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Minimality and ergodicity of a generic analytic foliation of $\mathbb{C}^2$

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Abstract. It is well known that a generic polynomial foliation of $\mathbb{C}^2$ is minimal and ergodic. In this paper we prove an analogous result for analytic foliations.

1. *Introduction*

This article is devoted to the study of analytic foliations of $\mathbb{C}^2$. Recall that (by the Oka–Cartan theory) any analytic foliation of $\mathbb{C}^2$ is a foliation by (complex-time) trajectories of some analytic vector field

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y), \end{cases} \tag{1}$$

where $f, g \in \mathcal{O}(\mathbb{C}^2)$, and the functions f and g have no common factor (so that the singularities are isolated).

Our main result is the following theorem, which ensures that the behavior of leaves of a *generic* analytic foliation is in some sense 'chaotic'. Its statement naturally extends the statements of analogous results that are already known for *polynomial* foliations of $\mathbb{C}^2$ and $\mathbb{C}P^2$; we will give a brief review of these results in §1.1.

THEOREM. (Main result) *A generic analytic foliation of* $\mathbb{C}^2$ *is minimal and ergodic.*

We shall recall the definitions of minimality and ergodicity in §2. The genericity here is understood as follows. The space of analytic foliations of $\mathbb{C}^2$ can be equipped with a natural (Baire) topology 'of uniform convergence on non-singular compacts'; that is, two foliations are said to be close if their tangent direction fields are uniformly close in a ball of large radius except for small neighborhoods of singular points (the precise definition will be given in §2). Now, in our result, we interpret the genericity in a topological way: the set of foliations that we construct is a *residual* one (i.e. a countable intersection of open dense sets).

One can also work directly with vector fields instead of foliations. Indeed, an analytic vector field is given by a pair of functions that are holomorphic in $\mathbb{C}^2$. Their space is therefore naturally equipped with the topology of uniform convergence on compacts, and we can then use the topological interpretation of genericity (residual set) for vector fields. In fact, the arguments used in the proof of the main result also lead to the following.

THEOREM. *The foliation, corresponding to a generic analytic vector field in $\mathbb{C}^2$, is minimal and ergodic.*

We do not show the proof for this case, as it is identical to that of the previous (main) theorem.

Note that one cannot replace 'residual' by 'open dense' in these theorems, owing to the following example communicated to us by D. Novikov.

Example 1. (D. Novikov) There exists a dense set of analytic foliations of the complex plane $\mathbb{C}^2$, with each of these foliations possessing a non-dense leaf.

We present Novikov's construction in §7.

1.1. *Background.* Polynomial foliations of $\mathbb{C}^2$ (that is, foliations by complex-time trajectories of a *polynomial* vector field) have been intensively studied since the times of Poincaré. Among other reasons, the interest of mathematicians in this topic has been motivated by its close relation to Hilbert's 16th problem. While we cannot present here all the known results on the topic, we will recall a few of them that are most relevant to the subject of this paper.

The following result, describing the 'chaotic' behavior of a typical polynomial foliation, is due to Khudai-Verenov [**11**], Ilyashenko [**8, 9**], Shcherbakov [**15**] and Nakai [**13**].

THEOREM. (Khudai-Verenov, Ilyashenko, Shcherbakov, Nakai) *For any given degree $d \geq 2$, a typical polynomial foliation of $\mathbb{C}^2$ of this degree is minimal and ergodic, as well as topologically rigid.*

The genericity here (in the strongest version of this theorem, which is due to Shcherbakov) is understood in an algebraic sense. Specifically, the statement holds for foliations belonging to an intersection of a complement of a nowhere-dense real-analytic subset of codimension no less than two and of a complement of a real algebraic subset of codimension no less than one in the space of degree d foliations. (Formally speaking, in the paper [**15**] ergodicity is not mentioned; however, it can be obtained by the same arguments as those used by Ilyashenko in [**8, 9**] or by Loray and Rebelo in [**12**].)

A natural related topic is the study of polynomial foliations of $\mathbb{C}P^n$. However, the notion of degree is slightly different in this case: a foliation of $\mathbb{C}P^n$ has the *geometric* (or *projective*) degree d if, in *any* affine chart, it is given by a vector field of degree no more than d. For this definition, a generic polynomial vector field of degree d in $\mathbb{C}^n$ generates a degree $d+1$ foliation of $\mathbb{C}P^n$. Such foliations are untypical among all degree $d+1$ foliations: they are specified by the fact that the infinite hyperplane is invariant. Thus, the issues of studying given degree polynomial vector fields in $\mathbb{C}^n$ and given projective degree polynomial foliations of $\mathbb{C}P^n$ are different.

Concerning the latter problem, Loray and Rebelo [**12**] have recently obtained local genericity of these properties for polynomial foliations on $\mathbb{C}P^n$ of a given projective degree. They consider the space of (complex one-dimensional) foliations of $\mathbb{C}P^n$ of a given projective degree, and construct in this space a ('small') open set, each foliation from which is minimal and ergodic (as well as topologically rigid). At the moment, the question of *global* genericity of minimality, ergodicity and topological rigidity among *projective* foliations (that is, if such foliations form, for instance, a dense set) remains open.

The results stated above (for both polynomial and projective foliations) are based on study of the monodromy group at infinity. After projectivizing a typical polynomial foliation, one sees that the infinite line is tangent to the foliation, except for some number of singularities. So, because of these singularities, this line becomes a non-simply connected leaf with a rich fundamental group; thus one can hope that the corresponding monodromy group is also rich. Studying the properties of this group is the key idea in the results of Khudai-Verenov, Ilyashenko, Shcherbakov, and Nakai mentioned above. The result of Loray and Rebelo also contains the idea of a series of perturbations of foliations having an 'integrable flag' of invariant 'infinite' planes.

This article, as it was already said, is devoted to the study of *analytic* foliations of $\mathbb{C}^2$; investigation of their topological properties has begun only recently [**1, 3, 4**]. One of the difficulties is that the infinity-line approach (standard in the study of the polynomial case) cannot be applied directly: generic holomorphic functions have essential singularities all along the infinite line.

Our main result can be considered as an analogue, for the analytic foliations case, of the aforementioned theorems of Khudai-Verenov, Ilyashenko, Shcherbakov, Nakai, and Loray and Rebelo.

1.2. *Plan of the proof: idea of the construction.* In order to obtain the desired result, we prove that for any compact set $K \subset \mathbb{C}^2$, there exists an open and dense set $\mathcal{U}(K)$ of foliations, such that each foliation from $\mathcal{U}(K)$ is minimal and ergodic on K (see precise definitions below). The theorem will then follow from a countable exhaustion of the plane $\mathbb{C}^2$ by compact sets: the intersection of the constructed open dense sets gives us the desired residual set.

The construction of these sets is split into a few steps (up to some technical details such as handling the singularities), and the general plan of the proof goes as follows.

- *Black box.* Construct a 'black box', a 'mechanism' providing minimality and ergodicity on a small cross-section, in a way that is stable under small perturbations. We refer to this mechanism as a 'black box' by analogy with physics: once it is constructed, we are no longer interested in how it works.
- *Plug in.* Show that if every leaf from some compact set intersects a 'black box' cross-section, then the dynamics on this compact set is stably minimal and ergodic (see definitions below).
- *Open dense set.* Given a foliation and a compact set $K \subset \mathbb{C}^2$, construct a foliation, arbitrarily close to the initial one, which possesses a 'black box' cross-section and is minimal on all of $\mathbb{C}^2$. By virtue of the previous step, this implies that for any compact

set K there is an open and dense set $\mathcal{U}(K)$ of foliations, minimal and ergodic on this compact set.

- *Intersection.* By taking the intersection of the open dense sets corresponding to a countable family of compact sets, exhausting $\mathbb{C}^2$, we conclude the proof.

2. *Notation and definitions*

An analytic foliation of $\mathbb{C}^2$ is a foliation by complex-time trajectories of some analytic vector field

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y), \end{cases} \tag{2}$$

where the functions $f(x, y)$, $g(x, y)$ are analytic on $\mathbb{C}^2$ and have isolated common zeros. More precisely (see [**7**]), an analytic foliation on a complex manifold is *locally* given by a vector field of the form (2). The fields we use in different neighborhoods then differ (on the intersection of their domains) by multiplication by a non-zero holomorphic function. Therefore, we have a multiplicative cocycle. But, owing to the Oka–Cartan theory, as $\mathbb{C}^2$ has no corresponding homologies, any foliation of $\mathbb{C}^2$ is in fact given by a *global* vector field (and thus defined up to multiplication by a non-zero holomorphic in $\mathbb{C}^2$ function).

To describe a foliation one does not need the couple (f, g), but only the direction $[f : g]$ of the tangent (complex) directions to the leaves. So, a natural and standard way to define a topology on the space of foliations is the following one. A basis of neighborhoods of a foliation $\mathcal{F}$ is formed by

$$U_{R,\varepsilon,\delta} = \big\{\mathcal{G} \mid \mathcal{G} \text{ is non-singular in } K_{\varepsilon,R} := B_R(0, 0) \setminus U_\varepsilon(\mathrm{Sing}(\mathcal{F})) \\ \text{and the tangent direction fields of } \mathcal{F} \text{ and } \mathcal{G} \text{ are } \delta\text{-close on } K_{\varepsilon,R}\big\},$$

where $\varepsilon, \delta > 0$, $0 < R < \infty$, and $B_R(0, 0)$ and $\mathrm{Sing}(\mathcal{F})$ stand, respectively, for the ball of radius R centered at the origin and for the set of singular points of $\mathcal{F}$.

We call a property *typical* for analytic foliations if it holds on a residual set in this space.

As mentioned already in the introduction, one can also work directly with vector fields generating foliations. An analytic vector field is given by a pair of functions holomorphic in $\mathbb{C}^2$ (that we suppose not to have a common factor for the singularities to be isolated), and the space of such pairs is naturally equipped with the (metrizable) topology of uniform convergence on compacts. It is easy to check that the map associating a foliation to a vector field is continuous in the sense of these topologies. In other words, close vector fields generate close foliations.

Let us recall the definitions of minimality and ergodicity of a foliation.

Definition 1. A (singular) foliation is called *minimal* if all its leaves are dense. A foliation is called *ergodic* if any measurable saturated set (i.e. one consisting only of entire leaves) has either zero or full Lebesgue measure.

A standard way to study a foliation is to consider its holonomy maps. We now recall how these maps are defined.

Definition 2. Let $\mathcal{F}$ be a foliation of $\mathbb{C}^2$, and let $\gamma : [0, s] \to \mathbb{C}^2$ be a path staying in the same leaf of $\mathcal{F}$. Let discs T_0 and T_1 be cross-sections to $\mathcal{F}$, passing through $\gamma(0)$

and $\gamma(s)$, respectively. Then, for an initial point x in T_0 sufficiently close to $\gamma(0)$, leafwise curves starting at x, staying close to γ and coming to T_1, arrive at a well-defined point of T_1—which is called the image of x under the holonomy along γ. This defines *the holonomy map* Δ_γ *along* γ, acting from a neighborhood of $\gamma(0)$ in T_0 to a neighborhood of $\gamma(s)$ in T_1.

Using the holonomy maps, we will reduce the study of a foliation to the study of a system of maps. We will therefore need the definitions of minimality and ergodicity for multiple-map dynamical systems. The following definitions (applicable to a non-invariant subset) are slight modifications of the standard ones.

Definition 3. Let $g_1, \ldots, g_s : D \to D$ be mappings of a compact set D to itself, and let $K \subset D$ be a compact set. A dynamical system $(D; g_1, \ldots, g_s)$ is called *minimal on* K if for any $x \in K$, the closure of its 'forward' orbit $G^+(x)$ contains K. Here $G^+(x) := \{(g_{j_1} \circ g_{j_2} \circ \cdots \circ g_{j_n})(x) \mid n \in \mathbb{N},\ j_1, \ldots, j_n \in \{1, \ldots, s\}\}$.

In order to give the definition of ergodicity, we need the following notion (which translates the notion of saturation for a set in a foliated space to the language of its transversal section).

Definition 4. A set $A \subset D$ is *saturated* for a system $(D; g_1, \ldots, g_s)$ if for any two points $x, y \in D$ such that $y = g_j(x)$ for some j, the point x belongs to A if and only if the point y does.

Now, the definitions of ergodicity and stable ergodicity are given in the same way.

Definition 5. Dynamical system $(D; g_1, \ldots, g_s)$ is called *ergodic on* $K \subset D$ if for any saturated subset $A \subset D$, its intersection with K is of either zero or full (in K) Lebesgue measure.

Definition 6. The system $(D; g_1, \ldots, g_s)$ is called *stably minimal* (respectively *stably ergodic*) *on* K if the system $(D; \widetilde{g}_1, \ldots, \widetilde{g}_s)$ is minimal (respectively ergodic) on K for any maps $\widetilde{g}_1, \ldots, \widetilde{g}_s : D \to D$ sufficiently close to the corresponding maps g_j.

3. *Auxiliary ('black box') construction*

In this section we recall two lemmas that give sufficient conditions for stable minimality and stable ergodicity of a multiple-map dynamical system. Once stated, they give us a tool to prove stable minimality and stable ergodicity on a subset for a foliation. The only thing we have to do is pass from the study of foliations to the study of holonomy maps on a cross-section (see Lemma 3). Both of these lemmas are common knowledge; however, as they are stated slightly differently from the way they appear in the papers cited, we provide their proofs in §8 ('Technical proofs') for completeness.

A small technical remark we would like to make is that, in these lemmas, multiple-map dynamical systems are considered; so in order to apply the lemmas, we have to use cross-sections and holonomy maps in such a way that all the considered holonomy maps are defined on the entire chosen cross-section and map it into itself.

The first lemma provides a sufficient condition for minimality in the context of multiple-map dynamical systems. To the best of our knowledge, it goes back to the work of Hutchinson [**5**].

LEMMA 1. *Suppose the maps $g_1, \ldots, g_s : D \to D$, which are diffeomorphisms on their images, are contracting on D and the images of interior points of some compact set $K \Subset D$ under these maps cover K: $\cup g_j(\overset{\circ}{K}) \supset K$. Then the system $(D; g_1, \ldots, g_s)$ is stably minimal on K.*

Its proof, as well as the proof of the next lemma, is postponed to §8.

Definition 7. If, for compact sets $D \supset K$ and maps g_j, the assumptions of Lemma 1 hold, we say that the system $(D; g_1, \ldots, g_s)$ possesses the property of *controlled contraction on K*.

The second (and also standard) lemma that we will need is one which helps to establish local ergodicity for multiple-map dynamical systems. The ideas it is based on go back to [**16, 17**]; in the case of group actions on the circle it can be found in [**14**, Proposition 4.3, Remark 4.6].

LEMMA 2. *Suppose that a system $(D, \{g_i\}_{i=1}^s)$ of conformal maps $\{g_i\}_{i=1}^s$ on a compact set $D \subset \mathbb{C}$ possesses on a subset $K \Subset D$ the property of controlled contraction. Then this system is stably ergodic on K.*

4. *Stable minimality in a compact domain under the 'black box' condition*

This section applies the preceding two lemmas to the case of foliations. Suppose that we are given an analytic foliation $\mathcal{F}_0$ and a compact subset X (of positive Lebesgue measure) which does not contain the singularities of $\mathcal{F}_0$. Consider a cross-section D to $\mathcal{F}_0$, and let $K \Subset D$ be a compact subset. Denote the interior of K by $\overset{\circ}{K}$. Then the following lemma holds.

LEMMA 3. *Suppose that there exist holonomy maps $g_1, \ldots, g_s : D \to D$ such that the system $(D; g_1, \ldots, g_s)$ possesses the controlled contraction property on K, and that any leaf passing through a point of X intersects $\overset{\circ}{K}$. Then all the foliations sufficiently close to $\mathcal{F}_0$ are minimal and ergodic on X.*

Proof. Note that for foliations sufficiently close to $\mathcal{F}_0$, the set D is still a cross-section. Let us prove that any leaf passing through a point of X will still intersect K. Indeed, for a $\mathcal{F}_0$-leafwise path from some point x to $\overset{\circ}{K}$, its small deformations provide $\mathcal{F}_0$-leafwise paths to $\overset{\circ}{K}$ from all points sufficiently close to x.

Thus, as K is a compact set, every point $x \in X$ hits the interior $\overset{\circ}{K}$ along a path in $\mathcal{F}_0$ of bounded length, which stays at a bounded-from-below distance from the singularities of $\mathcal{F}_0$. Hence, the same holds for sufficiently small perturbations of $\mathcal{F}_0$. Finally, small perturbations preserve the non-singularity of X. Therefore, the property that X is a non-singular compact set, every point of which is connected to some point of $\overset{\circ}{K}$ by a leafwise path (of bounded length), survives under small perturbations.

The maps g_j are holonomy maps along paths of bounded length, hence they depend continuously on a foliation, and thus the minimality on K is preserved by small perturbations by virtue of Lemma 1.

Any point $x \in X$ is connected by a leafwise path to a point of K. The dynamics on K is minimal, thus K is contained in the closure of the leaf L_x passing through the point x; so any leaf that intersects K is contained in $\bar{L}_x$. In particular, any point $y \in X$ belongs to $\bar{L}_x$ (because L_y intersects K). In other words, $\bar{L}_x \supset X$. As $x \in X$ is arbitrary, the foliation is minimal on X.

Ergodicity on X is automatically implied by Lemma 2. Let A be any saturated subset of $\mathbb{C}^2$ that intersects X in a set of positive (two-dimensional) Lebesgue measure. Then A intersects the cross-section K in a set A_K of positive Lebesgue measure on this cross-section. The dynamics on K is ergodic, so A_K is of full Lebesgue measure on the cross-section. Then $K \setminus A_K$ has zero Lebesgue measure on the cross-section and thus $X \setminus A$ has zero Lebesgue measure in $\mathbb{C}^2$. □

Remark 1. The 'intersection' condition of Lemma 1 automatically holds if the foliation $\mathcal{F}_0$ is minimal.

5. *Open dense set of foliations with a 'black box' cross-section*

In this part we will construct an open dense set of foliations with a 'black box' cross-section (a cross-section with controlled contraction). To do that, we construct one foliation with such a cross-section and then apply Viro gluing to obtain an open dense set.

LEMMA 4. *There exists a polynomial foliation $\mathcal{F}_0$ given by a vector field v_0,*

$$\begin{cases} \dot{x} = f_0(x, y) \\ \dot{y} = g_0(x, y), \end{cases}$$

with a cross-section D, subset $K \Subset D$ and holonomy maps $g_i : D \to D$ that possess the property of controlled contraction on K.

Proof. Consider the vector field of the form

$$\begin{cases} \dot{x} = P(x) \\ \dot{y} = \varepsilon y + Q(x), \end{cases}$$

where P, Q are polynomials and P has only prime roots:

$$P(x) = (x - a_1) \ldots (x - a_n), \quad a_j \neq a_k.$$

If $P(c) \neq 0$, then the unit disc $D_c = \{c\} \times \{y : |y| \leq 1\}$ is transversal to this foliation. The holonomy maps from the vertical plane $P_c = \{c\} \times \mathbb{C}$ onto itself by any path $\gamma \subset \mathbb{C} \setminus \{a_1, \ldots, a_n\}$ are affine:

$$g_{\gamma,\varepsilon}(y) = A_{\gamma,\varepsilon} y + B_{\gamma,\varepsilon}.$$

A direct computation provides the coefficient $A_{\gamma,\varepsilon}$:

$$A_{\gamma,\varepsilon} = \exp\left\{\varepsilon \int_\gamma \frac{1}{P(z)}\, dz\right\}.$$

On the other hand, it is clear that $B_{\gamma,\varepsilon} \xrightarrow[\varepsilon\to 0]{} B_{\gamma,0}$. From the differential equation

$$\begin{cases} \dot{x} = P(x) \\ \dot{y} = Q(x), \end{cases}$$

one can easily see that $B_{\gamma,0}$ is given by

$$B_{\gamma,0} = \int_{\gamma} \frac{Q(z)}{P(z)}\, dz.$$

Let us find the polynomial P such that the residues of $1/P$ in three of its roots ($x = 0, \ \pm 1$) are equal to i:

$$P(x) = i \cdot x(x-1)\,(x+1)\,(\tfrac{3}{2}x^2 - 1).$$

Now find a polynomial Q such that the residues of Q/P in these three roots are δ, $e^{2\pi i/3}\delta$ and $e^{4\pi i/3}\delta$, where δ is sufficiently small. For $\varepsilon = 0$, the images of the interior of the unit disc $K = D_c$ under the holonomy maps cover this disc. Hence this property holds also for sufficiently small ε.

Furthermore, for small real ε the corresponding coefficients are exponents of negative numbers (namely, of $2\pi i \cdot i\varepsilon = -2\pi\varepsilon$) and thus are less then one. Hence, there exists a sufficiently large transversal disc $D = D_R = \{(c, y) : |y| \le R\}$ such that the holonomy maps g_j are contracting maps of D into itself, i.e. $g_j(D) \subset D$. □

LEMMA 5. *The set of foliations for which there exist a cross-section D, subset $K \Subset D$ and holonomy maps $\{g_j\}$ satisfying the controlled contraction property is open and dense in the space of analytic foliations of $\mathbb{C}^2$.*

Proof. This set is clearly open, so we need to check that it is dense. For any given foliation $\mathcal{F}$, we will construct a foliation $\mathcal{G}$ close to $\mathcal{F}$ and having the desired property.

Let the foliation $\mathcal{F}$ be given by an analytic vector field w. First, let us approximate $\mathcal{F}$ by a polynomial foliation $\mathcal{G}_0$ by truncating the Taylor series of w.

In order to construct a foliation that is close to $\mathcal{G}_0$ and at the same time inherits some properties of the foliation $\mathcal{F}_0$ constructed in Lemma 4, we will apply Viro's gluing. This procedure, first invented by Viro [**18**] for construction of algebraic curves simultaneously similar to different ones, was then generalized by Itenberg and Shustin [**10**] to the vector fields case (one can also find an interesting application of this method in [**2**]). For completeness, here we repeat the arguments explicitly.

Recall that the foliation $\mathcal{F}_0$ is given by a polynomial vector field v_0. Consider a compact set X_0 without singular points of v_0, containing a transversal disc $K = D_c$ and the trajectories traced by this disc under the holonomy transformations along paths γ_j. Then for any vector field sufficiently close to v_0 on X_0, the corresponding foliation possesses the property of controlled contraction.

By shifting the origin, we may suppose that the projection of X_0 on the x-axis does not contain 0. Let

$$v_0(x, y) = P_0(x, y)\frac{\partial}{\partial x} + Q_0(x, y)\frac{\partial}{\partial y}.$$

Also, denote the vector field corresponding to $\mathcal{G}_0$ by

$$u(x, y) = P_1(x, y)\frac{\partial}{\partial x} + Q_1(x, y)\frac{\partial}{\partial y},$$

and let $N := \deg \mathcal{G}_0 = \max(\deg P_1, \deg Q_1)$.

Now, consider the family of vector fields

$$u_\varepsilon(x, y) = \varepsilon x^{N+3}\biggl(P_0(\varepsilon x, \varepsilon y)\frac{\partial}{\partial x} + Q_0(\varepsilon x, \varepsilon y)\frac{\partial}{\partial y}\biggr) + \underbrace{\biggl(P_1(x, y)\frac{\partial}{\partial x} + Q_1(x, y)\frac{\partial}{\partial y}\biggr)}_{u(x,y)}.$$

On one hand, as $\varepsilon \to 0$, this field tends to u. On the other hand, after a change of variables $\tilde{x} = \varepsilon x$, $\tilde{y} = \varepsilon y$, this family becomes

$$\begin{aligned}\tilde{u}_\varepsilon(\tilde{x}, \tilde{y}) &= \varepsilon\biggl(\frac{\tilde{x}}{\varepsilon}\biggr)^{N+3}\biggl[P_0(\tilde{x}, \tilde{y})\varepsilon\frac{\partial}{\partial \tilde{x}} + Q_0(\tilde{x}, \tilde{y})\varepsilon\frac{\partial}{\partial \tilde{y}}\biggr] \\ &\quad + \biggl[P_1\biggl(\frac{\tilde{x}}{\varepsilon}, \frac{\tilde{y}}{\varepsilon}\biggr)\varepsilon\frac{\partial}{\partial \tilde{x}} + Q_1\biggl(\frac{\tilde{x}}{\varepsilon}, \frac{\tilde{y}}{\varepsilon}\biggr)\varepsilon\frac{\partial}{\partial \tilde{y}}\biggr] \\ &= \frac{1}{\varepsilon^{N+1}}\biggl[\tilde{x}^{N+3}\underbrace{\biggl(P_0(\tilde{x}, \tilde{y})\frac{\partial}{\partial \tilde{x}} + Q_0(\tilde{x}, \tilde{y})\frac{\partial}{\partial \tilde{y}}\biggr)}_{v_0(\tilde{x},\tilde{y})} \\ &\quad + \biggl(\varepsilon^{N+2}P_1\biggl(\frac{\tilde{x}}{\varepsilon}, \frac{\tilde{y}}{\varepsilon}\biggr)\frac{\partial}{\partial \tilde{x}} + \varepsilon^{N+2}Q_1\biggl(\frac{\tilde{x}}{\varepsilon}, \frac{\tilde{y}}{\varepsilon}\biggr)\frac{\partial}{\partial \tilde{x}}\biggr)\biggr].\end{aligned}$$

As $\deg P_1, \deg Q_1 \le N$, we have

$$\varepsilon^{N+2}P_1\biggl(\frac{\tilde{x}}{\varepsilon}, \frac{\tilde{y}}{\varepsilon}\biggr) \to 0, \quad \varepsilon^{N+2}Q_1\biggl(\frac{\tilde{x}}{\varepsilon}, \frac{\tilde{y}}{\varepsilon}\biggr) \to 0,$$

and hence the vector field $\varepsilon^{N+1}\tilde{u}_\varepsilon$ tends to $v_1(\tilde{x}, \tilde{y}) := \tilde{x}^{N+3}v_0(\tilde{x}, \tilde{y})$.

On the compact set X_0, vector fields v_0 and v_1 are proportional with a non-zero coefficient (the projection of X_0 on the x-axis does not intersect zero). Thus, for sufficiently small ε, the foliation given by u_ε in the coordinates $(\tilde{x}, \tilde{y})$ is sufficiently close on X_0 to the foliation given by the vector field v_0. Hence, for any sufficiently small ε, the foliation u_ε possesses a triple (D, K, g_i) with the controlled contraction property. We have therefore constructed a foliation, possessing the controlled contraction property, which is arbitrarily close to the foliation $\mathcal{G}_0$ and thus arbitrarily close to the foliation $\mathcal{F}$. □

6. *Construction of the desired residual set*

To complete the proof of the main result, it suffices to launch an exhaustion procedure. Some technical precautions are required, however, because in previous steps we have worked with non-singular compact sets, and the placement of singularities depends on the foliation; so we need the following definition.

Definition 8. A foliation $\mathcal{F}$ is called *ε-good* if it is minimal and ergodic on the compact set X_ε, defined as the ball of radius $1/\varepsilon$ centered at the origin, from which ε-neighborhoods of the singular points of $\mathcal{F}$ have been removed.

LEMMA 6. *The set of ε-good foliations contains an open dense set.*

Proof. By Lemma 5, for any foliation $\mathcal{F}$ there exists a close polynomial foliation $\mathcal{G}_1$ which has a cross-section with controlled contraction. By the theorem of Khudai-Verenov, Ilyashenko, Shcherbakov and Nakai stated in the introduction, there is a small perturbation $\mathcal{G}_2$ of the foliation $\mathcal{G}_1$ in the space of polynomial foliations of a fixed degree that is minimal in the whole of $\mathbb{C}^2$. In particular, it will be minimal on $X_{\varepsilon/2}$. By Lemma 3, any foliation $\mathcal{G}'$ sufficiently close to $\mathcal{G}_2$ (including $\mathcal{G}_2$ itself) is minimal and ergodic on $X_{\varepsilon/2}(\mathcal{G}_2) \supset X_\varepsilon(\mathcal{G}')$—hence all the sufficiently close foliations are ε-good. □

We are now ready to conclude the proof of our theorem.

Proof of the main result. Note that a foliation of $\mathbb{C}^2$ is minimal and ergodic if and only if it is ε-good for any $\varepsilon > 0$. By Lemma 6, for all $\varepsilon > 0$ the set of ε-good analytic foliations is open and dense. So, taking the sequence $\varepsilon_n = 1/n \to 0$ and intersecting the corresponding open and dense sets, we obtain the desired residual set. □

7. *D. Novikov's example*

In this section we describe a construction of a dense set of foliations that are not minimal. Moreover, these foliations have an invariant complex line. This example is due to D. Novikov (private communication).

Consider a foliation $\mathcal{F}$ and let it be given by a vector field

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y). \end{cases}$$

Let us choose a sufficiently large $a \in \mathbb{R}$ and let $f(x, y) = P(x, y)/(x - a)$. Then the function f is analytic in the bidisc of radius $a/2$ centered at the origin; thus the function f can be approximated in this bidisc by a polynomial. Denote the ε-approximating polynomial by $R_\varepsilon(x, y)$; then $(x - a) \cdot R_\varepsilon(x, y)$ approximates $P(x, y)$ on the same bidisc. By choosing first a sufficiently large and then ε sufficiently small, we obtain a vector field $((x - a) \cdot R_\varepsilon, Q)$ arbitrarily close to the initial field (P, Q). The corresponding foliations are therefore also arbitrarily close. On the other hand, the line $x = a$ is invariant for the new vector field $((x - a) \cdot R_\varepsilon, Q)$, and thus the corresponding foliation is not minimal.

8. *Technical proofs*

Proof of Lemma 1. Note that the conditions in this lemma are stable under small perturbations. The property of the g_j being contractions is clearly stable. The images $g_j(\mathring{K})$ form an open cover of a compact set K and hence the same holds for sufficiently close $\tilde{g}_j$. Thus it suffices to prove the minimality of the initial system $(K; g_1, \ldots, g_s)$.

Let us first introduce some notation: for a word $\omega = (\omega_1 \dots \omega_n)$, $\omega_1, \dots, \omega_n \in \{1, \dots, s\}$, denote by g_ω the composition

$$g_\omega = g_{\omega_1} \circ \cdots \circ g_{\omega_n}.$$

As $\bigcup g_j(\overset{\circ}{K}) \supset K$, we have, in particular, that

$$\bigcup_{j=1}^{s} g_j(K) \supset K.$$

After iterations we obtain that for any $n \in \mathbb{N}$,

$$\bigcup_{\omega_1,\dots,\omega_n \in \{1,\dots,s\}} g_{\omega_1,\dots,\omega_n}(K) \supset K,$$

i.e. all the possible images of K under n iterations cover K. On the other hand, the maps g_j are contracting, so the diameters of these images tend to zero as n tends to infinity. Hence, for sufficiently large n, the compact set K is covered by sets of small diameter.

We now prove that the orbit of any point $x \in K$ is dense. For given $b \in K$ and $\varepsilon > 0$, let us find $\omega = \omega_1, \dots, \omega_n$ such that $\rho(g_\omega(x), b) < \varepsilon$. Indeed, for sufficiently large n all the images $g_{\omega_1 \dots \omega_n}(K)$ under the length-n words are of diameter less then ε (owing to the fact that g_j are contractions). For any such n, there exists a word $\omega = \omega_1, \dots, \omega_n$ such that $g_\omega(K) \ni b$. So then $g_\omega(K) \ni b$, $g_\omega(K) \ni g_\omega(x)$ and $\operatorname{diam}(g_\omega(K)) < \varepsilon$, and thus $\rho(g_\omega(x), b) < \varepsilon$. □

Proof of Lemma 2. Assume the contrary. Let the set A be a measurable saturated set with positive measure of intersection $\mu_L(A \cap \overset{\circ}{K}) > 0$; then the set $A \cap \overset{\circ}{K}$ has a density (Lebesgue) point x_0.

Note that as the compact set K is covered by the open images $g_j(\overset{\circ}{K})$, there exists $\varepsilon_0 > 0$ such that for any point $x \in K$, for some j the map g_j^{-1} is defined in the neighborhood $U_{\varepsilon_0}(x)$. (In particular, $U_{\varepsilon_0}(x) \subset D$.)

We will construct $\omega_n \in \{1, \dots, s\}$ and $x_n \in K$ step by step: for any $n \in \mathbb{N}$ we choose ω_n in such a way that the map $g_{\omega_n}^{-1}$ is defined in $U_{\varepsilon_0}(x_n)$, and let $x_{n+1} = g_{\omega_n}^{-1}(x_n)$. Note that for all n we have $g_{\omega_1 \dots \omega_n}(U_{\varepsilon_0}(x_n)) \subset D$, because all the maps g_j are contracting.

Consider the compositions $g_{\omega_n}^{-1} \circ \cdots \circ g_{\omega_1}^{-1}$. As the derivatives of such compositions in the point x_0 grow exponentially, by the Distortion Lemma (see, for example, [**17**]) the quotient of derivatives of direct maps $g_{\omega_1} \circ \cdots \circ g_{\omega_n}$ on the corresponding balls $U_{\varepsilon_0}(x_n)$ is bounded uniformly in n. Hence, the quotient of maximal and minimal distances from x_0 to the boundary of the image $B_n := g_{\omega_1} \circ \cdots \circ g_{\omega_n}(U_{\varepsilon_0}(x_n))$ is uniformly bounded. These distances tend to 0, hence the proportion of points of A in B_n tends to 1 (for x is a density point of A). Owing to the invariance of A and (again) to the boundedness of the derivatives quotient, the proportion of points of A in $U_{\varepsilon_0}(x_n)$ also tends to 1. By choosing a convergent subsequence of points $x_{n_k} \to y$, we find a ball $U_{\varepsilon_0}(y)$ in which the proportion of points of A is equal to 1—that is, Lebesgue-almost every point of the ball belongs to A. Now the minimality (Lemma 1) implies that the set A has full measure in $\overset{\circ}{K}$. □

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