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ON LITTLEWOOD'S CONSTANTS

D. BELIAEV AND S. SMIRNOV

Abstract

In two papers, Littlewood studied seemingly unrelated constants: (i) the best α such that for any polynomial f, of degree n, the areal integral of its spherical derivative is at most const $\cdot n^{\alpha}$, and (ii) the extremal growth rate β of the length of Green's equipotentials for simply connected domains. These two constants are shown to coincide, thus greatly improving known estimates on α.

1. *Introduction*

In this paper, we study the growth rate as $n \to \infty$ of the quantity

$$A_n = \sup \int_{\mathbb{D}} \frac{|g'|}{1+|g|^2}\, dm,$$

where the supremum is taken over all polynomials g of degree n, $\mathbb{D}$ is the unit disc $\{|z| < 1\}$, and m denotes the two-dimensional Lebesgue measure. We are interested in the best α such that $A_n \lesssim n^{\alpha}$ (which means that for every $\varepsilon > 0$, there is a constant C_ε with $A_n \leqslant C_\varepsilon n^{\alpha+\varepsilon}$). In [**16**], Littlewood observed that $0 \leqslant \alpha \leqslant 1/2$, and he conjectured that $\alpha < 1/2$. The problem of determining the best possible α appears under the number 4.18 in Hayman's problem list [**11**].

It is easy to show that there is a constant c such that for any rational function g of degree n,

$$\int_{\mathbb{D}} \frac{|g'|}{1+|g|^2}\, dm \leqslant c\sqrt{n}.$$

Note that the integrand is a modulus of the spherical derivative g'_σ of g (that is, the derivative with respect to the spherical metric), and that in $\mathbb{D}$ the spherical measure dm_σ is comparable to the Lebesgue measure dm. So our integral can be estimated by

$$\begin{aligned}
\int_{\mathbb{D}} |g'_\sigma|\, dm_\sigma &\leqslant \int_{\mathbb{C}} |g'_\sigma|\, dm_\sigma \\
&\leqslant \left(\int_{\mathbb{C}} |g'_\sigma|^2 dm_\sigma\right)^{1/2} \left(\int_{\mathbb{C}} dm_\sigma\right)^{1/2} \\
&= (2\pi n)^{1/2}(2\pi)^{1/2} \\
&= 2\pi\sqrt{n}.
\end{aligned}$$

Here we use the fact that a rational function of degree n maps the complex sphere to itself n-to-1, so the area of the image is n times bigger than the area of the sphere. In particular, this argument shows that $\alpha \leqslant 1/2$.

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Littlewood's conjecture was proved in [**14**] by Lewis and Wu: improving upon the work of Eremenko and Sodin [**8**], they obtained an explicit upper estimate $\alpha < 1/2 - 2^{-264}$. Later, Eremenko obtained in [**7**] a positive lower bound on α. Following the work by Eremenko and Sodin [**8**] and Lewis and Wu [**14**], we exploit a connection between this problem and the extremal behavior of the harmonic measure.

Our main result is that α is related to the growth rate of the length of Green's lines. In the case of simply connected domains Ω, we define β_Ω as

$$\limsup_{\varepsilon\to 0} \frac{\log \operatorname{length}\{z : G(z) = \varepsilon\}}{\log 1/\varepsilon},$$

where G is Green's function with a pole at infinity, and we define

$$\beta = \sup \beta_\Omega,$$

where the supremum is taken over all simply connected domains Ω. In the non-simply connected case, one needs a more elaborate definition, and we use the multifractal analysis technique.

For a given domain Ω, we define the *packing spectrum* $\pi_\Omega(t)$ as

$$\sup\left\{q : \text{for all } \delta > 0, \text{ there exists a } \delta\text{-packing}\{B\} \text{ with } \sum \operatorname{diam}(B)^t \omega(B)^q \geqslant 1\right\},$$

where ω is the harmonic measure in Ω, and a δ-packing is a collection of disjoint open sets whose diameters do not exceed δ. Note that this definition is valid for any domain with compact boundary, and for $t = 1$ it is analogous to β_Ω (see [**18**] for the proof of $\beta_\Omega = \pi_\Omega(1)$ for simply connected Ω).

We define the *universal spectrum* $\pi(t)$ as the supremum of $\pi_\Omega(t)$ over all planar domains Ω with compact boundary.

The main result of this paper is the following theorem.

MAIN THEOREM. *For any positive ε, there exists a constant $c = c(\varepsilon)$ such that*

$$A_n \leqslant cn^{\pi(1)+\varepsilon}.$$

Equivalently,

$$\alpha \leqslant \pi(1).$$

REMARK. Our proof actually implies that for $t \in [0, 2]$,

$$\int_{\mathbb{D}} \left(\frac{|g'|}{1 + |g|^2}\right)^t \leqslant cn^{\pi(2-t)+\varepsilon}.$$

Also of interest to us are $\pi_p(t)$ and $\pi_{p,sc}(t)$, which are, respectively, the suprema of $\pi_\Omega(t)$ over all the domains of attraction to infinity for polynomial mappings, and over the simply connected domains of attraction to infinity for polynomial mappings (see [**5**] for background material on complex dynamics). It is clear that $\pi_{p,sc} \leqslant \pi_p \leqslant \pi$, but *a priori* they might differ.

In [**7**], Eremenko essentially proved that $\pi_{p,sc}(1) \leqslant \alpha$. (He works under the assumption that the polynomials are hyperbolic, but this can easily be avoided.) Binder, Makarov, and Smirnov show in [**4**] that $\pi_{p,sc}(t) = \pi_p(t)$ for $t > 0$. Recently, Binder and Jones announced a proof of the identity $\pi_p(t) = \pi(t)$, which, together with our theorem, completes the circle:

$$\alpha \leqslant \pi(1) = \pi_p(1) = \pi_{p,sc}(1) \leqslant \alpha.$$

There is yet another growth (or, rather, decay) rate that is related to α; this was also studied by Littlewood. The growth rate γ of coefficients of univalent functions in $\mathbb{D}^-$ is defined by

$$\gamma := \sup_{\phi} \limsup_{n\to\infty} \frac{\log|b_n|}{\log n} + 1,$$

where the first supremum is taken over all functions

$$\phi(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$$

that are univalent in $\mathbb{D}^-$. Littlewood proved (see [**15**, **17**]) that $\beta \geqslant \gamma$. Much later, Carleson and Jones [**6**] showed that $\gamma = \beta$. Summing it all up, we arrive at the following corollary.

Corollary. *We have*

$$\alpha = \beta = \gamma = \pi(1) = \pi_{p,sc}(1). \tag{1.1}$$

The corollary uses the as yet unpublished result of Binder and Jones mentioned above. Note that a well-known conjecture (see [**6**, **18**, **13**, **20**]) states that $\pi(t) = (2-t)^2/4$ for $|t| \leqslant 2$; in particular, $\alpha = \beta = \gamma = 1/4$. The best published estimates to date for β are

$$0.17 \overset{[17]}{<} \beta \overset{[8]}{<} 0.4884,$$

so the same estimates hold for α, which is a significant improvement over the previously known

$$1.11 \cdot 10^{-5} \overset{[2]}{\leqslant} \alpha \overset{[14]}{\leqslant} 1/2 - 2^{-264}.$$

Recently, Hedenmalm and Shimorin released a preprint [**12**] with the estimate $\beta < 0.46$. The authors have also recently obtained an estimate from below: $\beta > 0.23$ (see [**2**, **3**]).

1.1. *Connection to the value distribution of entire functions*

Before giving the proof of our main theorem, we would like to note that the reason for Littlewood's interest in this problem was the following striking corollary to his conjecture (see [**16**]).

Littlewood's conditional theorem. *Assume that $\alpha < 1/2$. If f is an entire function of order $0 < \rho < \infty$, then for any $0 < \theta < 1/2 - \alpha$ there is a 'small' set S such that 'almost all' roots of any equation $f(z) = w$ lie in S. In other words,*

$$\frac{\operatorname{Area}(S \cap B(R))}{\operatorname{Area}(B(R))} = O\left(\frac{1}{R^{2\theta\rho}}\right), \qquad R \to \infty,$$

and for all w,

$$\frac{\#\{z \in B(R) \setminus S : f(z) = w\}}{\#\{z \in B(R) : f(z) = w\}} = O\left(\frac{1}{R^{\rho(1/2-\alpha-\theta)}}\right), \qquad R \to \infty,$$

where $B(R)$ is a disc of radius R, centered at the origin.

2. *Proof of the main theorem*

It is a standard fact (for details, see [**18**] and [**9**]) that $\pi(t)$ is finite, convex, and strictly decreasing on $[0,2]$. Hence, for any small δ, we can choose ε so small that

$$\pi(1-2\varepsilon)-\pi(1)<\delta.$$

We will also use a more elaborate fact (which follows from multifractal formalism and fractal approximation – see [**18**]), that there is a constant $\mathrm{const}(t,\varepsilon)$, such that for any disjoint collection of cubes $\{Q\}$ of size at most 1, one has

$$\sum_Q \omega(Q)^{\pi(t)} l(Q)^{t+\varepsilon} < \mathrm{const}(t,\varepsilon).$$

Let g be a polynomial of degree at most n, and a_i its zeros. Consider a set where $|g|$ is big; that is, $|g| \geqslant n$. We can easily estimate the integral over this set by

$$\begin{aligned}\int_{\mathbb{D}\cap\{|g|\geqslant n\}} \frac{|g'|}{1+|g|^2} &\leqslant n^{-1}\int_{\mathbb{D}}\frac{|g'|}{|g|} = n^{-1}\int_{\mathbb{D}}|(\log g)'| \\ &= n^{-1}\int_{\mathbb{D}}\left|\sum_{i=1}^{n}\frac{1}{z-a_i}\right| \leqslant n^{-1}\cdot 2\pi n = 2\pi.\end{aligned} \tag{2.1}$$

Now consider a complementary set, where $|g|$ is small, which is contained inside the disc of radius $3/2$:

$$\Omega := \left\{z : |g(z)|<n, |z|<\tfrac{3}{2}\right\},$$

and let $W=\{Q_j\}$ be a Whitney decomposition of Ω. We note that

$$d\mu(z) = 4n^{-1}\frac{|g'(z)|^2}{(1+|g(z)|^2)^2}\,dx\,dy$$

is the Riesz measure associated with the nonnegative subharmonic function

$$u = \frac{\log(1+|g|^2)}{n}.$$

Then, by the Riesz representation theorem,

$$\mu(B(z,r)) \leqslant \frac{c}{2\pi}\int_0^{2\pi}(u(z+2re^{i\theta})-u(z))\,d\theta. \tag{2.2}$$

Hence, for every cube Q_j, we have

$$\mu(Q_j) \leqslant c\frac{\log(1+n)}{n}.$$

Fix a cube Q_j such that $Q_j\cap\mathbb{D}\neq\emptyset$, and denote by ξ_j a point at $\partial\Omega$ such that $d(\xi_j,Q_j)\leqslant 2l(Q_j)$. Then, from (2.2),

$$\begin{aligned}\mu(Q_j) &\leqslant \frac{c}{2\pi}\int_0^{2\pi}(u(\xi_j+8l(Q_j)e^{i\theta})-u(\xi_j))d\theta \\ &\leqslant \max_{z\in B(\xi_j,8l(Q_j))} c[u(z)-u(\xi_j)]^+.\end{aligned} \tag{2.3}$$

Denote by $G(z)$ the Green function for $\mathbb{C}\setminus\overline{\Omega}$ with a pole at infinity. Extend G to a continuous subharmonic function in $\mathbb{C}$ by setting $G=0$ on Ω.

By the maximum principle, for the domain $\mathbb{C}\setminus\overline{\Omega}$ we obtain $G(z)\geqslant \log|2z/3|$ for any $z\in\mathbb{C}\setminus\Omega$; hence

$$G(z)\geqslant \log\tfrac{4}{3}\qquad \text{for } |z|=2.$$

By the maximum principle, for the domain $2\mathbb{D}\setminus\overline{\Omega}$ we have

$$u(z)-u(\xi_j)\leqslant M_2G(z)\Big(\log\tfrac{4}{3}\Big)^{-1}\qquad \text{for } |z|\leqslant 2,$$

where $M_2=\max_{|z|=2}u(z)$.

If we let z_j be a center of Q_j, and ω a harmonic measure on $\mathbb{C}\setminus\overline{\Omega}$ with a pole at infinity, then by the previous inequality and (2.3) we have

$$\begin{aligned}\mu(Q_j)&\leqslant \max_{z\in B(\xi_j,8l(Q_j))} c[u(z)-u(\xi_j)]^+\\ &\leqslant c\Big(\log\tfrac{4}{3}\Big)^{-1}M_2\max_{z\in B(\xi_j,8l(Q_j))}G(z)\\ &\leqslant cM_2\max_{z\in B(z_j,16l(Q_j))}G(z).\end{aligned}$$

By Harnack's inequality, the right-hand side is less than

$$cM_2 3\int_{\partial B(z_j,32l(Q_j))}G(z)\frac{|dz|}{2\pi 32l(Q_j)},$$

which, by the Riesz representation formula, is equal to

$$cM_2\int_0^{32l(Q_j)}\frac{\omega(B(z_j,t))}{t}\,dt\leqslant cM_2\omega(B(z_j,32l(Q_j))).$$

So, finally, we have

$$\mu(Q_j)\leqslant \operatorname{const} M_2\omega(B(z_j,32l(Q_j))).\tag{2.4}$$

By Schwarz's inequality we have

$$\begin{aligned}\int_\Omega\frac{|g'|}{1+|g|^2}\,dx\,dy&=n^{1/2}\sum_{Q_j\in W}n^{-1/2}\int_{Q_j}\frac{|g'|}{1+|g|^2}dxdy\\ &\leqslant\frac{1}{2}n^{1/2}\sum_{Q_j\in W}\Big(\int_{Q_j}\frac{4|g'|^2}{n(1+|g|^2)^2}\,dx\,dy\Big)^{1/2}\Big(\int_{Q_j}dx\,dy\Big)^{1/2}\\ &\leqslant\frac{1}{2}n^{1/2}\sum_{Q_j\in W}\mu(Q_j)^{1/2}l(Q_j)\\ &=\frac{1}{2}n^{1/2}\sum_{Q_j\in W}\mu(Q_j)^{1/2-\pi(1-2\varepsilon)}\mu(Q_j)^{\pi(1-2\varepsilon)}l(Q_j)\\ &\leqslant Cn^{1/2}\Big(\frac{\log(1+n)}{n}\Big)^{1/2-\pi(1-2\varepsilon)}\sum_{Q_j\in W}\mu(Q_j)^{\pi(1-2\varepsilon)}l(Q_j)\\ &\leqslant Cn^{\pi(1)+\delta}\sum_{Q_j\in D}\mu(Q_j)^{\pi(1-2\varepsilon)}l(Q_j),\end{aligned}$$

where ε is a small positive number, C is a constant, and D is the family of all dyadic squares with side length less then 32, that intersect Ω.

By (2.4), we can estimate the last sum as follows:

$$\begin{aligned}
&Cn^{\pi(1)+\delta} \sum_{Q_j \in D} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j) \\
&\quad \leqslant CM_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \sum_{l(Q_j)=1/2^k} \omega(Q_j)^{\pi(1-2\varepsilon)} l(Q_j)^{(1-2\varepsilon)+\varepsilon} l(Q_j)^{\varepsilon} \\
&\quad = CM_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \frac{1}{2^{k\varepsilon}} \sum_{l(Q_j)=1/2^k} \omega(Q_j)^{\pi(1-2\varepsilon)} l(Q_j)^{(1-2\varepsilon)+\varepsilon} \\
&\quad \leqslant CM_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \frac{1}{2^{k\varepsilon}} \ \mathrm{const}(\varepsilon) \\
&\quad \leqslant n^{\pi(1)+\delta} \,\mathrm{const}(\varepsilon) M_2.
\end{aligned} \tag{2.5}$$

Now assume that the following dichotomy holds: *For any* $\varepsilon > 0$, *there exists a constant* $\mathrm{const}(\varepsilon)$ *such that for any polynomial* g *of degree* n, *we have*

$$M_2 \leqslant \mathrm{const}(\varepsilon) n^{\varepsilon}, \tag{2.6}$$

or

$$|\Omega| \leqslant 1/n. \tag{2.7}$$

Thus, if (2.6) holds, then the desired estimate follows from (2.5):

$$\int_{\Omega} \frac{|g'|}{1+|g|^2} \leqslant \mathrm{const}\, n^{\pi(1)+\varepsilon+\delta}.$$

But both ε and δ can be made arbitrarily small, so we have the desired estimate.

If (2.7) holds, then an even better estimate follows from Schwartz's inequality:

$$\begin{aligned}
\int_{\Omega} \frac{|g'|}{1+|g|^2} &\leqslant \left(\int_{\Omega} \frac{|g'|^2}{(1+|g|^2)^2}\right)^{1/2} \left(\int_{\Omega} 1\right)^{1/2} \\
&\leqslant \left(\int_{\mathbb{C}} |g'_{\sigma}|\right)^{1/2} \sqrt{|\Omega|} \\
&\leqslant \sqrt{2\pi n|\Omega|} \\
&\leqslant \sqrt{2\pi}.
\end{aligned}$$

Therefore it remains only to prove the dichotomy.

Proof of the dichotomy. Assume that $M_2 > n^{\varepsilon}$. Recalling the definition that

$$M_2 = \sup_{|z|=2} \frac{\log(1+|g|^2)}{n},$$

we deduce that $\sup_{|z|=2} |g| > \exp(n^{1+\varepsilon})$, so the set Ω where $|g| < n$ cannot have big measure.

We can write g as $g = PQ$, where

$$P(z) = \lambda \prod_{|a_i|>4} (z - a_i), \qquad Q(z) = \lambda \prod_{|a_i|\leqslant 4} (z - a_i).$$

Let m be the degree of P; then

$$\log\Bigg(|\lambda|\prod_{|a_i|>4}|a_i|\Bigg)-m\log 2\leqslant\log|P(z)|$$
$$\leqslant\log\Bigg(|\lambda|\prod_{|a_i|>4}|a_i|\Bigg)+m\log 2,$$

when $|z|\leqslant 2$. Since $|Q(z)|<6^n$ for $|z|\leqslant 2$, it follows that

$$\begin{aligned}\inf_{z\in\Omega}\log|P(z)| &\geqslant \sup_{|z|=2}\log|P(z)|3^{-n}\\ &\geqslant \log\big(\exp(n^{1+\varepsilon})3^{-n}6^{-n}\big)\\ &\geqslant \tfrac{1}{2}n^{1+\varepsilon},\end{aligned}$$

if n is sufficiently large. Since $\log|PQ|=\log|g|<n$ in Ω, we can write

$$\begin{aligned}\log|Q(z)| &\leqslant \log|PQ|-\log|P|\\ &\leqslant n-\tfrac{1}{2}n^{1+\varepsilon}\\ &\leqslant -\tfrac{1}{4}n^{1+\varepsilon}, \qquad z\in\Omega,\end{aligned}$$

if n is large enough. Therefore, Ω is contained in the union of disks $\{z:|z-a_i|\leqslant \exp(-n^{\varepsilon}/4)\}$. Hence

$$|\Omega|\leqslant n\pi\exp\big(-\tfrac{1}{2}n^{\varepsilon}\big)\leqslant 1/n \qquad \text{when } n \text{ is sufficiently large.}$$

This proves the dichotomy for $n>N(\varepsilon)$. For degree n bounded from above by $N(\varepsilon)$, the dichotomy is easy to prove by a compactness argument (and, anyway, it suffices to prove that the estimate holds for polynomials of sufficiently large degree). This completes the proof of the main theorem. □

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References

1. I. N. Baker and G. M. Stallard, 'Error estimates in a calculation of Ruelle', *Complex Variables Theory Appl.* 29 (1996) 141–159.
2. D. Beliaev, 'Harmonic measure on random fractals', PhD thesis, Royal Institute of Technology, 2005.
3. D. Beliaev and S. Smirnov, 'Harmonic measure on fractal sets', *Proc. 4th European Congress of Math.*, to appear.
4. I. Binder, N. Makarov and S. Smirnov, 'Harmonic measure and polynomial Julia sets', *Duke Math. J.* 117 (2003) 343–365.
5. L. Carleson and T. Gamelin, *Complex dynamics*, Universitext: Tracts in Math. (Springer, New York, 1993).
6. L. Carleson and P. Jones, 'On coefficient problems for univalent functions and conformal dimension', *Duke Math. J.* 66 (1992) 169–206.
7. A. Eremenko, 'Lower estimate in Littlewood's conjecture on the mean spherical derivative of a polynomial and iteration theory', *Proc. Amer. Math. Soc.* 112 (1991) 713–715.
8. A. Eremenko and M. Sodin, 'A proof of the conditional Littlewood theorem on the distribution of the values of entire functions', *Izv. Akad. Nauk SSSR Ser. Mat.* 51 (1987) 421–428, 448 (in Russian); *Math. USSR-Izv.* 30 (1988) 395–402 (English translation).
9. K. Falconer, *Techniques in fractal geometry* (John Wiley & Sons, Ltd., Chichester, 1997).
10. A. Z. Grinshpan and Ch. Pommerenke, 'The Grunsky norm and some coefficient estimates for bounded functions', *Bull. London Math. Soc.* 29 (1997) 705–712.

11. W. K. Hayman, *Research problems in function theory* (The Athlone Press, University of London, London, 1967).
12. H. Hedenmalm and S. Shimorin, 'Weighted Bergman spaces and the integral means spectrum of conformal mappings', Duke Math. J., to appear.
13. P. Kraetzer, 'Experimental bounds for the universal integral means spectrum of conformal maps', *Complex Variables Theory Appl.* 31 (1996) 305–309.
14. J. Lewis and J. M. Wu, 'On conjectures of Arakelyan and Littlewood', *J. Analyse Math.* 50 (1988) 259–283.
15. J. Littlewood, 'On the coefficients of schlicht functions', *Quart. J. Math. Oxford Ser.* 9 (1938) 14–20.
16. J. Littlewood, 'On some conjectural inequalities, with applications to the theory of integral functions', *J. London Math. Soc.* 27 (1952) 387–393.
17. J. Littlewood and R. Paley, 'A proof that an odd schlicht function has bounded coefficients', *J. London Math. Soc.* 7 (1932) 167–169.
18. N. Makarov, 'Fine structure of harmonic measure', *St.Petersburg Math. J.* 10 (1999) 217–268.
19. Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grund. Math. Wiss. 299 (Springer, Berlin, 1992).
20. Ch. Pommerenke, 'The integral means spectrum of univalent functions', *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) 237 (1997) (in English, with Russian summary); *Anal. Teor. Chisel i Teor. Funkts.* 14 119–128, 229; *J. Math. Sci.* (*New York*) 95 (1999) 2249–2255 (English translation).

Dmitri Beliaev and
Stanislav Smirnov
Kungliga Tekniska Högskolan
Inst. för Matematik
100 44 Stockholm
Sweden

beliaev@math.kth.se
stas@math.kth.se

Stanislav Smirnov
Université de Genève
Section de Mathématiques
2-4, rue du Lièvre
Case postale 240
1211 Genève 24
Switzerland

Stanislav.Smirnov@math.unige.ch