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Locally finite approximation of Lie groups. II

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In an earlier paper [**10**], we constructed a 'locally finite approximation away from a given prime p' of the classifying space BG of a Lie group with finite component group. Such an approximation consists of a locally finite group $\mathfrak{g}$ and a homotopy class of maps $\phi: B\mathfrak{g} \rightarrow BG$ which in particular induces an isomorphism in cohomology with finite coefficients of order prime to p. The usefulness of such a construction is that it reduces various homotopy-theoretic questions concerning the space BG to the corresponding questions concerning $B\pi$ for finite subgroups π. For example, we demonstrated in [**10**] how H. Miller's proof of the Sullivan conjecture concerning maps from $B\pi \rightarrow X$, where π is a finite group and X is a finite-dimensional complex, can be extended to maps $BG \rightarrow X$ for G a Lie group with finite component group.

In this paper, we elaborate on this philosophy by studying the relationship between the Lie group G and a specific locally finite approximation which will be denoted by Γ_G (see Proposition 1·1). Our interest lies in comparing the structure of the families of finite subgroups of G and of Γ_G. With such additional information concerning the locally finite approximation $\phi: B\Gamma_G \rightarrow BG$, we then deduce further homotopy theoretic properties of BG. In particular, Theorem 2·3 asserts that the generalized cohomology of BG is isomorphic to the inverse limit of the generalized cohomology of $B\pi$ as π ranges over the finite subgroups of G provided that the coefficients of the generalized cohomology theory are finite.

This paper consists of three sections. The first compares the families of finite subgroups of G and Γ_G. Roughly speaking, Γ_G is obtained by reducing modulo the prime p an integral model of the complexification of a compact form of G. Although we do not prove that the categories of prime-to-p subgroups of G and Γ_G are equivalent except in the special case in which $G = GL_n\mathbb{C}$, we demonstrate in Theorem 1·4 that the category of finite, prime to p subgroups of Γ_G and of an intermediate 'integral' group $G(R)$ are equivalent for any Lie group G with finite component group. We then provide a sufficiently good comparison of the finite subgroups of $G(R)$ and G to facilitate our analysis in section 2 of the generalized cohomology of BG. Specializing our Theorem 2·3 to stable cohomotopy, we obtain a generalization to Lie groups of G. Carlsson's proof of Segal's Burnside Ring Conjecture for finite groups [**6**] (closely related to results of M. Fesbach [**9**]); specializing to complex K-theory, we obtain a comparison of the complex representation rings of G and the finite subgroups $\pi \subset G$. In section 3, we return to the study of maps from BG, proving a result concerning maps between classifying spaces of Lie groups similar to a recent theorem of A. Zabrodsky [**20**].

1. *Comparison of* $P(\Gamma_G)$ *with* $P(G)$

Our purpose in this section is to compare the categories of finite, prime-to-p subgroups of a Lie group G and Γ_G, some 'locally finite approximation away from p' of G. Theorem 1·4 asserts that 'reduction mod p' determines an equivalence of categories of finite, prime-to-p subgroups of Γ_G and an intermediate discrete group $G(R)$. Proposition 1·7 then provides a comparison of the finite subgroups of this $G(R)$ and the Lie group G. These two theorems do not imply that the categories of finite, prime-to-p subgroups of Γ_G and G are equivalent, but do provide a sufficiently strong comparison of these categories to permit our applications in sections 2 and 3. Such an equivalence is likely (for example, we verify this equivalence for $G = GL_n\mathbb{C}$ in Proposition 1·8), and we hope to consider this possibility further in a future paper by providing the required strengthening of Proposition 1·7.

We begin by establishing our notational conventions and recalling in Proposition 1·1 our construction of a locally finite approximation away from p of a Lie group G. Throughout this paper, all Lie groups considered will be assumed to have finite component group, $\pi_0(G)$. For such a Lie group G, the quotient of G by a maximal compact subgroup K is contractible and the complex form of K (which we shall denote $G\mathbb{C}$) has reductive connected component. As described in the proof of [**10**], 2·1, there exists a ring A of S-integers in a number field (depending on G) and a group scheme G_A smooth over A such that the Lie group of complex points $G_A(\mathbb{C})$ of G_A is the complex Lie group $G\mathbb{C}$.

For future reference, we summarize the relationship between the Lie group G and the locally finite group $G_A(\mathbb{F})$ of points of G_A in the algebraic closure $\mathbb{F}$ of the prime field $\mathbb{F}_p$.

PROPOSITION 1·1. *Let G be a Lie group with finite component group. Then there exists a ring A of S-integers in a number field and a group scheme G_A smooth over A with reductive connected component, such that for all sufficiently large primes p there is a map*

$$\phi: B\Gamma_G \to BG \quad \textit{with} \quad \Gamma_G = G_A(\mathbb{F})$$

which is a locally finite approximation away from p of G. (In particular, $\phi_\#: \Gamma_G \to \pi_0(G)$ is surjective; and for any finite $\pi_0(G)$-module M of order prime to p,

$$\phi^*: H^*(BG, M) \to H^*(\Gamma_G, \phi^* M)$$

is an isomorphism; cf. [**10**], 1·1.) *Moreover, if R is any subring of $\mathbb{C}$ containing A and mapping to $\mathbb{F}$, then Γ_G and G are related by a chain of homomorphisms*

$$\Gamma_G \leftarrow G_A(R) \to G\mathbb{C} \leftarrow K \to G$$

which in a natural way give rise to $\phi: B\Gamma_G \to BG$; cf. [**10**], 2·5.

The following definition formalizes the category of finite subgroups of a given group.

Definition 1·2. For any set of primes P and any (topological) group H, we denote by $P(H)$ the category whose objects are finite subgroups of H of order divisible only by primes in P and whose morphisms from the finite subgroup π to the finite subgroup π' are homomorphisms $\pi \to \pi'$ of the form int(h) (sending $x \in \pi$ to $h^{-1}xh \in \pi'$) for some $h \in H$.

Our first step in our comparison of categories of finite subgroups is the following easy consequence of [**4**], 2·8.

PROPOSITION 1·3. *Let G be a Lie group with finite component group and let $K \subset G$ be a maximal compact subgroup. Then $K \subset G$ induces an equivalence of categories*

$$P(K) \simeq P(G)$$

for any set of primes P. In particular, the maps $G\mathbb{C} \leftarrow K \rightarrow G$ of (1·1) *induce equivalences of categories $P(G\mathbb{C}) \simeq P(K) \simeq P(G)$ for P the set of all primes except the chosen prime p.*

Proof. To prove that $K \subset G$ induces an essential surjection $P(K) \rightarrow P(G)$, it suffices to observe that any finite subgroup π of G is contained in some maximal compact subgroup of G and that any two maximal compact subgroups of G are conjugate (cf. [**4**], 2·7, for example): thus, π is conjugate in G to a subgroup of K. Because $K \subset G$ is an inclusion, the functor $P(K) \rightarrow P(G)$ is faithful. Thus, it suffices to prove that this functor is full; in other words, for each pair of subgroups π and π' in $P(K)$ and for each $g \in G$ such that $\mathrm{int}(g)(\pi) \subset \pi'$, there exists some $k \in K$ such that $\mathrm{int}(k) = \mathrm{int}(g): \pi \rightarrow \pi'$. This is verified in [**4**], 2·8. ∎

We recall that a strict Hensel local ring is a local domain that satisfies Hensel's lemma and has separably closed residue field (cf. [**15**], p. 38). Examples of such rings are the direct limits of all etale neighbourhoods of a geometric point on a scheme; other examples are complete local domains with separably closed residue fields. For our purposes, the example of primary interest is the ring of Witt vectors of an algebraically closed field k in characteristic p, a complete discrete valuation ring of characteristic 0 with residue field k.

The following theorem is the central step in our comparison of finite subgroups of G and of a locally finite approximation of G.

THEOREM 1·4. *Let R be a noetherian strict Hensel local ring with residue field k and let G_R be a smooth R-group scheme. Then for any set of primes P excluding the residue characteristic of k, reduction modulo p (i.e. $G_R(R) \rightarrow G_R(k)$) induces an equivalence of categories $P(G_R(R)) \simeq P(G_R(k))$, sending finite P-primary subgroups of $G_R(R)$ isomorphically onto their images in $G_R(k)$.*

Our proof of Theorem 1·4 consists of two lemmas proved below. We first prove in Lemma 1·5 that any finite P-subgroup $L \subset G_R(k)$ lifts to a finite P-subgroup of $G_R(R)$ by considering an R-scheme Φ_R whose R-points are $\mathrm{Hom}_{\mathrm{grps}}(L, G_R(R))$ and whose k-points are $\mathrm{Hom}_{\mathrm{grps}}(L, G_R(k))$. Once we verify the smoothness of Φ_R, we shall conclude the required lifting, thereby proving the essential surjectivity of

$$P(G_R(R)) \rightarrow P(G_R(k)).$$

In fact, we shall have proved that $G_R(R) \rightarrow G_R(k)$ restricts to an injection on any finite P-subgroup of $G_R(R)$, so that $P(G_R(R)) \rightarrow P(G_R(k))$ is faithful. To prove that this functor is full, we consider in Lemma 1·6 another R-scheme Ψ_R associated to a pair L and L' of subgroups of $G_R(R)$, with $\Psi_R(R)$ equal to $\mathrm{transp}_{G_R(R)}(L, L')$ and $\Psi_R(k)$ equal to $\mathrm{transp}_{G_R(k)}(L, L')$. (We remind the reader that the transporter $\mathrm{transp}_H(M, M') \subset H$ of one subgroup $M \subset H$ into another subgroup $M' \subset H$ consists of those $h \in H$ with $\mathrm{int}(h)(M) \subset M'$.) Consequently, by proving the smoothness of Ψ_R, we shall conclude the surjectivity of $\Psi_R(R) \rightarrow \Psi_R(k)$ and thus the fullness of the functor

$$P(G_R(R)) \rightarrow P(G_R(k)).$$

In what follows, if L is a discrete group and S a scheme, then L_S will denote the associated constant S-group scheme. For an arbitrary scheme X_S over S, one has a natural bijection $\mathrm{Hom}_S(L_S, X_S) \simeq \mathrm{Map}(L, X_S(S))$. If $S = \mathrm{Spec}\, A$, then we often write X_A for X_S; if $T = \mathrm{Spec}\, B$ is an S-scheme, we often write $X_A(B)$ for

$$X_S(T) = \mathrm{Hom}_S(T, X_S).$$

LEMMA 1·5. *Let G_R be a smooth R-group scheme, R a noetherian, strict Hensel local ring with residue field k. Let L be a finite group of order prime to the characteristic of k. Then the functor Φ_R on R-schemes defined by $\Phi_R(S) = \mathrm{Hom}_{S\text{-grp}}(L_S, G_S)$ is representable by a smooth group scheme. Consequently,* $\mathrm{Hom}_{\mathrm{grp}}(L, G_R(R)) \to \mathrm{Hom}_{\mathrm{grp}}(L, G_R(k))$ *is surjective and $G_R(R) \to G_R(k)$ restricts to an injection on any finite subgroup of order prime to the characteristic of k.*

Proof. We can write Φ_R as a fibre product of functors

$$\mathrm{Hom}_R(L_R, G_R) \quad \text{and} \quad \mathrm{Hom}_R(L_R \times_R L_R, G_R)$$

which are representable by suitable products over R of copies of G_R. Since G_R is locally finitely presented, Φ_R is representable by a locally finitely presented R-scheme. To prove the smoothness of Φ_R, it suffices to prove that it is 'formally smooth': namely, for every affine R-scheme S and every subscheme $S_0 \subset S$ defined by a nilpotent ideal, $\Phi_R(S) \to \Phi_R(S_0)$ is surjective (cf. [**17**], XI·1·1). Since L_R is a constant group scheme, the Hochschild cohomology groups $H^*(L_S, F)$ and $H^*(L_{S_0}, F_0)$ are isomorphic to the Eilenberg–Mac Lane cohomology groups $H^*(L, F(S))$ and $H^*(L, F_0(S_0))$ whenever F and F_0 are quasicoherent $\mathcal{O}_S$- and $\mathcal{O}_{S_0}$-modules provided with an L-action (cf. [**8**], II·3·4·1). Because the order of L is invertible in R and thus in $F(S)$ and $F_0(S_0)$, we conclude the vanishing of these cohomology groups in positive degrees. As shown in [**17**], III·2·1–2·3, this not only implies that any morphism of S_0-groups $u_0\colon L_{S_0} \to G_{S_0}$ can be lifted to a morphism of S-groups $u\colon L_S \to G_S$, thereby proving the surjectivity of

$$\Phi_R(S) \to \Phi_R(S_0),$$

but also that for any two liftings $u, u'\colon L_S \to G_S$ of $u_0\colon L_{S_0} \to G_{S_0}$ there exists some $g \in G_R(S)$ such that $u' = \mathrm{int}(g) \circ u$ and the image of g in $G_R(S_0)$ is the identity element.

The smoothness of Φ_R thereby implies the surjectivity of

$$\mathrm{Hom}_{\mathrm{grp}}(L, G_R(R)) \simeq \Phi_R(R) \to \Phi_R(k) \simeq \mathrm{Hom}_{\mathrm{grp}}(L, G_R(k))$$

(cf. [**15**], 1·3·24b and 1·4·2d). To complete the proof of the lemma, let $L \subset G_R(R)$ be a finite group of order prime to the characteristic of k which lies in the kernel of the reduction map $G_R(R) \to G_R(k)$. Let $\mathfrak{m}$ denote the maximal ideal of R and let $R^\wedge$ denote the $\mathfrak{m}$-adic completion of R. Since $G_R(R) \subset G_R(R^\wedge) = \varprojlim G_R(R/\mathfrak{m}^n)$, we conclude for n sufficiently large that the composition $L \to G_R(R) \to G_R(R/\mathfrak{m}^n)$ is injective. Because the ideal $\mathfrak{m}/\mathfrak{m}^n$ is nilpotent in $R/\mathfrak{m}^n$, this contradicts the uniqueness (up to conjugation) of the lifting of the trivial map $L \to G_R(k)$ as verified above for $S = \mathrm{Spec}\, R/\mathfrak{m}^n$ and $S_0 = \mathrm{Spec}\, k$. ∎

One consequence of Lemma 1·5 is that any inclusion $L \subset G_R(k)$ determines an embedding $u\colon L_R \to G_R$ of schemes for any finite group L of order prime to the characteristic of k. Namely, any lifting $u\colon L_R \to G_R$ of the inclusion $u_0\colon L \subset G_R(k)$ must be an embedding, as seen by applying Nakayama's Lemma to the kernel of u. Thus, the following lemma completes the proof of Theorem 1·4.

LEMMA 1·6. *Let G_R be a smooth R-group scheme, R a noetherian, strict Hensel local ring with residue field k. Let L and M be finite groups of order prime to characteristic of k, and let $u_R: L_R \to G_R$ and $v_R: M_R \to G_R$ be given embeddings. Define the subfunctor Ψ_R of G_R on R-schemes by $\Psi_R(S) = \mathrm{transp}_G(L_S, M_S) = \{g \in G_R(S), \mathrm{int}(g) \circ u_S: L_S \to G_S$ factors through $v_S\}$. Then Ψ_R is representable by a smooth R-scheme. Consequently, $\Psi_R(R) \to \Psi_R(k)$ is surjective.*

Proof. Because L_R is locally free over R, Ψ_R is representable by a locally finitely presented scheme over R (cf. [**17**], VIII·6·5e). As argued in the proof of Lemma 1·5, to prove the smoothness of Ψ_R it suffices to prove the surjectivity of $\Psi_R(S) \to \Psi_R(S_0)$ for any affine R-scheme S and closed subscheme $S_0 \to S$ defined by a nilpotent ideal. Let $g' \in G_R(S_0)$ be such that $\mathrm{int}(g') \circ u_{S_0} = v_{S_0} \circ \phi$ for some (uniquely determined) $\phi: L_{S_0} \to M_{S_0}$. Using the smoothness of G_R, choose some $g \in G_R(S)$ mapping to $g' \in G_R(S_0)$. As argued in the proof of (1·5) with G_R replaced by M_R, ϕ lifts to some $\phi^{\sim}: L_S \to M_S$ together with some $g'' \in M_R(S)$ mapping to $e \in M_R(S_0)$ such that

$$\mathrm{int}(g'') \circ \mathrm{int}(g') \circ u_S = v_S \circ \phi^{\sim}.$$

Consequently, $g''g' \in \Psi_R(S)$ maps to $g \in \Psi_R(S_0)$, thereby proving the surjectivity of $\Psi_R(S) \to \Psi_R(S_0)$. The surjectivity of $\Psi_R(R) \to \Psi_R(k)$ now follows from the consequent smoothness of Ψ_R. ∎

To complete our comparison of $P(\Gamma_G)$ and $P(G)$, we must investigate the functor $P(G_A(R)) \to P(G\mathbb{C})$, induced by an embedding $R \to \mathbb{C}$, where G_A is a smooth group scheme over the ring of S-integers in a number field as in (1·1) and R is a strict Hensel local ring containing A with residue field $\mathbb{F}$ (e.g. the Witt vectors of $\mathbb{F}$). Although the comparison provided by Proposition 1·7 below is the weakest in the chain, it will prove sufficient for applications to generalized cohomology.

PROPOSITION 1·7. *Assume the notation and hypotheses of* (1·1). *Let P be a set of primes such that R contains all nth roots of unity whenever the prime divisors of n lie in P. If G_R contains an R-split maximal torus, then $P(G_A(R)) \to P(G\mathbb{C})$ is a faithful functor which is essentially surjective on subgroups of $P(G\mathbb{C})$ which are of prime power order.*

Proof. Because $G_A(R) \to G_A(\mathbb{C}) = G\mathbb{C}$ is a monomorphism, $P(G_A(R)) \to P(G\mathbb{C})$ is faithful. To prove the asserted essential surjectivity, we must show that any subgroup of $P(G\mathbb{C})$ of prime power order is conjugate to a subgroup of $G_A(R)$. Let q be a prime in P, and let L be a finite q-group. Because L normalizes a maximal torus of $G\mathbb{C}$ ([**19**], 5·17), we may assume that L lies in the group of complex points $N\mathbb{C} = N_A(\mathbb{C})$ of the normalizer of some maximal, R-split torus

$$T_R = T_A \times_{\mathrm{Spec}\,A} \mathrm{Spec}\,R \quad \text{of} \quad G_R = G_A \times_{\mathrm{Spec}\,A} \mathrm{Spec}\,R$$

with group of complex points the maximal torus $T\mathbb{C}$ of $G\mathbb{C}$. Because $T\mathbb{C}/T_R(R)$ is uniquely q-divisible and $W = N\mathbb{C}/T\mathbb{C}$ is isomorphic to $N_R(R)/T_R(R)$, $N_R(R) \subset N\mathbb{C}$ contains a maximal q-torsion subgroup of $N\mathbb{C}$.

Thus, it suffices to prove that any two maximal q-torsion subgroups of $N\mathbb{C}$ are conjugate. Let $Q \subset W$ be a q-Sylow subgroup and let H be a maximal q-torsion subgroup of $N\mathbb{C}$ with image Q in W. Then the inclusion $H \subset N\mathbb{C}$ fits in the following map of extensions, where $T_q = H \cap T\mathbb{C}$ is the q-torsion subgroup of $T\mathbb{C}$ and the extension class of $T_q \to H^{\sim} \to Q$ is uniquely determined by $Q \to W$ and the isomorphism $H^2(Q, T\mathbb{C}) \simeq H^2(Q, T_q)$:

$$\begin{array}{ccccc} T_q & \to & H^{\sim} & \to & Q \\ \downarrow & & \downarrow & & \downarrow \\ T\mathbb{C} & \to & N\mathbb{C} & \to & W \end{array}$$

with $T_q \to T\mathbb{C}$ the canonical inclusion, $Q \to W$ the given inclusion, and $f(H^{\sim}) = H$. Because the q-Sylow subgroups of W are conjugate, we see that some conjugate H' of any other maximal q-torsion subgroup of $N\mathbb{C}$ determines a similar map of extensions with middle arrow $f: H^{\sim} \to N\mathbb{C}$ such that $f'(H^{\sim}) = H'$. The two maps $f, f': H^{\sim} \to N\mathbb{C}$ differ by a cocyle $Q \to T\mathbb{C}$ which factors up to coboundary through T_q. By standard homological algebra, we conclude the existence of an automorphism Ψ of $H^{\sim}$ over Q and an inner automorphism $\mathrm{int}(g)$ of $N\mathbb{C}$ over W such that $f \circ \Psi = \mathrm{int}(g) \circ f'$. In particular, the images of f and f' are conjugate as required. ∎

As we see in the following proposition, Proposition 1·3, Theorem 1·4, and Proposition 1·7 are sufficient to imply an equivalence of categories $P(\Gamma_G) \simeq P(G)$ in the special case of $G = GL_n(\mathbb{C})$ (or, of course, the compact form U_n of $GL_n(\mathbb{C})$).

PROPOSITION 1·8. *Let $G = GL_n(\mathbb{C})$, let R denote the Witt vectors of $\mathbb{F}$ provided with some embedding $R \subset \mathbb{C}$, and let P denote any set of primes excluding p. Then the maps of* (1·1)

$$\Gamma_G \leftarrow G_{\mathbb{Z}}(R) \to G\mathbb{C} \leftarrow K \to G$$

induce equivalences of categories

$$P(\Gamma_G) \simeq P(G_{\mathbb{Z}}(R)) \simeq P(G\mathbb{C}) \simeq P(K) \simeq P(G).$$

Proof. After applying Proposition 1·3, Theorem 1·4, and Proposition 1·7, there remains to prove that the functor $P(G_{\mathbb{Z}}(R)) \to P(G\mathbb{C})$ is essentially surjective and full. We identify a finite subgroup $\pi \subset G\mathbb{C}$ with a complex representation ρ_π, so that subgroups π and π' are conjugate in G if and only if the corresponding representations ρ_π and $\rho_{\pi'}$ are equivalent. We recall ([**18**], theorem 24) that any complex representation ρ_π of the finite group π is equivalent to an F-representation (and only one, up to F-equivalence) defined over any characteristic 0 field F containing the nth roots of unity, where n equals the order of π. Thus, if F denotes the field of fractions of R, then $\pi \subset G\mathbb{C}$ is conjugate to a subgroup of $G_{\mathbb{Z}}(F) \subset G\mathbb{C}$ whenever p does not divide the order of π. By choosing an $R[\pi]$-lattice inside the resulting $F[\pi]$-space representing ρ_π (i.e. an 'R-form' for ρ_π), we obtain a conjugate of π contained in $G_{\mathbb{Z}}(R)$. This implies that $P(G_{\mathbb{Z}}(R)) \to P(G\mathbb{C})$ is essentially surjective.

To prove that $P(G_{\mathbb{Z}}(R)) \to P(G\mathbb{C})$ is full, we must verify for any two subgroups π and π' of $G_{\mathbb{Z}}(R)$ of order relatively prime to p that a map $\mathrm{int}(g): \pi \to \pi^g \subset \pi'$ is a map in $P(G_{\mathbb{Z}}(R))$. Clearly, it suffices to assume that $\pi^g = \pi'$, in which case $\pi \subset G_{\mathbb{Z}}(R)$ and $\mathrm{int}(g): \pi \to \pi' \subset G_{\mathbb{Z}}(R)$ correspond to $\mathbb{C}$-equivalent representations which we must prove R-equivalent. As seen above, these R-forms are F-equivalent. The R-equivalence now follows from the observation that $R[\pi]$ is a maximal order in $F[\pi]$ because the order of π is invertible in R (cf. [**7**], 27·1), so that F-equivalence implies R-equivalence ([**7**], example 26·11). ∎

2. *Applications to cohomology theories*

We utilize the results of section 1 comparing $P(\Gamma_G)$ to $P(G)$ to study the generalized cohomology $h^*(BG)$ of BG. Of particular interest is Theorem 2·3 below which asserts that $h^*(BG)$ can be computed as an inverse limit of $h^*(B\pi)$ for π ranging over finite subgroups of G whenever $h^*(\ )$ satisfies a suitable finiteness property. This sharpens a theorem of D. Quillen asserting that mod-p cohomology of BG can be detected modulo nilpotent elements on elementary abelian subgroups of G [**16**]. Theorem 2·3 is another example of extending a known property of $B\Gamma$ for all locally finite groups Γ to BG for G a Lie group by using a locally finite approximation $\phi\colon B\Gamma_G \to BG$. Specializing Theorem 2·3 to various cohomology theories provides several interesting corollaries: in the case of stable cohomotopy $\pi_s^*(\ )$, we obtain a version of the Segal Conjecture for Lie groups similar to results of M. Fesbach [9]; in the case of K-theory $K^*(\ )$, we give a comparison of the representation rings of G and its finite subgroups π.

Before applying generalized cohomology theories to $B\pi$ with $\pi \in P(G)$, we record the following consequence of the comparisons of section 1, applicable to very general functors on $P(G)$. This generality will be employed in section 3 when we consider mapping complexes with source BG.

PROPOSITION 2·1. *Let G be a Lie group with finite component group, let q be a prime, and let $\phi\colon B\Gamma_G \to BG$ be a locally finite approximation away from some prime $p \neq q$ induced up to homotopy by a chain of maps as in* (1·1). *Then for any functor*

$$\Phi\colon \mathbf{Grp}^{\mathrm{op}} \to \mathbf{Sets}$$

(*i.e. a set-valued, contravariant functor on the category of groups*), *this chain of maps determines an injection*

$$\varprojlim_{\pi \in \{q\}(G)} \Phi(\pi) \to \varprojlim_{\tau \in \{q\}(\Gamma_G)} \Phi(\tau)$$

where $\{q\}$ is the singleton set consisting of the prime q.

Proof. By Proposition 1·3, Theorem 1·4, and Proposition 1·7, we conclude that it suffices to prove that if $F\colon \mathbf{C} \to \mathbf{D}$ is an essentially surjective inclusion of subcategories of $\mathbf{Grp}$, then the map $\varprojlim_{d \in \mathbf{D}} \Phi(d) \to \varprojlim_{c \in \mathbf{C}} \Phi(c)$ induced by F is injective. Because $\varprojlim \Phi(d) \subset \prod \Phi(d)$ and $\varprojlim \Phi(c) \subset \prod \Phi(c)$ where the products are indexed by isomorphism classes of objects in $\mathbf{D}$ and $\mathbf{C}$ respectively, the required injectivity follows from the observation that essential surjectivity (i.e. surjectivity on isomorphism classes of objects) implies the injectivity of $\prod \Phi(d) \to \prod \Phi(c)$. ∎

Of course in the special case of $G = GL_n\mathbb{C}$, Proposition 1·8 enables us to conclude that

$$\varprojlim_{\pi \in P(G)} \Phi(\pi) \to \varprojlim_{\tau \in P(\Gamma_G)} \Phi(\tau)$$

is a bijection for any set of primes P not containing p. For more general Lie groups, Proposition 2·1 when employed in conjunction with a generalized cohomology theory $h^*(\ )$ will provide isomorphisms for sets of primes P not containing p. The essential property we employ of such a general cohomology theory is transfer, which reduces many questions concerning $B\pi$ to questions concerning $B\pi_{(q)}$ for primes $q \in P$, where $\pi_{(q)}$ is a q-Sylow subgroup of π.

In the following lemma, we observe that the generalized cohomology (satisfying a finiteness condition) of $B\Gamma$ for any countable, locally finite group Γ is determined by the finite subgroups of Γ. This will be applied in Theorem 2·3 to prove the analogous statement for BG for G a Lie group.

LEMMA 2·2. *Let P be a set of primes, let $\mathbb{Z}_P{}^\wedge = \prod \mathbb{Z}_q{}^\wedge$ with the product indexed by primes $q \in P$, and let $h^*(\ )$ be a generalized cohomology theory with $h^j(S^0)$ a finitely generated $\mathbb{Z}_P{}^\wedge$-module for all j. For any countable locally finite group Γ, there is a natural isomorphism $h^*(B\Gamma) \simeq \varprojlim h^*(B\pi)$, where the limit is indexed by $\pi \in P(\Gamma)$.*

Proof. We write Γ as a union of finite subgroups, $\Gamma = \bigcup \pi_n$. Using the Milnor exact sequence (cf. [**1**], III·8·1), we readily verify that $h^j(B\pi_n)$ is compact for each j and n (namely, $h^j(B\pi_n)$ is the inverse limit of the finitely generated $\mathbb{Z}_P{}^\wedge$-modules $h^j(X_{n,k})$ with $X_{n,k}$ the k-skeleton of $B\pi_n$). Thus, using the Milnor exact sequence again, we conclude that $h^*(B\Gamma) \simeq \varprojlim h^*(B\pi_n)$. We recall (cf. [**11**]) that for any finite subgroup π $h^*(B\pi)$ is isomorphic to $\varprojlim h^*(B\tau)$, where the limit is indexed by $\tau \in P(\pi)$, whenever $h^j(S^0)$ is a finitely generated $\mathbb{Z}_P{}^\wedge$-module for each j. Thus, the lemma follows from the observation that $P(\Gamma) = \bigcup P(\pi_n)$ and the fact that inverse limits commute. ∎

The following theorem should be compared with a theorem of M. Feschbach describing $h^*(BG)$ in terms of 'stable elements' in the generalized cohomology of finite subgroups of a normalizer of a maximal torus of G ([**9**], 2·2).

THEOREM 2·3. *Let P be a set of primes and let $h^*(\ )$ be a generalized cohomology theory with $h^j(S^0)$ a finitely generated $\mathbb{Z}_P{}^\wedge$-module for all j. For any Lie group G with finitely many components, there is a natural isomorphism*

$$h^*(BG) \simeq \varprojlim_{\pi \in P(G)} h^*(B\pi).$$

Proof. We recall that $h^*(\ )$ is a product of generalized cohomology theories,

$$h^*(\ ) \simeq \prod h_q^*(\ ),$$

with the product indexed by $q \in P$ and with $h_q^j(S^0)$ a finitely generated $\mathbb{Z}_q{}^\wedge$-module for all j (because the spectrum representing $h^*(\ )$ splits as a product of its q-adic completions). Thus, we may assume that for some $q \in P$ each $h^j(S^0)$ is a finitely generated $\mathbb{Z}_q{}^\wedge$-module. Let $\phi: B\Gamma_G \to BG$ denote a locally finite approximation away from some prime p different from q as in Proposition 1·1. We consider the following commutative diagram:

$$\begin{array}{ccccc} h^*(BG) & \to & \varprojlim_{\pi \in P(G)} h^*(B\pi) & \to & \varprojlim_{\pi' \in \{q\}(G)} h^*(B\pi') \\ \downarrow & & \downarrow & & \downarrow \\ h^*(B\Gamma_G) & \to & \varprojlim_{\tau \in P(\Gamma_G)} h^*(B\tau) & \to & \varprojlim_{\tau' \in \{q\}(\Gamma_G)} h^*(B\tau'). \end{array} \qquad (2·3·1)$$

The left vertical map is induced by the locally finite approximation; this map is an isomorphism because $h^*(\ )$ is an inverse limit of cohomology theories with finite q-primary coefficients. The two lower horizontal maps are isomorphisms by Lemma 2·2. The right vertical map is an injection by Proposition 2·1. An easy diagram chase verifies that, to prove that the upper left horizontal map is an isomorphism as asserted, it suffices to prove that the upper right horizontal map is an injection. This latter fact is proved by observing that Lemma 2·2 implies that $h^*(B\pi)$ restricts injectively to $h^*(B\pi')$ for any q-Sylow subgroup π' of π. ∎

Specializing Theorem 2·3 to $h^*(\ ) = H^*(\ ,\mathbb{Z}/n)$, we conclude the following sharpening of Quillen's theorem asserting that mod-p cohomology modulo nilpotents is detected by restricting to finite elementary abelian p-groups [**16**]. Of course, Corollary 2·4 together with such a detection result modulo nilpotents in the special case of finite groups (now provable in an efficient, algebraic manner; cf. [**3**], for example) implies Quillen's detection result for general compact Lie groups.

COROLLARY 2·4. *Let G be a Lie group with finitely many components, n a positive integer, and P a set of primes containing the prime divisors of n. Then the natural restriction map is an isomorphism*:

$$H^*(BG, \mathbb{Z}/n) \simeq \varprojlim_{\pi \in P(G)} H^*(B\pi, \mathbb{Z}/n).$$

The reader can easily verify that a very similar proof to that of Theorem 2·3 applies to finite, local coefficient cohomology as asserted in Corollary 2·5 below. One considers a commutative diagram obtained from (2·3·1) by replacing $h^*(\ )$ with $H^*(\ ,A)$ and one observes that the analogue of Lemma 2·2 is easily implied by the classical Cartan–Eilenberg 'stable element' theorem ([5], XII·10·1).

COROLLARY 2·5. *Let G be a Lie group with finite component group $\pi_0(G)$, A a finite $\pi_0(G)$-module, and P a set of primes containing the prime divisors of the order of A. Then the natural restriction map is an isomorphism*:

$$H^*(BG, A) \simeq \varprojlim_{\pi \in P(G)} H^*(B\pi, A).$$

In the following proposition, we investigate a consequence of Theorem 2·3 in the special case in which $h^*(\ )$ equals mod-n stable cohomotopy, $h^*(\ ) = \pi_s^*(\ , \mathbb{Z}/n)$. Following our general philosophy, we utilize a known result for the classifying space of finite groups (the affirmation of the Segal Conjecture [6]) to obtain a corresponding result for the classifying spaces of Lie groups. Proposition 2·6 is closely related to results of [9]. Following the usual convention, we let $A(\pi)$ denote the Burnside ring of a finite group π and $A^\wedge(\pi)$ denote the completion of this Burnside ring with respect to its augmentation ideal topology.

PROPOSITION 2·6. *Let G be a Lie group with finitely many components, n a positive integer, and P a set of primes containing the prime divisors of n. Let $\pi_s^*(\ )$ denote unreduced stable cohomotopy (i.e. $\pi_s^j(X)$ denotes the group of pointed homotopy classes of maps $X_+ = X \coprod pt \to \Omega^\infty\Sigma^\infty S^j$). Then*

$$\pi_s^0(BG) \otimes \mathbb{Z}/n \simeq \varprojlim_{\pi \in P(G)} (A^\wedge(\pi) \otimes \mathbb{Z}/n)$$

$$\pi_s^1(BG) = 0$$

$$\pi_s^j(BG) \simeq H^{j-1}(BG, \mathbb{Z}^\wedge/\mathbb{Z}) \quad \text{for} \quad j > 1.$$

Proof. The affirmative solution to the Segal Conjecture for finite groups [6] implies for any finite group π that $\pi_s^0(B\pi)$ is isomorphic to $A^\wedge(\pi)$ and $\pi^j(B\pi) = 0$ for $j > 0$, so that $\pi_s^0(B\pi, \mathbb{Z}/n) \simeq A^\wedge(\pi) \otimes \mathbb{Z}/n$ and $\pi_s^j(B\pi, \mathbb{Z}/n) = 0$ for $j > 0$. Theorem 2·3 therefore implies that $\pi_s^0(BG, \mathbb{Z}/n) \simeq \varprojlim (A^\wedge(\pi) \otimes \mathbb{Z}/n)$, where the limit is indexed by $\pi \in P(G)$, and that $\pi_s^j(BG, \mathbb{Z}/n) = 0$ for $j > 0$. This implies that

$$\pi_0(\mathrm{map}_*(BG, (\Omega^\infty\Sigma^\infty S^j)^\wedge) \simeq \varprojlim \tilde{\pi}_s^j(BG, \mathbb{Z}/n) = 0 \quad (j > 0).$$

Considering pointed maps from BG to the fibration

$$K(\mathbb{Z}^\wedge/\mathbb{Z}, j-1) \to \Omega^\infty\Sigma^\infty S^j \to (\Omega^\infty\Sigma^\infty S^j)^\wedge,$$

we conclude that $\tilde{H}^{j-1}(BG, \mathbb{Z}^\wedge/\mathbb{Z}) \simeq \tilde{\pi}_s^j(BG)$ for any $j > 0$. Because $\pi_s^j(S^0) = 0$ for

$j > 0, \pi_s^j(\ ) \simeq \tilde{\pi}_s^j(\ )$ so that $\pi_s^j(BG) \simeq \tilde{H}^{j-1}(BG, \mathbb{Z}^\wedge/\mathbb{Z})$ for $j > 0$. In particular, $\pi_s^1(BG) = 0$. The vanishing of $\pi_s^1(BG)$ immediately implies the isomorphism

$$\pi_s^0(BG) \otimes \mathbb{Z}/n \simeq \pi_s^0(BG, \mathbb{Z}/n),$$

completing the proof of the proposition. ∎

We consider another special case of Theorem 2·3 in the following proposition, namely that of unreduced (complex) K-theory. In this case, we employ Theorem 2·3 with $h^*(\ ) = K^*(\ , \mathbb{Z}/n)$, together with results of [**2**] relating the (complex) representation rings of compact groups to their K-theory, to relate the representation rings of BG and $B\pi$ as π ranges over the finite subgroups of the compact Lie group G.

PROPOSITION 2·7. *Let G be a compact Lie group, n a positive integer, P a set of primes containing the prime divisors of n. Let $R(\ )$ denote the functor sending a compact group to its complex representation ring, and let $R^\wedge(G)$ denote the completion of $R(G)$ with respect to its augmentation ideal topology. Then the restriction map induces an isomorphism*

$$R^\wedge(G) \otimes \mathbb{Z}/n \simeq \varprojlim_{\pi \in P(G)} (R^\wedge(\pi) \otimes \mathbb{Z}/n).$$

Proof. Because $K^0(BH) \simeq R^\wedge(H)$ and $K^1(BH) = 0$ for any compact Lie group H [**2**], $K^0(BH, \mathbb{Z}/n) \simeq R^\wedge(H) \otimes \mathbb{Z}/n$. Thus, the required isomorphism follows immediately from the isomorphism $K^0(BG, \mathbb{Z}/n) \simeq \varprojlim K^0(B\pi, \mathbb{Z}/n)$ given by Theorem 2·3. ∎

3. *Applications to mapping complexes*

In [**10**], we employed the existence of locally finite approximations to a Lie group G in order to study the mapping complex of (pointed) maps from BG to a finite dimensional complex X. The key input into our analysis in addition to the existence of such approximations was H. Miller's affirmation of the Sullivan Conjecture asserting that the mapping complex of (pointed) maps from $B\pi$ to a simply connected, finite dimensional complex is weakly contractible for any finite group π [**14**]. In the preceding section, our study of generalized cohomology of BG can be equally viewed as the study of the homotopy groups of the mapping complex of maps from BG to the infinite loop spaces representing generalized cohomology theories. Our analysis used the infinite loop space structure of the target space of such a mapping complex in order to employ a transfer argument (necessary in the proof of Lemma 2·2).

In this section, we give some additional results concerning mapping complexes with source BG. The goal of our analysis is some understanding of mapping complexes for which both source and target are classifying spaces of compact Lie groups.

THEOREM 3·1. *Let G be a Lie group with finite component group, P a set of primes, X a simply connected space whose loop space ΩX is homotopy equivalent to a finite-dimensional complex, and $X^\wedge$ the Sullivan P-completion of X. Let $f: BG \to X^\wedge$ satisfy the condition that the restriction of f to $B\pi$ is homotopically trivial for every finite q-subgroup $\pi \subset G$ with $q \in P$. Then f is itself homotopically trivial.*

Proof. Since $X^\wedge$ is homotopy equivalent to the product $\prod X_q{}^\wedge$ of the Sullivan $\{q\}$-completions for all primes $q \in P$, it suffices to assume that $P = \{q\}$. Let $\phi: B\Gamma_G \to BG$

be a locally finite approximation away from some prime $p \neq q$, and write $\Gamma_G = \bigcup \gamma_n$ with each γ_n finite. We consider the following commutative diagram:

$$\begin{array}{ccc} [BG, X^\wedge] & \longrightarrow & \varprojlim_{\pi \in \{q\}(G)} [B\pi, X^\wedge] \\ \downarrow & & \downarrow \\ [B\Gamma_G, X^\wedge] \to \varprojlim_n [B\gamma_n, X^\wedge] & \to & \varprojlim_{\pi \in \{q\}(\Gamma_G)} [B\pi, X^\wedge] \end{array}$$

Using obstruction theory, we conclude that the fact that $\phi\colon B\Gamma_G \to BG$ induces an isomorphism in mod-q cohomology implies that the left vertical map is a bijection. By Proposition 2·1, the right vertical map is an injection. The Milnor exact sequence implies that the lower left horizontal map is a bijection because the homotopy groups of each component of each mapping space $\mathrm{map}_*(B\gamma_n, X^\wedge)$ are compact.

Consequently, to prove the theorem it suffices to prove that the lower right horizontal map sends only the base point to the base point. For this, it suffices to prove for each n that any map $f_n\colon B\gamma_n \to X^\wedge$ whose restrictions to every q-subgroup $\pi \subset \gamma_n$ is homotopically trivial is itself homotopically trivial; in other words, we may assume G is a finite group, $G = \gamma$. Let $\sigma \subset \gamma$ be a q-Sylow subgroup and let $B(\gamma/\sigma)$ denote the orbit space $\gamma \backslash E(\gamma/\sigma)$ as defined in [**14**], section 9. This space is $\mathbb{Z}_{(q)}$-acyclic; moreover, the natural surjective map $\lambda\colon B\gamma \to B(\gamma/\sigma)$ has the property that the pre-image $\lambda^{-1}(x)$ of any $x \in B(\gamma/\sigma)$ is homotopy equivalent to $B\pi$ for some q-subgroup π of γ.

To prove that $f\colon B\gamma \to X^\wedge$ is homotopically trivial, it suffices to prove that f extends to a map $B(\gamma/\sigma) \to X^\wedge$ because $B(\gamma/\sigma)$ is $\mathbb{Z}_{(q)}$-acyclic and $X_q^\wedge$ is $H^*(\ ;\mathbb{Z}_{(q)})$ local. Because $f\colon B\gamma \to X^\wedge$ restricts to a homotopically trivial map $f_|\colon \lambda^{-1}(x) \to X^\wedge$ for each $x \in B(\gamma/\sigma)$, f extends by [**20**], 1·5, to $B(\gamma/\sigma)$ provided that $\mathrm{map}_*(\lambda^{-1}(x), X^\wedge)$ has contractible trivial component $\mathrm{map}_*(\lambda^{-1}(x), X^\wedge)_0$ for each $x \in B(\gamma/\sigma)$. Finally, this contractibility is given by [**10**], 2·1, in view of the equivalence

$$\Omega(\mathrm{map}_*(\lambda^{-1}(x), X^\wedge)_0) \simeq \mathrm{map}_*(\lambda^{-1}(x), \Omega X^\wedge)$$

and the assumed finite dimensionality of ΩX. ∎

The following lemma, to be employed in the proof of Proposition 3·3, is perhaps of some independent interest. The lemma is a consequence of Theorem 1·4 and the construction of the locally finite approximation $\phi\colon B\Gamma_G \to BG$.

LEMMA 3·2. *Let G be a Lie group with finite component group $\pi_0(G)$. For any subgroup $Q \subset \pi_0(G)$ of prime power order, there exists a finite subgroup $\pi \subset G$ of prime power order such that Q is contained in the image of the composition $\pi \subset G \to \pi_0(G)$.*

Proof. Let $\phi\colon B\Gamma_G \to BG$ be a locally finite approximation away from some prime p so that $\Gamma_G \to \pi_0(G)$ is surjective (by [**10**], 1·1), and let $\delta \subset \Gamma_G$ map onto $\pi_0(G)$. Let q be the unique prime dividing $|Q|$ and let $\tau \subset \delta$ denote a q-Sylow subgroup of δ, so that $\tau \subset \Gamma_G$ has image in $\pi_0(G)$ containing Q. Using Theorem 1·4, find

$$\tau_R \subset G_R(R)$$

mapping isomorphically onto $\tau \subset \Gamma_G$ and let $\pi \subset G$ denote the (isomorphic) image of

τ_R. The lemma is implied by the homotopy commutativity of the following diagram

$$\begin{array}{ccccc} B\tau & \simeq & B\tau_R & \simeq & B\pi \\ \downarrow & & \downarrow & & \downarrow \\ B\Gamma_G & \leftarrow & BG_R(R) & \rightarrow & BG \\ \downarrow & & & & \downarrow \\ B\Gamma_G & & \xrightarrow{\quad\phi\quad} & & BG \end{array}$$

implicit in the construction of ϕ: $B\Gamma_G \rightarrow BG$ (cf. [**10**], 1·6).

Applying Theorem 3·1 and Lemma 3·2, we obtain the following criterion for a map f: $BG \rightarrow BH$ to be homotopically trivial. The interested reader should compare Proposition 3·3 in light of our Proposition 2·7 to a criterion of A. Zabrodsky [**20**] given in terms of the triviality of the map induced by f on complex K-theory.

PROPOSITION 3·3. *Let G and H be Lie groups with $\pi_0(G)$ finite. Let f: $BG \rightarrow BH$ be a map whose restriction $f|_{B\pi}$: $B\pi \rightarrow BH$ is homotopically trivial for every subgroup $\pi \subset G$ of prime power order. Then f is itself homotopically trivial.*

Proof. We first prove that f: $BG \rightarrow BH$ factors though f^0: $BG \rightarrow BH^0$, where H^0 is the connected component of H, by proving that $f_{\#}$: $\pi_0(G) \rightarrow \pi_0(H)$ is trivial. For this, it clearly suffices to prove that $f_{\#}$ is trivial upon restriction to any subgroup of prime power order. This is immediately implied by Lemma 3·2 and our hypothesis concerning $f|_{B\pi}$.

Because our hypotheses on f imply that $f^0|_{B\pi}$ is also homotopically trivial for every subgroup $\pi \subset G$ of prime power order, we may apply Theorem 3·1 to f^0 to conclude that $(f^0)^\wedge$: $BG \rightarrow (BH^0)^\wedge$ is homotopically trivial. Consequently, f^0 is a phantom map (cf. [**12**], [**13**]). On the other hand, the vanishing of $H^j(BG, \pi_{j+1}(BH^0) \otimes \mathbb{Q})$ for every $j \geqslant 0$ implies that every phantom map $BG \rightarrow BH^0$ is homotopically trivial. ∎

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