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Symmetry-Free, p-Robust Equilibrated Error Indication for the hp-Version of the FEM in Nearly Incompressible Linear Elasticity

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Dedicated to Ernst P. Stephan on the occasion of his 65th birthday

Abstract — We consider the extension of the p-robust equilibrated error estimator due to Braess, Pillwein and Schöberl to linear elasticity. We derive a formulation where the local mixed auxiliary problems do not require symmetry of the stresses. The resulting error estimator is p-robust, and the reliability estimate is also robust in the incompressible limit if quadratics are included in the approximation space. Extensions to other systems of linear second-order partial differential equations are discussed. Numerical simulations show only moderate deterioration of the effectivity index for a Poisson ratio close to $\frac{1}{2}$.

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1. Introduction

In the application of finite element methods to engineering problems, it is essential not only to develop methods converging with an *a priori* determined rate, but also to give *a posteriori* error indicators [3, 14, 15, 18, 25, 32, 36]; see also the references in [4]. These are used to gauge the total error of the current approximation, thus serving as a stopping criterion in calculations, as well as to steer an adaptive mesh generation, where only those elements with high estimated error are refined. Equilibrated error estimators, cf., e.g., [2, 13, 20], have seen a lot of interest in the last years, as they have particularly good properties. On the one hand, they usually do not have generic constants in the reliability bound, hence providing a precise estimate of the accuracy of the current approximation. Furthermore, it was recently proved by Braess, Pillwein and Schöberl in [12] that the equilibrated error estimator based on the solution of dual problems on node patches introduced by Braess and Schöberl in [13] is *p-robust*, i.e., does not suffer from the p-gap observed in [16, 23] for residual error estimators.

While the implementation of equilibrated error estimators for the Laplace problem is straightforward, the extension to linear elasticity is not. The issue stems from the symmetry

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requirement on the stresses. In [3,4,25], Arnold–Winther elements [6,7] and Arnold–Douglas–Gupta elements [5], respectively, are used to obtain an equilibrated error estimator for the lowest-order case. Both of these finite element classes consist of symmetric functions and allow higher orders of approximation. However, their implementation for higher polynomial degrees is not straightforward, and the dimensions of the local approximation spaces are very high. While a mixed formulation using a weak symmetry is more easily implemented, stability results in the *hp*-context are still lacking (see [26,27] for recent results with arbitrary and non-uniform, but bounded p).

Therefore, we take a different approach. Instead of deriving a mixed formulation directly using a variation of the Prager–Synge principle as a guideline, we interpret the results in [12] as representability of the residual by a polynomial. Hence, we obtain an error indicator by applying this methodology to the different components of the displacement field separately.

This yields a p-robust error estimator, but the reliability constant depends on the coercivity constant of the bilinear form. Therefore, we modify the error estimator to be more suitable for applications in linear elasticity. The advantages of this modified estimator is that if the polynomial degree is at least two everywhere, then the reliability estimate does not contain the global coercivity constant, but depends only on the Korn constants of the node patches. Hence, while our proof assumes shape regular meshes, our results apply to discretisations of anisotropic geometries, such as beams, by shape-regular meshes, where the local Korn constants are markedly smaller than the global constants, see [17, 21] and references therein for a discussion of Korn constants for different geometries.

Furthermore, we are able to prove that if quadratics are included in the approximation space, the reliability estimate is robust in the nearly incompressible case, i.e., the reliability bound does not degenerate if the Lamé parameter λ tends to infinity (or, equivalently, the Poisson ratio ν tends to $\frac{1}{2}$). Such a property, combined with the p-robustness of the error indicator, is particularly attractive in the p-context, as p- and *hp*-methods are known to be free of locking, see [8, 9, 29–31, 33].

Recently, Kim derived in [19] an equilibrated error estimator in linear elasticity using a nonsymmetric stress tensor approximation. His approach, however, takes the route of equilibrated fluxes, is for lowest-order polynomials, and cannot be directly generalised to arbitrary polynomial degrees. The functional approach by Repin, see [28], also leads to *a posteriori* error estimates for elasticity problems based on the Prager–Synge principle where the symmetry of the stress tensors is not required. The analysis of residual error estimation for conforming and non-conforming h-FEM by Carstensen in [14] gives the first mathematical indication that λ-robust error estimation is possible. The influence of (local) Korn inequalities on error estimation is studied in detail in [3, 4, 15].

This paper is structured as follows. In Section 2, we formulate our approach for general vector-valued problems. Section 3 adapts the error estimator to linear elasticity. Section 4 contains computations for a model problem on the L-shaped domain, which confirms that the proposed error estimator yields useful error bounds for the *hp*-finite element analysis in linear elasticity.

2. Equilibrated Error Indication for Vector-Valued Problems

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded, polygonal domain. We aim to provide an error estimate for the *hp*-FEM approximation to a system of K second-order partial differential equations on Ω.

The space $\mathrm{H}^1(\Omega)$ is the standard Sobolev space of square integrable functions with square integrable derivatives endowed with the norm

$$\|v\|_{\mathrm{H}^1(\Omega)} := \Big(\|v\|^2_{\mathrm{L}^2(\Omega)} + \sum_{j=1}^{d} \|v_{,j}\|^2_{\mathrm{L}^2(\Omega)}\Big)^{1/2}, \tag{1}$$

where $\|v\|_{\mathrm{L}^2(\Omega)} := \big(\int_\Omega v(\vec{x})^2 \mathrm{d}\vec{x}\big)^{1/2}$ denotes the usual norm in the Hilbert space $\mathrm{L}^2(\Omega)$, and $v_{,j} := \partial v / \partial x_j$.

Fix $K \in \mathbb{N}$ and denote $\Gamma := \partial\Omega$. For each $k = 1, \ldots, K$, let $\Gamma = \Gamma_{\mathrm{D},k} \cup \Gamma_{\mathrm{N},k}$, where we assume that both $\Gamma_{\mathrm{D},k}$ and $\Gamma_{\mathrm{N},k}$ are closed, that they have disjoint interior, and that $\Gamma_{\mathrm{D},k}$ has positive measure in Γ. In particular, mixed boundary conditions are admissible. We consider a variational problem set in the space $\vec{V} := V_1 \times \cdots \times V_K$, where for $k = 1, \ldots, K$, $V_k := \{v_k \in \mathrm{H}^1(\Omega) : v_k|_{\Gamma_{\mathrm{D},k}} = 0 \text{ for } k = 1, \ldots, K\}$ is the closed subspace of $\mathrm{H}^1(\Omega)$ obtained by prescribing (component-wise) homogeneous Dirichlet boundary conditions on $\Gamma_{\mathrm{D},k}$. Here and in the following, we write $\vec{v} = (v_k)_{k=1,\ldots,K}$ for $\vec{v} \in \vec{V}$. The norm on $\vec{V}$ is defined by

$$\|\vec{v}\|_{\vec{V}} := \Big(\sum_{k=1}^{K} \|v_k\|^2_{\mathrm{H}^1(\Omega)}\Big)^{1/2}. \tag{2}$$

Denoting $(D\vec{v})_{kj} := v_{k,j}$, we observe that

$$\|\vec{v}\|^2_{\vec{V}} = \|\vec{v}\|^2_{(\mathrm{L}^2(\Omega))^K} + \|D\vec{v}\|^2_{(\mathrm{L}^2(\Omega))^{K\times d}}. \tag{3}$$

Here, we set

$$\|\vec{v}\|_{(\mathrm{L}^2(\Omega))^K} := \Big(\sum_{k=1}^{K} \|v_k\|^2_{\mathrm{L}^2(\Omega)}\Big)^{1/2}, \tag{4}$$

$$\|\boldsymbol{\tau}\|_{(\mathrm{L}^2(\Omega))^{K\times d}} := \Big(\int_\Omega \tau_{kj}(\vec{x})\tau_{kj}(\vec{x}) \mathrm{d}\vec{x}\Big)^{1/2} \tag{5}$$

for $\boldsymbol{\tau} = (\tau_{kj})_{k=1,\ldots,K, j=1,\ldots,d} \colon \Omega \to \mathbb{R}^{K\times d}$, where we apply the Einstein convention of summing over repeated indices.

Remark 1. In the context of linear elasticity, a standard boundary condition is to assume that along a part Γ_{C} of the boundary, the body is in frictionless contact with a non-deforming obstacle. This corresponds to assuming homogeneous Dirichlet conditions in the normal and homogeneous Neumann conditions in the tangential direction. If we suppose that Γ_{C} is parallel to a coordinate axis (2D) or a coordinate plane (3D), then this type of boundary condition is contained in the above setup if we ensure $\Gamma_{\mathrm{C}} \subseteq \Gamma_{\mathrm{D},k_0} \cap \bigcap_{k\neq k_0} \Gamma_{\mathrm{N},k}$ where $k_0 \in \{1, \ldots, K\}$ and the normal vector on Γ_{C} points in the k_0-th coordinate direction.

Let $a \colon \vec{V} \times \vec{V} \to \mathbb{R}$ be the bilinear form

$$a(\vec{u}, \vec{v}) := \int_\Omega \alpha_{ijkl}(\vec{x}) u_{i,j}(\vec{x}) v_{k,l}(\vec{x}) + \beta_{ijk}(\vec{x}) u_{i,j}(\vec{x}) v_k(\vec{x}) + \gamma_{ik}(\vec{x}) u_i(\vec{x}) v_k(\vec{x}) \mathrm{d}\vec{x}. \tag{6}$$

The coefficient functions α_{ijkl}, β_{ijk}, $\gamma_{ik} \colon \Omega \to \mathbb{R}$, $i, k = 1, \ldots, K$, $j, l = 1, \ldots, d$, are assumed to be piecewise constant with respect to the triangulation; this will be made precise below.

Let furthermore $\ell\colon \vec{V} \to \mathbb{R}$ be a continuous linear functional,

$$\ell(\vec{v}) := \int_\Omega \vec{f}(\vec{x}) \cdot \vec{v}(\vec{x}) \mathrm{d}\vec{x} + \sum_{k=1}^{K} \int_{\Gamma_{\mathrm{N},k}} g_k(\vec{x}) v_k(\vec{x}) \mathrm{d}s_{\vec{x}}, \tag{7}$$

prescribing component-wise Neumann boundary conditions on $\Gamma_{\mathrm{N},k}$. The functions $\vec{f}$ and $\vec{g}_k$, $k = 1, \ldots, K$, are assumed to be piecewise polynomials, and this is made precise below. This setup is of interest in the application of our results to problems in linear elasticity, as there, prescribing Dirichlet boundary conditions in some and Neumann boundary conditions in other components corresponds to contact boundary conditions.

The continuous problem now reads as follows: Find $\vec{u} \in \vec{V}$ such that

$$a(\vec{u}, \vec{v}) = \ell(\vec{v}) \quad \text{for all } \vec{v} \in \vec{V}. \tag{8}$$

We shall now formulate a finite element approximation with respect to the triangulation $\mathcal{T}_N$. We fix the parameters of the *hp*-finite element space as follows. For N, let $\mathcal{T}_N$ be a partition of Ω into closed triangles ($d = 2$) or simplices ($d = 3$); an extension to quadrilateral or hexahedral meshes should be possible, see also [1]. Let $p_{N,T} \in \mathbb{N}$ be the polynomial degree on element $T \in \mathcal{T}_N$, and let $\mathbb{P}^p$ be the space of polynomials of total degree less than or equal to p. Similarly as in [12], we suppose that for every $T \in \mathcal{T}_N$, $\alpha_{ijkl}|_T$, $\beta_{ijk}|_T$, $\gamma_{ik}|_T \in \mathbb{R}$ for $i, k = 1, \ldots, K$, $j, l = 1, \ldots, d$, and we assume $f_k|_T \in \mathbb{P}^{p_{N,T}}$, and $g_k|_{\partial T \cap \Gamma_{N,k}} \in \mathbb{P}^{p_{N,T}}$ for $k = 1, \ldots, K$. Our estimates will depend implicitly on the shape regularity constants of the triangulation, which therefore has to be assumed bounded for $N \to \infty$, but on the mesh width and approximation order only in the manner stated below. The approximation space $\vec{V}_N$ is defined by

$$\vec{V}_N := \bigl\{\vec{v} \in \vec{V} : v_k|_T \in \mathbb{P}^{p_{N,T}} \text{ for } T \in \mathcal{T}_N \text{ and } k = 1, \ldots, K\bigr\}. \tag{9}$$

The discrete problem is then to find $\vec{u}_N \in \vec{V}_N$ such that

$$a(\vec{u}_N, \vec{v}_N) = \ell(\vec{v}_N) \quad \text{for all } \vec{v}_N \in \vec{V}_N. \tag{10}$$

We suppose that both the continuous and discrete problems have unique solutions u and u_N for every $\ell \in \vec{V}^*$, and that these satisfy the bounds

$$\|\vec{u}\|_{\vec{V}} \leqslant C_a \|\ell\|_{\vec{V}^*} \quad \text{and} \quad \|\vec{u_N}\|_{\vec{V}} \leqslant C_N \|\ell\|_{\vec{V}^*} \tag{11}$$

It is our aim to determine a representation of the residual $r_N \in \vec{V}^*$, which is defined through

$$r_N(\vec{v}) := a(\vec{u} - \vec{u}_N, \vec{v}) = \ell(\vec{v}) - a(\vec{u}_N, \vec{v}). \tag{12}$$

This will allow us to bound $\|\vec{u} - \vec{u}_N\|_{\vec{V}}$ due to

$$\|\vec{u} - \vec{u}_N\|_{\vec{V}} \leqslant C_a \|r_N\|_{\vec{V}^*}. \tag{13}$$

We recall that $r_N(\vec{v}_N) = 0$ for $\vec{v}_N \in \vec{V}_N$ by Galerkin orthogonality. With $\vec{e}_k$ the canonical basis of $\mathbb{R}^K$, we have $\vec{v} = \sum_{k=1}^K v_k \vec{e}_k$, where $v_k \in V_k$. We define the component residual $r_{N,k} \in V_k^*$ by

$$r_{N,k}(v) := r_N(v\vec{e}_k) \quad \text{for } v \in V_k; \tag{14}$$

the postulated continuity of $r_{N,k}$ follows from the continuity of r_N. The results of [12] prove that for every $k = 1, \ldots, K$, there exists $\vec{\sigma}^{\Delta}_{N,k}\colon \Omega \to \mathbb{R}^d$ with the following three properties:

1. $\vec{\sigma}^{\Delta}_{N,k}|_T \in \mathrm{RT}^{p_{N,T}+1}(T)$ for all $T \in \mathcal{T}_N$, where
$$\mathrm{RT}^p(T) := \left\{\tau : \tau(\vec{x}) = q_T + s_T\vec{x} \text{ with } q_T \in (\mathbb{P}^p)^d,\ s_T \in \mathbb{P}^p\right\} \tag{15}$$
is the standard Raviart–Thomas space, see [11, p. 148],
2. $(\vec{\sigma}^{\Delta}_{N,k}, \vec{\nabla} v)_{(\mathrm{L}^2(\Omega))^d} = r_{N,k}(v)$ for all $v \in V_k$, and
3. $\|\vec{\sigma}^{\Delta}_{N,k}\|_{(\mathrm{L}^2(T))^d} \leqslant C\|r_{N,k}\|_{(\mathrm{H}^1(\omega_T)/\mathbb{R})^*}$, where $C > 0$ is a constant depending only on the shape regularity of the mesh but independent of the mesh resolution and the polynomial degree, and $\omega_T := \bigcup_{T\cap T' \neq \emptyset} T'$ is the element patch of T, i.e., the union over all elements sharing a node with T.

Here, $\mathrm{H}^1(\omega_T)/\mathbb{R}$ denotes the space of H^1-functions on ω_T factored by the constant functions, and $(\mathrm{H}^1(\omega_T)/\mathbb{R})^*$ its dual space. Setting $(\boldsymbol{\sigma}^{\Delta}_N)_{kj} := \vec{\sigma}^{\Delta}_{N,kj}$, we define the local equilibrated error indicator by

$$\eta_{N,T} := \|\boldsymbol{\sigma}^{\Delta}_N\|_{(\mathrm{L}^2(T))^{K\times d}}, \quad T \in \mathcal{T}_N. \tag{16}$$

The global error indicator reads $\eta_N := \left(\sum_{T\in\mathcal{T}_N} \eta^2_{N,T}\right)^{1/2} = \|\boldsymbol{\sigma}^{\Delta}_N\|_{(\mathrm{L}^2(\Omega))^{K\times d}}$.

Theorem 1. *The equilibrated error indicator $(\eta_{N,T})_{T\in\mathcal{T}_N}$ defined above is reliable and efficient. More precisely,*

$$\|\vec{u} - \vec{u}_N\|_{\vec{V}} \leqslant C_a \eta_N \tag{17}$$

and

$$\eta_{N,T} \leqslant C\Big(\sum_{k=1}^{K} \|r_{N,k}\|^2_{(\mathrm{H}^1(\omega_T)/\mathbb{R})^*}\Big)^{1/2}. \tag{18}$$

The reliability and efficiency constants are independent of the local mesh width and the local polynomial degree.

We stress that the reliability constant only depends on the stability estimate for the continuous, but not the discrete problem. Hence, even if an unstable discretisation is employed, our estimates apply and yield a reliable and efficient error indicator.

Proof. We estimate

$$\begin{aligned}\|\vec{u} - \vec{u}_N\|_{\vec{V}} &\leqslant C_a\|r_N\|_{\vec{V}^*} = C_a \sup_{0\neq\vec{v}\in\vec{V}} \frac{r_N(\vec{v})}{\|\vec{v}\|_{\vec{V}}} = C_a \sup_{0\neq\vec{v}\in\vec{V}} \frac{\sum_{k=1}^K r_{N,k}(v_k)}{\|\vec{v}\|_{\vec{V}}} \\ &= C_a \sup_{0\neq\vec{v}\in\vec{V}} \frac{\sum_{k=1}^K (\vec{\sigma}^{\Delta}_{N,k}, \vec{\nabla} v_k)_{(\mathrm{L}^2(\Omega))^d}}{\|\vec{v}\|_{\vec{V}}} \leqslant C_a\|\boldsymbol{\sigma}^{\Delta}_N\|_{(\mathrm{L}^2(\Omega))^{K\times d}} = C_a\eta_N.\end{aligned} \tag{19}$$

Hence, the proposed error indicator is reliable with a generic constant that is independent of the mesh resolution and the polynomial degree. Similarly,

$$\Big(\sum_{k=1}^{K} \|\vec{\sigma}^{\Delta}_{N,k}\|^2_{(\mathrm{L}^2(T))^d}\Big)^{1/2} \leqslant C\Big(\sum_{k=1}^{K} \|r_{N,k}\|^2_{(\mathrm{H}^1(\omega_T)/\mathbb{R})^*}\Big)^{1/2}, \tag{20}$$

which proves local efficiency. □

Remark 2. If we assume that a is symmetric and coercive, we can choose to work in the energy norm $\|\vec{v}\|_a := a(\vec{v},\vec{v})^{1/2}$ on $\vec{V}$ instead of the standard norm. The reliability estimate then becomes

$$\|\vec{u} - \vec{u}_N\|_a \leqslant c_a^{-1/2}\eta_N, \tag{21}$$

where $c_a := \inf_{0\neq\vec{v}\in\vec{V}} a(\vec{v},\vec{v})/\|\vec{v}\|^2_{\vec{V}}$ is the coercivity constant of a.

3. An Application to Linear Elasticity

The error indicator defined above is clearly applicable to problems in linear elasticity. Here, $K = d$, and

$$a(\vec{u},\vec{v}) := \int_\Omega \mathbb{C}(\vec{x})\boldsymbol{\varepsilon}(\vec{u})(\vec{x}) : \boldsymbol{\varepsilon}(\vec{v})(\vec{x})\mathrm{d}\vec{x} \tag{22}$$

with $\mathbb{C}(\vec{x})$ the Hooke tensor, $\boldsymbol{\varepsilon}(\vec{v})_{ij} := \frac{1}{2}(v_{i,j} + v_{j,i})$ the small strain tensor, and $\mathbf{p} : \mathbf{q} := p_{ij}q_{ij}$ for $\mathbf{p}$, $\mathbf{q} \in \mathbb{R}^{d\times d}$. A particular advantage is that we do not need to worry about the symmetry condition on the strains and stresses as was done in [25] for lowest order approximations, which is difficult in hp-methods (but see [26] for some results on the generalisation of Arnold–Falk–Winther elements to meshes with varying, but bounded polynomial degrees). In this case, however, we can improve our results by modifying the error indicator slightly.

Assume that the material is isotropic and, for simplicity, homogeneous; an extension to $\mathbb{C}$ piecewise constant with respect to the partition $\mathcal{T}_N$ is possible. Then, $\mathbb{C}_{ijkl}\tau_{kl} = \lambda\tau_{kk}\delta_{ij} + 2\mu\tau_{ij}$ with the Lamé constants $\lambda, \mu > 0$. Furthermore, we can find an inverse $\mathbb{C}^{-1}$ of $\mathbb{C}$, i.e., $\mathbb{C}\mathbb{C}^{-1}\tau = \mathbb{C}^{-1}\mathbb{C}\tau = \tau$, and this inverse is given explicitly by

$$\mathbb{C}^{-1}_{ijk\ell}\tau_{kl} = -\frac{\lambda}{2\mu(d\lambda + 2\mu)}\tau_{kk}\delta_{ij} + \frac{1}{2\mu}\tau_{ij}. \tag{23}$$

It is our aim to show that given $p_{N,T} \geqslant 2$ for all $T \in \mathcal{T}_N$, we do not have to include the global Korn constant in the reliability estimate, but can instead use the maximal local Korn constant, which is expected to be considerably smaller for beam problems and other strongly anisotropic geometries, see [17, 21]. Moreover, we will establish that the reliability bound of the error indicator is independent of λ, i.e., it is robust for nearly incompressible materials. The modified local error indicator is defined by

$$\eta_{N,T} := \Big(\sum_{z\in\mathcal{N}_N} \|\boldsymbol{\sigma}^\Delta_{Nz}\|^2_{\mathrm{L}^2_{\mathbb{C}^{-1}}(T)}\Big)^{1/2}, \quad T \in \mathcal{T}_N, \tag{24}$$

where $\|\boldsymbol{\tau}\|_{\mathrm{L}^2_\gamma(\Omega)} := \big(\int_\Omega \alpha_{ijkl}\tau_{ij}(\vec{x})\tau_{kl}(\vec{x})\mathrm{d}\vec{x}\big)^{1/2}$. Here, $\boldsymbol{\sigma}^\Delta_{Nz}$ has support only in ω_z and satisfies

$$(\boldsymbol{\sigma}^\Delta_{Nz}, D\vec{v})_{(\mathrm{L}^2(\omega_z))^{d\times d}} = r_N(\vec{v}\varphi_z), \quad \vec{v} \in \vec{V}, \tag{25}$$

where φ_z denotes the hat function at a node $z \in \mathcal{N}_N$ with $\mathcal{N}_N$ the set of all nodes of the partition $\mathcal{T}_N$, and $\omega_z := \bigcup_{z\in T} T = \operatorname{supp}\varphi_z$ denotes the node patch of $z \in \mathcal{N}_N$. Hence, all but $d+1$ terms in the sum in (24) vanish. The existence of $\boldsymbol{\sigma}^\Delta_{Nz}$ with its rows in a broken Raviart–Thomas space, i.e., in practically computable form, is established under our assumptions in [12]. We stress that it is possible to calculate $\boldsymbol{\sigma}^\Delta_{Nz}$ disregarding any symmetry assumptions. The global error indicator again reads

$$\eta_N := \Big(\sum_{T\in\mathcal{T}_N} \eta^2_{N,T}\Big)^{1/2} = \Big(\sum_{z\in\mathcal{N}_N} \|\boldsymbol{\sigma}^\Delta_{Nz}\|^2_{\mathrm{L}^2_{\mathbb{C}^{-1}(\omega_z)}}\Big)^{1/2}. \tag{26}$$

This error indicator is still efficient by the results in [12], and its reliability constants have better properties than the direct application of the indicator suggested in Section 2, as will be seen below.

Remark 3. Similarly as in [12], we determine $\boldsymbol{\sigma}_{Nz}^{\Delta}$ in a broken Raviart–Thomas space by solving the mixed variational formulation that results from the minimisation of $\frac{1}{2}\|\boldsymbol{\sigma}\|^2_{\mathrm{L}^2_{\mathbb{C}^{-1}}(\omega_z)}$ under the constraint

$$(\boldsymbol{\sigma}, D\vec{v})_{(\mathrm{L}^2(\omega_z))^{d\times d}} = r_N(\vec{v}\varphi_z) \quad \text{for } \vec{v} \in \vec{V}. \tag{27}$$

In contrast to [3, 25], however, we do not search for $\boldsymbol{\sigma}_{Nz}^{\Delta}$ in a space of symmetric tensors, which drastically simplifies the implementation.

Theorem 2. *Assume that the polynomial degree p is everywhere greater than or equal to 2. Then, the equilibrated error indicator $(\eta_{N,T})_{T\in\mathcal{T}_N}$ for problems from linear elasticity is reliable with a constant robust with respect to approximation order and Lamé parameters. More precisely, the estimate*

$$a(u-u_N, u-u_N)^{1/2} \leqslant C\eta_N \tag{28}$$

holds true, where $C>0$ is independent of the polynomial degrees p and the Lamé parameters λ and μ, but may depend on the local Korn constants of the node patches.

Remark 4. Strictly speaking, it would be necessary to also prove λ-independent efficiency of the equilibrated error indicator to ensure its performance in the nearly incompressible case. This means proving results analogous to [12, Lemma 3, Theorem 5] for the case of linear elasticity, i.e.,

1. given $\vec{r}_T \in (\mathbb{P}^p)^2$, there exists $\boldsymbol{\sigma}_T$ with rows in RT^p such that $-\operatorname{Div}\boldsymbol{\sigma}_T = \vec{r}_T$ and

$$\|\boldsymbol{\sigma}_T\|_{\mathrm{L}^2_{\mathbb{C}^{-1}}(T)} \leqslant C \sup_{0\neq\vec{v}\in\vec{V}} \frac{\int_T \vec{r}_T(\vec{x})\cdot\vec{v}(\vec{x})\mathrm{d}\vec{x}}{\|\vec{v}\|_a}, \tag{29}$$

and

2. given $\vec{R}_{\partial T}$ on ∂T such that $\vec{R}_{\partial T}|_E \in (\mathbb{P}^p)^2$ for every edge E of ∂T, there exists $\boldsymbol{\sigma}_T$ with rows in RT^p such that $\boldsymbol{\sigma}_T\vec{n} = \vec{R}_{\partial T}$ on ∂T, where $\vec{n}$ is the outer normal vector to T, $\operatorname{Div}\boldsymbol{\sigma}_T = 0$ on T, and

$$\|\boldsymbol{\sigma}_T\|_{\mathrm{L}^2_{\mathbb{C}^{-1}}(T)} \leqslant C \sup_{0\neq\vec{v}\in\vec{V}} \frac{\int_{\partial T} \vec{R}_{\partial T}(\vec{x})\cdot\vec{v}(\vec{x})\mathrm{d}s_{\vec{x}}}{\|\vec{v}\|_a}. \tag{30}$$

Here, the constants $C>0$ have to be independent of T and p, and $\|\vec{v}\|_a := a(\vec{v},\vec{v})^{1/2}$ denotes the energy norm. Our numerical calculations show a λ-robust behaviour of the error indicator. Furthermore, we stress that this issue vanishes if we choose $\boldsymbol{\sigma}_{Nz}^{\Delta}$ as the minimiser of $\frac{1}{2}\|\boldsymbol{\sigma}\|^2_{\mathrm{L}^2_{\mathbb{C}^{-1}}(\omega_z)}$ under the constraint (27) in the space of $d\times d$-tensors row-wise in the broken H(div)-space. Then, λ-robust stability is satisfied. Hence, the problem stems from the difficulty of finding a "good enough" approximation of the minimiser in a finite-dimensional space of polynomials.

Proof. For every $z \in \mathcal{N}_N$, let $\vec{\rho}_z$ denote a rigid body motion with $\vec{\rho}_z\varphi_z \in \vec{V}$. Clearly, $r_N(\vec{\rho}_z\varphi_z) = 0$, as $\min_{T\in\mathcal{T}_N} p_{N,T} \geqslant 2$ and $\vec{\rho}_z\varphi_z$ is a polynomial of total degree at most 2. The Cauchy–Schwarz inequality implies that

$$\int_\Omega p_{ij}(\vec{x})q_{ij}(\vec{x})\mathrm{d}\vec{x} \leqslant \|\mathbf{p}\|_{\mathrm{L}^2_{\mathbb{C}^{-1}}(\Omega)}\|\mathbf{q}\|_{\mathrm{L}^2_{\mathbb{C}}(\Omega)} \quad \text{for } \mathbf{p},\mathbf{q}\colon \Omega \to \mathbb{R}^{d\times d}. \tag{31}$$

Thus, as $\sum_{z\in\mathcal{N}_N}\varphi_z = 1$,

$$\begin{aligned} r_N(\vec{v}) &= \sum_{z\in\mathcal{N}_N} r_N(\vec{v}\varphi_z) = \sum_{z\in\mathcal{N}_N} r_N((\vec{v}-\vec{\rho}_z)\varphi_z) \\ &= \sum_{z\in\mathcal{N}_N} (\boldsymbol{\sigma}^\Delta_{Nz}, D(\vec{v}-\vec{\rho}_z))_{(\mathrm{L}^2(\omega_z))^{d\times d}} \\ &\leqslant \Big(\sum_{z\in\mathcal{N}_N}\|\boldsymbol{\sigma}^\Delta_{Nz}\|^2_{\mathrm{L}^2_{\mathbb{C}^{-1}}(\omega_z)}\Big)^{1/2}\Big(\sum_{z\in\mathcal{N}_N}\|D(\vec{v}-\vec{\rho}_z)\|^2_{\mathrm{L}^2_{\mathbb{C}}(\omega_z)}\Big)^{1/2}. \end{aligned} \tag{32}$$

We shall now apply the Korn inequality to estimate the final terms. Assume first that z is an unconstrained node. Then, $\vec{\rho}_z$ can be an arbitrary rigid body motion. As $\rho_{zk,k} = 0$ for a rigid body motion $\vec{\rho}_z$ and $\boldsymbol{\varepsilon}(\vec{v})_{kk} = v_{k,k}$, it follows that

$$\|D(\vec{v}-\vec{\rho}_z)\|^2_{\mathrm{L}^2_{\mathbb{C}}(\omega_z)} = \big(\lambda\|\boldsymbol{\varepsilon}(\vec{v})_{kk}\|^2_{\mathrm{L}^2(\omega_z)} + 2\mu\|D(\vec{v}-\vec{\rho}_z)\|^2_{(\mathrm{L}^2(\omega_z))^{d\times d}}\big). \tag{33}$$

The Korn inequality implies that there exists $c_{\omega_z} > 0$ such that

$$\inf\|D(\vec{v}-\vec{\rho}_z)\|^2_{(\mathrm{L}^2(\omega_z))^{d\times d}} \leqslant c^{-1}_{\omega_z}\|\boldsymbol{\varepsilon}(\vec{v})\|^2_{(\mathrm{L}^2(\omega_z))^{d\times d}}, \tag{34}$$

where the infimum is taken over all rigid body motions $\vec{\rho}_z$. Hence,

$$\|D(\vec{v}-\vec{\rho}_z)\|^2_{\mathrm{L}^2_{\mathbb{C}}(\omega_z)} \leqslant c^{-1}_{\omega_z}\big(\lambda\|\boldsymbol{\varepsilon}(\vec{v})_{kk}\|^2_{\mathrm{L}^2(\omega_z)} + 2\mu\|\boldsymbol{\varepsilon}(\vec{v})\|_{(\mathrm{L}^2(\omega_z))^{d\times d}}\big) = c^{-1}_{\omega_z}\|\boldsymbol{\varepsilon}(\vec{v})\|^2_{\mathrm{L}^2_{\mathbb{C}}(\omega_z)}. \tag{35}$$

We stress that c_{ω_z} is independent of the specific size of ω_z, but only depends on its shape. Therefore, we expect $c^{-1}_{\omega_z}$ to stay bounded and small for typical sequences of shape-regular finite element meshes.

A similar argument is possible if z is a constrained node. If $z \in \Gamma_{\mathrm{D},k}$, $k = 1,\ldots,d$, we necessarily have $\vec{\rho}_z = 0$, but we can apply the Korn inequality for constrained functions (in this case, $\vec{v}$ vanishes along an outer edge of ω_z). If $z \in \Gamma_{\mathrm{D},k}$ for some, but not all $k = 1,\ldots,d$, we need to apply the respective Korn inequality, which is possible as all rigid body motions $\vec{\rho}_z$ that have to be factored out (translations in free directions, rotations within free planes) satisfy $\vec{\rho}_z\varphi_z \in \vec{V}$. Altogether,

$$\begin{aligned} (\boldsymbol{\sigma}^\Delta_N, D\vec{v})_{(\mathrm{L}^2(\Omega))^{d\times d}} &\leqslant \big(\max_{z\in\mathcal{N}_N} c^{-1/2}_{\omega_z}\big)\eta_N\Big(\sum_{z\in\mathcal{N}_N}\|\boldsymbol{\varepsilon}(\vec{v})\|^2_{\mathrm{L}_{\mathbb{C}^2}(\omega_z)}\Big)^{1/2} \\ &\leqslant \big((d+1)\max_{z\in\mathcal{N}_N} c^{-1/2}_{\omega_z}\big)\eta_N\|\boldsymbol{\varepsilon}(\vec{v})\|_{\mathrm{L}_{\mathbb{C}^2}(\Omega)} = \big((d+1)\max_{z\in\mathcal{N}_N} c^{-1/2}_{\omega_z}\big)\eta_N\|v\|_a. \end{aligned} \tag{36}$$

Thus, we obtain robustness of the reliability constant with respect to λ and μ, and dependence only on local Korn constants instead of the global one. □

Remark 5. At first glance, the dependence on local Korn constants in Theorem 2 might seem bad, because it appears difficult to check in practice. However, if the mesh is refined using newest vertex bisection [10,24,35], it is known that only certain patch types are produced by the refinement. Hence, with a good initial mesh, it is ensured that the local Korn constants will behave nicely.

4. Numerical Examples

To confirm our theoretical results, we consider the test case given in [3, Example 1], i.e., the approximation of the Mode 1 singularity function of linear elasticity, see also [34, Section 10.1.2]. The geometry Ω is the L-shaped domain given as the union of the three squares with end points

$$\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-2\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \tag{37}$$

$$\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \quad \text{and} \tag{38}$$

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}. \tag{39}$$

The material is assumed isotropic and homogeneous with the Lamé parameters λ and μ. We approximate the displacement field

$$\vec{u} = \frac{1}{2\mu} r^{\alpha} \begin{pmatrix} (3-4\nu-\beta(\alpha+1))\cos(\alpha\theta) - \alpha\cos((\alpha-2)\theta) \\ (3-4\nu+\beta(\alpha+1))\sin(\alpha\theta) + \alpha\sin((\alpha-2)\theta) \end{pmatrix}, \tag{40}$$

where $\alpha = 0.544483737$ and $\beta = 0.543075579$, and $\nu = \lambda/(2(\lambda+\mu))$ is the Poisson ratio. This problem is chosen such that $-\operatorname{Div} \mathbb{C}\varepsilon(\vec{u}) = 0$ on Ω, where Div is the row-wise divergence operator. Furthermore, $\mathbb{C}\varepsilon(\vec{u})\vec{n} = 0$ along the edges from $\binom{-1}{-1}$ to $\binom{0}{0}$ and $\binom{0}{0}$ to $\binom{-1}{1}$. Here, $\vec{n}$ denotes the outer normal vector to Ω along these two edges. We hence assume, as in [3, Example 1], homogeneous Neumann conditions on this part of the boundary, and inhomogeneous Dirichlet conditions matching the exact solution on the remainder.

Let us comment on the impact of data oscillations in this example. The volume term and the Neumann boundary data vanish, whence these terms do not lead to additional errors. The inhomogeneous Dirichlet boundary data is not polynomial. Hence, we incur data oscillations there. As the Dirichlet boundary is away from the singularity of $\vec{u}$ at $\binom{0}{0}$, we see, however, that the given function is smooth, and any approximation error will be negligible for reasonable polynomial degrees and mesh resolutions. We therefore ignore any error in the approximation of the Dirichlet data in our error indicator.

We consider two choices of parameters. In both, we fix the Young modulus $E = 100$. In the first, we take the Poisson ratio moderately high, $\nu = 0.3$. In the second, we choose $\nu = 0.49999$, which is reasonably close to the incompressible limit. Recall that the Lamé parameters can be found from E and ν through the formulas

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}. \tag{41}$$

Let us give some general comments on the chosen methods. We compare h-uniform methods of polynomial degree $p = 1$ and $p = 4$, an h-adaptive method of polynomial degree $p = 4$, a p-uniform method, an hp-adaptive strategy along the lines of [16] (leading to an aspect ratio of 0.5 at the singularity), and an hp-method on *a priori* generated, geometrically graded meshes with aspect ratio $\sigma = 0.3$ and minimal polynomial degree 3 on the elements closest to the singularity, see [29, p. 170]. It is known that apart from the h-uniform method with $p = 1$, all of these methods are stable for nearly incompressible materials, cf. [8]. Our numerical experiments agree with these results, see Figures 1 and 3. Furthermore, the geometrically graded *hp*-meshes yield an exponential convergence in the degrees of freedom for both choices of the Poisson ratio. All calculations were performed using `maiprogs` [22]. To illustrate our results, we plot the exact and estimated errors and the effectivity indices, i.e., the ratios between estimated and exact errors.

4.1. Poisson Ratio $\nu = 0.3$

The results are presented in Figures 1 and 2. The automatic *hp*-adaptivity is only slightly better than the *h*-adaptive strategy due to the high polynomial degree, and because in the *hp*-adaptivity, we only do *h*-refinements by cutting the element in half. While our error estimates contain a generic constant, namely the maximum of the Korn constants of the node patches, the effectivity indices in Figure 2 show that our indicator only slightly overestimates the true error, at most by a factor of about 1.3 for the uniform *p*-version.

4.2. Poisson Ratio $\nu = 0.49999$

The results are depicted in Figures 3 and 4. The effectivity indices increase compared to the case $\nu = 0.3$, and this appears related to the lack of a λ-robust efficiency estimate. If the grid is fine enough in the uniform or adaptive *h*-version with $p = 4$ or in the adaptive *hp*-version or the polynomial degree is high enough in the uniform *p*-version, the error is only overestimated by a factor of less than 3.5, and the results for the *hp*-version with geometrically graded meshes is similar, the factor being 3.8. The error of the uniform *h*-version with $p = 1$ is overestimated by a factor of more than 100. This seems to be related to low polynomial degrees, and corresponds well to our result in Theorem 2 that states that the error estimator behaves better when higher order ansatz functions are used. Here, the *hp*-adaptive strategy yields better results than the *h*-adaptive method with polynomial degree 4.

5. Conclusions

We have presented a simple approach to extend the equilibrated error estimation strategy of [12, 13] to linear elasticity in such a way that the restrictive symmetry condition on the stresses can be dropped. While the efficiency constant is always *p*-robust, we have proved that the reliability constant is robust in the Poisson ratio and only depends on local Korn constants if quadratics are included in the finite element space. We have established that these ideas are applicable to a large class of elliptic systems. Numerical calculations confirming our theoretical analysis have been shown.

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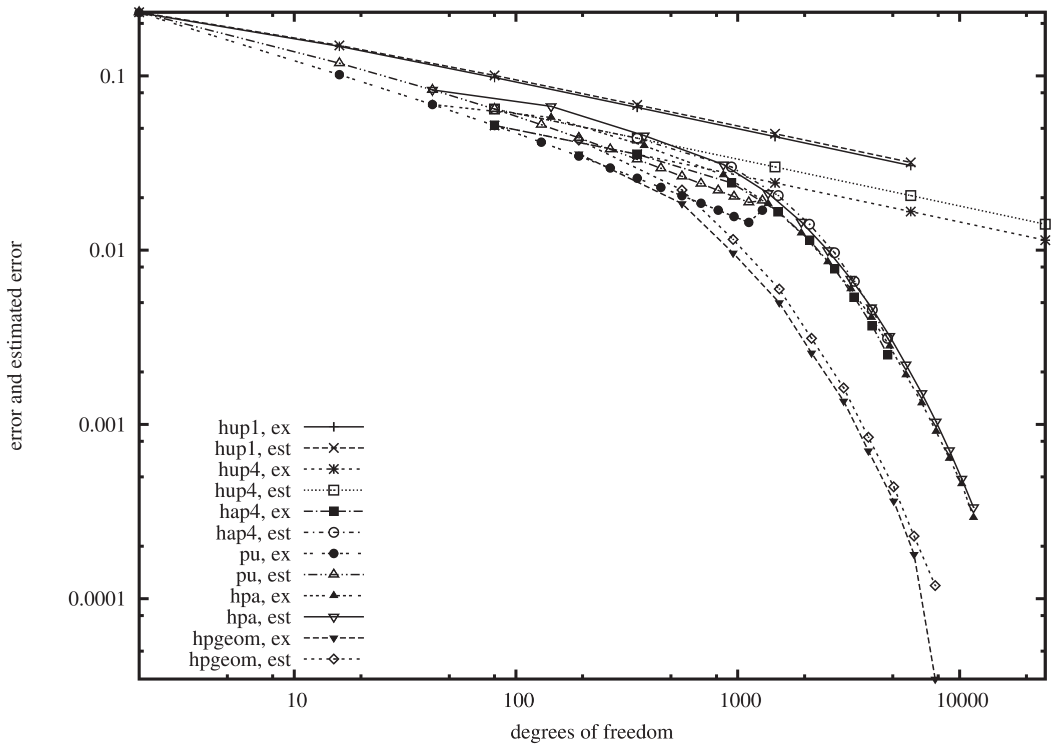

Figure 1. Poisson ratio $\nu = 0.3$, exact and estimated errors. Here, `ex` stands for the exact error and `est` for the error estimator, `hup1` for the uniform h-version with polynomial degree 1, `hup4` for the uniform h-version with polynomial degree 4, `hap4` for the adaptive h-version with polynomial degree 4, `pu` for the uniform p-version, `hpa` for the adaptive hp-version and `hpgeom` for the hp-version on geometrically graded meshes.

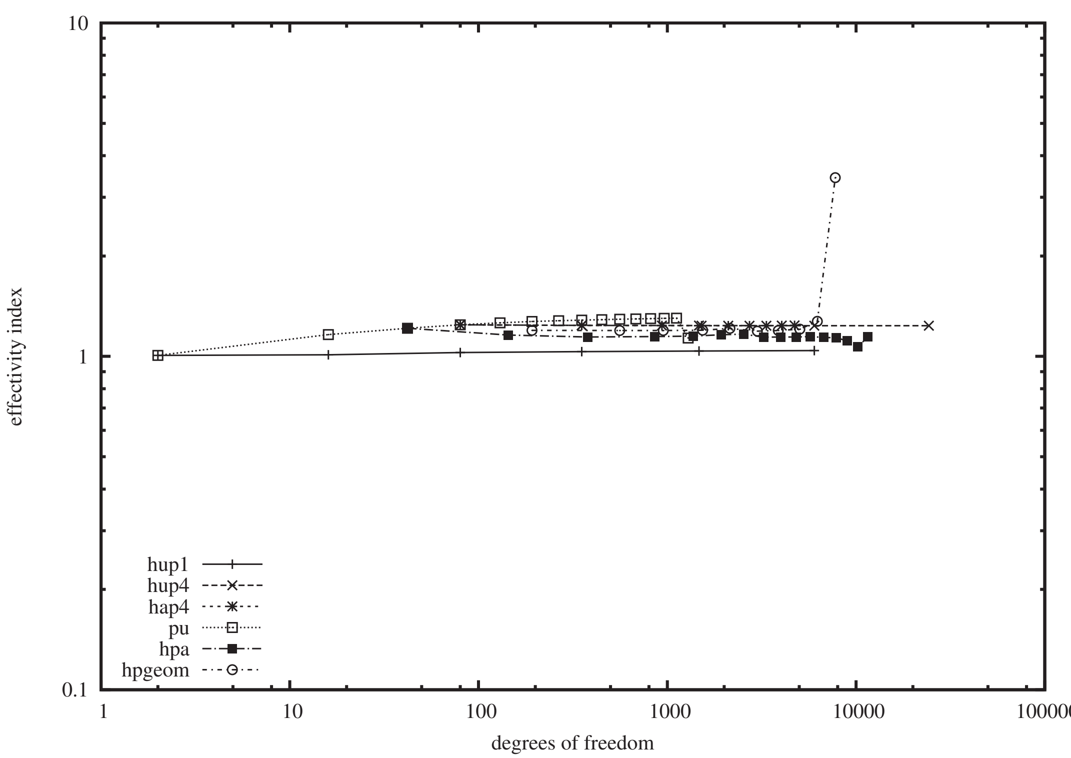

Figure 2. Poisson ratio $\nu = 0.3$, effectivity indices.

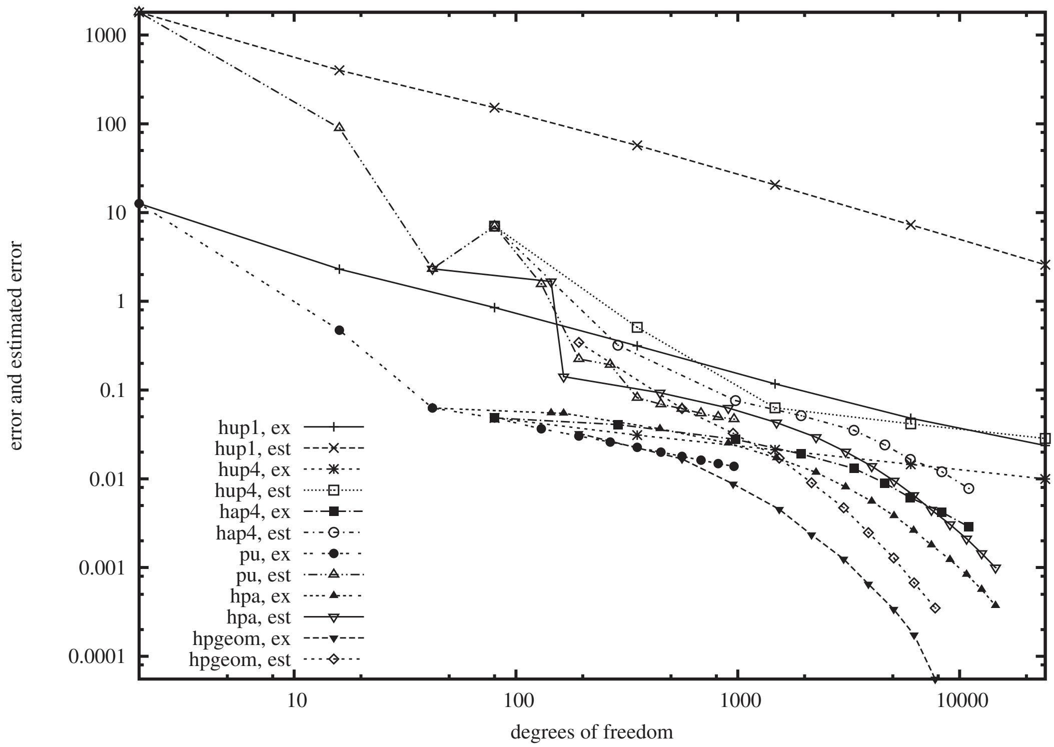

Figure 3. Poisson ratio $\nu = 0.49999$, exact and estimated errors.

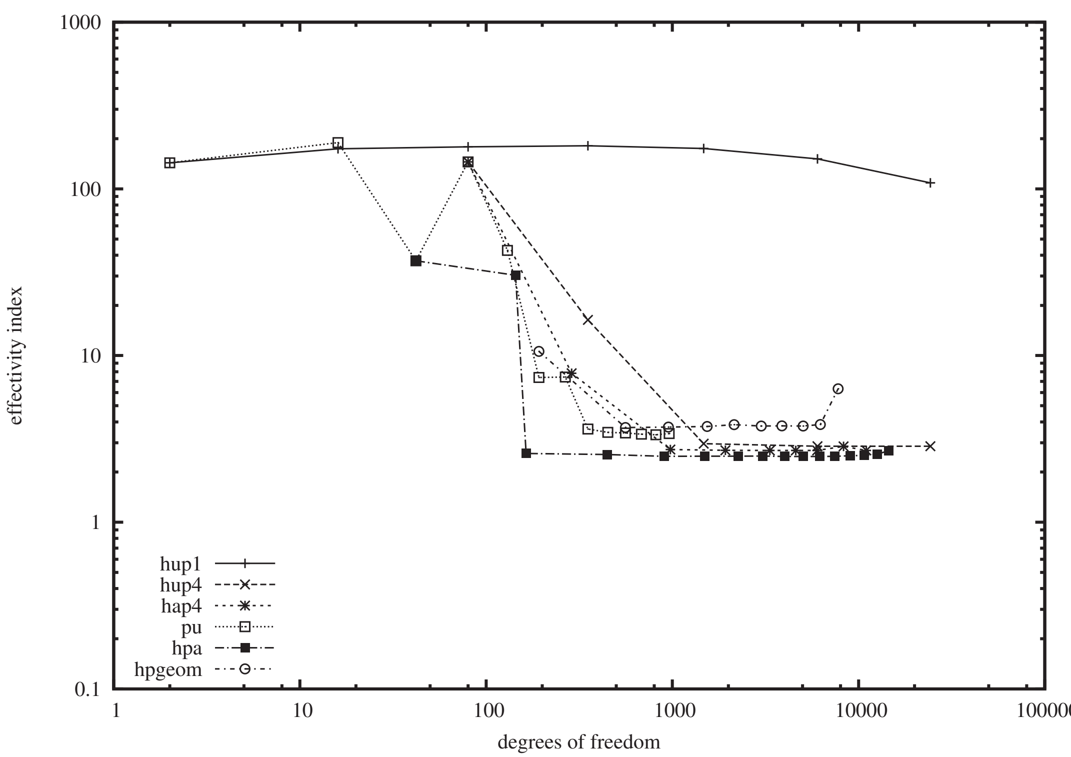

Figure 4. Poisson ratio $\nu = 0.49999$, effectivity indices.

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