{(\sigma_i + q)\beta - \sigma_1}{2\sigma_1(\sigma_i + q)\beta}.$$

Hence

$$(\gamma_i - \varepsilon)(\sigma_i + q) \geq \frac{\sigma_1 + q}{2\sigma_1} - \frac{1}{2\beta} - \varepsilon(\sigma_n + q)$$

and so

$$|u|_q \leq C\,|\lambda|^{-(1/q)+((\sigma_1+q)/2\sigma_1 q)-1/(2\beta q)-(\varepsilon/q)(\sigma_n+q)}.$$

Setting $\alpha = (1+\varepsilon)/t$ and noting that $t \geq 1$, we find that the exponent of $|\lambda|$ is greater than

$$\frac{q-\sigma_1}{2\sigma_1 q} - \frac{\varepsilon}{q}(\sigma_n + q + \tfrac{1}{2}).$$

This proves the desired inequality for $q \geq \max\{1, 2-\sigma_1\}$ and the proof is completed by induction as in part (a).

(iii) By Steps (i) and (ii), we know that the result is true for all $q \in [1, \infty)$. Furthermore, in the Sobolev embedding (5.13) we can allow $q = \infty$ provided that $p > \frac{1}{2}N$. The result now follows using (5.15) as in Step (i).

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Département de Mathématiques
École Polytechnique Fédérale de Lausanne
CH-1015 Lausanne-Ecublens
Switzerland