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Relaxation for some dynamical problems

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Synopsis

In this article, we study the functional $H(u)=\int_0^T\int_\Omega[\frac{1}{2}|\partial u/\partial t(x,t)|^2-F(\nabla_x u)]\,dx\,dt$ where $\Omega\subset\mathbb{R}^n$ is a bounded open set and $u:\Omega\times(0,T)\to\mathbb{R}^m$ and when $F:\mathbb{R}^{nm}\to\mathbb{R}$ fails to be quasiconvex. We show that with respect to strong convergence of $\partial u/\partial t$ and weak convergence of $\nabla_x u$, the above functional behaves as $\bar{H}(u)=\int_0^T\int_\Omega[\frac{1}{2}|\partial u/\partial t|^2-QF(\nabla_x u)]\,dx\,dt$ where QF is the lower quasiconvex envelope of F.

Introduction

Consider the functional

$$H(u)=\int_0^T\int_\Omega\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}(x,t)\right|^2-F(\nabla_x u(x,t))\right]dx\,dt \tag{0.1}$$

where

(i) $\Omega\subset\mathbb{R}^n$ is a bounded open set and $T>0$,

(ii) $u(x,t):\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^m$ and thus

$$\nabla u=\left(\nabla_x u;\frac{\partial u}{\partial t}\right)\in\mathbb{R}^{nm}\times\mathbb{R}^m,$$

(iii) $F:\mathbb{R}^{nm}\to\mathbb{R}$ is continuous and satisfies

$$a+b|A|^p\leqq F(A)\leqq c+d|A|^p \tag{0.2}$$

for some $a,c\in\mathbb{R}$, $d\geqq b>0$, $p>1$ and for every $A\in\mathbb{R}^{nm}$.

We can associate with (0.1) the functional

$$E(u)=\int_\Omega F(\nabla_x u(x))\,dx \tag{0.3}$$

where $u(x,t)\equiv u(x)$.

Before describing the results of this article, one should remark that the functional H is rarely studied from the point of view of the calculus of variations, since H has undesirable properties such as unboundedness from below. However, we will see that this is not a major obstacle to obtaining a relaxation theorem.

The usual hypothesis imposed on F in (0.1) or in (0.3) is the so called *quasiconvexity* condition, i.e.

$$\int_D F(A+\nabla\varphi(x))\,dx\geqq F(A)\text{ meas }D \tag{0.4}$$

for every bounded domain $D \subset \mathbb{R}^n$, $A \in \mathbb{R}^{nm}$ and $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^m)$ (i.e. φ is locally Lipschitz and $\varphi = 0$ on $\partial\Omega$).

Note that in the case $m = 1$ (or $n = 1$), the notion of quasiconvexity reduces to the ordinary convexity condition. Furthermore, (0.4) implies the well known *Legendre-Hadamard* condition, i.e.

$$\sum_{\alpha,\beta=1}^{m} \sum_{i,j=1}^{n} \frac{\partial^2 F(A)}{\partial A_{i\alpha}\, \partial A_{j\beta}} \lambda_i \lambda_j \mu_\alpha \mu_\beta \geqq 0 \tag{0.5}$$

for every $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ and $A \in \mathbb{R}^{nm}$.

If one assumes F to be quasiconvex and thus (0.5), then the Euler equations associated with (0.1) (respectively (0.3)) are *hyperbolic* (respectively *elliptic*). In the case $m = 1$, the Euler equations reduce to a single equation, i.e.

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div} f(\operatorname{grad}_x u) = 0, \tag{0.1'}$$

$$\operatorname{div} f(\operatorname{grad}_x u) = 0, \quad \text{if} \quad u(x, t) \equiv u(x), \tag{0.3'}$$

where $f = \operatorname{grad} F$ (provided F is C^1). Thus, in this case, the quasiconvexity of F is equivalent to the monotonicity of f.

The purpose of this article is to study (0.1) when F fails to be quasiconvex, therefore the Euler equations are of *mixed type* (hyperbolic-elliptic). We will show that if QF denotes the lower *quasiconvex envelope* of F, i.e.

$$QF = \sup \{\Phi \leqq F : \Phi \text{ quasiconvex}\}$$

and

$$\bar{H}(u) = \int_0^T \int_\Omega \left[\frac{1}{2}\left|\frac{\partial u}{\partial t}(x, t)\right|^2 - QF(\nabla_x u(x, t))\right] dx\, dt, \tag{0.6}$$

then the following theorem holds.

THEOREM. *For every $\bar{u} \in \mathcal{W} = \{u(x, t) : \nabla_x u \in L^p_{nm}(\Omega \times (0, T)),\ \partial u/\partial t \in L^2_m(\Omega \times (0, T))\}$, there exists a sequence $u^\nu \in \mathcal{W}$ so that*

$$\begin{cases} u^\nu(x, t) = \bar{u}(x, t), & (x, t) \in \partial\Omega \times (0, T), \\ u^\nu(x, 0) = \bar{u}(x, 0), & x \in \Omega, \\ \dfrac{\partial u^\nu}{\partial t}(x, 0) = \dfrac{\partial \bar{u}}{\partial t}(x, 0), & x \in \Omega, \\ \nabla_x u^\nu \rightharpoonup \nabla_x \bar{u} & \text{in} \quad L^p_{nm}(\Omega \times (0, T)), \\ \dfrac{\partial u^\nu}{\partial t} \to \dfrac{\partial \bar{u}}{\partial t} & \text{in} \quad L^2_m(\Omega \times (0, T)), \\ H(u^\nu) \to \bar{H}(\bar{u}). \end{cases} \tag{0.7}$$

Furthermore, in the case $m = 1$, if F is C^1 and if we let $f = \operatorname{grad} F$ and $\tilde{f} = \operatorname{grad} QF$, then

$$\frac{\partial^2 u^\nu}{\partial t^2} - \operatorname{div} f(\operatorname{grad}_x u^\nu) \to \frac{\partial^2 \bar{u}}{\partial t^2} - \operatorname{div} \tilde{f}(\operatorname{grad}_x \bar{u}) \text{ in the sense of distributions} \tag{0.8}$$

(where $\rightharpoonup$ denotes weak convergence in L^p).

The proof of the theorem is based on that of [**1**] valid for the functional E and on a representation theorem (Theorem 5) for the quasiconvex envelope QF.

By rephrasing the theorem, one can say that with respect to weak convergence, the functionals H and $\bar{H}$ are "equivalent"; note however that the Euler equations associated with $\bar{H}$ are hyperbolic in contrast to those associated with H. Besides, in the case $m=1$, one can consider the weak solutions of the equation associated with $\bar{H}$ if they exist (which is obviously an open problem), as generalized solutions of $(0, 1')$ in the sense of (0.7) and (0.8).

The article is divided into three sections. The first recalls basic facts about quasiconvex functions and relaxation theorems. The second section gives the proof of the above theorem. In the last section, we see how to apply the above result to nonlinear conservation laws of mixed type

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma(u)_x = 0, \end{cases}$$

where σ is not necessarily an increasing function of its argument.

1. Quasiconvex envelopes and the relaxation theorem

We recall first some elementary properties of quasiconvex functions (the notion of quasiconvexity was introduced by Morrey [**5**], [**6**]). For more details about the history and the proofs of the next theorems, see [**2**].

THEOREM 1. *Let $F: \mathbb{R}^{nm} \to \mathbb{R}$ be continuous and let*

$$I(u, \Omega) = \int_\Omega F(\nabla u(x))\, dx. \tag{1.1}$$

(i) *Let F satisfy*

$$a \leqq F(u) \leqq b + c\,|u|^p$$

for every $u \in \mathbb{R}^{nm}$ and for some $a, b \in \mathbb{R}$, $c \geqq 0$ and $p \geqq 1$. Then I is (sequentially) weakly lower semicontinuous in $W^{1,p}$, i.e.

$$\liminf_{\nu \to \infty} I(u^\nu, \Omega) \geqq I(u, \Omega) \tag{1.2}$$

for every $u^\nu \rightharpoonup u$ in $W^{1,p}$ if and only if F is quasiconvex, i.e.

$$\int_D F(u + \nabla\varphi(x))\, dx \geqq F(u) \operatorname{meas} D \tag{1.3}$$

for every bounded domain $D \subset \mathbb{R}^n$, $u \in \mathbb{R}^{nm}$ and $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^m)$.

(ii) *If F is quasiconvex, then it is rank one convex, i.e.*

$$F(\lambda u + (1-\lambda)v) \leqq \lambda F(u) + (1-\lambda)F(v) \tag{1.4}$$

for every $\lambda \in [0, 1]$, $u, v \in \mathbb{R}^{nm}$ with $\operatorname{rank}(u - v) \leqq 1$. Furthermore, if F is C^2

then (1.4) *is equivalent to the Legendre-Hadamard* (*or ellipticity*) *condition*

$$\sum_{i,j,\alpha,\beta} \frac{\partial^2 F(u)}{\partial u_{i\alpha}\, \partial u_{j\beta}} \lambda_i \lambda_j \mu_\alpha \mu_\beta \geqq 0 \tag{1.5}$$

for every $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, $u \in \mathbb{R}^{nm}$.

(iii) *If* $n = 1$ *or* $m = 1$, *then* F *is quasiconvex if and only if* F *is convex.*

(iv) *If* $n = m$ *and there exists* $G: \mathbb{R} \to \mathbb{R}$ *continuous and such that*

$$F(u) = G(\det u),$$

then F *is quasiconvex if and only if* G *is convex.*

Remarks. (i) One has in general

$$F \text{ convex} \Rightarrow F \text{ quasiconvex} \Rightarrow F \text{ rank one convex},$$

the converse being false for the first implication and an open problem for the second.

(ii) The case $n = 1$ corresponds in (0.3) to the case of a system of ordinary differential equations while that of $m = 1$ corresponds to the case of a single equation.

A special class of quasiconvex functions are the so called *quasiaffine functions* (i.e. functions Φ such that Φ and $-\Phi$ are quasiconvex).

THEOREM 2. *The following properties are equivalent:*

(i) Φ *is quasiaffine,*

(ii) *let* $\operatorname{adj}_s u$ *denote the matrix of all* $s \times s (1 \leqq s \leqq \inf\{n, m\})$ *subdeterminants of the matrix* $u \in \mathbb{R}^{nm}$, *then*

$$\Phi(u) = A + \sum_{s=1}^{\inf\{n,m\}} \langle B_s ; \operatorname{adj}_s u \rangle_{\sigma(s)} \tag{1.6}$$

where

$$\sigma(s) = \binom{m}{s}\binom{n}{s} = \frac{m!\, n!}{s!\,(m-s)!\; s!\,(n-s)!},$$

$\langle .,. \rangle_{\sigma(s)}$ *denotes the scalar product in* $\mathbb{R}^{\sigma(s)}$, $A \in \mathbb{R}$ *and* $B_s \in \mathbb{R}^{\sigma(s)}$ *are constants.*

We now show that quasiconvex functions must be Lipschitz, analogous to convex functions [**4**, Corollary 2.4, Chap. I]).

PROPOSITION 3. Let $F: \mathbb{R}^{nm} \to \mathbb{R}$ be quasiconvex. Then F is locally Lipschitz.

Proof. Step 1: We first show that F is continuous. Observe that there is no loss of generality if we prove the continuity at $u = 0$ and if we assume that $F(0) = 0$.

For fixed $\varepsilon \in (0, \frac{1}{2})$, we want to find $\delta > 0$ such that

$$|v| \leqq \delta \Rightarrow |F(v)| \leqq \varepsilon. \tag{1.7}$$

Let us put

$$B_\eta = \{x \in \mathbb{R}^n : |x| < \eta\}.$$

We then define

$$\chi(x)=\begin{cases}1 & \text{if} \quad x\in B_{1-\varepsilon},\\ \dfrac{-|x|+1}{\varepsilon} & \text{if} \quad x\in B_1-B_{1-\varepsilon},\end{cases}$$

and let $\varphi\in W_0^{1,\infty}(B_1;\mathbb{R}^m)$ be defined as

$$\varphi(x)=-v\chi(x)x. \tag{1.8}$$

Since F is quasiconvex we have that

$$\begin{aligned} F(v)\text{ meas } B_1 &\leqq \int_{B_1} F(v+\nabla\varphi(x))\,dx\\ &=\int_{B_{1-\varepsilon}} F(v+\nabla\varphi(x))\,dx+\int_{B_1-B_{1-\varepsilon}} F(v+\nabla\varphi(x))\,dx\\ &\leqq F(0)\text{ meas } B_{1-\varepsilon}+\int_{B_1-B_{1-\varepsilon}} F(v+\nabla\varphi(x))\,dx\\ &\leqq \int_{B_1-B_{1-\varepsilon}} F(v+\nabla\varphi(x))\,dx, \end{aligned} \tag{1.9}$$

since $F(0)=0$.

Observe then that there exist K and K' such that

$$|v+\nabla\varphi|\leqq|v|+|\nabla\varphi|\leqq|v|\left(1+\frac{K'}{\varepsilon}\right)\leqq\frac{K}{\varepsilon}|v|,\text{ almost everywhere in } B_1. \tag{1.10}$$

We then define

$$a=\sup\{F(z)\colon |z|\leqq 1\}, \tag{1.11}$$

and let

$$\delta=\varepsilon/K$$

so that

$$|v|\leqq\delta\Rightarrow K/\varepsilon\,|v|\leqq 1\Rightarrow|v+\nabla\varphi|\leqq 1\text{ almost everywhere}$$

and hence

$$F(v+\nabla\varphi(x))\leqq a\text{ almost everywhere in } B_1-B_{1-\varepsilon}. \tag{1.12}$$

On combining (1.9) and (1.12), we obtain

$$F(v)\leqq a\frac{\text{meas }(B_1-B_{1-\varepsilon})}{\text{meas } B_1}\leqq na\varepsilon. \tag{1.13}$$

Similarly, if we choose

$$\psi(x)=v\chi(x)x$$

and use the quasiconvexity of F, we have

$$F(0)\text{ meas } B_1=0\leqq\int_{B_1}F(\nabla\psi(x))\,dx\leqq F(v)\text{ meas } B_{1-\varepsilon}+\int_{B_1-B_{1-\varepsilon}}F(\nabla\psi(x))\,dx. \tag{1.14}$$

As in (1.10)–(1.12), we also find that

$$|v| \leqq \delta \Rightarrow F(\nabla\psi(x)) \leqq a$$

and thus since $\varepsilon \in (0, \frac{1}{2})$,

$$F(v) \geqq -a \frac{\text{meas}\,(B_1 - B_{1-\varepsilon})}{\text{meas}\, B_{1-\varepsilon}} \geqq -\frac{na\varepsilon}{(1-\varepsilon)^n} \geqq -2^n na\varepsilon. \tag{1.15}$$

By combining (1.13) and (1.15), we obtain

$$|v| \leqq \varepsilon/K \Rightarrow |F(v)| \leqq 2^n na\varepsilon. \tag{1.16}$$

Step 2: We are now in a position to show that F is actually locally Lipschitz. Let

$$\Omega = \{x \in \mathbb{R}^{nm} : |v| < 1\}$$

and let

$$m = \inf\{F(v) : v \in \Omega\}, \tag{1.17}$$

$$M = \sup\{F(v) : v \in \Omega\}. \tag{1.18}$$

For $v \in \Omega$, we define

$$G(u) = F(u+v) - F(v), \tag{1.19}$$

so that if $u, u+v \in \Omega$ we have

$$G(u) \leqq M - m \quad \text{for} \quad u \in \Omega$$

and using (1.16) we get that

$$|u| \leqq \varepsilon/K \Rightarrow |G(u)| \leqq (M-m)2^n n\varepsilon. \tag{1.20}$$

On letting $\bar{u} = u + v$ and choosing $\varepsilon = K\,|\bar{u} - v|$, we get from (1.20) that

$$|\bar{u} - v| \leqq \varepsilon/K \Rightarrow |F(\bar{u}) - F(v)| \leqq K(M-m)2^n n\,|\bar{u} - v|. \tag{1.21}$$

Now let $w \in \Omega$ and consider the segment $[v, w] \subset \Omega$ and choose points $u_1, \ldots, u_N$ on $[v, w]$ such that

$$v = u_1, \ldots, u_N = w$$

with

$$|u_k - u_{k-1}| \leqq \varepsilon/K, \quad k = 1, \ldots, N. \tag{1.22}$$

By using (1.21) and (1.22), we get

$$|F(u_k) - F(u_{k-1})| \leqq K(M-m)2^n n\,|u_k - u_{k-1}|,$$

and thus

$$|F(w) - F(v)| \leqq K(M-m)2^n n\,|w - v|$$

for every $v, w \in \Omega$. $\square$

We now turn our attention to quasiconvex envelopes of F and we mention here a result obtained in [**1**]. Let

$$QF=\sup\{\Phi\leqq F\colon \Phi \text{ quasiconvex}\}. \tag{1.23}$$

THEOREM 4. *Let $D\subset\mathbb{R}^n$ be a bounded domain and suppose that*

$$a\leqq F(u)\leqq b(1+|u|^p), \tag{1.24}$$

$a\in\mathbb{R}$, $b\geqq 0$, $p\geqq 1$ *and* $u\in\mathbb{R}^{nm}$. *Then*

$$QF(u)=\inf_{\varphi}\left\{\frac{1}{\text{meas } D}\int_D F(u+\nabla\varphi(x))\,dx\colon \varphi\in W_0^{1,\infty}(D;\mathbb{R}^m)\right\}.$$

Remark. The question of whether $F\in C^1$ implies that $QF\in C^1$ is still open, as Proposition 3 just ensures that QF is locally Lipschitz. However if $n=1$ or $m=1$, Theorem 1 shows that the notions of convexity and quasiconvexity are equivalent, thus $QF=F^{**}$ (where F^{**} is the convex envelope of F) and in this case a direct application of the Hahn-Banach Theorem shows that F^{**} is also C^1.

We now obtain a more explicit form of the quasiconvex envelope in some special cases.

THEOREM 5. *Let $u(x,t)\colon\mathbb{R}^n\times\mathbb{R}^N\to\mathbb{R}^m$, $\nabla u(x,t)=(\nabla_x u,\nabla_t u)$ and let*

$$F(\nabla u(x,t))=G(\nabla_x u(x,t))+H(\nabla_t u(x,t)) \tag{1.25}$$

where $G\colon\mathbb{R}^{nm}\to\mathbb{R}$, $H\colon\mathbb{R}^{Nm}\to\mathbb{R}$ are continuous and satisfy hypotheses of the type (1.24). *Then*

$$QF=QG+QH. \tag{1.26}$$

Proof. The proof is in three steps.

Step 1. We first show that

$$QF\leqq QG+H. \tag{1.27}$$

Step 2. A similar proof to that of Step 1, but inverting the rôles of x and t will give

$$QF\leqq G+QH. \tag{1.28}$$

Step 3. Assume that Step 1 (and thus Step 2) have been established and let us demonstrate the theorem. Apply Step 2 to

$$\Psi=QG+H$$

to get

$$Q\Psi=Q(QG+H)\leqq QG+QH. \tag{1.29}$$

By using (1.27) and (1.29), we obtain

$$QG+QH\leqq Q(QF)=QF\leqq Q(QG+H)\leqq QG+QH$$

and thus the result.

It therefore remains to show (1.27). Let $u\in\mathbb{R}^{nm}$ and $v\in\mathbb{R}^{Nm}$ and use Theorem

4 to get

$$QF(u, v) = \inf_{\varphi} \left\{ \int_D \int_\Omega [G(u + \nabla_x \varphi(x, t)) + H(v + \nabla_t \varphi(x, t))] \, dx \, dt : \varphi \in W_0^{1,\infty}(D \times \Omega; \mathbb{R}^m) \right\} \tag{1.30}$$

where $D \subset \mathbb{R}^n$, $\Omega \subset \mathbb{R}^N$ are unit hypercubes.

Let $\varepsilon > 0$ be fixed, then Theorem 4 implies that there exist $\sigma \in W_0^{1,\infty}(D; \mathbb{R}^m)$ such that

$$\int_D G(u + \nabla_x \sigma(x)) \, dx \leqq \varepsilon + QG(u). \tag{1.31}$$

On extending σ by periodicity from D to $\mathbb{R}^n$, we trivially have that for $\nu \in \mathbb{N}$,

$$\int_D G(u + \nabla_x \sigma(\nu x)) \, dx \leqq \varepsilon + QG(u). \tag{1.32}$$

Let $\Omega_\nu \subset \Omega$ be a hypercube with the same centre as Ω and such that

$$\operatorname{dist}(\Omega; \Omega_\nu) = 1/\nu. \tag{1.33}$$

We then define $\psi \in W_0^{1,\infty}(\Omega)$, $0 \leqq \psi(t) \leqq 1$ such that

$$\psi(t) = \begin{cases} 1 & \text{if} \quad t \in \Omega_\nu \subset \Omega \subset \mathbb{R}^N, \\ 0 & \text{if} \quad t \in \partial\Omega. \end{cases}$$

We now choose

$$\varphi(x, t) = \frac{1}{\nu} \sigma(\nu x) \psi(t) \tag{1.34}$$

and observe that $\varphi \in W_0^{1,\infty}(D \times \Omega; \mathbb{R}^m)$. On using (1.30), we get

$$QF(u, v) \leqq \int_D \int_\Omega G(u + \psi(t) \nabla_x \sigma(\nu x)) \, dx \, dt + \int_D \int_\Omega H\left(v + \frac{1}{\nu} \sigma(\nu x) \otimes \operatorname{grad} \psi(t)\right) dx \, dt, \tag{1.35}$$

where $\sigma(\nu x) \otimes \operatorname{grad} \psi(t) \in \mathbb{R}^{Nm}$ denotes the tensor product.

We next use (1.32) to estimate separately the two terms on the right hand side of (1.35), recalling that meas D = meas $\Omega = 1$. Thus we have

$$\begin{aligned} &\int_D \int_\Omega G(u + \psi(t) \nabla_x \sigma(\nu x)) \, dx \, dt \\ &\quad = \int_{\Omega_\nu} \int_D G(u + \nabla_x \sigma(\nu x)) \, dx \, dt + \int_{\Omega - \Omega_\nu} \int_D G(u + \psi(t) \nabla_x \sigma(\nu x)) \, dx \, dt \\ &\quad \leqq \operatorname{meas} \Omega_\nu (\varepsilon + QG(u)) + \operatorname{meas} (\Omega - \Omega_\nu) \sup \{ |G(u + \psi(t) \nabla_x \sigma(\nu x))| \}. \end{aligned} \tag{1.36}$$

Similarly,

$$\int_D \int_\Omega H\left(v + \frac{1}{\nu} \sigma(\nu x) \otimes \operatorname{grad} \psi(t)\right) dx \, dt$$

$$= H(v) \text{ meas } \Omega_\nu + \int_D \int_{\Omega - \Omega_\nu} H\Big(v + \frac{1}{\nu}\sigma(\nu x) \otimes \text{grad } \psi(t)\Big)\, dx\, dt$$

$$\leqq H(v) \text{ meas } \Omega_\nu + \text{meas } (\Omega - \Omega_\nu) \sup \Big\{\Big|H\Big(v + \frac{1}{\nu}\sigma(\nu x) \otimes \text{grad } \psi(t)\Big)\Big|\Big\}. \quad (1.37)$$

By combining (1.36) and (1.37) and choosing ν sufficiently large, we deduce that

$$QF(u, v) \leqq QG(u) + H(v) + 2\varepsilon.$$

Since ε is arbitrary, we have indeed obtained (1.27) and thus the theorem is proved. □

We conclude this section with the relaxation theorem established in [**1**].

THEOREM 6. *Let $\Omega \subset \mathbb{R}^q$ be a bounded open set with Lipschitz boundary. Let $\varphi : \mathbb{R}^{qs} \to \mathbb{R}$ be continuous and such that*

$$a + b\,|A|^\beta \leqq \varphi(A) \leqq c + d\,|A|^\beta \quad (1.38)$$

for every $A \in \mathbb{R}^{qs}$, for some $a, c \in \mathbb{R}$, $d \geqq b > 0$ and $\beta > 1$. Then for every $\bar{u} \in W^{1,\beta}(\Omega; \mathbb{R}^s)$, there exists $\{u^\nu\}$, $u^\nu \in W^{1,\beta}(\Omega; \mathbb{R}^s)$ such that

$$u^\nu = \bar{u} \quad \text{on} \quad \partial\Omega, \quad (1.39)$$

$$u^\nu \rightharpoonup \bar{u} \quad \text{in} \quad W^{1,\beta}, \quad (1.40)$$

$$\int_\Omega \varphi(\nabla u^\nu(x))\, dx \to \int_\Omega Q\varphi(\nabla \bar{u}(x))\, dx. \quad (1.41)$$

Remark. In fact, one can allow a much weaker coercivity condition than (1.38). One only needs

$$a + \sum_{j=1}^J b_j\,|\Phi_j(A)|^{\beta_j} \leqq \varphi(A) \leqq c + \sum_{j=1}^J d_j\,|\Phi_j(A)|^{\beta_j} \quad (1.42)$$

for every $A \in \mathbb{R}^{qs}$, for some $a, c \in \mathbb{R}$, $J \geqq 1$ (an integer), $\beta_j > 1$, $d_j \geqq b_j > 0$ and where $\Phi_j : \mathbb{R}^{qs} \to \mathbb{R}$, $j = 1, \dots, J$ are quasiaffine functions. Then the conclusions of the theorem hold with (1.40) replaced by

$$\Phi_j(\nabla u^\nu) \rightharpoonup \Phi_j(\nabla \bar{u}) \quad \text{in } L^{\beta_j}, \quad j = 1, \dots, J. \quad (1.43)$$

2. Main result

We now restate the hypotheses.

(i) Let $0 \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $\Omega = 0 \times (0, T) \subset \mathbb{R}^{n+1}$ where $T > 0$.

(ii) Let $F : \mathbb{R}^{nm} \to \mathbb{R}$ be continuous and such that

$$a + b\,|A|^p \leqq F(A) \leqq c + d\,|A|^p \quad (2.1)$$

for every $A \in \mathbb{R}^{nm}$ and for some $a, c \in \mathbb{R}$, $d \geqq b > 0$ and $p > 1$.

(iii) Let $\bar{p} = \min\{2, p\} > 1$ and let

$$\mathcal{W} = \Big\{u \in W^{1,\bar{p}}(\Omega; \mathbb{R}^m) : \frac{\partial u}{\partial t} \in L^2_m(\Omega), \nabla_x u \in L^p_{nm}(\Omega)\Big\}.$$

We then have the main theorem.

THEOREM 7. For every $\bar{u} \in \mathcal{W}$, there exist $u^\nu \in \mathcal{W}$ so that

$$u^\nu = \bar{u} \quad \text{on} \quad \partial\Omega, \tag{2.2}$$

$$\nabla_x u^\nu \rightharpoonup \nabla_x \bar{u} \quad \text{in } L^p_{nm}(\Omega), \tag{2.3}$$

$$\frac{\partial}{\partial t} u^\nu \to \frac{\partial}{\partial t} \bar{u} \quad \text{in } L^2_m(\Omega), \tag{2.4}$$

$$\int_\Omega F(\nabla_x u^\nu(x,t))\,dx\,dt \to \int_\Omega QF(\nabla_x \bar{u}(x,t))\,dx\,dt. \tag{2.5}$$

Remarks. (i) Obviously if F satisfies coercivity conditions similar to (1.42) instead of (2.1) then the theorem remains valid provided (2.3) is replaced by the natural weak convergence, i.e. (1.43).

(ii) (2.4) and (2.5) imply in particular that $H(u^\nu) \to \bar{H}(\bar{u})$.

(iii) An elementary change in Theorem 6 allows the replacement of (2.2) by the more interesting conditions

$$\begin{cases} u^\nu(x,t) = \bar{u}(x,t), & (x,t) \in \partial 0 \times (0,T), \\ u^\nu(x,0) = \bar{u}(x,0), & x \in 0, \\ \dfrac{\partial}{\partial t} u^\nu(x,0) = \dfrac{\partial}{\partial t} \bar{u}(x,0), & x \in 0. \end{cases}$$

Proof. Let

$$\varphi(\nabla u(x,t)) = \varphi\left(\nabla_x u, \frac{\partial}{\partial t} u\right) = \frac{1}{2}\left|\frac{\partial}{\partial t} u\right|^2 + F(\nabla_x u), \tag{2.6}$$

$$I(u) = \int_\Omega \varphi(\nabla u(x,t))\,dx\,dt, \tag{2.7}$$

$$\bar{I}(u) = \int_\Omega Q\varphi(\nabla u(x,t))\,dx\,dt = \int_\Omega \left[\frac{1}{2}\left|\frac{\partial}{\partial t} u\right|^2 + QF(\nabla_x u)\right]dx\,dt, \tag{2.8}$$

where we have used Theorem 5 in (2.8).

We then apply Theorem 6 to such a φ. We deduce that there exist $u^\nu \in \mathcal{W}$ so that

$$\begin{cases} u^\nu = \bar{u} \quad \text{on} \quad \partial\Omega, & (2.9) \\ \nabla_x u^\nu \rightharpoonup \nabla_x \bar{u} \quad \text{in } L^p_{nm}(\Omega), & (2.10) \\ \dfrac{\partial}{\partial t} u^\nu \rightharpoonup \dfrac{\partial}{\partial t} \bar{u} \quad \text{in } L^2_m(\Omega), & (2.11) \\ I(u^\nu) \to \bar{I}(\bar{u}). & (2.12) \end{cases}$$

Observe furthermore that since the functions $u \to |u|^2$ ($\mathbb{R}^m \to \mathbb{R}$) and $QF: \mathbb{R}^{nm} \to \mathbb{R}$ are quasiconvex, they are weakly lower semicontinuous (Theorem 1) and thus

$$\varliminf_{\nu\to\infty} \int_\Omega \left|\frac{\partial}{\partial t} u^\nu(x,t)\right|^2 dx\,dt \geqq \int_\Omega \left|\frac{\partial}{\partial t} \bar{u}(x,t)\right|^2 dx\,dt, \tag{2.13}$$

$$\varliminf_{\nu\to\infty} \int_\Omega F(\nabla_x u^\nu(x,t))\,dx\,dt \geqq \int_\Omega QF(\nabla_x \bar{u}(x,t))\,dx\,dt. \tag{2.14}$$

On combining (2.12) with (2.13) and (2.14) we get that (2.13) and (2.14) are actually equalities; thus the theorem is proved. □

We now conclude this section by a refinement of Theorem 7, in the case $m=1$, in particular the Euler equations are reduced to a single equation and $QF=F^{**}$.

COROLLARY 8. *Let Ω be as above, $F:\mathbb{R}^n\to\mathbb{R}$ be C^1 and let $f=\operatorname{grad} F$, $\tilde{f}=\operatorname{grad} QF$. We suppose furthermore that*

$$|f(A)|\leqq a+b\,|A|^{p-1}, \tag{2.15}$$

$$F(A)\geqq c+d\,|A|^p, \tag{2.16}$$

where $a, b\geqq 0$, $c\in\mathbb{R}$ $d>0$ and $p>1$. If $\bar{u}\in\mathcal{W}$, then there exist $u^\nu\in\mathcal{W}$ such that

$$u^\nu=\bar{u}\quad on\ \partial\Omega, \tag{2.17}$$

$$\operatorname{grad}_x u^\nu\rightharpoonup\operatorname{grad}_x\bar{u}\quad in\ L_n^p(\Omega), \tag{2.18}$$

$$\frac{\partial}{\partial t}u^\nu\to\frac{\partial}{\partial t}\bar{u}\quad in\ L^2(\Omega), \tag{2.19}$$

$$\frac{\partial^2 u^\nu}{\partial t^2}-\operatorname{div}_x f(\operatorname{grad}_x u^\nu)\to\frac{\partial^2\bar{u}}{\partial t^2}-\operatorname{div}_x\tilde{f}(\operatorname{grad}_x\bar{u})\ \textit{in the sense of distributions.} \tag{2.20}$$

Proof. From Theorem 7, there exist $v^\nu\in\mathcal{W}$ such that

$$\begin{cases} v^\nu=\bar{u}\quad \text{on } \partial\Omega, & (2.21)\\ \operatorname{grad}_x v^\nu\rightharpoonup\operatorname{grad}_x\bar{u}\quad \text{in } L_n^p(\Omega), & (2.22)\\ \dfrac{\partial}{\partial t}v^\nu\to\dfrac{\partial}{\partial t}\bar{u}\quad \text{in } L^2(\Omega), & (2.23)\\ I(v^\nu)\equiv\displaystyle\int_\Omega F(\operatorname{grad}_x v^\nu(x,t))\,dx\,dt\to\bar{I}(\bar{u})\equiv\int_\Omega F^{**}(\operatorname{grad}_x\bar{u}(x,t))\,dx\,dt. & (2.24)\end{cases}$$

Observe that since F^{**} is convex, then so is $\bar{I}$. Furthermore, I and $\bar{I}$ are Gâteaux differentiable and we denote their differential by I' and $\bar{I}'$ respectively.

Let $\bar{p}=\min\{2,p\}>1$ and

$$V=\left\{u\in W_0^{1,\bar{p}}(\Omega);\frac{\partial u}{\partial t}\in L^2(\Omega),\operatorname{grad}_x u\in L_n^p(\Omega)\right\}.$$

Let V' be the dual of V and $\langle\cdot\,;\cdot\rangle$ the bilinear canonical form on $V'\times V$. We then define for every $v\in V$,

$$J(v)=I(v+\bar{u})-\bar{I}(\bar{u})-\langle\bar{I}'(\bar{u});v\rangle, \tag{2.25}$$

$$\bar{J}(v)=\bar{I}(v+\bar{u})-\bar{I}(\bar{u})-\langle\bar{I}'(\bar{u});v\rangle. \tag{2.26}$$

It is then easy to see that, since $\bar{I}$ is convex, for every $v\in V$,

$$J(v)\geqq\bar{J}(v)\geqq 0=\bar{J}(0)=\min\{\bar{J}(u):u\in V\}. \tag{2.27}$$

Furthermore, from (2.21)–(2.24), we have

$$J(v^\nu-\bar{u})\to 0=\min\{\bar{J}(u):u\in V\}, \tag{2.28}$$

and thus

$$\min\{\bar{J}(u): u\in V\}=\inf\{J(u): u\in V\}=0. \tag{2.29}$$

We finally apply [**4**, Corollary 6.1, p. 30] to the functional J and to the sequence $v^\nu-\bar{u}$ to obtain the result. □

3. Applications

In this section, we show how to apply Corollary 8 to nonlinear conservation laws of the type

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma(u)_x = 0, \end{cases} \tag{3.1}$$

where σ is not necessarily an increasing function. We only assume that there exist $F:\mathbb{R}\to\mathbb{R}$ C^1 with $F'(z)=\sigma(z)$ and such that

$$a+b|z|^p \leqq F(z) \leqq c+d\,|z|^p \tag{3.2}$$

and every $z\in\mathbb{R}$ and for some $a, c\in\mathbb{R}$, $d\geqq b>0$ and $p>1$.

Therefore, a function σ of the type shown in Figure 1 is admissible. (For another approach to the study of equations (3.1), see Shearer [**7**].)

By setting $u=w_x$ and $v=w_t$, it is immediately seen that (3.1) is equivalent to

$$w_{tt}-\sigma(w_x)_x=0. \tag{3.3}$$

The above equation governs the one-dimensional motion of a homogeneous nonlinear elastic material under zero body forces; $w(x,t)$ denotes the displacement at time t of a particle having position x in a reference configuration. We also have that

$$F(z)=\int_0^z \sigma(u)\,du \tag{3.4}$$

is the stored energy function and since σ is not a monotone function then F is not convex.

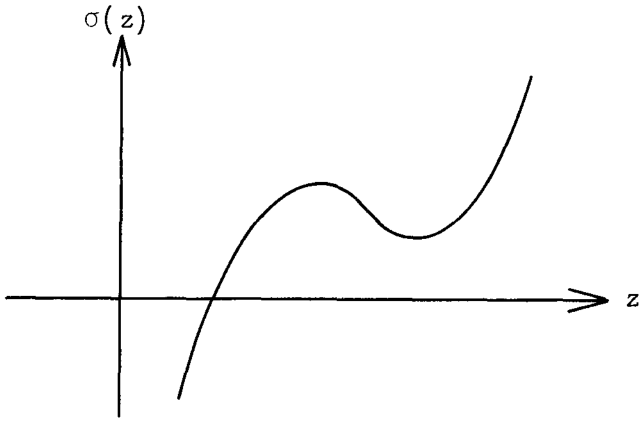

Figure 1

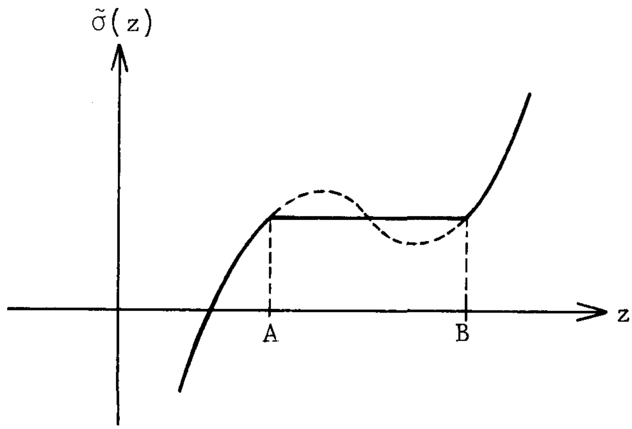

Figure 2

We then define QF, the quasiconvex envelope of F, which is in this case (i.e. $m=n=1$ and Theorem 2) the convex envelope, F^{**} of F. We also have that F^{**} is C^1, since F is C^1 (c.f. Remark following Theorem 4). We may then define

$$\tilde{\sigma}(z)=(F^{**})'(z). \tag{3.5}$$

More precisely, if σ is as in Figure 1, then $\tilde{\sigma}$ appears as in Figure 2 and satisfies the Maxwell condition, i.e.

$$\int_A^B \tilde{\sigma}(z)\,dz=\int_A^B \sigma(z)\,dz. \tag{3.6}$$

(The line $\{\tilde{\sigma}(z)=\sigma(A)\}$ is usually called the Maxwell line.)

We then immediately get from Corollary 8 the following result.

THEOREM 9. *Let* $T>0$, $\alpha,\beta\in\mathbb{R}$ *with* $\alpha<\beta$, $\bar{p}=\min\{2,p\}$ *and* $\mathcal{W}=\{u\in W^{1,\bar{p}}((\alpha,\beta)\times(0,T));\ \partial u/\partial t\in L^2,\ \partial u/\partial x\in L^p\}$, *then for every* $\bar{w}\in\mathcal{W}$, *there exist* $w^\nu\in\mathcal{W}$ *so that:*

$$\begin{cases} w^\nu(\alpha,t)=\bar{w}(\alpha,t),\ w^\nu(\beta,t)=\bar{w}(\beta,t) \text{ for every } t\in(0,T),\\ w^\nu(x,0)=\bar{w}(x,0),\ \dfrac{\partial}{\partial t}w^\nu(x,0)=\dfrac{\partial}{\partial t}\bar{w}(x,0) \text{ for every } x\in(\alpha,\beta),\\ w^\nu_x=u^\nu\rightharpoonup\bar{w}_x=\bar{u} \quad \text{in } L^p((\alpha,\beta)\times(0,T)),\\ w^\nu_t=v^\nu\rightarrow\bar{w}_t=\bar{v} \quad \text{in } L^2((\alpha,\beta)\times(0,T)),\\ w^\nu_{tt}-\sigma(w^\nu_x)_x\rightarrow\bar{w}_{tt}-\tilde{\sigma}(\bar{w}_x)_x \text{ in the sense of distributions.} \end{cases}$$

Remarks. (i) Returning to (3.1), one may rewrite the last equation of the above theorem as

$$\begin{cases} u^\nu_t-v^\nu_x=\bar{u}_t-\bar{v}_x=0,\\ v^\nu_t-\sigma(u^\nu)_x\rightarrow\bar{v}_t-\tilde{\sigma}(\bar{u})_x \text{ in the sense of distributions.} \end{cases}$$

(ii) The problem of solving the relaxed equations (i.e. (3.1) with σ replaced by $\tilde{\sigma}$) is still open. For recent results on system of equations of the above type with $\tilde{\sigma}$ allowed to have only one inflection point, see Di Perna [**3**].

References

1 B. Dacorogna. Quasiconvexity and relaxation of non convex problems in the calculus of variations. *J. Funct. Anal.* **46** (1982), 102–118.
2 B. Dacorogna. Weak continuity and weak lower semicontinuity of nonlinear functionals. *Lecture Notes in Mathematics* 922 (Berlin: Springer, 1982).
3 R. J. Di Perna. Convergence of approximate solutions to conservation laws. *Arch. Rational Mech. Anal.* **82** (1983), 27–70.
4 I. Ekeland and R. Témam. *Analyse convexe et problèmes variationnels* (Paris: Dunod, Gauthier-Villars, 1976).
5 C. B. Morrey. Quasiconvexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.* **2** (1952), 25–53.
6 C. B. Morrey. *Multiple integrals in the calculus of variations* (Berlin: Springer, 1966).
7 M. Shearer. Admissibility criteria for shock wave solutions of a system of conservations laws of mixed type. *Proc. Roy. Soc. Edinburgh Sect. A* **93** (1983), 233–244.

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