RAREFACTION INDICES

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§1. *Introduction.* There are at least two indices used to measure the size of bounded sets of $\mathbb{R}^n$ of zero measure—Hausdorff dimension (see [**4**] for a definition), and the density index [**7**].

The object of this paper is to consider the relation between these two indices and to set up a general theory of such parameters.

It is interesting to note that they coincide on some more or less "regular" sets; for instance in $\mathbb{R}$, on the triadic Cantor set, or on any symmetric perfect set with a constant ratio ([**4**], [**7**]). To explain further the notion of regularity we first define an "irregularity coefficient" of a set, and then look for rarefaction indices which coincide on so-called regular sets, *i.e.* those whose coefficient is zero.

By definition, a "rarefaction index" (a term due to E. Borel) is a map α from the set of bounded subsets of $\mathbb{R}^n$, into $\mathbb{R}$, with the following two properties:

(1) $E \subset E'$ implies $\alpha(E) \leqslant \alpha(E')$ (monotonicity);

(2) $\alpha(E) = \alpha(h(E))$ for any homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ with the following property:

(*) On each compact set $K \subset \mathbb{R}^n$, the ratio

$$\frac{\log |h(x+\xi)-h(x)|}{\log |\xi|}$$

converges towards 1 uniformly when the norm of the vector ξ converges towards 0. (In particular, this condition is satisfied if h is a similarity, or a diffeomorphism.)

For the sake of simplicity, we consider in this paper the bounded borelian subsets of $\mathbb{R}$; the same methods easily generalize to $\mathbb{R}^n$. We use a net of 2^n-meshes, a classical method ([**1**], [**2**], [**5**], [**7**]) for constructing set coverings. By definition, a 2^n-mesh is an interval

$$\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[, \qquad k \in \mathbb{Z}.$$

If $x \in \mathbb{R}$, $n \in \mathbb{N}$, x belongs to one 2^n-mesh, denoted $u_n(x)$.

It is known [**2**] that calculating the Hausdorff dimension using mesh coverings gives the same result as if we use arbitrary intervals. The same holds for the density index [**7**]; we can take either n-meshes or 2^n-meshes.

The only disadvantage of this procedure is the following one: during a calculation or in the statement of a result, some irregularities may appear at the endpoints of

[MATHEMATIKA, **27** (1980), 46–57]

meshes, *i.e.* at points of

$$A = \left\{ \frac{k}{2^n} \,\middle|\, n \in \mathbb{N},\ k \in \mathbb{Z} \right\};$$

A is countable and dense in $\mathbb{R}$. Therefore we will prefer to work in the set $\mathbb{R}$-A, with induced topology, thus avoiding minor difficulties due to A. We remark that, if $E \subset R$, the Hausdorff dimension of E is equal to that of $E-A$.

Hence let $\mathscr{B}$ be the set of bounded non-empty borelian subsets of $\mathbb{R}-A$.

Let $\mathscr{M}$ be the set of finite σ-additive Borel measures **[6]**.

In §2 we give, with the help of measures of $\mathscr{M}$, the definition of the irregularity coefficient $r(E)$ of a set E in $\mathscr{B}$, and that of two important families of sets: "regular" sets, and "almost regular" sets.

In §3 we define the "regular" rarefaction indices, which take the same values on regular sets, and "σ-stable" ones; the indices which are both regular and σ-stable take the same value on almost regular sets.

An important example of a regular and σ-stable index is the Hausdorff dimension; one way of finding dim E can be stated as follows:

If E in $\mathscr{B}$, μ in $\mathscr{M}$ and a in $\mathbb{R}$ are such that $\mu(E) > 0$, and that for all x in E,

$$\underline{\lim} \frac{\log \mu(u_n(x))}{\log |u_n(x)|} = a,$$

then $a =$ dim E. (*$|u|$ denotes the length of interval u; here* $|u_n(x)| = 2^{-n}$.)

We show in §4 that this result, due to Billingsley, can be proved faster than in **[1]** with the help of results on Hausdorff measures due to Rogers and Taylor **[5]**.

§2. *The irregularity coefficient.*

Notation. For every E in $\mathscr{B}$, we denote by $\mathscr{M}(E)$ the set of all measure μ in $\mathscr{M}$ such that

(a) E is included in the support of μ,

(b) $\mu(E) > 0$.

Property (a) is equivalent to: $(\forall x)(\forall n)\Big(x \in E \;\&\; n \in \mathbb{N} \Rightarrow \mu(u_n(x)) > 0\Big)$.

If E is in $\mathscr{B}$ then $E \subset [a, b]$, and, if we let $\omega(2^n, E)$ denote the number of 2^n-meshes which meet E, then

$$\omega(2^n, E) \leqslant (b-a)2^n + 2.$$

LEMMA 1. *For each E in $\mathscr{B}$, for each $c > 1$, there exists a measure μ in $\mathscr{M}(E)$, and N in $\mathbb{N}$, such that for all x in E, and for all $n \geqslant \mathbb{N}$,*

$$1-c \leqslant \frac{\log \mu(u_n(x))}{\log |u_n(x)|} \leqslant c.$$

Proof. It suffices to prove this lemma in the case where E is included in a 1-mesh.

For each n in $\mathbb{N}$, let F_n be a set of $\omega(2^n, E)$ points of E, so that each 2^n-mesh meeting E contains one point of F_n.

Let $F = \bigcup_n F_n$. To each y of F, let us associate $N(y)$, the smallest integer n such that $y \in F_n$, and define a measure μ, for all y in F, by

$$\mu(\{y\}) = 2^{-cN(y)}.$$

For each n, there are at most $\omega(2^n, E)$ points of F of measure 2^{-cn}, so

$$\mu(F) \leqslant \sum_{n=0}^{\infty} \omega(2^n, E) 2^{-cn} \leqslant +\infty ,$$

and

$$\operatorname{Supp} \mu = \bar{F} = \bar{E} ,$$

therefore $\mu \in \mathscr{M}(E)$.

Let N be the smallest integer such that $1-c \leqslant \dfrac{\log \mu(F)}{\log 2^{-N}}$.

Let x be in E and let $n \geqslant N$, then $u_n(x)$ contains a point y of F_n, such that $N(y) \leqslant n$, so

$$1-c \leqslant \frac{\log \mu(u_n(x))}{\log 2^{-n}} \leqslant \frac{\log \mu(\{y\})}{\log 2^{-n}} \leqslant c .$$

Definition 1. *We call "irregularity coefficient" the map* $r : \mathscr{B} \to I = [0, 1]$, *such that for each E in* $\mathscr{B}$:

$$r(E) = 2 \inf \left\{ \varepsilon > 0 \,|\, \text{there exist } \mu_\varepsilon \text{ in } \mathscr{M}(E),\ a_\varepsilon \text{ in } \mathbb{R} \text{ and } N_\varepsilon \text{ in } \mathbb{N} \text{ such that for all } n \geqslant \mathbb{N}_\varepsilon,\ \text{for all } x \text{ in } E \text{ we have } \left| \frac{\log \mu_\varepsilon(u_n(x))}{\log |u_n(x)|} - a_\varepsilon \right| \leqslant \varepsilon \right\}.$$

This map has its value in I, because by Lemma 1, for each E in $\mathscr{B}$, for each $c > 1$, we can construct a measure μ in $\mathscr{M}(E)$ such that for all n in $\mathbb{N}$, and for all x in E:

$$\left| \frac{\log \mu(u_n(x))}{\log |u_n(x)|} - \tfrac{1}{2} \right| \leqslant c - \tfrac{1}{2} .$$

Definition 2. *E in* $\mathscr{B}$ *is said to be "regular" if* $r(E) = 0$.

We will see later (Theorem 1) that for regular sets, a_ε can be taken to be constant in Definition 1.

Let us give some examples of regular sets:

Example 1. Let $E = \{x\}$ with $x \in \mathbb{R} - A$.
With any measure μ on E, and $a = 0$, we obtain

$$\lim \left| \frac{\log \mu(u_n(x))}{\log |u_n(x)|} - a \right| = 0,$$

so $r(E) = 0$. The same result holds for all finite sets.

Example 2. Let $E = I - A$: take the Lebesgue measure, and $a = 1$. E is also a regular set.

Example 3. Let $(k_n)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{N}$, such that

$$k_1 \geqslant 2, \quad k_{n+1} - k_n \geqslant 2.$$

We can define a symmetrical perfect set E in $I - A$ (see [4]) such that

$$\omega(2^{k_n}, E) = 2^n,$$

and a measure μ on E such that, for all x in E,

$$\mu(u_{k_n}(x)) = 2^{-n}.$$

For each m in $\mathbb{N}$ with $m > k_2$, there exists n in $\mathbb{N}$ such that

$$k_{n-1} < m \leqslant k_n,$$

so for all x in E we have $\mu(u_m(x)) = \mu(u_{k_n}(x)) = 2^{-n}$, and

$$\frac{n}{k_n} \leqslant \frac{\log \mu(u_m(x))}{\log |u_m(x)|} \leqslant \frac{n}{k_{n-1}} = \frac{n-1}{k_{n-1}} \frac{n}{n-1}.$$

So E is regular if the sequence

$$\left(\frac{n}{k_n} \right)_{n \in \mathbb{N}}$$

converges (for example take $k_n = 2n$). We will see later that this condition is necessary.

THEOREM 1. *If E in $\mathscr{B}$ is regular, then the sequence*

$$\left(\frac{\log \omega(2^n, \mathrm{E})}{\log 2^n} \right)_{n \in \mathbb{N}}$$

converges.

Proof. For each $\varepsilon > 0$, there exist μ_ε in $\mathscr{M}(E)$, a_ε in $\mathbb{R}$, and N_ε in $\mathbb{N}$ such that:

for each 2^n-mesh u_i meeting E $(i = 1, 2, \ldots, \omega(2^n, E))$, we have

$$2^{-n(a_\varepsilon+\varepsilon)} \leqslant \mu_\varepsilon(u_i) \leqslant 2^{-n(a_\varepsilon-\varepsilon)}.$$

If $\|\mu_\varepsilon\|$ denotes the total mass of μ_ε, we have by summing over i:

(1) $\omega(2^n, E)2^{-n(a_\varepsilon+\varepsilon)} \leqslant \|\mu_\varepsilon\|$;

(2) $\omega(2^n, E)2^{-n(a_\varepsilon-\varepsilon)} \geqslant \mu_\varepsilon(E)$.

(1) and (2) imply

$$(3) \quad a_\varepsilon - \varepsilon \leqslant \underline{\lim} \frac{\log \omega(2^n, E)}{\log 2^n} \leqslant \overline{\lim} \frac{\log \omega(2^n, E)}{\log 2^n} \leqslant a_\varepsilon + \varepsilon.$$

As this is true for all $\varepsilon > 0$, the theorem is proved.

We write for all E in $\mathscr{B}$:

$$\underline{d}(E) = \underline{\lim} \frac{\log \omega(2^n, E)}{\log 2^n}$$

$$\bar{d}(E) = \overline{\lim} \frac{\log \omega(2^n, E)}{\log 2^n}.$$

$\bar{d}(E)$ is called "logarithmic density of E" [7].

We have just seen in (3) that the inequality

$$\bar{d}(E) - \underline{d}(E) \leqslant 2\varepsilon$$

is true for all $\varepsilon > \frac{1}{2}r(E)$, so

$$\bar{d}(E) - \underline{d}(E) \leqslant r(E),$$

which will be useful for the sequel.

Our interest in $\underline{d}$ and $\bar{d}$ derives from

PROPOSITION 1. *$\underline{d}$ and $\bar{d}$ are two rarefaction indices, in the sense of* §1.

Proof. The monotonicity property is easy to verify. For property (2) we can argue as follows.

Let E be in $\mathscr{B}$, let h be a homeomorphism verifying condition (*). For each interval u, $h(u)$ is an interval, whose extremities are the images of those of u. E being bounded, for each $\varepsilon > 0$, there exists N such that if u is a 2^n-mesh meeting E, $n \geqslant N$, we have: $|h(u)| < 2^{-n(1-\varepsilon)}$, so $h(u)$ is covered by a number of 2^n-meshes which is less than $2^{1+n\varepsilon}$.

Therefore $\omega(2^n, h(E)) \leqslant \omega(2^n, E) \,.\, 2^{1+n\varepsilon}$.

From this inequality, and the fact that ε is arbitrarily small, we deduce:

$$\underline{d}(h(E)) \leqslant \underline{d}(E), \quad \text{and} \quad \bar{d}(h(E)) \leqslant \bar{d}(E).$$

For the reverse inequalities, we use the following result

LEMMA 2. *h satisfies* (*), *if, and only if,* h^{-1} *satisfies* (*).

It suffices to replace h by h^{-1} and E by $h(E)$ in the previous inequalities to reverse them. So the proposition is proved.

Proof of Lemma 2. Let $K \subset \mathbb{R}$ be a compact set. $h^{-1}(K)$ is compact and by uniform continuity on K, it follows that, for each $\varepsilon' > 0$, there is an η, such that, for all x in K and all ξ with $x+\xi \in K$ and $|\xi| < \eta$, we have

$$|h^{-1}(x+\xi)-h^{-1}(x)| < \varepsilon'.$$

On the other hand by (*), for all $\varepsilon > 0$, there is an $\varepsilon' > 0$, such that, for all y in $h^{-1}(K)$ and all ζ with $y+\zeta \in h^{-1}(K)$ and $|\zeta| < \varepsilon'$, we have

$$\left|\frac{\log |h(y+\zeta)-h(y)|}{\log |\zeta|} - 1\right| < \frac{\varepsilon}{2}.$$

If we take $y = h^{-1}(x)$ and $\zeta = h^{-1}(x+\xi)-h^{-1}(x)$, we have:

$$\left|\frac{\log |\xi|}{\log |h^{-1}(x+\xi)-h^{-1}(x)|} - 1\right| < \frac{\varepsilon}{2},$$

so

$$\left|\frac{\log |h^{-1}(x+\xi)-h^{-1}(x)|}{\log |\xi|} - 1\right| < \varepsilon,$$

which was to be proved.

To close this section, we define a new class of sets.

DEFINITION 3. *An E in $\mathscr{B}$ is said to be "almost regular", if for all $\varepsilon > 0$, there exists a sequence $(E_n)_{n \in \mathbb{N}}$ in $\mathscr{B}$ such that*

$$E = \bigcup_n E_n, \quad \textit{and} \quad r(E_n) \leqslant \varepsilon, \quad \textit{for all } n \textit{ in } \mathbb{N}.$$

It is clear that a regular set is almost regular. We give in the next chapter an example of an almost regular, but non-regular set (Example 4), and a set that is not almost regular (Example 5).

§3. *Properties of rarefaction indices.*

DEFINITION 4. *A rarefaction index α is said to be "regular", if*

$$\alpha - \underline{d} = O(r),$$

where $O(r)$ is a function $\mathscr{B} \to I$ for which there exist $\varepsilon_0 > 0$ and K in $\mathbb{R}^+$, such that for all E in $\mathscr{B}$ with $r(E) \leqslant \varepsilon_0$, we have

$$|O(r)(E)| \leqslant Kr(E).$$

We remark that any regular index can be used for $\underline{d}$ in Definition 4. On regular sets r is zero, so regular indices take the same value on regular sets.

If α is a rarefaction index, and if $(E_n)_{n \in \mathbb{N}}$ is a sequence in $\mathscr{B}$, we have by monotonicity $\alpha(\bigcup_n E_n) \geqslant \sup_n \alpha(E_n)$. Equality does not always hold, as we shall see in a later example. But there exist indices for which equality always holds, whence a new definition.

DEFINITION 5. *A rarefaction index is "σ-stable" if, for every sequence $(E_n)_{n \in \mathbb{N}}$ with uniformly bounded E_n,*

$$\alpha\left(\bigcup_n E_n\right) = \sup_n \alpha(E_n).$$

For example, the Hausdorff dimension is σ-stable. We will prove in §4 that it is a rarefaction index.

Remark 1. Clearly there is a natural extension of σ-stable indices to unbounded subsets of $R - A$.

Remark 2. Each index α gives rise to a σ-stable index $\hat{\alpha}$, in the following manner:

Let E be in $\mathscr{B}$. To each decomposition of E into a countable infinity of sets E_n such that $E = \bigcup_n E_n$, and $E_n \in \mathscr{B}$, for all $n \in \mathbb{N}$, we associate the number $\sup_n \alpha(E_n)$. $\hat{\alpha}(E)$ is defined as the infimum of the set $\{\sup_n \alpha(E_n)\}$, over all decompositions of E.

PROPOSITION 2. *If α is a rarefaction index, $\hat{\alpha}$ is a σ-stable rarefaction index. If α is regular, so is $\hat{\alpha}$.*

Proof. The properties (1) $\hat{\alpha}$ is monotonic, (2) $\hat{\alpha}(E) = \hat{\alpha}(h(E))$ if h has property (*), are immediate.

(3) $\hat{\alpha}$ is σ-stable. Let $\varepsilon > 0$, and $E = \bigcup_n E_n$. For each n, there exists a sequence $(E_{n,k})_k$ such that

$$E = \bigcup_k E_{n,k}, \quad \text{and} \quad \sup_k \alpha(E_{n,k}) - \varepsilon \leqslant \hat{\alpha}(E_n).$$

As

$$\hat{\alpha}(E) \leqslant \sup_{n,k} \alpha(E_{n,k}),$$

and ε is arbitrarily small, the inequality

$$\hat{\alpha}(E) \leqslant \sup_n \alpha(E_n)$$

follows. The reverse inequality is obvious.

(4) $\hat{\alpha}$ is regular, if α is. There is an $\varepsilon_0 > 0$, and a $K > 0$, such that for all E with $r(E) < \varepsilon_0$, and for all ε in $]\frac{1}{2}r(E), \frac{1}{2}\varepsilon_0]$, there exist μ in $\mathcal{M}(E)$, a in $\mathbb{R}$ and N in $\mathbb{N}$ with the property that for all $n \geqslant N$ and for all x in E

$$\left|\frac{\log \mu(u_n(x))}{\log 2^{-n}} - a\right| \leqslant \varepsilon \quad \text{and} \quad |\alpha(E) - a| \leqslant K\varepsilon .$$

We consider such a set E, and (E_n) a sequence in $\mathcal{B}$ such that $E = \bigcup_n E_n$. There is an $n_0 \in \mathbb{N}$ such that $\mu(E_{n_0}) > 0$, so μ is in $\mathcal{M}(E_{n_0})$, and therefore $r(E_{n_0}) \leqslant 2\varepsilon$, and $|\alpha(E_{n_0}) - a| \leqslant K\varepsilon$. So $\alpha(E) - \alpha(E_{n_0}) \leqslant 2K\varepsilon$, which shows that $\alpha(E) - \hat{\alpha}(E) \leqslant Kr(E)$, and concludes the proof.

We have seen that regular rarefaction indices coincide on regular sets; this follows directly from the definitions. If we add the assumption of σ-stability, we have

THEOREM 2. *Rarefaction indices which are both regular and σ-stable take the same value on each almost regular set.*

Proof. We shall see that if E is almost regular and α is regular and σ-stable, then

$$\alpha = \hat{\underline{d}} .$$

For each $\varepsilon > 0$ there exists a sequence (E_n) in $\mathcal{B}$ such that $E = \bigcup E_n$, and $r(E_n) \leqslant \varepsilon$, for all n.

As α and $\hat{\underline{d}}$ are regular, there exist $\varepsilon_0 > 0$ and K in $\mathbb{R}^+$ such that, when $\varepsilon < \varepsilon_0$ we have, for all n,

$$|\alpha(E_n) - \underline{d}(E_n)| \leqslant K\, r(E_n) , \qquad |\hat{\underline{d}}(E_n) - \underline{d}(E_n)| \leqslant K\, r(E_n) .$$

So, for such ε, and, for all n in $\mathbb{N}$, $|\alpha(E_n) - \hat{\underline{d}}(E_n)| \leqslant 2K\varepsilon$. $\hat{\underline{d}}$ and α are σ-stable, whence $|\alpha(E) - \hat{\underline{d}}(E)| \leqslant 2K\varepsilon$, for all $\varepsilon > 0$, which proves the theorem.

Example 4. The set $E = \left\{\frac{1}{k} \,\middle|\, k \in \mathbb{N}\right\}$ is countable. Each subset consisting of a single point $\left\{\frac{1}{k}\right\}$ is regular, and $d\left(\left\{\frac{1}{k}\right\}\right) = 0$, so $\hat{d}(E) = 0$. E is almost regular.

On the other hand $\underline{d}(E) = \frac{1}{2}$, see [7]. So E is non-regular.

Almost regular sets intervene in the following result.

PROPOSITION 3. *Let α be a regular and σ-stable, and let E in $\mathcal{B}$ be such that there exist μ in $\mathcal{M}(E)$ and a in $\mathbb{R}$ with the following property. For all x in E,*

$$\lim_n \frac{\log \mu(u_n(x))}{\log |u_n(x)|} = a .$$

Then $a = \alpha(E)$.

Proof. In fact such a set is almost regular: it suffices to set, for each $\varepsilon > 0$,

$$E_k(\varepsilon) = \left\{x \in E \mid n \geqslant k \Rightarrow \left|\frac{\log \mu(u_n(x))}{\log |u_n(x)|} - a\right| \leqslant \varepsilon\right\}.$$

Then $E_1(\varepsilon) \subset \ldots \subset E_k(\varepsilon)$, and $E = \bigcup_k E_k(\varepsilon)$. As $\mu(E) > 0$, there exists K such that $\mu(E_k(\varepsilon)) > 0$ for $k \geqslant K$, so $\mu \in \mathscr{M}(E_k(\varepsilon))$. Therefore $r(E_k(\varepsilon)) \leqslant 2\varepsilon$, and E is almost regular.

By Theorem 2, we deduce that regular and σ-stable indices take a common value on E. Finally, this value is precisely a: we have $|d(E_k(\varepsilon)) - a| \leqslant \varepsilon$ for $k \geqslant K$, and by regularity of α there exist $\varepsilon_0 > 0$ and L in $\mathbb{R}^+$ such that, when $\varepsilon < \varepsilon_0$,

$$|\underline{d}(E_k(\varepsilon)) - \alpha(E_k(\varepsilon))| < L\varepsilon;$$

so $|\alpha(E_k(\varepsilon)) - a| < (L+1)\varepsilon$. As α is σ-stable,

$$|\alpha(E) - a| \leqslant (L+1)\varepsilon, \quad \text{for each } \varepsilon > 0.$$

Example 5 of a set that is not almost regular.

Let E be a symmetrical perfect set as in Example 3, such that the sequence $\left(\frac{n}{k_n}\right)_{n \in \mathbb{N}}$ is not convergent. We easily see that

$$\hat{\underline{d}}(E) \leqslant \underline{d}(E) = \underline{\lim} \frac{n}{k_n}, \quad \text{and} \quad \hat{\bar{d}}(E) \leqslant \bar{d}(E) = \overline{\lim} \frac{n}{k_n}.$$

We will prove that $\hat{\bar{d}}(E) \geqslant \bar{d}(E)$, which implies that $\hat{\underline{d}}(E) \neq \hat{\bar{d}}(E)$, so that E is not almost regular, by Theorem 2.

Let (E_n) be a sequence of $\mathscr{B}$, $E = \bigcup_n E_n$. There exists n_0 such that $\mu(E_{n_0}) > 0$. Now, for all $\varepsilon > 0$ and for all N in $\mathbb{N}$, there is an $n \geqslant N$, such that, for all x in E_{n_0},

$$\frac{\log \mu(u_{k_n}(x))}{\log |u_{k_n}(x)|} \geqslant \bar{d}(E) - \varepsilon.$$

Hence

$$\mu(u_{k_n}(x)) \leqslant 2^{-k_n(\bar{d}(E) - \varepsilon)}.$$

By summing on all 2^{k_n}-meshes meeting E_{n_0} we obtain

$$\omega(2^{k_n}, E_{n_0}) 2^{-k_n(\bar{d}(E) - \varepsilon)} \geqslant \mu(E_{n_0}) > 0.$$

So

$$\bar{d}(E_{n_0}) \geqslant \overline{\lim} \frac{\log \omega(2^{k_n}, E_{n_0})}{\log 2^{k_n}} \geqslant \bar{d}(E) - \varepsilon,$$

for all $\varepsilon > 0$. So $\hat{\bar{d}}(E) \geqslant \bar{d}(E)$.

§4. *Hausdorff dimension.* The examples of rarefaction indices that we gave in the preceding chapters are $\underline{d}$, $\bar{d}$, $\hat{\underline{d}}$, $\hat{\bar{d}}$. Another example is the Hausdorff dimension, denoted dim, which is also a regular rarefaction index. Before proving this, we give some notation.

For each $\rho > 0$, let $\Omega(E, \rho)$ be the set of all countable coverings $\mathscr{R} = \{u_i\}$ of E, where each u_i is a mesh, and $|u_i| \leqslant \rho$.

Recall that for each $a \in I$ the Hausdorff measure of E is

$$a\text{-}m(E) = \lim_{\rho \to 0} \inf \{\sum |u_i|^a | \{u_i\} \in \Omega(E, \rho)\},$$

and

$$\dim E = \inf \{a \in I | a\text{-}m(E) = 0\} = \sup \{a \in I | a\text{-}m(E) = +\infty\}.$$

THEOREM 3. *The function* dim : $\mathscr{B} \to I$ *is a σ-stable and regular rarefaction index.*

Proof. (1) The properties of monotonicity, and σ-stability, are well known.

(2) Let h be an homeomorphism verifying (*), with E in $\mathscr{B}$ and a in I. For each $\varepsilon > 0$ there exists $\rho > 0$ such that for each mesh u meeting E, with $|u| < \rho$ we have $|h(u)| < |u|^{1-\varepsilon}$. If n is the integer such that

$$2^{-n-1} < |u|^{1-\varepsilon} \leqslant 2^{-n}$$

the interval $h(u)$ is covered by at most two 2^n-meshes, say v_1 and v_2, with $|v_i| \leqslant 2|u|^{1-\varepsilon}$, so that

$$|v_1|^a + |v_2|^a \leqslant 2^{1+a}|u|^{a(1-\varepsilon)}.$$

Let $\mathscr{R}$ be in $\Omega(E, \rho)$, then there exists $\mathscr{R}'$ in $\Omega(h(E), 2\rho^{1-\varepsilon})$, such that

$$\sum_{v \in \mathscr{R}'} |v|^a \leqslant 2^{1+a} \sum_{u \in \mathscr{R}} |u|^{a(1-\varepsilon)}.$$

As ρ is arbitrarily small, the inequality

$$a\text{-}m(h(E)) \leqslant 2^{1+a} \, . \, a(1-\varepsilon)\text{-}m(E)$$

follows, and then $\dim h(E) \leqslant (1/(1-\varepsilon)) \dim E$, for each $\varepsilon > 0$. Hence $\dim h(E) \leqslant \dim E$.

The reverse inequality comes from the fact that h^{-1} satisfies (*).

(3) dim is regular. It suffices to prove the inequality

$$|\dim E - \underline{d}(E)| \leqslant r(E),$$

for each E in $\mathscr{B}$. Let E be in $\mathscr{B}$ and let $\varepsilon > \frac{1}{2}r(E)$. There exist μ in $\mathscr{M}(E)$, a in $\mathbb{R}$ and N in $\mathbb{N}$ such that, for all $n \geqslant N$ and all x in E,

$$|u_n(x)|^{a+\varepsilon} \leqslant \mu(u_n(x)) \leqslant |u_n(x)|^{a-\varepsilon}.$$

Let $\rho < 2^{-N}$, and R be in $\Omega(E, \rho)$.

Let $R = \{u_i\}$. Without loss of generality, we may assume that the meshes u_i are disjoint, and that each of them contains a point of E, and thus

$$|u_i|^{a+\varepsilon} \leqslant \mu(u_i) \leqslant |u_i|^{a-\varepsilon}.$$

Summation on i yields

$$\sum |u_i|^{a+\varepsilon} \leqslant \|\mu\|, \qquad \sum |u_i|^{a-\varepsilon} \geqslant \mu(E).$$

This is true for each $R \in \Omega(E, \rho)$, and each $\rho > 0$, so

$$(a+\varepsilon)-m(E) \leqslant \|\mu\|, \qquad (a-\varepsilon)-m(E) \geqslant \mu(E).$$

Therefore $a-\varepsilon \leqslant \dim E \leqslant a+\varepsilon$.

The same inequalities are satisfied by $\underline{d}(E)$ (Th. 1, inequalities (3)), so our assertion is proved.

Now we can say that dim takes the same values as $\underline{d}$ on regular sets, and as $\hat{\underline{d}}$ on almost regular sets.

Thus Hausdorff dimension verifies the assumption of Prop. 3. But we can prove a stronger result which seems particular to this index.

PROPOSITION 4. *Let E in $\mathscr{B}$ be such that there exist μ in $\mathscr{M}(E)$ and a in $\mathbb{R}$ satisfying*

$$\underline{\lim}_n \frac{\log \mu(u_n(x))}{\log |u_n(x)|} = a,$$

for all x in E. Then $a = \dim E$.

Proof. This follows from Theorems 2.1 and 2.2 of Billingsley [1]. We can also apply the results of Rogers and Taylor [5] on Hausdorff measure, to obtain a different proof, as follows.

(1) Let $b > a$. We have, for all x in E,

$$\underline{\lim}_n \frac{\log \mu(u_n(x))}{\log |u_n(x)|^b} < 1,$$

hence

$$\overline{\lim}_n \frac{\mu(u_n(x))}{|u_n(x)|^b} = +\infty.$$

If we put $h(t) = t^b$, this last value is called by Rogers and Taylor upper "h-density" relative to the sequence of mesh nets, denoted $\mathscr{M}\bar{D}_h\mu(x)$.

So for each k in $\mathbb{N}$, $E \subset E_k^* = \{x \mid \mathscr{M}\bar{D}_h\mu(x) > k\}$. There exist real λ_1 such that

$$b\text{-}m(E_k^*) \leqslant \frac{\lambda_1 \|\mu\|}{k}$$

([5], Lemma 2A). So b-$m(E) = 0$, and $\dim E \leqslant b$, for each $b > a$. Hence $\dim E \leqslant a$.

(2) Let $b < a$. We now have, if $h(t) = t^b$, then $\mathscr{M}\bar{D}_h\mu(x) = 0$, so $E \cap E_k^* = 0$. By Lemma 3 of [5], we deduce that

$$b\text{-}m(E) \geqslant \frac{\mu(E)}{k\lambda_2},$$

for some constant λ_2. So b-$m(E) > 0$: $\dim E \geqslant a$.

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