SOME PROPERTIES OF THE EXOTIC MULTIPLICATIONS ON THE THREE-SPHERE

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[Received 14 June 1968]

1. Introduction

IT is well known that up to homotopy the three-sphere S^3 admits twelve distinct multiplications or H-space structures [see (**2**), or Lemma 2]. In this note we establish some properties of these different multiplications on S^3 by elementary group-theoretic arguments.

We work with path-connected topological spaces with base point and homotopy classes of base-point-preserving maps. As usual $[A, B]$ stands for the collection of such homotopy classes of A into B. For convenience we do not distinguish notationally between a map and its homotopy class. If (G, m) and (H, n) are two H-spaces then a map $f\colon (G, m) \to (H, n)$ is called an *H-map* if $n(f\times f) = fm \in [G\times G, H]$. If the H-space (G, m) has a two-sided homotopy inverse then a (two-fold) *commutator map* $\phi\colon G\times G \to G$ can be defined by $\phi(x, y) = (xy)(x^{-1}y^{-1})$, where the multiplication is denoted by juxtaposition and the homotopy inverse by the exponent '-1'. A *k-fold commutator map* $\phi_k\colon G^k \to G$ is then inductively defined by $\phi_1 =$ the identity map 1, $\phi_2 = \phi$, and

$$\phi_k = \phi(\phi_{k-1}\times 1).$$

The *homotopical nilpotency* of (G, m) is said to be equal to l if ϕ_{l+1} is nullhomotopic and ϕ_l is not. We write $\mathrm{nil}(G, m) = l$ for $\phi_l \neq 0$ and $\phi_{l+1} = 0$, where 0 is the constant map. In the present paper we determine the H-maps and the homotopical nilpotency of S^3 for all possible multiplications.

If m_0 is the standard multiplication on S^3 then it is shown in Lemma 2 that the twelve multiplications on S^3 can be written as

$$m_t = m_0\phi^t \in [S^3\times S^3,\ S^3].$$

Here $t = 0, 1,\ldots, 11$, ϕ denotes the commutator map, and the exponent and juxtaposition are taken with respect to the group structure in $[S^3\times S^3, S^3]$ induced by m_0. Finally we denote by N the map of S^3 into itself of degree N.

† The first author was supported by a Dartmouth College Faculty Fellowship and the second author partially by the National Science Foundation (NSF-GP5874).

THEOREM A. *The map* N: $(S^3, m_r) \to (S^3, m_t)$ *of degree* N *is an* H*-map if and only if* $N^2(2t+1) \equiv N(2r+1) \bmod 24$.

We also prove

THEOREM B. *If* $r = 1, 4, 7,$ *or* 10, *then* $\mathrm{nil}(S^3, m_r) = 2$. *If* $r = 0, 2, 3, 5, 6, 8, 9,$ *or* 11, *then* $\mathrm{nil}(S^3, m_r) = 3$.

In the case of the standard multiplication ($r = t = 0$) Theorem A was proved by us in (**1**). S. Y. Husseini informs us that he and J. Stasheff have obtained Theorem A by different methods (unpublished). Theorem B was proved by G. J. Porter for m_0 in (**5**). Some of the steps in the proof of Theorem B can be found in (**3**), 175–6 and (**4**), § 10, where they are established in another way.

Our method of proof consists of examining the group-theoretic properties of $[S^3 \times S^3, S^3]$. With this method it is also possible to retrieve rather easily some known theorems on the multiplications of S^3 (see Remark 1). A similar discussion of the group $[S^7 \times S^7, S^7]$ leads to results on the multiplications of S^7. It would be interesting in our opinion to extend our theorems to other H-spaces.

2. Proof of Theorem A

We write S for S^3 and consider the set $[A, S]$ for any space A as a group, the group operation being induced from the standard multiplication m_0 of S. We write this group multiplicatively even though we denote the identity of the group by 0. The commutator of two elements a and b is $(a, b) = aba^{-1}b^{-1}$.

LEMMA 1. *The group* $[S \times S, S]$ *is nilpotent of class* $\leqslant 2$, *and so for any* $a, b \in [S \times S, S]$ *and integer* $n > 0$,

$$(ab)^n = (b, a)^{\frac{1}{2}n(n-1)} a^n b^n.$$

Proof. The first assertion is a result of G. W. Whitehead [see (**6**), Lemma 2.14 and § 3], and the second is an identity which is known to hold in a group of nilpotency class $\leqslant 2$.

LEMMA 2. *The twelve homotopy classes of multiplications on* S *can be written as* $m_t = m_0 \phi^t \in [S \times S, S]$, $t = 0, \ldots, 11$, *where* $\phi \in [S \times S, S]$ *is the commutator map. Moreover,* ϕ *has order* 12.

Proof. Consider the cofibre sequence

$$S \vee S \xrightarrow{j} S \times S \xrightarrow{q} S \wedge S \approx S^6.$$

This gives a long exact sequence

$$\longrightarrow [\Sigma(S\times S), S] \xrightarrow{\Sigma j^*} [\Sigma(S\vee S), S] \longrightarrow [S^6, S] \xrightarrow{q^*} [S\times S, S] \xrightarrow{j^*} [S\vee S, S],$$

where Σ denotes reduced suspension. It is well known and easily proved that Σj^* and j^* are epimorphisms. Thus we get a short exact sequence of groups and homomorphisms

$$0 \longrightarrow \pi_6(S) \xrightarrow{q^*} [S\times S, S] \xrightarrow{j^*} \pi_3(S)\oplus\pi_3(S) \longrightarrow 0.$$

A multiplication m on S is an element of $[S\times S, S]$ such that

$$j^*(m) = \mathbf{1}\oplus\mathbf{1},$$

where $\mathbf{1}\in\pi_3(S)$ is the homotopy class of the identity map. Thus the collection of multiplications is just the coset of $\operatorname{Ker} j^*$ which contains m_0. Hence it suffices to show that $\operatorname{Im} q^*$ is a cyclic group of order 12 generated by ϕ. Now it is known that $\pi_6(S)$ is a cyclic group of order 12 generated by $\langle\mathbf{1},\mathbf{1}\rangle$, the Samelson product of $\mathbf{1}\in\pi_3(S)$ with itself [see (**3**) 176]. But by the definition of the Samelson product $q^*\langle\mathbf{1},\mathbf{1}\rangle = \phi$. This completes the proof of the lemma.

Proof proper of Theorem A. First note that $m_0 = p_1p_2$ and $\phi = (p_1, p_2)$ in $[S\times S, S]$ where $p_i\in[S\times S, S]$ is the projection on to the ith factor. Assume that $N>0$. Then

$$\begin{aligned} m_t\circ(\mathrm{N}\times\mathrm{N}) &= (p_1p_2\phi^t)\circ(\mathrm{N}\times\mathrm{N})\\ &= (\mathrm{N}p_1)(\mathrm{N}p_2)(\mathrm{N}p_1,\mathrm{N}p_2)^t\\ &= p_1^Np_2^N(p_1,p_2)^{N^2t}\end{aligned}$$

since the commutator is biadditive in a group of nilpotency $\leqslant 2$. On the other hand,

$$\begin{aligned}\mathrm{N}\circ m_r &= (p_1p_2\phi^r)^N\\ &= (\phi^r, p_1p_2)^{\frac12N(N-1)}(p_1p_2)^N\phi^{rN}, \quad\text{by Lemma 1,}\\ &= (p_1p_2)^N\phi^{rN}\quad\text{since } (\phi^r, p_1p_2) = 0\\ &= (p_2,p_1)^{\frac12N(N-1)}p_1^Np_2^N\phi^{rN},\quad\text{by Lemma 1,}\\ &= p_1^Np_2^N(p_1,p_2)^{rN-\frac12N(N-1)}.\end{aligned}$$

Thus N is an H-map if and only if $\phi^{N^2t} = \phi^{rN-\frac12N(N-1)}$. Since ϕ has order 12 by Lemma 2, this is the case if and only if

$$N^2(2t+1)\equiv N(2r+1) \bmod 24.$$

If $N<0$ write $N = (-N)(-1)$ and repeat the above argument.

3. Proof of Theorem B

We begin with a simple lemma on homotopy inverses.

LEMMA 3. *The map* $-\mathbf{1} \in [S, S]$ *of degree* -1 *is a two-sided identity for every multiplication on* S.

Proof. A left inverse $\nu_l \in [S, S]$ for m_l is characterized by the equation $\nu_l . \mathbf{1} = 0$ in $[S, S]$, where the dot '.' is the operation in $[S, S]$ induced by m_l. But this operation coincides with the usual group operation in the homotopy group $[S, S] = \pi_3(S)$. Thus $\nu_l = -\mathbf{1}$ is the unique solution of the equation. Similarly the right inverne for m_l is $-\mathbf{1}$.

Next we introduce the following notation: $\phi_k^{(r)}$ is the k-fold commutator map with respect to the multiplication m_r. It is well known that each $\phi_k^{(r)}$ induces a unique homotopy class $\psi_k^{(r)}$: $\wedge^k S \to S$ with $\psi_k^{(r)} \circ q_k = \phi_k^{(r)}$, where $\wedge^k S$ is the smashed product $S \wedge \ldots \wedge S$ (k times) $\approx S^{3k}$ and q_k is the projection of $S \times \ldots \times S$ on to $\wedge^k S$. For $\alpha \in \pi_p(S)$ and $\beta \in \pi_q(S)$ denote the Samelson product relative to m_r by $\langle \alpha, \beta \rangle_r \in \pi_{p+q}(S)$. We write ϕ_k for $\phi_k^{(0)}$, $\phi^{(r)}$ for $\phi_2^{(r)}$, ψ_k for $\psi_k^{(0)}$, $\psi^{(r)}$ for $\psi_2^{(r)}$, and $\langle \alpha, \beta \rangle$ for $\langle \alpha, \beta \rangle_0$.

LEMMA 4. (*a*) $\phi^{(r)} = \phi^{2r+1} \in [S \times S, S]$;

(*b*) $\psi^{(r)} = \psi^{2r+1} \in [S^6, S]$;

(*c*) $\langle \alpha, \beta \rangle_r = (2r+1) \langle \alpha, \beta \rangle$ *for* $\alpha \in \pi_p(S)$ *and* $\beta \in \pi_q(S)$.

Proof. Clearly (*b*) implies (*c*) since $\langle \alpha, \beta \rangle_r = \psi^{(r)}(\alpha \wedge \beta)$. Also (*b*) is an immediate consequence of (*a*). Thus it suffices to prove (*a*): Write '.' for the operation in $[S \times S, S]$ obtained from m_r and use juxtaposition for the standard multiplication in $[S \times S, S]$. Then, by Lemma 3,

$$\phi^{(r)} = (p_1 \cdot p_2) \cdot (p_1^{-1} \cdot p_2^{-1}),$$

where p_1 and p_2 are the two projections $S \times S \to S$ and the exponent '-1' denotes the inverse in $[S \times S, S]$ with respect to the standard multiplication in $[S \times S, S]$. Then

$$\begin{aligned} \phi^{(r)} &= p_1 p_2 (p_1, p_2)^r \cdot p_1^{-1} p_2^{-1} (p_1^{-1}, p_2^{-1})^r \\ &= p_1 p_2 \phi^r p_1^{-1} p_2^{-1} \phi^r (p_1 p_2 \phi^r, p_1^{-1} p_2^{-1} \phi^r)^r \\ &= \phi^{2r+1} (p_1 p_2 \phi^r, p_1^{-1} p_2^{-1} \phi^r)^r. \end{aligned}$$

But by repeatedly using the biadditivity of commutators in the group $[S \times S, S]$ of nilpotency class $\leqslant 2$ one easily sees that

$$(p_1 p_2 \phi^r, p_1^{-1} p_2^{-1} \phi^r) = 0.$$

This completes the proof.

Proof proper of Theorem B. We first note that $\phi_k^{(r)} = 0$ if and only if $\psi_k^{(r)} = 0$. Now $\psi_2^{(r)} = \psi^{(r)} = \langle \mathbf{1}, \mathbf{1}\rangle_r = (2r+1)\langle \mathbf{1}, \mathbf{1}\rangle$ for $\mathbf{1} \in \pi_3(S)$. But $\langle \mathbf{1}, \mathbf{1}\rangle \in \pi_6(S)$ is an element of order 12. Therefore,

$$\psi_2^{(r)} \neq 0 \quad \text{for all } r.$$

Next,

$$\begin{aligned} \psi_3^{(r)} &= \langle\langle \mathbf{1}, \mathbf{1}\rangle_r, \mathbf{1}\rangle_r \\ &= (2r+1)\langle(2r+1)\langle \mathbf{1}, \mathbf{1}\rangle, \mathbf{1}\rangle \\ &= (2r+1)^2\langle\langle \mathbf{1}, \mathbf{1}\rangle, \mathbf{1}\rangle. \end{aligned}$$

But $\langle\langle \mathbf{1}, \mathbf{1}\rangle, \mathbf{1}\rangle \in \pi_9(S^3) \approx Z_3$ is an element of order 3 [see for instance (**2**) § 3]. Therefore $\psi_3^{(r)}$ (and consequently $\phi_3^{(r)}$) is trivial if and only if $r = 1$, 4, 7, or 10. Thus $\mathrm{nil}(S, m_r) = 2$ for $r = 1$, 4, 7, or 10. Finally note that $\langle\langle\langle \mathbf{1}, \mathbf{1}\rangle_r, \mathbf{1}\rangle_r, \mathbf{1}\rangle_r$ is a multiple of $\langle\langle\langle \mathbf{1}, \mathbf{1}\rangle, \mathbf{1}\rangle, \mathbf{1}\rangle$. To complete the proof of Theorem B it suffices to show that this latter Samelson product is trivial.

LEMMA 5. *If* $\mathbf{1} \in \pi_3(S)$ *is the homotopy class of the identity map then* $\langle\langle\langle \mathbf{1}, \mathbf{1}\rangle, \mathbf{1}\rangle, \mathbf{1}\rangle = 0 \in \pi_{12}(S)$.

Proof. Let $\theta = \langle\langle\langle \mathbf{1}, \mathbf{1}\rangle, \mathbf{1}\rangle, \mathbf{1}\rangle$ and $\eta = \langle\langle \mathbf{1}, \mathbf{1}\rangle, \langle \mathbf{1}, \mathbf{1}\rangle\rangle$. First note that $3\langle\langle \mathbf{1}, \mathbf{1}\rangle, \mathbf{1}\rangle = 0$ by the Jacobi identity and so $3\langle\langle \alpha, \beta\rangle, \gamma\rangle = 0$ for any elements α, β, γ. Thus $3\eta = 0$ and $3\theta = 0$. But $\eta = -\eta$ by anti-commutativity, and so $2\eta = 0$. Hence it follows that $\eta = 0$. Now apply the Jacobi identity to the elements $\langle \mathbf{1}, \mathbf{1}\rangle$, $\mathbf{1}$, and $\mathbf{1}$ to obtain $2\theta + \eta = 0$. Thus $2\theta = 0$. But we observed earlier that $3\theta = 0$. Therefore $\theta = 0$.

Remark 1. It is possible with our methods to retrieve some known theorems on the multiplications of S. We can prove that all multiplications on S satisfy the Moufang identity [see (**4**) § 9] and can determine which of them are homotopy-associative [see (**2**) Theorem 1.3]. These results require an examination of the group $[S \times S \times S, S]$, which is nilpotent of class $\leqslant 3$. For example, m_r is homotopy-associative if and only if $q_1 \cdot (q_2 \cdot q_3) = (q_1 \cdot q_2) \cdot q_3$ in $[S \times S \times S, S]$, where q_i is the projection on to the ith factor. This equation, together with facts on commutators in groups of nilpotency class $\leqslant 3$, reduces to a simple identity in $[S \times S \times S, S]$ in terms of m_0. We note that the multiplications on S which are homotopy-associative are precisely those which in Theorem B were shown to be of homotopical nilpotency 3.

Remark 2. The definition of homotopical nilpotency for a non-homotopy-associative H-space depended on the bracketing in the definition of a commutator map. It follows from our proof of Theorem B that the homotopical nilpotency with respect to one bracketing is the same as that with respect to any other bracketing of the commutator.

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