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Continuity of homomorphisms and derivations from algebras of approximable and nuclear operators

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1. Let $\mathfrak{A}$ be a Banach algebra. We say that homomorphisms from $\mathfrak{A}$ are continuous if every homomorphism from $\mathfrak{A}$ into a Banach algebra is automatically continuous, and that derivations from $\mathfrak{A}$ are continuous if every derivation from $\mathfrak{A}$ into a Banach $\mathfrak{A}$-bimodule is automatically continuous.

We recall that a Banach space $\mathfrak{X}$ is a Banach $\mathfrak{A}$-bimodule if $\mathfrak{X}$ is a bimodule with respect to continuous bilinear maps $(a,x)\mapsto a.x$ and $(a,x)\mapsto x.a$ from $\mathfrak{A}\times\mathfrak{X}$ into $\mathfrak{X}$. (See [3, §9], for example.) A derivation from an algebra $\mathfrak{A}$ into an $\mathfrak{A}$-bimodule $\mathfrak{X}$ is a linear map $D:\mathfrak{A}\to\mathfrak{X}$ such that

$$D(ab) = a.Db+Da.b \quad (a,b\in\mathfrak{A}). \tag{1·1}$$

For example, let $\mathfrak{A}$ be a Banach algebra and let ϕ be a character on $\mathfrak{A}^{\#}$, the algebra formed by adjoining an identity to $\mathfrak{A}$, so that $\phi:\mathfrak{A}^{\#}\to\mathbb{C}$ is a non-zero homomorphism. Then $\mathbb{C}$ is a Banach $\mathfrak{A}$-bimodule for the operations given by

$$a.z = z.a = \phi(a)z \quad (a\in\mathfrak{A}, z\in\mathbb{C}),$$

and a derivation from $\mathfrak{A}$ to $\mathbb{C}$ is a linear functional d on $\mathfrak{A}$ such that

$$d(ab) = \phi(a)d(b)+\phi(b)d(a) \quad (a,b\in\mathfrak{A}). \tag{1·2}$$

Linear functionals of this form are said to be point derivations at ϕ on $\mathfrak{A}$. We say that point derivations from $\mathfrak{A}$ are continuous if every linear functional d on $\mathfrak{A}$ which satisfies (1·2) for some character ϕ on $\mathfrak{A}^{\#}$ is automatically continuous.

Clearly, if derivations from $\mathfrak{A}$ are continuous, then point derivations from $\mathfrak{A}$ are continuous. It is well known (e.g., [6]) that, if all homomorphisms from a Banach algebra $\mathfrak{A}$ are continuous, then all derivations from $\mathfrak{A}$ are continuous; indeed, given a discontinuous derivation from $\mathfrak{A}$, it is easy to construct a discontinuous homomorphism from $\mathfrak{A}$. It is also well known that the converses of these two results are false in general. For example, all point derivations on the disc algebra $A(\bar{\mathbb{D}})$ are continuous, but there are discontinuous derivations from $A(\bar{\mathbb{D}})$ ([4]), and all derivations from the Banach algebras $C(\Omega)$ of continuous functions on a compact space Ω are continuous, but there are discontinuous homomorphisms from $C(\Omega)$ for each infinite space Ω, at least if the continuous hypothesis (CH) be assumed ([5], [8]).

There is an immediate method of obtaining discontinuous point derivations. For let $\mathfrak{A}$ be an algebra, and write $\mathfrak{A}^{[2]} = \{ab : a, b \in \mathfrak{A}\}$ and

$$\mathfrak{A}^2 = \text{lin}\,\mathfrak{A}^{[2]} = \left\{\sum_{j=1}^{n} a_j b_j : a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathfrak{A}\right\}.$$

Now suppose that $\mathfrak{A}$ is a non-unital Banach algebra and that $\mathfrak{A}^2$ has infinite codimension in $\mathfrak{A}$. Take d to be a discontinuous linear functional on $\mathfrak{A}^{\#}$ such that $d(e) = 0$ and $d \mid \mathfrak{A}^2 = 0$ (the existence of such a linear functional follows from the axiom of choice). Then d is a discontinuous point derivation at the character ϕ on $\mathfrak{A}^{\#}$ with $\phi \mid \mathfrak{A} = 0$.

These three automatic continuity questions of when homomorphisms, derivations, and point derivations from a Banach algebra are continuous have been extensively studied for various special classes of Banach algebras, and in particular they have been studied for certain Banach algebras of operators on a Banach space. For the sake of later comparison, we summarize the known results for the algebra $\mathscr{B}(E)$ of all bounded linear operators on a Banach space E. The first part of the following theorem is a seminal result of B. E. Johnson. Two Banach spaces E and F are isomorphic, written $E \simeq F$, if there is a continuous linear bijection from E onto F.

1·1. Theorem. (i) [**11**] *Suppose that E is a Banach space such that $E \simeq E \oplus E$. Then all homomorphisms from $\mathscr{B}(E)$ are continuous.*

(ii) [**20**] *There exists a Banach space E_1 such that there is a discontinuous point derivation from $\mathscr{B}(E_1)$.*

(iii) [**6**] *There exists a Banach space E_2 such that all derivations from $\mathscr{B}(E_2)$ are continuous, but such that (with the assumption of* CH*) there are discontinuous homomorphisms from $\mathscr{B}(E_2)$.* ∎

The purpose of the present note is to discuss these questions for the classes of approximable operators $\mathscr{A}(E)$ and nuclear operators $\mathscr{N}(E)$ (see below) on a Banach space E. Our main result will be given in Section 4: there is a Banach space E such that there is a discontinuous point derivation from $\mathscr{A}(E)$. This implies, of course, that there is a discontinuous homomorphism from the algebra $\mathscr{A}(E)$. To set this result in context, we shall establish in Section 2 some positive results about the automatic continuity of derivations and homomorphisms from algebras $\mathscr{A}(E)$ for which the Banach space E has certain additional properties. The methods in Section 2 are modifications of known techniques in automatic continuity theory.

We conclude this introduction by recalling the definitions of various Banach algebras and some notations which will appear in this work. For further details, see [**10**] or [**16**], for example.

Let E be a Banach space. The (topological) dual space of E is denoted by E'.

Let E and F be Banach spaces. Then $\mathscr{B}(E,F)$ is the Banach space of bounded linear maps from E into F. The tensor product $E' \otimes F$ is identified with a subspace of $\mathscr{B}(E,F)$ by the map which associates with the element $x' \otimes y$ of $E' \otimes F$ the operator

$$x' \otimes y : x \mapsto x'(x)\,y, \quad E \to F.$$

The set $\mathscr{F}(E,F)$ is the subspace of finite-rank operators in $\mathscr{B}(E,F)$, so that

$$\mathscr{F}(E,F) = \left\{\sum_{j=1}^{m} x'_j \otimes y_j : y_1, \ldots, y_m \in F, x'_1, \ldots, x'_m \in E'\right\};$$

clearly $\mathscr{F}(E,F)$ is linearly isomorphic to $E' \otimes F$. We write $\mathscr{F}(E)$ for $\mathscr{F}(E,E)$; $\mathscr{F}(E)$ is an ideal in $\mathscr{B}(E)$, and it is the minimum non-zero ideal.

1·2. *Definition.* Let E and F be Banach spaces. The closure of $\mathscr{F}(E,F)$ with respect to the operator norm on $\mathscr{B}(E,F)$ is called the set of approximable operators from E to F.

We denote the set of approximable operators by $\mathscr{A}(E,F)$, and write $\mathscr{A}(E)$ for $\mathscr{A}(E,E)$. Thus, $\mathscr{A}(E)$ is the minimum closed ideal in $\mathscr{B}(E)$. Every ideal in $\mathscr{A}(E)$ contains $\mathscr{F}(E)$, and so $\mathscr{A}(E)$ is topologically simple, in the sense that the only closed ideals in $\mathscr{A}(E)$ are $\{0\}$ and $\mathscr{A}(E)$ itself. Clearly $\mathscr{A}(E,F)$ is contained in $\mathscr{K}(E,F)$, the Banach space of compact operators from E to F.

Let E and F be Banach spaces. An operator $T \in \mathscr{B}(E,F)$ is *nuclear* if there exist $\{x'_n : n \in \mathbb{N}\}$ in E' and $\{y_n : n \in \mathbb{N}\}$ in F with

$$\sum_{j=1}^{\infty} \|x'_j\| \, \|y_j\| < \infty \quad \text{and} \quad T(x) = \sum_{j=1}^{\infty} x'_j(x)\, y_j \quad (x \in E),$$

so that $T = \sum_{j=1}^{\infty} x'_j \otimes y_j$ in $\mathscr{B}(E)$. The set of such nuclear operators is a linear space, denoted by $\mathscr{N}(E,F)$. For $T \in \mathscr{N}(E,F)$, the nuclear norm ν is given by

$$\nu(T) = \inf\left\{\sum_{j=1}^{\infty} \|x'_j\| \, \|y_j\| : T = \sum_{j=1}^{\infty} x'_j \otimes y_j\right\}.$$

Then $(\mathscr{N}(E,F), \nu)$ is a Banach space with $\nu(T) \geqslant \|T\|$ $(T \in \mathscr{N}(E,F))$,

$$\mathscr{F}(E,F) \subset \mathscr{N}(E,F) \subset \mathscr{A}(E,F),$$

and $\mathscr{F}(E,F)$ is ν-dense in $\mathscr{N}(E,F)$. We write $\mathscr{N}(E)$ for $\mathscr{N}(E,E)$. It is easy to check that $\mathscr{N}(E)$ is an ideal in $\mathscr{B}(E)$, that its norm satisfies

$$\nu(RTS) \leqslant \|R\| \, \nu(T) \, \|S\| \quad (R, S \in \mathscr{B}(E), T \in \mathscr{N}(E)),$$

and that $(\mathscr{N}(E), \nu)$ is a Banach algebra.

Let E and F be Banach spaces, and let $p \in [1, \infty)$. An operator $T \in \mathscr{B}(E,F)$ is p-summing if there is a constant $c \geqslant 0$ such that, for each finite set $\{x_1, \ldots, x_n\}$ in E, we have

$$\left(\sum_{j=1}^{n} \|Tx_j\|^p\right)^{1/p} \leqslant c \sup\left\{\left(\sum_{j=1}^{n} |x'(x_j)|^p\right)^{1/p} : x' \in E', \|x'\| = 1\right\};$$

the minimum of the constants c such that this inequality holds is $\pi_p(T)$, the *p-summing norm* of T, and the set of p-summing operators is denoted by $\Pi_p(E,F)$, with $\Pi_p(E) = \Pi_p(E,E)$. The space $(\Pi_p(E,F), \pi_p)$ is a Banach space, and

$$\mathscr{N}(E,F) \subset \Pi_1(E,F) \subset \Pi_2(E,F).$$

A Banach space E has the approximation property if, for each compact subset K of E and each $\epsilon > 0$, there exists $T \in \mathscr{F}(E)$ with $\|Tx - x\| < \epsilon$ $(x \in K)$, and E has the bounded approximation property if there exists $M > 0$ such that, for each compact subset K of E and each $\epsilon > 0$, there exists $T \in \mathscr{F}(E)$ with $\|T\| \leqslant M$ and $\|Tx - x\| < \epsilon$ $(x \in K)$. See [**15**, 1*e*], for example. In the case where E has the approximation property, we have $\mathscr{A}(E) = \mathscr{K}(E)$; the converse to this is an open question.

2. In this section we prove some positive results on the automatic continuity of homomorphisms and derivations from $\mathscr{A}(E)$. We first recall some standard notions of automatic continuity theory.

Let E and F be Banach spaces, and let $T:E\to F$ be a linear map (not necessarily continuous). The separating space of T is

$$\mathfrak{S}(T) = \{y\in F: \text{there exists } (x_n) \text{ in } E \text{ with } x_n\to 0 \text{ and } Tx_n\to y\}.$$

We see that $\mathfrak{S}(T)$ is a closed linear subspace of F, and $\mathfrak{S}(T) = \{0\}$ if and only if T is continuous.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let $\theta:\mathfrak{A}\to\mathfrak{B}$ be a homomorphism. The (left) continuity ideal of θ is by definition

$$\mathscr{I}(\theta) = \{a\in\mathfrak{A} : \theta(a)\,\mathfrak{S}(\theta) = \{0\}\}.$$

It is a standard that $\mathscr{I}(\theta)$ is an ideal in $\mathfrak{A}$ and that

$$\mathscr{I}(\theta) = \{a\in\mathfrak{A} : b\mapsto\theta(ab) \text{ is continuous}\}.$$

Let $\mathfrak{X}$ be a Banach $\mathfrak{A}$-bimodule, and let $D:\mathfrak{A}\to\mathfrak{X}$ be a derivation. The (left) continuity ideal of D is by definition

$$\mathscr{I}(D) = \{a\in\mathfrak{A} : a\,.\,\mathfrak{S}(D) = \{0\}\},$$

and it is again standard that $\mathscr{I}(D)$ is a closed ideal in $\mathfrak{A}$ and that

$$\mathscr{I}(D) = \{a\in\mathfrak{A} : b\mapsto D(ab) \text{ is continuous}\}.$$

The stability lemma for derivations (e.g., [**2**]) asserts that, for each sequence (a_n) in A, the sequence

$$(\overline{a_1\ldots a_n\,.\,\mathfrak{S}(D)} : n\in\mathbb{N})$$

of closed subspaces of $\mathfrak{X}$ is decreasing and eventually constant.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let $\theta:\mathfrak{A}\to\mathfrak{B}$ be a homomorphism. Suppose that (a_n) and (b_n) are sequences in $\mathfrak{A}$ such that $a_m b_n = 0$ $(m\neq n)$. Then it follows from the main boundedness theorem of Bade and Curtis ([**1**]) that there is a constant C such that

$$\|\theta(a_n b_n)\| \leqslant C\|a_n\|\,\|b_n\| \quad (n\in\mathbb{N}).$$

We shall also require some results about bounded approximate identities and factorization. Let $\mathfrak{A}$ be a Banach algebra. A bounded left approximate identity in $\mathfrak{A}$ is a bounded net (e_λ) in $\mathfrak{A}$ such that $e_\lambda a\to a$ for each $a\in\mathfrak{A}$. Let $\mathfrak{A}$ be a Banach algebra with a bounded left approximate identity. Then a form of Cohen's factorization theorem ([**3**, 11·12]) asserts that $\mathfrak{A}^{[2]} = \mathfrak{A}$ and indeed that, for each sequence (a_n) in $\mathfrak{A}$ with $a_n\to 0$, there exist $b\in\mathfrak{A}$ and (c_n) in $\mathfrak{A}$ with $c_n\to 0$ such that $a_n = bc_n$ $(n\in\mathbb{N})$. By [**7**, theorem 2·6 (i)], the Banach algebra $\mathscr{A}(E)$ has a bounded left approximate identity if and only if E has the bounded approximation property.

2·1. THEOREM. *Let E be a Banach space with the bounded approximation property. Then derivations from $\mathscr{A}(E)$ are continuous.*

Proof. We may suppose that E is infinite dimensional. Let $\mathfrak{X}$ be a Banach $\mathscr{A}(E)$-bimodule, let $D:\mathscr{A}(E)\to\mathfrak{X}$ be a derivation, and set $\mathfrak{S} = \mathfrak{S}(D)$.

To obtain a contradiction, assume that $\mathscr{I}(D) = \{0\}$. Let (x_k) be a basic sequence in

E, and take F to be the closed linear span of $\{x_n : n \in \mathbb{N}\}$, so that (x_n) is a basis of F. Take (x'_k) in F' such that $x'_k(x_j) = \delta_{k,j}$ $(k, j \in \mathbb{N})$, and extend each λ_k to an element of E'. For $n \in \mathbb{N}$, set

$$P_n(x) = \sum_{j=1}^{n} x'_j(x)\, x_j \quad (x \in E),$$

so that $P_n(E) = \mathrm{lin}\{x_1, \ldots, x_n\}$ and $P_n \in \mathscr{F}(E)$. Clearly, we have

$$P_m P_n = P_{\min\{m,n\}} \quad (m, n \in \mathbb{N}). \tag{2·1}$$

Now take (ζ_k) in $\mathbb{R}^+ \backslash \{0\}$ with $\sum_{k=1}^{\infty} \zeta_k \|P_k\| < \infty$, and set $T = \sum_{k=1}^{\infty} \zeta_k P_k$, so that $T \in \mathscr{A}(E)$ and $TP_n = P_n T$ $(n \in \mathbb{N})$. Finally set

$$S_n = T - TP_n \quad (n \in \mathbb{N}),$$

so that (S_n) is a sequence in $\mathscr{A}(E)$. For $n \in \mathbb{N}$, we have $S_1 \ldots S_n = T^{n-1} S_n$, and so

$$P_n S_1 \ldots S_n = 0 \quad (n \in \mathbb{N}). \tag{2·2}$$

Also $P_{n+1} S_1 \ldots S_n = T^n (P_{n+1} - P_n)$, and so $P_{n+1} S_1 \ldots S_n x_{n+1} = (\sum_{k=n+1}^{\infty} \zeta_k)^n x_{n+1}$, showing that

$$P_{n+1} S_1 \ldots S_n \neq 0 \quad (n \in \mathbb{N}). \tag{2·3}$$

Now set

$$\mathfrak{S}_n = \overline{S_1 \ldots S_n . \mathfrak{S}} \; (n \in \mathbb{N}).$$

By (2·2), $P_{n+1} . \mathfrak{S}_{n+1} = \{0\}$. By (2·3) and the assumption, $P_{n+1} S_1 \ldots S_n \notin \mathscr{I}(D)$, and so $P_{n+1} . \mathfrak{S}_n \neq \{0\}$. Thus, for each $n \in \mathbb{N}$, $\mathfrak{S}_{n+1} \subsetneq \mathfrak{S}_n$, a contradiction of the stability lemma. We conclude that $\mathscr{I}(D) \neq \{0\}$.

Since $\mathscr{A}(E)$ is topologically simple and $\mathscr{I}(D)$ is a closed ideal in $\mathscr{A}(E)$, necessarily $\mathscr{I}(D) = \mathscr{A}(E)$.

Now let $T_n \to 0$ in $\mathscr{A}(E)$. Since E has the bounded approximation property, $\mathscr{A}(E)$ has a bounded left approximate identity, and so, by Cohen's factorization theorem, there exist $R \in \mathscr{A}(E)$ and (S_n) in $\mathscr{A}(E)$ with $S_n \to 0$ such that $T_n = RS_n$ $(n \in \mathbb{N})$. Since $R \in \mathscr{I}(D)$, the map $S \mapsto D(RS)$, $\mathscr{A}(E) \to \mathfrak{X}$, is continuous, and so $D(T_n) = D(RS_n) \to 0$ in $\mathfrak{X}$. Thus D is continuous. ∎

We can prove the stronger result that homomorphisms from $\mathscr{A}(E)$ are continuous under a stronger hypothesis on E; the argument is a modification of Johnson's proof for $\mathscr{B}(E)$.

2·2. Theorem. *Let E be a Banach space with the bounded approximation property and such that $E \simeq E \oplus E$. Then homomorphisms from $\mathscr{A}(E)$ are continuous.*

Proof. Since $E \simeq E \oplus E$, there are sequences (P_n) and (Q_n) of idempotents in $\mathscr{B}(E)$ such that $I_E = P_1 + Q_1$ (where I_E is the identity operator), and such that, for each $n \in \mathbb{N}$,

$$P_n Q_n = Q_n P_n = 0, \quad P_n = P_{n+1} + Q_{n+1} \quad \text{and} \quad \mathscr{A} P_n \mathscr{A} = \mathscr{A} Q_n \mathscr{A},$$

where $\mathscr{A} = \mathscr{A}(E)$. Clearly (Q_n) is an orthogonal sequence in $\mathscr{A}$.

Let θ be a homomorphism from $\mathscr{A}$ into a Banach algebra, and set

$$K = \{T \in \mathscr{B}(E) : \mathscr{A} T \mathscr{A} \subset \mathscr{I}(\theta)\}.$$

We claim that there exists $k \in \mathbb{N}$ such that $Q_k \in K$. For assume towards a contradiction that $Q_n \notin K$ $(n \in \mathbb{N})$. Then there exist (R_n) and (S_n) in $\mathscr{A}$ with $R_n Q_n S_n \notin \mathscr{I}(\theta)$ $(n \in \mathbb{N})$, and so there exists (A_n) in $\mathscr{A}$ with

$$\|\theta(R_n Q_n S_n A_n)\| \geqslant n\|R_n Q_n\| \, \|Q_n S_n\| \, \|A_n\| \quad (n \in \mathbb{N}).$$

But this contradicts the main boundedness theorem (applied with $a_n = R_n Q_n$ and $b_n = Q_n S_n A_n$). Thus the claim holds.

We see successively that $P_k, P_{k-1}, Q_{k-1}, \ldots, I_E$ belong to K, and so $\mathscr{A}^2 \subset \mathscr{I}(\theta)$. Since $\mathscr{A}$ has a bounded left approximate identity, it follows that $\mathscr{A} = \mathscr{I}(\theta)$, and then that θ is continuous as before. ∎

3. In this section, we shall show that, in distinction to the situation for $\mathscr{A}(E)$, there are discontinuous point derivations on $\mathscr{N}(E)$ for each infinite-dimensional Banach space E.

The notion of an operator ideal is the central topic of the book [**16**] of Pietsch. The product $\mathfrak{A} \circ \mathfrak{B}$ of two operator ideals is defined in [**16**, 3·1·1]; we shall require only the following special cases of the definition. Let E be a Banach space. Then $(\mathscr{N} \circ \mathscr{N})(E)$ (respectively, $(\Pi_2 \circ \Pi_2)(E)$) is the set of operators T in $\mathscr{B}(E)$ such that there are a Banach space F and operators $R \in \mathscr{N}(E,F)$ and $S \in \mathscr{N}(F,E)$ (respectively, with $R \in \Pi_2(E,F)$ and $S \in \Pi_2(F,E)$) with $T = R \circ S$.

Let $p \in (0,1]$. A p-norm on a linear space E is a map $\|\cdot\| : E \to \mathbb{R}^+$ with the properties of a norm, save that the triangle inequality becomes

$$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p \quad (x, y \in E).$$

3·1. LEMMA. *Let E be an infinite-dimensional Banach space, and set $\mathscr{N} = \mathscr{N}(E)$.*
(i) *The set $(\mathscr{N} \circ \mathscr{N})(E)$ is an ideal in $\mathscr{B}(E)$ with $\mathscr{N}^2 \subset (\mathscr{N} \circ \mathscr{N})(E) \subset \mathscr{N}$.*
(ii) *$\mathscr{N}^2$ is dense in $\mathscr{N}$.*
(iii) *$(\mathscr{N} \circ \mathscr{N})(E)$ is a complete p-normed space for some $p \in [\frac{1}{2}, 1)$.*
(iv) *Either $(\mathscr{N} \circ \mathscr{N})(E) = \mathscr{N}$ or $\mathscr{N}^2$ has infinite codimension in $\mathscr{N}$.*

Proof. (i) That $(\mathscr{N} \circ \mathscr{N})(E)$ is an ideal is [**16**, 3·1·2]; certainly $\mathscr{N}^{[2]} \subset (\mathscr{N} \circ \mathscr{N})(E)$, and so $\mathscr{N}^2 \subset (\mathscr{N} \circ \mathscr{N})(E)$.

(ii) Clearly $\mathscr{N}^2 \supset \mathscr{F}(E)^2 = \mathscr{F}(E)^{[2]} = \mathscr{F}(E)$, and so $\mathscr{N}^2$ is dense in $\mathscr{N}$.

(iii) This is [**16**, Theorem 7·1·2].

(iv) Suppose that $\mathscr{N}^2$ has finite codimension in $\mathscr{N}$. Then by (i), $(\mathscr{N} \circ \mathscr{N})(E)$ has finite codimension in $\mathscr{N}$. Since $(\mathscr{N} \circ \mathscr{N})(E)$ is a complete p-normed space, it follows from the open mapping theorem that $(\mathscr{N} \circ \mathscr{N})(E)$ is closed in $(\mathscr{N}, \nu)$. By (i) and (ii), $(\mathscr{N} \circ \mathscr{N})(E)$ is dense in $(\mathscr{N}, \nu)$, and so $(\mathscr{N} \circ \mathscr{N})(E) = \mathscr{N}$. ∎

In fact, $(\Pi_2 \circ \Pi_2)(E) \subset \mathscr{N}(E)$ for each Banach space E ([**13**]).

Let E be a Banach space. For an operator $T \in \mathscr{B}(E)$ such that each non-zero element of the spectrum of T is an eigenvalue, we denote by $(\lambda_n(T))$ the sequence of eigenvalues of T ordered by their magnitude in absolute value and counted according to their multiplicity.

3·2. LEMMA. *Let E be a Banach space. Suppose that $T \in (\Pi_2 \circ \Pi_2)(E)$. Then each non-zero element of $\sigma(T)$ is an eigenvalue and $(\lambda_n(T))$ is ℓ^1-summable.*

Proof. This result is essentially due to Grothendieck [**9**], who proved it for operators $T: E \to E$ which admit a factorization $T = S \circ R$, where $R \in \mathscr{N}(E, l_2)$ and $S \in \mathscr{B}(l_2, E)$. That these operators are exactly the operators in $(\Pi_2 \circ \Pi_2)(E)$ was observed in [**13**].

The result is also given in [**12**, proposition 2·2] (and that result gives a bound for the sum $\sum_{n=1}^{\infty}|\lambda_n(T)|$), and, with a different proof, due to Pietsch, in [**14**, 2·b·5]. ▮

3·3. LEMMA. *Let E be a Banach space. Then the following are equivalent:*
(*a*) *for each* $T\in\mathcal{N}(E)$, *the sequence* $(\lambda_n(T))$ *is* l^1-*summable*;
(*b*) *E is isomorphic to a Hilbert space.*

Proof. This is [**12**, Theorem 3·11]; see also [**14**, 4·b·11]. ▮

3·4. LEMMA. *Let E be an infinite-dimensional Banach space. Then* $\mathcal{N}(E)^2$ *has infinite codimension in* $\mathcal{N}(E)$.

Proof. Set $\mathcal{N}=\mathcal{N}(E)$. To obtain a contradiction, assume that $\mathcal{N}^2$ has finite codimension in $\mathcal{N}$. Then, by Lemma 3·1 (iv), $(\mathcal{N}\circ\mathcal{N})(E)=\mathcal{N}$, and so

$$\mathcal{N}\subset(\Pi_2\circ\Pi_2)(E).$$

By Lemma 3·2, for each $T\in\mathcal{N}$, $(\lambda_n(T))$ is ℓ^1-summable, and so, by Lemma 3·3, E is isomorphic to a Hilbert space, say H.

It is standard that an operator T on H belongs to $\mathcal{N}$ if and only if there exist orthonormal sequences (x_n) and (y_n) in H and $(\sigma_n)\in l^1$ such that

$$T(x)=\sum_{n=1}^{\infty}\sigma_n\langle x,x_n\rangle \quad (x\in H),$$

and that the scalars σ_n are the approximation numbers

$$\sigma_n=\sigma_n(T)=\inf\{\|T-S\|:\operatorname{rank}S<n\}.$$

(See [**14**, 1·b·4, 1·b·13, and 1·d·12(*c*)], for example.) We have

$$\left.\begin{aligned}\sigma_{m+n-1}(S+T)&\leqslant\sigma_m(S)+\sigma_n(T)\\ \sigma_{m+n-1}(ST)&\leqslant\sigma_m(S)\,\sigma_n(T)\end{aligned}\right\}\quad(m,n\in\mathbb{N},\ S,T\in\mathcal{N}),$$

([**14**, 1·d·14 and 1·d·15], [**16**, 11·8·2 and 11·9·2], [**17**, 2·3·3 and 2·3·12]). It follows that

$$\left.\begin{aligned}\sum_{n=1}^{\infty}\sigma_n(S+T)&\leqslant 2\sum_{n=1}^{\infty}(\sigma_n(S)+\sigma_n(T))\\ \sum_{n=1}^{\infty}\sigma_n(ST)&\leqslant 2\sum_{n=1}^{\infty}\sigma_n(S)\,\sigma_n(T)\end{aligned}\right\}\quad(S,T\in\mathcal{N}).$$

Now take $R_1,\ldots,R_k,\ S_1,\ldots,S_k\in\mathcal{N}$, and set $T=\sum_{j=1}^{k}R_jS_j$. Then

$$\sum_{n=1}^{\infty}\sigma_n(T)\leqslant 2^k\sum_{j=1}^{k}\sum_{n=1}^{\infty}\sigma_n(R_j)\,\sigma_n(S_j),$$

and so $(\sigma_n(T))\in l^{1/2}$. Certainly there exists $T\in\mathcal{N}$ with $(\sigma_n(T))\in l^1\backslash l^{1/2}$ because H is infinite dimensional. Thus $\mathcal{N}^2\neq\mathcal{N}$, and so $\mathcal{N}^2$ has infinite codimension in $\mathcal{N}$. ▮

The following result now follows from the above lemmas.

3·5. THEOREM. *Let E be an infinite-dimensional Banach space. Then* $\mathcal{N}(E)^2$ *has infinite codimension in* $\mathcal{N}(E)$, *and there exist discontinuous point derivations on* $\mathcal{N}(E)$. ▮

The above theorem implies that, for each infinite-dimensional Banach space E, the Banach algebra $\mathscr{N}(E)$ has neither a bounded left approximate identity nor a bounded right approximate identity, a result already proved by Selivanov ([**21**, Corollary 6]).

4. In this section, we shall show by example that Theorem 2·1 does not extend to all Banach spaces E. Indeed we shall exhibit Banach spaces P such that $\mathscr{A}(P)^2$ has infinite codimension in $\mathscr{A}(P)$.

The spaces P are the famous spaces of Pisier, constructed as counter-examples to a conjecture of Grothendieck that, if two Banach spaces E and F are such that their injective and projective tensor products $E \check{\otimes} F$ and $E \hat{\otimes} F$ coincide, then either E or F must be finite dimensional. In fact, P is an infinite-dimensional Banach space such that $P \check{\otimes} P = P \hat{\otimes} P$ both algebraically and topologically. We list some properties of the space P; we shall not use clause (iii) of the result.

4·1. THEOREM [**18**; **19**, Chapter 10]. *There is a Banach space P with the following properties*:

(i) *P is separable and infinite dimensional*;

(ii) *every approximable operator on P is nuclear*;

(iii) *both P and P' are of cotype* 2. ∎

In fact Pisier showed that every Banach space E which is separable and of cotype 2 is contained in such a space P. A Banach space with properties (i)–(iii) of the above theorem is said to be a *Pisier space*. It follows from (i) and (ii) that P does not have the approximation property.

We can now establish our main example.

4·2. THEOREM. *Let P be a Pisier space. Then there is a discontinuous point derivation on the Banach algebra $\mathscr{A}(P)$.*

Proof. Set $\mathscr{A} = \mathscr{A}(P)$. By Theorem 4·1 (ii), $\mathscr{A} = \mathscr{N}(P)$. By Theorem 3·5, $\mathscr{A}^2$ has infinite codimension in $\mathscr{A}$ and there are discontinuous point derivations on $\mathscr{A}$. ∎

We do not know whether or not there is a Pisier space with $P \simeq P \oplus P$. However, it can be shown that a Banach space E which is the l^2-sum of a Pisier space has the properties that $E \simeq E \oplus E$ and that there are discontinuous point derivations on $\mathscr{A}(E)$.

Let P be a Pisier space. We suspect that $\mathscr{K}(P)^2$ has infinite codimension in $\mathscr{K}(P)$, and hence that there are discontinuous point derivations on $\mathscr{K}(P)$. However we do not have a proof of this result.

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