COMPOSITIO MATHEMATICA

Gaudin subalgebras and stable rational curves

Leonardo Aguirre, Giovanni Felder and Alexander P. Veselov

Compositio Math. **147** (2011), 1463–1478.

doi:10.1112/S0010437X11005306

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

Compositio Math. 147 (2011) 1463–1478
doi:10.1112/S0010437X11005306

Gaudin subalgebras and stable rational curves

Leonardo Aguirre, Giovanni Felder and Alexander P. Veselov

Abstract

Gaudin subalgebras are abelian Lie subalgebras of maximal dimension spanned by generators of the Kohno–Drinfeld Lie algebra $\mathfrak{t}_n$. We show that Gaudin subalgebras form a variety isomorphic to the moduli space $\bar{M}_{0,n+1}$ of stable curves of genus zero with $n+1$ marked points. In particular, this gives an embedding of $\bar{M}_{0,n+1}$ in a Grassmannian of $(n-1)$-planes in an $n(n-1)/2$-dimensional space. We show that the sheaf of Gaudin subalgebras over $\bar{M}_{0,n+1}$ is isomorphic to a sheaf of twisted first-order differential operators. For each representation of the Kohno–Drinfeld Lie algebra with fixed central character, we obtain a sheaf of commutative algebras whose spectrum is a coisotropic subscheme of a twisted version of the logarithmic cotangent bundle of $\bar{M}_{0,n+1}$.

1. Introduction

The Kohno–Drinfeld Lie algebra $\mathfrak{t}_n$ $(n=2,3,\dots)$ over $\mathbb{C}$, see [Koh85, Dri90], is the quotient of the free Lie algebra on generators $t_{ij}=t_{ji}$, $i\neq j\in\{1,\dots,n\}$, by the ideal generated by the relations

$$[t_{ij},t_{kl}]=0 \quad \text{if } i,j,k,l \text{ are distinct},$$
$$[t_{ij},t_{ik}+t_{jk}]=0 \quad \text{if } i,j,k \text{ are distinct}.$$

This Lie algebra appears in [Koh85] as the holonomy Lie algebra of the complement of the union of the diagonals $z_i=z_j$, $i<j$, in $\mathbb{C}^n$. The universal Knizhnik–Zamolodchikov connection [Dri90] takes values in $\mathfrak{t}_n$.

In this paper we consider the abelian Lie subalgebras of maximal dimension contained in the linear span $\mathfrak{t}_n^1$ of the generators t_{ij}. Motivating examples are the algebras considered by Gaudin [Gau76, Gau83] in the framework of integrable spin chains in quantum statistical mechanics and the Jucys–Murphy subalgebras spanned by $t_{12}, t_{13}+t_{23}, t_{14}+t_{24}+t_{34},\dots$ appearing in the representation theory of the symmetric group (see [OV96, VO04] and references therein).

Our main result is the classification of Gaudin subalgebras. We show that they are parametrised by the moduli space $\bar{M}_{0,n+1}$ of stable curves of genus zero with $n+1$ marked points (Theorem 2.5). The Gaudin subalgebras parametrised by the open subset $M_{0,n+1}$ are the ones considered originally by Gaudin (with t_{ij} replaced by their image in certain representations of $\mathfrak{t}_n$). To prove this theorem, it is useful to represent $\bar{M}_{0,n+1}$ as a subvariety of a product of projective lines given by explicit equations. We give such a description, proving a variant of a theorem of Gerritzen *et al.* [GHv88], in Appendix A.

Gaudin subalgebras form a locally free sheaf of Lie algebras on $\bar{M}_{0,n+1}$. We describe this sheaf as a sheaf of first-order twisted logarithmic differential operators (Theorem 3.3). For an algebra homomorphism $U\mathfrak{t}_n\to A$ from the universal enveloping algebra of t_n to an associative

Received 24 May 2010, accepted in final form 29 October 2010, published online 26 April 2011.
2010 Mathematics Subject Classification 14H10 (primary), 14H70, 17B99, 20C30 (secondary).
Keywords: Gaudin models, Kohno–Drinfeld Lie algebras, stable curves, Jucys–Murphy elements.

algebra A, we then get a sheaf of commutative subalgebras $\mathcal{E}_A$ of the $\mathcal{O}_X$-algebra $A \otimes \mathcal{O}_X$ on $X = \bar{M}_{0,n+1}$. We show that its relative spectrum is a coisotropic subscheme of a Poisson variety, a twisted version of the logarithmic cotangent bundle of $\bar{M}_{0,n+1}$ (Corollary 4.3). For a large class of representations of $U\mathfrak{t}_n$ these spectra, or at least their part over $M_{0,n+1}$, have been recently described in algebro-geometric terms using the Bethe ansatz, see [Fre05, MTV09, MTVa, MTVb, FFR10, FFT10] and references therein, and shown to have a surprising connection with several other mathematical subjects. It will be interesting to relate these descriptions to the geometry of $\bar{M}_{0,n+1}$. This and possible generalisations to other root systems will be the subject of further investigation.

It is interesting to look at our result in the context of the relation between flag varieties and configuration spaces initiated by Atiyah [Ati00, Ati01, AB02], who was inspired by Berry and Robbins. Note that the usual flag variety $U(n)/T^n$ can be naturally viewed as the space of all Cartan subalgebras in the unitary Lie algebra $\mathfrak{u}(n)$. Our result and a parallel between Gaudin and Cartan subalgebras give another link between these two varieties.

The limiting behaviour of the algebras introduced by Gaudin, that in our approach are parametrised by the open subset $M_{0,n+1}$, has been studied in various contexts. In [Vin90], Vinberg studied the commutative subspaces of degree two of the universal enveloping algebra $U\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ in relation with Poisson commutative subalgebras of the Poisson algebra $S\mathfrak{g}$ of polynomial functions on the dual of $\mathfrak{g}$. In the case of sl_n, his result implies a set-theoretic description of all possible limits of Gaudin subalgebras. Limits of Gaudin subalgebras of $U(\mathfrak{g})^{\otimes n}$ were studied more recently in [CFR09, CFR10]. In [CFR10], it was noticed that Jucys–Murphy elements arise as limits of Gaudin Hamiltonians, see Remark 2.6.

It is important to mention that our result works over any field. In particular, it holds over reals, which is important for applications. It is known that the set of real points $\bar{M}_{0,n+1}(\mathbb{R}) \subset \bar{M}_{0,n+1}$ is a smooth real manifold, which can be glued of $n!/2$ copies of the Stasheff associahedron (see [Kap93]). This gives a very convenient geometric representation of all limiting cases of the real Gaudin subalgebras and related quantum integrable systems. In particular, Jucys–Murphy subalgebra corresponds to one of the vertices of the associahedron. The spectrum in this case was studied in detail by Vershik and Okounkov [OV96, VO04]. What happens at other vertices labelled by different triangulations of an n-gon is worthy of further investigation. As was explained in [FM03, CFR09, CFR10], the corresponding integrable systems have a nice geometric realisation as Kapovich–Millson bending flows [KM96].

2. Classification of Gaudin subalgebras

Since $\mathfrak{t}_n$ is defined by homogeneous relations, it is graded in positive degrees: $\mathfrak{t}_n = \bigoplus_{i \geqslant 1} \mathfrak{t}_n^i$, with $\mathfrak{t}_n^1 = \bigoplus_{i<j} \mathbb{C}t_{ij}$, $\mathfrak{t}_n^2 = \bigoplus_{i<j<k} \mathbb{C}[t_{ij}, t_{ik}]$. In particular,

$$\dim(\mathfrak{t}_n^1) = \frac{n(n-1)}{2}, \quad \dim(\mathfrak{t}_n^2) = \frac{n(n-1)(n-2)}{6}.$$

Definition 2.1. A *Gaudin subalgebra* of $\mathfrak{t}_n$ is an abelian subalgebra of maximal dimension contained in $\mathfrak{t}_n^1$.

We will prove that this maximal dimension is $n-1$. It follows from the maximality condition that the central element

$$c_n = \sum_{1 \leqslant i<j \leqslant n} t_{ij}$$

belongs to all Gaudin subalgebras.

Example 2.2. The *Jucys–Murphy elements* $t_{12}, t_{12}+t_{13}, \dots, \sum_{i=1}^{n-1} t_{in}$ are pairwise commutative and thus span a Gaudin subalgebra. They play an important role in the representation theory of the symmetric group, see Remark 2.6.

Example 2.3. The main class of examples is provided by the spaces [Gau76, Gau83]

$$G_n(z) = \left\{ \sum_{1 \leqslant i<j \leqslant n} \frac{a_i - a_j}{z_i - z_j}\, t_{ij}, a \in \mathbb{C}^n \right\}, \tag{1}$$

parametrised by $z \in \Sigma_n/\mathrm{Aff}$, where

$$\Sigma_n = \mathbb{C}^n \Big\backslash \bigcup_{i<j} \{z \in \mathbb{C}^n \mid z_i = z_j\}$$

is the configuration space of n distinct ordered points in the plane and Aff is the group of affine maps $z \mapsto az+b$, $a \neq 0$, acting diagonally on $\mathbb{C}^n$. This parameter space is isomorphic to the moduli space $M_{0,n+1} = ((\mathbb{P}^1)^{n+1} - \bigcup_{i<j}\{z_i = z_j\})/\mathrm{PSL}_2(\mathbb{C})$: the class of z is mapped to the class of $(z_1, \dots, z_n, \infty)$ in $M_{0,n+1}$.

LEMMA 2.4. *The dimension of $G_n(z)$ is $n-1$.*

Proof. The dimension is at most $n-1$ since there are n parameters $a_1, \dots, a_n$ defined up to a common shift. Taking $a = (1, \dots, 1, 0, \dots, 0)$ with the number of ones ranging from 1 to $n-1$, we obtain $n-1$ elements K_j which are linearly independent: $t_{j,j+1}$ appears in K_j with non-vanishing coefficient but not in K_i, $i \neq j$. □

The main result of this section is that Gaudin subalgebras are in one to one correspondence with points in the Knudsen compactification $\bar{M}_{0,n+1}$ of $M_{0,n+1}$, which is a non-singular irreducible projective variety defined over $\mathbb{Z}$ (see [Knu83]). More precisely, we have the following result.

THEOREM 2.5. *Gaudin subalgebras in $\mathfrak{t}_n$ form a non-singular subvariety of the Grassmannian $G(n-1, n(n-1)/2)$ of $(n-1)$-planes in $\mathfrak{t}_n^1$, isomorphic to $\bar{M}_{0,n+1}$.*

Remark 2.6. To prove this theorem, we only use the defining relations of $\mathfrak{t}_n$ and the fact that both the generators t_{ij}, $1 \leqslant i<j \leqslant n$, and the brackets $[t_{ij}, t_{ik}]$, $1 \leqslant i<j<k \leqslant n$, are linearly independent. Thus, our result holds for any quotient of $\mathfrak{t}_n$ with these properties. An important example is the image of $\mathfrak{t}_n$ in the group algebra $\mathbb{C}S_n$ of the symmetric group with commutator bracket, with t_{ij} sent to the transposition of i and j. An approach to the representation theory of S_n based on the simultaneous diagonalisation of the image of the Jucys–Murphy elements was proposed in [OV96, VO04]. Another interesting case is given by the homomorphism $\phi : \mathfrak{t}_n \mapsto U\mathfrak{o}(n)$ into the universal enveloping algebra of the Lie algebra of the orthogonal group sending t_{ij} to X_{ij}^2, where X_{ij}, $i<j$, are the standard generators of the Lie algebra $\mathfrak{o}(n)$. The image consists of (the complex versions of) quantum Hamiltonians of the corresponding Manakov tops [Man76].

The rest of this section is dedicated to the proof of Theorem 2.5.

Let D_n be the set of all distinct triples (i,j,k) of numbers between 1 and n (i.e. the set of injective maps $\{1,2,3\} \to \{1, \dots, n\}$). For $(i,j,k) \in D_n$, denote by $p_{ijk} : \mathfrak{t}_n^1 \to \mathbb{C}^3$ the linear map

$$\sum_{i<j} a_{ij} t_{ij} \to (a_{jk}, a_{ik}, a_{ij}),$$

where a_{ij} is extended to all pairs by the rule $a_{ji} = a_{ij}$. The map $p : D_n \to \mathrm{Hom}(\mathfrak{t}_n^1, \mathbb{C}^3)$, $(i, j, k) \mapsto p_{ijk}$ is equivariant under the natural action of S_3 on D_n and on $\mathbb{C}^3$.

We start with the following simple calculation, which was probably first done by Vinberg [Vin90].

Lemma 2.7. *Let $V \subset \mathfrak{t}_n^1$ be a Gaudin subalgebra. Then, for all $(i, j, k) \in D_n$, $p_{ijk}(V)$ contains $(1, 1, 1)$ and is at most two dimensional.*

Proof. By the S_3-equivariance, it is sufficient to prove the claim for $i < j < k$. The space $p_{ijk}(V)$ contains $p_{ijk}(c_n) = (1, 1, 1)$. Let $a = \sum_{i<j} a_{ij}t_{ij}$, $b = \sum_{i<j} b_{ij}t_{ij} \in \mathfrak{t}_n^1$. The commutator $[a, b]$ is a linear combination of the linearly independent elements $[t_{ij}, t_{jk}]$, $1 \leqslant i < j < k \leqslant n$. Then the equation $[a, b] = 0$ is equivalent to the system

$$a_{ij}b_{jk} - a_{ij}b_{ik} + a_{ik}b_{ij} - a_{ik}b_{jk} + a_{jk}b_{ik} - a_{jk}b_{ij} = 0,$$

$1 \leqslant i < j < k \leqslant n$. These equations are conveniently written in determinant form (cf. [Vin90, Proof of Theorem 1])

$$\det \begin{pmatrix} a_{jk} & b_{jk} & 1 \\ a_{ik} & b_{ik} & 1 \\ a_{ij} & b_{ij} & 1 \end{pmatrix} = 0. \tag{2}$$

Thus, $p_{ijk}(V)$ contains at most two linearly independent vectors. □

Thus, for each Gaudin subalgebra there exists an S_3-equivariant map $\ell : D_n \to (\mathbb{C}^3)^*$ sending (i, j, k) to a linear form ℓ_{ijk} vanishing on $(1, 1, 1)$ and such that

$$\ell_{ijk} \circ p_{ijk}|_V = 0. \tag{3}$$

If $V = G_n(z)$, (3) is satisfied with the linear forms

$$\ell_{ijk} = (z_j - z_k, z_k - z_i, z_i - z_j).$$

Conversely, we have the following result.

Lemma 2.8. *Let $\ell : D_n \to (\mathbb{C}^3)^*$, $(i, j, k) \mapsto \ell_{ijk}$, be an S_3-equivariant map such that $\ell_{ijk}(1, 1, 1) = 0$ for all (i, j, k). Then*

$$V = \bigcap_{ijk} \mathrm{Ker}(\ell_{ijk} \circ p_{ijk})$$

is an abelian Lie subalgebra.

Proof. The vanishing condition implies that $c_n \in V$. If $a, b \in V$, then $p_{ijk}(a)$, $p_{ijk}(b)$ and $(1, 1, 1)$ belong to a two-dimensional subspace of $\mathbb{C}^3$ and therefore obey (2) for all i, j, k. It follows as in the proof of Lemma 2.7 that $[a, b] = 0$. □

It remains to determine which systems of linear forms ℓ_{ijk} give commuting subspaces of maximal dimension. With respect to the basis t_{ij} of $\mathfrak{t}_n^1$, we can represent the linear forms $\ell_{ijk} \circ p_{ijk}$ as the rows of a matrix L, so that the corresponding commuting subspace is the kernel of L. The matrices arising in this way belong to the following set.

Definition 2.9. Let $n \geqslant 3$ and $\mathcal{L}_n$ be the space of matrices whose rows are labelled by triples in $D_n^+ = \{(i, j, k), 1 \leqslant i < j < k \leqslant n\}$, whose columns are labelled by pairs in $Z_n^+ = \{(i, j), 1 \leqslant i < j \leqslant n\}$ and such that:

(1) the matrix elements in the row labelled by $(i,j,k)\in D_n^+$ vanish except possibly those in the columns (j,k), (i,k), (i,j);
(2) each row has at least a non-vanishing matrix element;
(3) the sum of the matrix elements in each row is zero.

For example, matrices in $\mathcal{L}_4$ are of the form

$$\begin{array}{c} \\ 123\\ 124\\ 134\\ 234\end{array}\begin{array}{c}\begin{array}{cccccc}12 & 13 & 14 & 23 & 24 & 34\end{array}\\ \begin{pmatrix} c_{123} & b_{123} & 0 & a_{123} & 0 & 0\\ c_{124} & 0 & b_{124} & 0 & a_{124} & 0\\ 0 & c_{134} & b_{134} & 0 & 0 & a_{134}\\ 0 & 0 & 0 & c_{234} & b_{234} & a_{234}\end{pmatrix}\end{array}, \tag{4}$$

with non-zero rows and zero row sums.

Proposition 2.10. *Let $L\in\mathcal{L}_n$, $m\leqslant n$. Let L_m be the matrix obtained from L by taking the matrix elements labelled by $D_m^+\times Z_m^+\subset D_n^+\times Z_n^+$. Then $L_m\in\mathcal{L}_m$.*

Proof. The claim is an easy consequence of the definition. □

Lemma 2.11. *Let $L\in\mathcal{L}_n$ and L_m, $3\leqslant m\leqslant n$, be the submatrix with labels in $D_m^+\times Z_m^+$. Then*

$$\operatorname{rank}(L)\geqslant\frac{(n-1)(n-2)}{2},$$

with equality if and only if

$$\operatorname{rank}(L_m)=\frac{(m-1)(m-2)}{2}\quad \text{for all } m=3,\dots,n.$$

Proof. We claim that there exists a row index set $I=I_3\cup I_4\cup\cdots\cup I_n\subset D_n^+$ such that:

(1) for each m, the set I_m has $m-2$ elements; they are of the form (i,j,m) for some $i<j<m$;
(2) for each m, there are distinct indices $k_1,\dots,k_{m-1}\in\{1,\dots,m\}$ and an ordering $r_1,\dots,r_{m-2}$ of I_m such that the entry of row r_i in column (k_j,m) is zero for $i<j$ and non-zero if $i=j$.

The $(n-1)(n-2)/2$ rows of L labelled by I are then clearly linearly independent, and the same holds for the rows of L_m in $I_3\cup\cdots\cup I_m$ for all $m\leqslant n$. It is also clear that if a row of L labelled by $D_m^+\subset D_n^+$ is a linear combination of rows labelled by I, then it is a linear combinations of rows labelled by I_m. This proves the lemma assuming the existence of I.

To describe the construction of I, it is notationally convenient to think of D_n^+ as the set D_n/S_3 of subsets of three elements and thus to identify $(i,j,k)=(j,i,k)=(i,k,j)$. The row indices I_m can then be taken as $r_i=(\sigma_m(i),\sigma_m(i+1),m)$, for some permutation σ_m of $\{1,\dots,m-1\}$ such that the entry of the row r_j in the column $(\sigma_m(j+1),m)$ is non-zero. By property (1) of $\mathcal{L}_n$, this then implies that only r_j among $r_1,\dots,r_{m-1}$ has a non-zero entry in this column and we set $k_j=\sigma_m(j+1)$, as desired. It remains to prove that such a permutation exists. Let Γ_m be the complete graph with vertex set $\{1,\dots,m\}$. Pick an orientation $i\to j$ on each edge $\{i,j\}$ of Γ_m such that the entry in the row (i,j,m) and the column (j,m) is non-zero. Such an orientation exists since at least two of the entries in columns (i,j), (i,m) and (j,m) are non-zero. Then the claim follows from the following simple result of elementary graph theory.

Lemma 2.12. *For any orientation of the edges of a complete graph with k vertices, there exists an oriented path $\sigma(1)\to\sigma(2)\to\cdots\to\sigma(k)$ visiting each vertex exactly once.*

In graph theory, such a path is called Hamiltonian and this fact is known as the existence of a Hamiltonian path in any tournament [Red34].

The proof is by induction: for $k=1$ there is nothing to prove. Let Γ_k be the complete graph with vertex set $\{1,\ldots,k\}$. An orientation of its edges restricts to an orientation of the edges of $\Gamma_{k-1}\subset\Gamma_k$. If we have a path γ on Γ_{k-1} starting at a vertex i and ending at a vertex j, then either there is an edge $k\to i$ or $j\to k$ and we can complete γ to a path in Γ_k by adding it, or there exists a step $a\to b$ of γ that can be replaced by $a\to k\to b$ to obtain a path in Γ_k with the required property. □

Corollary 2.13. *Abelian subalgebras lying in $\mathfrak{t}_n^1$ have dimensions at most $n-1$.*

Indeed, the rank of a matrix in $\mathcal{L}_n$ is at least $(n-1)(n-2)/2$ and its kernel has dimension at most

$$\frac{n(n-1)}{2}-\frac{(n-1)(n-2)}{2}=n-1.$$

Lemma 2.14. *Suppose that $L\in\mathcal{L}_n$ has minimal rank $(n-1)(n-2)/2$, so that* Ker L *defines a Gaudin subalgebra. Denote the entries of row (i,j,k) in columns $(i,j),(i,k),(j,k)$ by a_{ijk},b_{ijk},c_{ijk}, respectively. Then*

$$(a_{ijk}:b_{ijk}:c_{ijk})=(b_{jik}:a_{jik}:c_{jik})=(a_{ikj}:c_{ikj}:b_{ikj}),\tag{5}$$

$$a_{ijk}+b_{ijk}+c_{ijk}=0\tag{6}$$

for all $(i,j,k)\in D_n$, and

$$b_{ijk}c_{ijl}b_{ikl}+c_{ijk}b_{ijl}c_{ikl}=0\tag{7}$$

for all $(i,j,k,l)\in V_n$.

Proof. The first two equations are a rephrasing of the S_3-equivariance and the condition $\ell_{ijk}(1,1,1)=0$. Consider the third equation. By possibly renumbering the vertices, we can assume that the four indices are 1, 2, 3, 4. By Lemma 2.11, the submatrix L_4 with labels in $D^+(4)\times Z^+(4)$ has rank three. This matrix has the form (4). Since the last row is non-zero, the upper left 3×3 minor vanishes. □

We can now conclude the proof of Theorem 2.5. Let Z_n be the subvariety of Gaudin subalgebras in the Grassmannian of $(n-1)$ planes in $\mathfrak{t}_n^1$. By Lemma 2.7, every Gaudin subalgebra V is contained in Ker L, for some $L\in\mathcal{L}_n$. Since, by Lemma 2.8, Ker L is an abelian subalgebra, it follows by maximality that V is actually equal to Ker L. Let Y_n be the subvariety of $(\mathbb{P}^2)^{D_n}$ defined by the equations (5)–(7) for homogeneous coordinates $(a_{ijk}:b_{ijk}:c_{ijk})$. By Lemma 2.14, we then have a map $Z_n\to Y_n$ which is clearly injective. Since the subalgebras $G_n(z)$ are contained in Z_n, Z_n has a component of dimension $n-2$. As we will presently see, Y_n is isomorphic to the non-singular irreducible projective $(n-2)$-dimensional variety $\bar{M}_{0,n+1}$ and therefore $Z_n\to Y_n$ is an isomorphism.

It remains to prove that $Y_n\simeq\bar{M}_{0,n+1}$. In order to do so, let us first notice that by (6), Y_n actually lies in $(\mathbb{P}^1)^{D_n}$, where $\mathbb{P}^1\subset\mathbb{P}^2$ is embedded as $(x:y)\mapsto(x-y:-x:y)$. It is easy to rewrite the remaining equations defining Y_n in these coordinates:

(1) $x_{ikj}x_{ijk}=y_{ikj}y_{ijk}$;

(2) $x_{jik}y_{ijk}=y_{ijk}y_{jik}-x_{ijk}y_{jik}$;

(3) $x_{ijk}y_{ijl}x_{ikl}=y_{ijk}x_{ijl}y_{ikl}$.

It turns out that, by a variant of a theorem of Gerritzen, Herrlich and van der Put, these are precisely the relations defining $\bar{M}_{0,n+1}$ as a subvariety of $(\mathbb{P}^1)^{D_n}$. We deduce this variant from the original theorem of Gerritzen, Herrlich and van der Put in Appendix A, see Theorem A.2.

3. The sheaf of Gaudin subalgebras

By Theorem 2.5, Gaudin subalgebras of $\mathfrak{t}_n$ form a family of vector spaces $\mathcal{G}_n$ on $\bar{M}_{0,n+1}$: the fibre at z is the Gaudin subalgebra corresponding to z. The purpose of this section is to identify this family in terms of the geometry of $\bar{M}_{0,n+1}$.

Consider first the Gaudin subalgebras parametrised by $M_{0,n+1}$. Let $\tilde{M}_{0,n+1} = \Sigma_n/\mathbb{C}$, where the group $\mathbb{C} \subset \mathrm{Aff}$ is the translation subgroup. The natural projection $p : \tilde{M}_{0,n+1} \to M_{0,n+1}$ is a principal $\mathbb{C}^\times$-bundle. The $\mathfrak{t}_n^1$-valued 1-form

$$\omega = \sum_{i<j} \frac{dz_i - dz_j}{z_i - z_j} t_{ij}$$

is a $\mathbb{C}^\times$-invariant element $\Omega^1(\tilde{M}_{0,n+1}) \otimes \mathfrak{t}_n^1$. The pairing with ω defines a map

$$T_z\tilde{M}_{0,n+1} \to \mathfrak{t}_n^1$$

from the tangent space at $z \in \tilde{M}_{0,n+1}$ to $\mathfrak{t}_n^1$, with image $G_n(z)$, see (1). By Lemma 2.4, this map is injective. Now $G_n(z) = G_n(z')$ if and only if $z' = \lambda z$ for some $\lambda \in \mathbb{C}^\times$. More precisely, the action of $\mathbb{C}^\times$ on $\tilde{M}_{0,n+1}$ lifts naturally to $T\tilde{M}_{0,n+1}$ and the invariance of ω implies that ω defines an injective bundle map.

$$\begin{array}{ccc} T\tilde{M}_{0,n+1}/\mathbb{C}^\times & \longrightarrow & M_{0,n+1} \times \mathfrak{t}_n^1 \\ \downarrow & & \downarrow \\ M_{0,n+1} & = & M_{0,n+1} \end{array}$$

Its image is a vector bundle with fibres $G_n(z)$, $z \in M_{0,n+1}$. Moreover, $T\tilde{M}_{0,n+1}/\mathbb{C}^\times$ is an extension of the tangent bundle to $M_{0,n+1}$: the kernel of the natural surjective bundle map $T\tilde{M}_{0,n+1}/\mathbb{C}^\times \to TM_{0,n+1}$ is spanned by the class of the Euler vector field

$$E = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i},$$

generating the $\mathbb{C}^\times$-action. Turning to the language of sheaves, more convenient when we pass to $\bar{M}_{0,n+1}$, we thus have an exact sequence of locally free sheaves on $X = M_{0,n+1}$:

$$0 \to \mathcal{O}_X \to \mathcal{G}_n \to T_X \to 0. \tag{8}$$

Here T_X is the sheaf of vector fields and $\mathcal{G}_n$ is the sheaf of t_n^1-valued functions whose value at each z lies in $G_n(z)$. For any open set $U \subset M_{0,n+1}$, $\mathcal{G}_n(U)$ may be identified via the map ω with the space of $\mathbb{C}^\times$-invariant vector fields on $p^{-1}(U)$.

As is well known, invariant vector fields can be identified with first-order twisted differential operators on the base manifold.

LEMMA 3.1. *Let $p : P \to X$ be a principal $\mathbb{C}^\times$-bundle on a smooth variety X and $L = P \times_{\mathbb{C}^\times} \mathbb{C}$ be the associated line bundle, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by multiplication. Then for each open set $U \subset X$, the Lie algebra $T_P(p^{-1}(U))^{\mathbb{C}^\times}$ is isomorphic to the Lie algebra $D^1_{L^\vee}(U)$ of first-order differential operators acting on sections of the dual line bundle $L^\vee$ (i.e. twisted by $L^\vee$).*

Proof. A section of $L^\vee$ on U is the same as a function $f : p^{-1}(U) \to \mathbb{C}$ such that $f(y \cdot \lambda) = \lambda f(y)$, $y \in p^{-1}(U)$, $\lambda \in \mathbb{C}^\times$. It is clear from this representation that $T_P(p^{-1}(U))^{\mathbb{C}^\times}$ acts on sections of $L^\vee$. Moreover, the infinitesimal generator E of the $\mathbb{C}^\times$-action acts by 1. Thus, upon choosing a local trivialisation of P, we may write any invariant vector field as $\xi + fE$, where ξ is a vector field on U and $f \in O_X(U)$. This invariant vector field acts on a section as the first-order differential operator $\xi + f$. □

As a consequence, we have a description of $\mathcal{G}_n$ as a sheaf of twisted first-order differential operators.

Proposition 3.2. *Gaudin subalgebras corresponding to points of $M_{0,n+1}$ form a locally free sheaf isomorphic to the sheaf $D^1_{L^\vee}$ of first-order differential operators on $M_{0,n+1}$ twisted by the line bundle $L^\vee$ dual to the associated bundle to $\tilde{M}_{0,n+1}$ via the identity character of $\mathbb{C}^\times$.*

Let us now extend this to $\bar{M}_{0,n+1}$.

Let us first recall some known facts about the geometry of $\bar{M}_{0,n+1}$ [Knu83, GHv88, Kee92]. This space can be defined as the closure in $(\mathbb{P}^1)^{D_n}$ of the image of the injective map

$$\mu : M_{0,n+1} = \Sigma_n/\mathrm{Aff} \to (\mathbb{P}^1)^{D_n}$$

sending the class of $z \in \Sigma_n$ to the collection of cross ratios involving the point at infinity

$$\mu_{ijk}(z) = \frac{z_i - z_k}{z_i - z_j} = \frac{(z_i - z_k)(\infty - z_j)}{(z_i - z_j)(\infty - z_k)}, \quad (i, j, k) \in D_n.$$

Moreover, the image is characterised by an explicit set of equations, see Theorem A.2.

The complement of $M_{0,n+1}$ in $\bar{M}_{0,n+1}$ is a normal crossing divisor $D = \bigcup D_S$, where the union is over all subsets S of $\{1, \ldots, n\}$ with at least two and at most $n-1$ elements. The irreducible component D_S is isomorphic to $\bar{M}_{0,m+1} \times \bar{M}_{0,n-m+2}$ and is the closure of the subvariety consisting of stable curves with one nodal point such that the marked points labelled by S are those on the component not containing the point labelled by $n+1$.

Local coordinates on a neighbourhood in $\bar{M}_{0,n+1}$ of a generic point of D_S are the cross ratios $\zeta_r = \mu_{ijr}$, $r \in S \smallsetminus \{i, j\}$, $z_s = \mu_{iks}$, $s \in S^c \smallsetminus \{k\}$, $t = \mu_{ijk}$, for any fixed $i, j \in S$, $k \notin S$ (S^c denotes the complement of S in $\{1, \ldots, n\}$). In these coordinates, D_S is given by the equation $t = 0$. The change of variables from these coordinates to the coordinates z_i of $M_{0,n+1}$ is as follows. We may assume that $S = \{1, \ldots, m\}$ and choose $i = 1$, $j = m$, $k = m+1$ (the general case can be obtained from this by permuting the coordinates). Then for generic $t \neq 0$ the point in $M_{0,n+1}$ with coordinates (ζ, z, t) is

$$[0, t\zeta_2, \ldots, t\zeta_{m-1}, t, z_2, \ldots, z_{n-m}, 1] \in M_{0,n+1} = \Sigma_n/\mathrm{Aff}. \tag{9}$$

To extend the exact sequence (8) to $\bar{M}_{0,n+1}$, we first show that $\tilde{M}_{0,n+1}$ extends to a principal $\mathbb{C}^\times$-bundle

$$p : \tilde{\bar{M}}_{0,n+1} \to \bar{M}_{0,n+1}.$$

This can be seen from the presentation of Theorem A.2. Let H be the kernel of the product map $(\mathbb{C}^\times)^{D_n} \to \mathbb{C}^\times$. Then $P_n = (\mathbb{C}^2 \smallsetminus \{0\})^{D_n}/H$ is a principal $\mathbb{C}^\times$-bundle on $(\mathbb{P}^1)^{D_n}$ and $\tilde{M}_{0,n+1}$ embeds into P_n (via $z \mapsto$ class of $((z_i - z_k, z_i - z_j)_{(i,j,k) \in D_n})$) as the restriction of P_n to the image of $M_{0,n+1}$ in $(\mathbb{P}^1)^{D_n}$. We then define $\tilde{\bar{M}}_{0,n+1}$ to be the restriction of P_n to $\bar{M}_{0,n+1} \subset (\mathbb{P}^1)^{D_n}$.

Recall that the locally free sheaf $T_X\langle -D \rangle$ of logarithmic vector fields on a variety X with a normal crossing divisor D consists of vector fields whose restriction to a generic point of D is tangent to D. It is dual to the sheaf $\Omega^1_X\langle D\rangle$ of logarithmic 1-forms, spanned over $\mathcal{O}_X$ by regular 1-forms and df/f, where $f\in\mathcal{O}_X$ with $f\neq 0$ on $X\smallsetminus D$, see [Del70, § II.3].

Theorem 3.3. *Let L be the associated line bundle $\tilde{\bar{M}}_{0,n+1}\times_{\mathbb{C}^\times}\mathbb{C}$ with the identity character of $\mathbb{C}^\times$. Gaudin subalgebras form a vector bundle on $\bar{M}_{0,n+1}$. As a locally free sheaf, it is isomorphic to the sheaf $D^1_{L^\vee}\langle -D\rangle$ of first-order differential operators on $\bar{M}_{0,n+1}$ twisted by $L^\vee$, whose symbol is logarithmic. In particular, there is an exact sequence of sheaves on $X=\bar{M}_{0,n+1}$:*

$$0\to\mathcal{O}_X\to\mathcal{G}_n\to T_X\langle -D\rangle\to 0.$$

The embedding of the trivial bundle $\mathcal{O}_X$ sends 1 to $c_n=\sum_{i<j}t_{ij}$.

Proof. Let us introduce the abbreviated notation $X=\bar{M}_{0,n+1}$, $p:\tilde{X}=\tilde{\bar{M}}_{0,n+1}\to X$. Let $\tilde{D}=p^{-1}(D)$ be the pull-back to $\tilde{X}$ of the divisor D. Then ω is a form in $\Omega^1(\tilde{X})\otimes\mathfrak{t}^1_n$ with logarithmic coefficients. Indeed, in the coordinates ζ,z,t of (9) and fibre coordinate $\lambda\in\mathbb{C}^\times$ around a generic point of $\tilde{D}$, ω has the local coordinate expression

$$\begin{aligned}\omega=&\sum_{1\leqslant i<j\leqslant m}t_{ij}\,d\log(\zeta_i-\zeta_j)+\sum_{1\leqslant i<j\leqslant m}t_{ij}\,d\log(t)\\&+\sum_{m<i<j\leqslant n}t_{ij}\,d\log(z_i-z_j)+\sum_{1\leqslant i<m<j\leqslant n}t_{ij}\,d\log(z_j)+c_n\,d\log(\lambda),\end{aligned}\tag{10}$$

with the understanding that $\zeta_0=0,\zeta_m=z_n=1$. Thus, ω may be paired with invariant logarithmic vector fields on $\tilde{X}$, which in turn may be identified by Lemma 3.1 with first-order differential operators on X, to give a map $D^1_{L^\vee}\langle -D\rangle\to\mathfrak{t}^1\otimes\mathcal{O}_X$ which is injective on $M_{0,n+1}$. We need to show that the map is injective on all of $X=\bar{M}_{0,n+1}$. Since the locus of non-injectivity is empty or of codimension one, it is sufficient to show that the map is injective as we approach a generic point of the divisor D. This is easy to check using (10).

The embedding of $\mathcal{O}_X$ sends 1 to $\langle\omega,E\rangle=c_n$. □

Thus, $\mathcal{G}_n$ is a sheaf of twisted first-order differential operators with regular singularities along D, see [Uen08, § 5.2].

Remark 3.4. The divisor class of the line bundle L can be easily computed by choosing a section. The result is

$$[L]=-\sum_{S\supset\{1,2\}}[D_S].$$

4. Coisotropic spectra

Suppose that $U\mathfrak{t}_n\to A$ is a homomorphism of unital algebras, e.g. $A=\mathrm{End}(V)$ for some representation V of $\mathfrak{t}_n$. Then we get a sheaf $\mathcal{E}_A$ of commutative subalgebras of A on $X=\bar{M}_{0,n+1}$ as the image of the symmetric $\mathcal{O}_X$-algebra $S\mathcal{G}_n$. By Theorem 3.3, $\mathcal{G}_n$ is naturally a sheaf of Lie algebras, so that $S\mathcal{G}_n$ is a sheaf of Poisson algebras.

Corollary 4.1. *Let $\varphi:U\mathfrak{t}^1_n\to A$ be an algebra homomorphism. Then the kernel of the induced map of sheaves of algebras $\varphi:S\mathcal{G}_n\to A\otimes\mathcal{O}_X$, is closed under Poisson brackets.*

Proof. This is basically a consequence of the fact that ω is closed. The map is defined by identifying $\mathcal{G}_n$ with the sheaf of $\mathbb{C}^\times$-invariants of $p_*T_{\tilde{X}}\langle -\tilde{D}\rangle$. It is the algebra homomorphism sending an invariant section ξ to $\omega_A(\xi) = \varphi(\omega(\xi))$. A Poisson bracket of monomials $\xi_1 \cdots \xi_k$, $\eta_1 \cdots \eta_l$, $\xi_i, \eta_j \in \mathcal{G}_n \simeq p_* T_{\tilde{X}}\langle -\tilde{D}\rangle^{\mathbb{C}^\times}$ is sent to

$$\begin{aligned}
&\varphi(\{\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_l\}) \\
&\quad = \sum_{i,j} \varphi([\xi_i, \eta_j]\xi_1 \cdots \hat{\xi}_i \cdots \xi_k \, \eta_1 \cdots \hat{\eta}_j \cdots \eta_l) \\
&\quad = \sum_{i,j} \omega_A([\xi_i, \eta_j]) \prod_{r \neq i} \omega_A(\xi_r) \prod_{s \neq j} \omega_A(\eta_s) \\
&\quad = \sum_{i,j} \prod_{r<i} \omega_A(\xi_r) \prod_{s<j} \omega_A(\eta_s) \omega_A([\xi_i, \eta_j]) \prod_{r>i} \omega_A(\xi_r) \prod_{s>j} \omega_A(\eta_s) \\
&\quad = \sum_{i,j} \prod_{r<i} \omega_A(\xi_r) \prod_{s<j} \omega_A(\eta_s)(\xi_i \omega_A(\eta_j) - \eta_j \omega_A(\xi_i)) \prod_{r>i} \omega_A(\xi_r) \prod_{s>j} \omega_A(\eta_s) \\
&\quad = \sum_{i,j} \prod_{r<i} \omega_A(\xi_r) \prod_{s<j} \omega_A(\eta_s) \xi_i \omega_A(\eta_j) \prod_{s>j} \omega_A(\eta_s) \prod_{r>i} \omega_A(\xi_r) \\
&\qquad - \sum_{i,j} \prod_{s<j} \omega_A(\eta_s) \prod_{r<i} \omega_A(\xi_r) \eta_j \omega_A(\xi_i) \prod_{r>i} \omega_A(\xi_r) \prod_{s>j} \omega_A(\eta_s) \\
&\quad = \sum_{i} \prod_{r<i} \omega_A(\xi_r) \xi_i \varphi(\eta_1 \cdots \eta_l) \prod_{r>i} \omega_A(\xi_r) \\
&\qquad - \sum_{j} \prod_{s<j} \omega_A(\eta_s) \eta_j \varphi(\xi_1 \cdots \xi_k) \prod_{s>j} \omega_A(\eta_s).
\end{aligned}$$

In this calculation, we use the fact that $\omega_A(\xi_r), \omega_A(\eta_s)$ commute with each other; the peculiar choice of ordering of factors is necessary since their derivatives $\xi_i \omega_A(\eta_j)$, $\eta_j \omega_A(\xi_i)$ do not necessarily commute with them.

It follows that if $\varphi(a) = \varphi(b) = 0$, then also $\varphi(\{a, b\}) = 0$, which is the claim. □

Recall that $\Omega^1_{\tilde{X}}\langle \tilde{D}\rangle$ is locally free and thus the sheaf of sections of a vector bundle, the logarithmic cotangent bundle. Let us denote by $T^*\tilde{X}\langle \tilde{D}\rangle$ the total space of this vector bundle. It is the relative spectrum of the symmetric algebra $ST_{\tilde{X}}\langle -\tilde{D}\rangle$, which is a sheaf of Poisson algebras. Thus, $T^*\tilde{X}\langle \tilde{D}\rangle$ is a Poisson variety; the Poisson structure restricts to the usual symplectic structure on the cotangent bundle of $\tilde{M}_{0,n+1}$. The group $\mathbb{C}^\times$ acts on $\tilde{X}$ preserving $\tilde{D}$. This action lifts to a Hamiltonian action on the logarithmic cotangent bundle with moment map $E \in \Gamma(\tilde{X}, T_{\tilde{X}}\langle -\tilde{D}\rangle) \subset \Gamma(\tilde{X}, ST_{\tilde{X}}\langle -\tilde{D}\rangle) = \mathcal{O}(T^*\tilde{X}\langle \tilde{D}\rangle)$.

Definition 4.2. Let $\alpha \in \mathbb{C}$. The *twisted logarithmic cotangent bundle $T^*X_\alpha\langle D\rangle$ with twist α* is the Hamiltonian reduction $E^{-1}(\alpha)/\mathbb{C}^\times$.

By construction, $T^*X_\alpha\langle D\rangle$ is a Poisson variety. For $\alpha = 0$, it is the logarithmic cotangent bundle of X. By definition, the regular functions on an open set $\tilde{U} = p^{-1}(U)$, $U \subset X$, are sections of $(ST_{\tilde{X}}\langle -D\rangle(\tilde{U}))^{\mathbb{C}^\times}/I(\tilde{U})$, where $I(\tilde{U})$ is the ideal generated by $E - \alpha$.

Then Corollary 4.1 can be reformulated as follows.

Corollary 4.3. *Let $U\mathfrak{t}^1_n \to A$ be an algebra homomorphism such that c_n is mapped to $\alpha 1$ and let $\mathcal{E}_A$ be the corresponding sheaf of commutative algebras on $X = \bar{M}_{0,n+1}$. Then the relative spectrum of $\mathcal{E}_A$ is a coisotropic subscheme of the twisted logarithmic cotangent bundle $T^*X_\alpha\langle D\rangle$.*

In particular, if A is finite dimensional, then the part of the spectrum over $X^0 = M_{0,n+1}$ is a Lagrangian subvariety of the symplectic manifold $T^*X^0_\alpha = (E^{-1}(\alpha) \cap T^*\tilde{X}^0)/\mathbb{C}^\times$.

Acknowledgements

We thank A. M. Vershik for inspiring discussions at an early stage of this project. We thank Pavel Etingof and Valerio Toledano Laredo for useful comments and stimulating discussions. The second author wishes to thank the Department of Mathematics of MIT for hospitality. The third author would like to thank the Institute for Mathematical Research at ETH Zurich for hospitality. We thank the referee for providing relevant references and correcting misprints. This work was partially supported by the Swiss National Science Foundation (Grant 200020-122126).

Appendix A. Stable curves of genus zero

Recall that a stable curve of genus zero with $r \geqslant 3$ marked points is a pair (C, S), where C is a connected projective algebraic curve of genus 0 whose singularities are ordinary double points and $S = (p_1, \ldots, p_r)$ is an ordered set of distinct non-singular points of C such that each irreducible component of C has at least three special (marked or singular) points. The genus zero condition means that the irreducible components are projective lines whose intersection graph is a tree. The moduli space $\bar{M}_{0,r}$ of stable rational curves with $r \geqslant 3$ marked points [Knu83] is a smooth algebraic variety of dimension $r - 3$ defined over $\mathbb{Q}$. It contains as a dense open set the quotient of the configuration space

$$M_{0,r} = \{z \in (\mathbb{P}^1)^r \mid z_i \neq z_j, \text{for all } i \neq j\} / \operatorname{PSL}_2$$

of r distinct labelled points on the projective line by the diagonal action of $\operatorname{Aut}(\mathbb{P}^1) \simeq \operatorname{PSL}_2$.

Here is a simple description of $\bar{M}_{0,r}$, due to Gerritzen *et al.* [GHv88]. For each distinct (i, j, k, l) in $\{1, \ldots, r\}$, let $\lambda_{ijkl} : M_{0,r} \to \mathbb{P}^1$ be the map sending the class of z to the cross ratio

$$\lambda_{ijkl}(z) = \frac{(z_i - z_l)(z_j - z_k)}{(z_i - z_k)(z_j - z_l)} \in \mathbb{C} \subset \mathbb{P}^1.$$

Let V_r be the set of distinct quadruples of integers between 1 and r. Then $\bar{M}_{0,r}$ is the closure of the image of the embedding

$$M_{0,r} \to (\mathbb{P}^1)^{V_r}, \quad z \to (\lambda_v(z))_{v \in V_r}$$

sending z to the system of cross ratios $\lambda_{ijkl}(z) \in \mathbb{P}^1 \smallsetminus \{0, \infty, 1\}$.

The cross ratios $\lambda_v = \lambda_v(z)$ of a point in $M_{0,r}$ obey the following relations:

(λ1) $\lambda_{jikl} = 1/\lambda_{ijkl}$ for all distinct i, j, k, l;

(λ2) $\lambda_{jkli} = 1 - \lambda_{ijkl}$ for all distinct i, j, k, l;

(λ3) $\lambda_{ijkl}\lambda_{ijlm} = \lambda_{ijkm}$ for all distinct i, j, k, l, m.

Theorem A.1 (Gerritzen *et al.* [GHv88]). *The subvariety of* $(\mathbb{P}^1)^{V_r}$ *defined by these relations, more precisely by their version for homogeneous coordinates* $\lambda_v = x_v/y_v$,

(1) $x_{jikl}x_{ijkl} = y_{jikl}y_{ijkl}$ *for all distinct* i, j, k, l,

(2) $x_{jkli}y_{ijkl} = y_{ijkl}y_{jkli} - x_{ijkl}y_{jkli}$ *for all distinct* i, j, k, l,

(3) $x_{ijkl}x_{ijlm}y_{ijkm} = y_{ijkl}y_{ijlm}x_{ijkm}$ *for all distinct* i, j, k, l, m,

is a fine moduli space of stable curves of genus zero. The dense open subvariety $M_{0,r}$ *is embedded via the cross ratios* $z \mapsto (x_v : y_v)_{v \in V_r}$, *with*

$$(x_{ijkl} : y_{ijkl}) = ((z_i - z_l)(z_j - z_k) : (z_i - z_k)(z_j - z_l)).$$

For our purpose, it is useful to have a more economical description of the moduli space by taking only cross ratios involving a distinguished marked point. Let $n = r + 1 \geqslant 2$ and D_n be the set of all distinct triples (i, j, k) of integers between 1 and n.[1] The cross ratios $\mu_{ijk}(z) = \lambda_{i,n+1,j,k}(z)$ obey:

(μ1) $\mu_{ikj} = 1/\mu_{ijk}$ for all distinct i, j, k;

(μ2) $\mu_{jik} = 1 - \mu_{ijk}$ for all distinct i, j, k;

(μ3) $\mu_{ijk}\mu_{ikl} = \mu_{ijl}$ for all distinct i, j, k, l.

The claim is that the homogeneous version of these relations defines $\bar{M}_{0,n+1}$.

Theorem A.2. *The moduli space $\bar{M}_{0,n+1}$ is isomorphic to the subvariety of $(\mathbb{P}^1)^{D_n}$ defined by the following equations.*

(1) $x_{ikj}x_{ijk} = y_{ikj}y_{ijk}$ *for all* $(i, j, k) \in D_n$.

(2) $x_{jik}y_{ijk} = y_{ijk}y_{jik} - x_{ijk}y_{jik}$ *for all* $(i, j, k) \in D_n$.

(3) $x_{ijk}x_{ikl}y_{ijl} = y_{ijk}y_{ikl}x_{ijl}$ *for all* $(i, j, k, l) \in V_n$.

The open subvariety $M_{0,n+1}$ is embedded via the cross ratios $z \mapsto (x_d : y_d)_{d\in D_n}$, with

$$(x_{ijk} : y_{ijk}) = ((z_i - z_k)(z_{n+1} - z_j) : (z_i - z_j)(z_{n+1} - z_k)).$$

Remark A.3. In [GHv88], $\bar{M}_{0,n+1}$ is considered as a scheme over $\mathbb{Z}$, being defined as a subscheme of $\prod_{v\in V_{n+1}} \mathrm{Proj}(\mathbb{Z}[x_v, y_v])$. Our proof applies also to this more general setting.

Proof. Let us denote by Y_n the subvariety of $(\mathbb{P}^1)^{D_n}$ defined by these relations. We have an obvious map $f : \bar{M}_{0,n+1} \to Y_n$, the projection onto the cross ratios $\mu_{ijk} = (x_{i,n+1,j,k} : y_{i,n+1,j,k})$, with $(i, j, k) \in D_n$. We show that this map is an isomorphism by constructing the inverse map $\mu \mapsto \lambda = g(\mu)$. If one of i, j, k, l is equal to $n + 1$, λ_{ijkl} is obtained using (λ1) and (λ2) from

$$\lambda_{i,n+1,k,l} = \mu_{ikl}. \tag{A1}$$

For $(i, j, k, l) \in V_n$, λ_{ijkl} is given by either

$$\lambda_{ijkl} = \frac{\mu_{ikl}}{\mu_{jkl}} \tag{A2}$$

or

$$\lambda_{ijkl} = \frac{\mu_{kij}}{\mu_{lij}}, \tag{A3}$$

depending on which of the two expressions is defined (i.e. not $0/0$ or ∞/∞).[2]

We first check that (A1)–(A3) correctly define a map $Y_n \to (\mathbb{P}^1)^{V_{n+1}}$. First of all, ($\lambda$1) and ($\lambda$2) say that the map $\lambda : V_{n+1} \to \mathbb{P}^1$ is equivariant under the natural action of S_4 on V_{n+1} and on $\mathbb{P}^1$ by fractional linear transformations. Since, by (μ1) and (μ2), μ is S_3-equivariant, (A1) defines consistently λ_v for v in the S_4-orbit of $(i, n + 1, k, l)$. Next, we claim that at least one of the ratios in (A2) and (A3) is defined. Indeed, suppose that (A2) is not defined because $\mu_{ikl} = \mu_{jkl} = 0$. Then, by ($\mu$1) and ($\mu$2), $\mu_{kil} = 1$ and $\mu_{klj} = \mu_{kjl}^{-1} = (1 - \mu_{jkl})^{-1} = 1$. Thus, by ($\mu$3), $\mu_{kij} = 1$ and (A3) is defined. Similarly, if $\mu_{ikl} = \mu_{jkl} = \infty$, then $\mu_{lij} = 1$. The same arguments shows

[1] Following [GHv88], we denote the sets of distinct pairs, triples and quadruples by the initials Z, D and V of the corresponding German or Dutch numerals.

[2] The notation $x_3 = x_1/x_2$ for $x_i = (x_i' : x_i'') \in \mathbb{P}^1$ means $x_3'x_2'x_1'' = x_1'x_2''x_3''$; this defines x_3 given x_1 and x_2 unless x_1 and x_2 are both $0 = (0 : 1)$ or $\infty = (1 : 0)$.

that (A2) is defined if (A3) is not. It remains to show that if the right-hand sides of both (A2) and (A3) are defined then they are equal. The following identity is useful for this purpose.

LEMMA A.4. *Suppose that $(\mu_d)_{d\in D_n}$ obey $(\mu 1)$, $(\mu 2)$ and let $(i,j,k)\in D_n$. Then either $\mu_{ijk},\mu_{jki},\mu_{kij}$ are a permutation of $0,\infty,1$ or none of them belongs to $\{0,\infty,1\}$. In the latter case, their product is -1 or, equivalently,*

$$\mu_{ijk}=-\frac{\mu_{jik}}{\mu_{kij}}.$$

Indeed, if $\mu_{ijk}=x$, then $\mu_{jki}=1/\mu_{jik}=1/(1-x)$ and $\mu_{kij}=1-1/x$, which implies the lemma.

This lemma combined with $(\mu 1)$ and $(\mu 3)$ implies the following two identities (holding whenever the expressions are defined):

$$\frac{\mu_{ikl}}{\mu_{jkl}}=\frac{\mu_{ikj}\mu_{ijl}}{\mu_{jki}\mu_{jil}}=\frac{\mu_{kij}}{\mu_{lij}},\qquad \frac{\mu_{kij}}{\mu_{lij}}=\frac{\mu_{kil}\mu_{klj}}{\mu_{lik}\mu_{lkj}}=\frac{\mu_{ikl}}{\mu_{jkl}}. \tag{A4}$$

Assume that both right-hand sides of (A2) and (A3) are defined. We have four cases: (a) μ_{ikl}, $\mu_{jkl}\notin\{0,\infty,1\}$. Then the second identity in (A4) proves that (A2) and (A3) agree. (b) $\mu_{kij},\mu_{lij}\notin\{0,\infty,1\}$. Here the first identity implies the claim. (c) $\mu_{ikl}=0$. Since, by $(\mu 3)$, $\mu_{ikl}=\mu_{ikj}\mu_{ijl}$, we have either $\mu_{ikj}=0$ or $\mu_{ijl}=0$. In the first case, $\mu_{kij}=1$ and therefore $\mu_{kjl}=\mu_{kji}\mu_{kil}=1\cdot 1=1$, implying that $\mu_{jkl}=0$, in contradiction with the assumption that the right-hand side of (A2) is defined. In the second case, $\mu_{lij}=1-1/\mu_{ijl}=\infty$ and thus (A3) gives $\lambda_{ijkl}=0$, in agreement with (A2). The cases where any of $\mu_{ikl},\mu_{jkl},\mu_{kij},\mu_{lij}$ are 0 or ∞ are treated in the same way. (d) $\mu_{ikl}=1$. Then $\mu_{kil}=0$ and thus $\mu_{kij}\mu_{kjl}=0$. The case $\mu_{kij}=0$ is covered by (c), so let $\mu_{kjl}=0$, whence $\mu_{jkl}=1$. Equation (A2) gives then $\lambda_{ijkl}=1$. By $(\mu 3)$, $\mu_{ikj}=\mu_{ikl}\mu_{ilj}=1\cdot\mu_{ilj}$, $(\mu 1)$ implies that $\mu_{kij}=\mu_{lij}$ and thus also (A3) gives $\lambda_{ijkl}=1$. The remaining case $\mu_{kij}=\mu_{lij}=1$ is treated in the same way by exchanging i,j and k,l.

Thus, g is a well-defined morphism $Y_n\to\bar M_{0,n+1}$ and by construction $g\circ f$ is the identity.

It remains to show that $\lambda=g(\mu)$ obeys the relations $(\lambda 1)$–$(\lambda 3)$. The first relation $(\lambda 1)$ is obviously satisfied. The relation $(\lambda 2)$ is satisfied by construction if one of i,j,k,l is equal to $n+1$. Also, by construction, we have

$$\lambda_{ijkl}=\lambda_{klij} \tag{A5}$$

for all distinct i,j,k,l. If $(i,j,k,l)\in V_n$ and $\mu_{jli},\mu_{kli}\notin\{0,\infty,1\}$, $(\mu 1)$–$(\mu 3)$ imply that

$$\begin{aligned}\lambda_{jkli}=\frac{\mu_{jli}}{\mu_{kli}}=\frac{1-\mu_{lji}}{\mu_{kli}}=\frac{1-\mu_{ljk}\mu_{lki}}{\mu_{kli}}=\frac{1-(1-\mu_{jlk})\mu_{lki}}{1-\mu_{lki}}\\ =1+\frac{\mu_{jlk}\mu_{lki}}{1-\mu_{lki}}=1-\frac{\mu_{ikl}}{\mu_{jkl}}=1-\lambda_{ijkl}.\end{aligned}$$

We now consider the degenerate cases. (a) If $\mu_{kli}=0$, then $\mu_{lki}=1$ and thus $\mu_{lji}=\mu_{ljk}\mu_{lki}=\mu_{ljk}$ and therefore also $\mu_{jli}=\mu_{jlk}$. Either $\mu_{jli}\neq 0$, so that $\lambda_{jkli}=\infty$ and $\lambda_{ijkl}=\mu_{ikl}/\mu_{jkl}=\infty\cdot\mu_{jlk}=\infty\cdot\mu_{jli}=\infty$ proving the identity; or $\mu_{jli}=0$, so that the second formula (A3) applies and we have

$$\lambda_{jkli}=\frac{\mu_{ljk}}{\mu_{ijk}}=\frac{\mu_{lji}}{\mu_{ijk}}=\frac{1}{\mu_{ijk}}=\mu_{ikj}.$$

On the other hand, since $\mu_{lij}=1/(1-\mu_{jli})=1$,

$$1-\lambda_{ijkl}=1-\frac{\mu_{kij}}{\mu_{lij}}=1-\mu_{kij}=\mu_{ikj},$$

proving the claim. (b) If $\mu_{kli}=1$, then $\mu_{ikl}=0=\mu_{lki}$. Then either $\mu_{jkl}=0$ and (a) with permuted indices gives $\lambda_{ijkl} = 1 - \lambda_{lijk}$, which with (A5) implies the claim; or $\mu_{jkl} \neq 0$ and $\lambda_{ijkl} = \mu_{ikl}/\mu_{jkl} = 0$. In this case, $\lambda_{jkli} = \mu_{jli}/1 = 1 - \mu_{lji} = 1 - \mu_{ljk}\mu_{lki} = 1 - \mu_{ljk} \cdot 0 = 1$, since $\mu_{ljk} = 1 - 1/\mu_{jkl} \neq \infty$. (c) If $\mu_{kli} = \infty$, then $\mu_{ilk} = 1$ and we are in case (b) up to permutation of i and k, so that we get $\lambda_{jilk} = 1 - \lambda_{kjil}$, which reduces to the claim by using ($\lambda 1$) and (A5). Thus, the cases where the denominator μ_{kli} belongs to $\{0, \infty, 1\}$ are covered by (a)–(c). (d) The case where the numerator μ_{jli} is in $\{0, \infty, 1\}$ is reduced to the previous case by the substitutions $i \leftrightarrow k$, $j \leftrightarrow l$. Indeed, (a)–(c) give $\lambda_{lijk} = 1 - \lambda_{klij}$, which reduces to ($\lambda 2$) by applying (A5). This completes the proof of ($\lambda 2$).

Finally, if μ is generic, ($\lambda 3$) follows from ($\mu 3$):

$$\lambda_{ijkl}\lambda_{ijlm} = \frac{\mu_{ikl}}{\mu_{jkl}}\frac{\mu_{ilm}}{\mu_{jlm}} = \lambda_{ijkm}. \tag{A6}$$

This formula applies more generally if one or both λ_{ijkl} and λ_{ijlm} are given by (A2): the only tricky case is if the left-hand side of (A6) is $0 \cdot \infty$, but in this case there is nothing to prove (see the footnote on page 1474). If both factors are given by (A3), we have (trivially)

$$\lambda_{ijkl}\lambda_{ijlm} = \frac{\mu_{kij}}{\mu_{lij}}\frac{\mu_{lij}}{\mu_{mij}} = \lambda_{ijkm}.$$

The remaining case is when for one factor, say λ_{ijkl}, (A3) is not defined and for the other, say λ_{ijlm}, (A2) is not defined. Then

$$\lambda_{ijkl} = \frac{\mu_{ikl}}{\mu_{jkl}}, \quad \lambda_{ijlm} = \frac{\mu_{lij}}{\mu_{mij}}.$$

If (A3) for λ_{ijkl} is $0/0$, i.e. $\mu_{lij} = 0$ and $\mu_{kij} = 0$, we have that $\mu_{mij} \neq 0$ (since λ_{ijlm} is assumed to be defined by (A3)) and $\lambda_{ijlm} = 0/\mu_{mij} = 0$. Also, $\lambda_{ijkm} = \mu_{kij}/\mu_{mij}$ is defined and equal to zero and ($\lambda 3$) is obeyed. Similarly, in the case where (A3) is $\lambda_{ijkl} = \infty/\infty$, ($\lambda 3$) is obeyed since $\lambda_{ijlm} = \lambda_{ijkm} = \infty$. □

References

Ati00 M. Atiyah, *The geometry of classical particles*, Surveys in Differential Geometry, vol. VII (International Press, Somerville, MA, 2000), 1–15; MR 1919420(2003h:55022).

Ati01 M. Atiyah, *Equivariant cohomology and representations of the symmetric group*, Chin. Ann. Math. Ser. B **22** (2001), 23–30, doi: 10.1142/S0252959901000048; MR 1823127(2002b:20017).

AB02 M. Atiyah and R. Bielawski, *Nahm's equations, configuration spaces and flag manifolds*, Bull. Braz. Math. Soc. (N.S.) **33** (2002), 157–176; MR 1940347(2004f:53024).

CFR09 A. Chervov, G. Falqui and L. Rybnikov, *Limits of Gaudin systems: classical and quantum cases*, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 029, 17; MR 2506183(2010f:82024).

CFR10 A. Chervov, G. Falqui and L. Rybnikov, *Limits of Gaudin algebras, quantization of bending flows, Jucys–Murphy elements and Gelfand–Tsetlin bases*, Lett. Math. Phys. **91** (2010), 129–150, doi: 10.1007/s11005-010-0371-y; MR 2586869.

Del70 P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, vol. 163 (Springer, Berlin, 1970), iii+133 (in French); MR 0417174(54#5232).

Dri90 V. G. Drinfel'd, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* Gal($\overline{\mathbf{Q}}/\mathbf{Q}$), Algebra i Analiz **2** (1990), 149–181 (in Russian); English transl., Leningrad Math. J., **2** (1991) 829–860; MR 1080203(92f:16047).

FM03 G. Falqui and F. Musso, *Gaudin models and bending flows: a geometrical point of view*, J. Phys. A **36** (2003), 11655–11676, doi: 10.1088/0305-4470/36/46/009; MR 2025867(2004i:37121).

FFR10 B. Feigin, E. Frenkel and L. Rybnikov, *Opers with irregular singularity and spectra of the shift of argument subalgebra*, Duke Math. J. **155** (2010), 337–363, doi: 10.1215/00127094-2010-057.

FFT10 B. Feigin, E. Frenkel and V. Toledano Laredo, *Gaudin models with irregular singularities*, Adv. Math. **223** (2010), 873–948, doi: 10.1016/j.aim.2009.09.007; MR 2565552.

Fre05 E. Frenkel, *Gaudin model and opers*, in *Infinite dimensional algebras and quantum integrable systems*, Progress in Mathematics, vol. 237 (Birkhäuser, Basel, 2005), 1–58; MR 2160841(2007e:17022).

Gau76 M. Gaudin, *Diagonalisation d'une classe d'Hamiltoniens de spin*, J. Phys. **37** (1976), 1089–1098; (in French, with English summary); MR 0421442(54#9446).

Gau83 M. Gaudin, *La fonction d'onde de Bethe*, in *Collection du Commissariat à l'Énergie Atomique: série Scientifique [Collection of the atomic energy commission: science series]* (Masson, Paris, 1983), xvi+331 (in French); MR 693905(85h:82001).

GHv88 L. Gerritzen, F. Herrlich and M. van der Put, *Stable n-pointed trees of projective lines*, Indag. Math. (Proc.) **91** (1988), 131–163; MR 952512(89i:14005).

KM96 M. Kapovich and J. Millson, *The symplectic geometry of polygons in Euclidean space*, J. Differential Geom. **44** (1996), 479–513; MR 1816048(2001m:53159).

Kap93 M. M. Kapranov, *The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation*, J. Pure Appl. Algebra **85** (1993), 119–142, doi: 10.1016/0022-4049(93)90049-Y; MR 1207505(94b:52017).

Kee92 S. Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. **330** (1992), 545–574; MR 1034665(92f:14003).

Knu83 F. F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), 161–199; MR 702953(85d:14038a).

Koh85 T. Kohno, *Série de Poincaré–Koszul associée aux groupes de tresses pures*, Invent. Math. **82** (1985), 57–75 (in French), doi: 10.1007/BF01394779; MR 808109(87c:32015a).

Man76 S. V. Manakov, *A remark on the integration of the Eulerian equations of the dynamics of an n-dimensional rigid body*, Funksional. Anal. i Priložen. **10** (1976), 93–94 (in Russian); MR 0455031(56#13272).

MTV09 E. Mukhin, V. Tarasov and A. Varchenko, *Schubert calculus and representations of the general linear group*, J. Amer. Math. Soc. **22** (2009), 909–940, doi: 10.1090/S0894-0347-09-00640-7; MR 2525775.

MTVa E. Mukhin, V. Tarasov and A. Varchenko, *Three sides of the geometric Langlands correspondence for gl_N Gaudin model and Bethe vector averaging maps*, available at arxiv.org/abs/0907.3266.

MTVb E. Mukhin, V. Tarasov and A. Varchenko, *Bethe algebra of the gl_{N+1} Gaudin model and algebra of functions on the critical set of the master function*, available at arxiv.org/abs/0910.4690.

OV96 A. Okounkov and A. Vershik, *A new approach to representation theory of symmetric groups*, Selecta Math. (N.S.) **2** (1996), 581–605, doi: 10.1007/PL00001384; MR 1443185(99g:20024).

Red34 L. Rédei, *Ein kombinatorischer Satz*, Acta Litteraria Szeged **7** (1934), 3943.

Uen08 K. Ueno, *Conformal field theory with gauge symmetry*, Fields Institute Monographs, vol. 24 (American Mathematical Society, Providence, RI, 2008); MR 2433154(2009k:81204).

VO04 A. M. Vershik and A. Yu. Okun'kov, *A new approach to representation theory of symmetric groups. II*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **307** (2004), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. **10**, 57–98, 281 (in Russian, with English and Russian summaries); English transl., J. Math. Sci. (N. Y.), **131**, 2005, 5471–5494; MR 2050688(2005c:20024).

Vin90 E. B. Vinberg, *Some commutative subalgebras of a universal enveloping algebra*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), 3–25, 221, (in Russian); English transl., Math. USSR-Izv. **36** (1991) 1–22; MR 1044045(91b:17015).

Leonardo Aguirre leonardo.aguirre@math.ethz.ch
Department of Mathematics, ETH Zurich, 8092 Zurich, Switzerland

Giovanni Felder giovanni.felder@math.ethz.ch
Department of Mathematics, ETH Zurich, 8092 Zurich, Switzerland

Alexander P. Veselov a.p.veselov@lboro.ac.uk
Department of Mathematical Sciences, Loughborough University,
Loughborough LE11 3TU, UK
and
Moscow State University, Moscow 119899, Russia