AUTOMORPHIC GROUP REPRESENTATIONS: A NEW PROOF OF BLATTNER'S THEOREM

P. DE LA HARPE AND R. J. PLYMEN

1. *Introduction*

There are many constructions which give rise to the hyperfinite II_1 factor $\mathfrak{A}$ of Murray and von Neumann [**12**]. For example:

(i) the infinite tensor product of a countable number of matrix algebras with respect to their traces;

(ii) the group measure space construction from an ergodic measure preserving transformation;

(iii) the left regular representation of a locally finite discrete group with infinite conjugacy classes;

(iv) the von Neumann algebra generated by the CAR algebra of an infinite-dimensional Euclidean space E in its trace representation.

To each of these ways of obtaining $\mathfrak{A}$ correspond automorphisms of $\mathfrak{A}$. The construction that will interest us is the fourth one. The automorphisms in question arise naturally in the following way. The vectors in E generate $\mathfrak{A}$ as a von Neumann algebra; the full orthogonal group $O(E)$ of E acts on E, hence on $\mathfrak{A}$ as automorphisms.

The normal subgroup G_2 of $O(E)$ which acts as inner automorphisms of $\mathfrak{A}$ is of particular interest. The group G_2 was determined by Blattner [**2**]. The normal subgroup G_1 of $O(E)$ which acts as inner automorphisms of the CAR algebra is also of interest, and was determined by Shale and Stinespring [**21**]. The purpose of this article is to give new, simpler proofs of these two results.

We quote a result of the first-named author according to which each element of a proper normal subgroup of $O(E)$ is a compact perturbation either of the identity operator I or of $-I$. We prove directly that G_1 and G_2 are proper normal subgroups of $O(E)$ and then exploit the spectral theory of compact operators. Although our proofs are new, they make liberal use of original ideas of Blattner, Shale and Stinespring. It is illuminating to prove the two basic results simultaneously. Our proofs are naive and constructive: corresponding to the spectral decomposition of an orthogonal operator in G_2, the implementing unitary operator in $\mathfrak{A}$ is explicitly constructed as an infinite product.

Let $\mathscr{C}$ denote the CAR algebra. Since the centres of $\mathfrak{A}$ and $\mathscr{C}$ comprise scalar multiples of the identity, each implementing unitary operator is determined up to such a scalar. Consequently we have projective unitary representations of G_1 and G_2 as follows:

$$G_2 \to U(\mathfrak{A})/U(1), \qquad G_1 \to U(\mathscr{C})/U(1)$$

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where $U(\mathfrak{A})$ and $U(\mathscr{C})$ denote the unitary groups in $\mathfrak{A}$ and $\mathscr{C}$ respectively. When E is of finite dimension $2n$, these representations become the spin representation

$$O(E) \to U(2^n)/U(1).$$

This aspect, already noted by Shale and Stinespring, has been developed by the present authors in [9] and [13].

In Section 2 we recall various definitions and facts concerning orthogonal groups and the CAR algebra. In Section 3, the production of an outer automorphism of $\mathfrak{A}$, and the above result on proper normal subgroups of $O(E)$, leads readily to the following result: a separable locally compact group acts faithfully by outer automorphisms of $\mathfrak{A}$. This result is due to Blattner [2], and, for the special case of a discrete group, to Suzuki [24]; see also Choda [25], and Nakamura and Takeda [26]. Sections 4 and 5 are devoted to the proofs of the two basic results. We have tried to make these proofs as self-contained as possible. We avoid the use of unbounded operators.

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2. *Various orthogonal groups and statement of the main results*

We first review basic facts on the Clifford algebra of a real Hilbert space: details can be found in any one of [2, 9, 10, 14, 21]. The classical theory of finite-dimensional Clifford algebras appears in Atiyah, Bott and Shapiro [1].

Let E be a real Hilbert space, with inner product denoted by $\langle . | . \rangle$. Let $\mathrm{Cl}\,(E)$ be the Clifford algebra of the quadratic form q on E defined by $q(e) = \langle e|e \rangle = \|e\|^2$. We denote by $X \mapsto X^*$ both the main anti-automorphism of $\mathrm{Cl}\,(E)$, namely that which restricts to the identity on E, and its semi-linear extension to the complexified algebra $\mathrm{Cl}\,(E)^{\mathbb{C}} = \mathrm{Cl}\,(E) \otimes_{\mathbb{R}} \mathbb{C}$. Both $\mathrm{Cl}\,(E)$ and $\mathrm{Cl}\,(E)^{\mathbb{C}}$ are $\mathbb{Z}_2$-graded algebras. It is easy to check that $\mathrm{Cl}\,(E)^{\mathbb{C}}$ admits a unique normalized central even linear form τ, which is called the *trace*. " Even " means that τ vanishes on the odd part of the grading, " central " means that $\tau(XY) = \tau(YX)$ and " normalized " that $\tau(1) = 1$.

If F is a subspace of E, there is a natural injection of $\mathrm{Cl}\,(F)^{\mathbb{C}}$ into $\mathrm{Cl}\,(E)^{\mathbb{C}}$; indeed, if F is closed in E, then $\mathrm{Cl}\,(E)^{\mathbb{C}}$ is the graded tensor product of $\mathrm{Cl}\,(F)^{\mathbb{C}}$ and $\mathrm{Cl}\,(F^{\perp})^{\mathbb{C}}$, where $F^{\perp}$ is the orthogonal complement to F. By uniqueness of the trace, the restriction of τ to $\mathrm{Cl}\,(F)^{\mathbb{C}}$ is the trace on $\mathrm{Cl}\,(F)^{\mathbb{C}}$.

When F has dimension $2n$, it is a standard fact that $\mathrm{Cl}\,(F)^{\mathbb{C}} \cong \mathbb{C}(2^n)$, the algebra of $2^n \times 2^n$ matrices over $\mathbb{C}$; see [1]. If E is infinite dimensional, then both $\mathrm{Cl}\,(E)^{\mathbb{C}}$ and its even part are simple algebras; both the centres coincide with scalar multiples of the identity.

From now on, E will always denote an infinite dimensional real Hilbert space. Note that, as we shall be interested in complexified algebras, we could equally as well have started with the negative definite quadratic form $e \mapsto -\|e\|^2$ [9].

A prehilbert space structure is defined on $\mathrm{Cl}\,(E)^{\mathbb{C}}$ by $\langle X|Y \rangle = \tau(XY^*)$; we denote by H_0 the complex Hilbert space obtained by completion. We denote the continuous extension of τ to H_0 by the same letter, and we write $\|.\|_2$ for the norm defined by the inner product on H_0; the natural inclusion of E in H_0 is an isometry. The left multiplication by any X in $\mathrm{Cl}\,(E)^{\mathbb{C}}$ extends to a bounded operator $\pi_0(X)$ on H_0; we denote its norm by $\|X\|_{\infty}$. The completion $\mathscr{C}(E)$ of $\mathrm{Cl}\,(E)^{\mathbb{C}}$ with respect to this norm is a uniformly hyperfinite C*-algebra known as the C*-Clifford algebra, or the CAR algebra over E. If 1 denotes again the unit in this algebra, the triple $(H_0, \pi_0, 1)$ can be identified with the cyclic representation of $\mathscr{C}(E)$ associated to τ by the Gelfand–Naimark–Segal construction. If E were finite dimensional, then π_0 would be the left regular representation of $\mathrm{Cl}\,(E)^{\mathbb{C}}$.

The weak closure $\mathfrak{A}(E)$ of the image of π_0 in the algebra of bounded operators on H_0 is a von Neumann algebra; indeed, it is easy to check that $\mathfrak{A}(E)$ is the hyperfinite factor of type II_1. We shall identify $\mathrm{Cl}\,(E)^{\mathbb{C}}$ and $\mathscr{C}(E)$ with subalgebras of $\mathfrak{A}(E)$, and the latter to a subspace (not closed) of H_0. The operator norm on $\mathfrak{A}(E)$ will again be denoted by $\|.\|_\infty$. Each algebra introduced so far is $\mathbb{Z}_2$-graded: $\mathscr{C}(E) = \mathscr{C}(E)^+ \oplus \mathscr{C}(E)^-$, $\mathfrak{A}(E) = \mathfrak{A}(E)^+ \oplus \mathfrak{A}(E)^-$.

The theorems we are concerned with set up a basic relationship between some groups of unitary operators in the II_1 factor $\mathfrak{A}(E)$ on the one hand, and some groups of unitary operators in the "real factor of type I_∞" on the other hand. We now describe this relationship.

Let $O(E)$ be the group of orthogonal operators on E, also called canonical transformations. From the universal properties of $\mathrm{Cl}\,(E)$, it follows that any R in $O(E)$ extends uniquely to an automorphism $\mathscr{C}(R)$ of $\mathscr{C}(E)$ and also to an automorphism $\mathfrak{A}(R)$ of $\mathfrak{A}(E)$; this defines homomorphisms form $O(E)$ to the groups of * automorphisms of $\mathscr{C}(E)$ and $\mathfrak{A}(E)$ respectively. The images consist of the so-called *special* automorphisms. Special automorphisms are called *Bogoliubov* automorphisms in mathematical physics. A * automorphism ϕ of $\mathscr{C}(E)$ (respectively of $\mathfrak{A}(E)$) is *inner* if there exists a unitary element U in $\mathscr{C}(E)$ (respectively in $\mathfrak{A}(E)$) with $\phi = \mathrm{Ad}\,(U)$, namely with $\phi(X) = UXU^*$ for all X in the algebra. The centres of both $\mathscr{C}(E)$ and $\mathfrak{A}(E)$ coincide with scalar multiples of the identity. If $\mathfrak{A}(R) = \mathrm{Ad}\,U$ with U unitary in $\mathfrak{A}(E)$ then we have

$$\mathrm{Ad}\,U = \mathfrak{A}\big((-I)R(-I)\big) = \alpha(\mathrm{Ad}\,U)\alpha^{-1} = \mathrm{Ad}\,\alpha(U)$$

where $\alpha = \mathfrak{A}(-I)$. Consequently $\alpha(U) = \lambda U$ with λ a scalar of modulus 1. But α has period 2 so $\lambda^2 = 1$. Therefore either $\alpha(U) = U$ or $\alpha(U) = -U$. When $\alpha(U) = U$, U lies in $\mathfrak{A}(E)^+$ and $\mathfrak{A}(R)$ is called *even*; when $\alpha(U) = -U$, U lies in $\mathfrak{A}(E)^-$ and $\mathfrak{A}(R)$ is called *odd*. Similar terminology applies to $\mathscr{C}(R)$.

Define $O^+(E)$ to be the set (not a group) of those R in $O(E)$ for which the eigenspace $\{e \in E : Re = -e\}$ is either infinite-dimensional, or finite even-dimensional; let $O^-(E)$ be the complement of $O^+(E)$ in $O(E)$. It is shown in Putnam and Wintner [17] that $O^+(E)$ is the image by the exponential map of the space of skew-adjoint operators on E. Let us denote by $C_1(E)$ (respectively $C_2(E)$, $C_\infty(E)$) the set of all nuclear (respectively Hilbert–Schmidt, compact) operators on E; each is clearly an ideal in the algebra of all bounded operators on E. If $p = 1, 2, \infty$, then the sets

$$\begin{aligned} SO(E)_p &= \{R \in O^+(E) : R-1 \in C_p(E)\} \\ O(E)_p &= \{R \in O(E) : R-1 \in C_p(E)\} \\ O^t(E)_p &= SO(E)_p \cup \{R \in O(E) : -R \in O^-(E) \quad \text{and} \quad R+1 \in C_p(E)\} \\ O^{\max}(E)_p &= O(E)_p \cup \{R \in O(E) : R+1 \in C_p(E)\} \end{aligned}$$

are all normal subgroups of $O(E)$ (the superscript t means "twisted"). The lattice of subgroups is

$$\begin{tikzcd} & O^{\max}(E)_p \arrow[dl, dash] \arrow[dr, dash] & \\ O(E)_p \arrow[dr, dash] & & O^t(E)_p \arrow[dl, dash] \\ & SO(E)_p & \end{tikzcd}$$

The quotient $O^{\max}(E)_p/SO(E)_p$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Quite specifically, let us denote by $O^b(E)_p$ the complement of $SO(E)_p$ in $O^t(E)_p$; then the complement of $SO(E)_p$ in $O(E)_p$ is $\{R \in O(E) : -R \in O^b(E)_p\}$ and the complement of $O(E)_p \cup O^t(E)_p$ in $O^{\max}(E)_p$ is $\{R \in O(E) : -R \in SO(E)_p\}$. Incidentally, $O(E)_1$ is denoted by $O(E; C_1)$ in [**9**] and $O(E)_2$ is defined in [**15**]. Also, $SO(E)_p$ is a subgroup of index two in $O(E)_p$ and $O^t(E)_p$; each of these is, in turn, a subgroup of index two in $O^{\max}(E)_p$.

If R lies in $O^{\max}(E)_\infty$, then $E \otimes_{\mathbb{R}} \mathbb{C}$ has an orthonormal basis of eigenvectors of $R \otimes 1$. This property is crucial in the following discussion.

We pause here for a short digression. The group $O(E)_1$ has a standard bi-invariant metric which assigns to two operators the nuclear norm of their difference. Using the same method as that used in Putnam and Wintner [**17**] one shows that the connected component of the topological group defined in this way is precisely $SO(E)_1$. It follows that $O^t(E)_1$ (respectively $O^{\max}(E)_1$) can be looked at as a topological group having two (respectively four) connected components, with $SO(E)_1$ again as the component of the identity. This remark carries over to the cases $p = 2$ and $p = \infty$ (and indeed to any p with $1 \leqslant p \leqslant \infty$ for the appropriate definitions in terms of the von Neumann–Schatten classes of operators $C_p(E)$). It is also true that the natural injection $O^{\max}(E)_1 \to O^{\max}(E)_2$ is continuous with dense range, as is the natural injection $O^{\max}(E)_2 \to O^{\max}(E)_\infty$. This ends the digression.

The main results are the following:

THEOREM 1 (Shale and Stinespring). *Let $R \in O(E)$. Then*

(i) *$\mathscr{C}(R)$ is inner if and only if $R \in O^t(E)_1$*

(ii) *$\mathscr{C}(R)$ is inner and even if and only if $R \in SO(E)_1$.*

THEOREM 2 (Blattner). *Let $R \in O(E)$. Then*

(i) *$\mathfrak{A}(R)$ is inner if and only if $R \in O^t(E)_2$*

(ii) *$\mathfrak{A}(R)$ is inner and even if and only if $R \in SO(E)_2$.*

In the original papers, Blattner's theorem came first [**2**]. Shale and Stinespring obtained their theorem as a corollary of a result which is itself close to Theorem 2 (see [**20**; Section 3]). In Sections 4 and 5 below, we shall prove the theorems independently of each other; we shall express the arguments simultaneously: most lemmas will have statements (i) and (ii).

The notations $\mathscr{C}(R)$, $\mathscr{C}(E)$, $\mathfrak{A}(R)$, $\mathfrak{A}(E)$ have been used because $\mathscr{C}$ and $\mathfrak{A}$ are *functors*. This aspect has been extensively developed by Schrader and Uhlenbrock n [19].

3. *Special automorphic group representations*

Let Aut $(\mathfrak{A})$ denote the group of automorphisms of $\mathfrak{A}(E)$ as a * algebra. Then Aut $(\mathfrak{A})$ is the group of automorphisms of $\mathfrak{A}(E)$ as a von Neumann algebra, because the ultraweak topology on $\mathfrak{A}(E)$ is determined by the normal states and these are determined by the order structure which is preserved by automorphisms (see [**6**; Chapter I, §4]). As E generates $\mathfrak{A}(E)$ as a von Neumann algebra, it follows that the map $R \mapsto \mathfrak{A}(R)$ is a monomorphism of $O(E)$ into the group

$$\mathrm{Aut}_E(\mathfrak{A}) = \{\phi \in \mathrm{Aut}\,(\mathfrak{A}) : \phi(E) \subset E\}.$$

It is easy to see that, for any ϕ in $\mathrm{Aut}_E(\mathfrak{A})$, the induced linear map on E is indeed an orthogonal operator; in other words, $O(E)$ is canonically isomorphic to the group $\mathrm{Aut}_E(\mathfrak{A})$ of special automorphisms of $\mathfrak{A}$.

Let Inn $(\mathfrak{A})$ denote the group of inner automorphisms of $\mathfrak{A}(E)$. It is a normal subgroup of Aut $(\mathfrak{A})$; indeed, if $\phi' = \mathrm{Ad}\,(U)$ for some unitary U in $\mathfrak{A}(E)$, then $\phi\phi'\phi^{-1} = \mathrm{Ad}\,\phi(U)$ for any ϕ in Aut $(\mathfrak{A})$. We shall denote by G_2 the inverse image of $\mathrm{Inn}_E(\mathfrak{A}) = \mathrm{Inn}(\mathfrak{A}) \cap \mathrm{Aut}_E(\mathfrak{A})$ under the map $R \mapsto \mathfrak{A}(R)$. It is clear that $\mathrm{Inn}_E(\mathfrak{A})$ is normal in $\mathrm{Aut}_E(\mathfrak{A})$ and that G_2 is normal in $O(E)$.

LEMMA 1. *The principal automorphism* $\alpha = \mathfrak{A}(-I)$ *of* $\mathfrak{A}(E)$ *is outer, so that* G_2 *is a proper normal subgroup of* $O(E)$.

Proof. Suppose, on the contrary, that $\alpha = \mathrm{Ad}\,U$ with U unitary in $\mathfrak{A}(E)$. Let N be the positive integers and let $(e_n)_{n \in N}$ be an orthonormal basis in E. Let $V_n = -\mathrm{Ad}\,e_n$. Then V_n is a symmetry (self-adjoint unitary) on H_0. So V_n is reflection in its maximal invariant subspace L_n. For any finite subset $S = \{n_1, \ldots, n_k\} \subset N$ with $n_1 < \ldots < n_k$, let $e_S = e_{n_1} \ldots e_{n_k}$. The elements e_S form an orthonormal basis of H_0. The following equation is easy to check:

$$V_n e_S = \begin{cases} e_S \text{ when } |S| \text{ odd and } n \notin S, \text{ or } |S| \text{ even and } n \in S \\ -e_S \text{ otherwise.} \end{cases}$$

It follows immediately that $\cap L_n = 0$. But for all n we have

$$V_n(U) = -e_n U e_n = \alpha(e_n) U e_n = U e_n U^* U e_n = U$$

so that $U \in \cap L_n$, a contradiction. This proof is based on an idea of Blattner [**2**; p. 273].

Our task is to identify G_2. As a first step, we quote a global algebraic fact: any proper normal subgroup of $O(E)$ is in $O^{\max}(E)_\infty$ (see Proposition 1R in [**8**]). So we have

LEMMA 2. G_2 *is a subgroup of* $O^{\max}(E)_\infty$.

Let Γ be a separable locally compact group. Let r_1 be a faithful continuous representation of Γ into some orthogonal group $O(E_1)$, with E_1 a real separable Hilbert space (for example, r_1 may be the left regular representation of Γ on real-valued L^2-functions). Let r be an infinite countable direct sum of copies of r_1; then r is a representation of Γ on a Hilbert space that we may identify with E. Since the direct sum is infinite, it is clear that $r(\gamma)$ lies in $O^{\max}(E)_\infty$ if and only if $r(\gamma) = 1$, that is if and only if γ is the identity in Γ. Following Blattner, we let ρ be the composite map : $\Gamma \xrightarrow{r} O(E) \to \mathrm{Aut}_E(\mathfrak{A})$. We have proved the following result of Blattner [**2**; p. 674].

COROLLARY 3. *Let Γ be a separable locally compact group. Then Γ has a special continuous automorphic group representation ρ such that $\rho(\gamma)$ is outer unless γ is the identity in Γ.*

By a special automorphic group representation we mean a homomorphism $\rho : \Gamma \to \mathrm{Aut}_E(\mathfrak{A})$; it is continuous if the map $\gamma \to \rho(\gamma)X$ of Γ into $\mathfrak{A}(E)$ is continuous for each X in $\mathfrak{A}(E)$, with the weak operator topology on $\mathfrak{A}(E)$. The special case of Corollary 3 when Γ is a countable discrete group was obtained by Suzuki [**24**]. Other proofs are due to Choda [**25**] and to Nakamura and Takeda [**26**].

4. *Proofs of Theorems* 1 *and* 2: *first part*

We shall write $U(\mathscr{C})$ (respectively $U(\mathfrak{A})$) for the group of unitary elements in $\mathscr{C}(E)$ (respectively $\mathfrak{A}(E)$) furnished with the norm topology (respectively strong operator topology). It is clear the $U(\mathscr{C})$ is a complete metric group, but the case of $U(\mathfrak{A})$ necessitates a lemma (which could be traced to Lemma 1.3.2 in Murray and von Neumann [**12**]). We denote by $\mathfrak{A}(E)_1$ the unit ball $\{X \in \mathfrak{A}(E) : \|X\|_\infty \leqslant 1\}$. Recall that $\mathfrak{A}(E)$ is identified with a subset of the Hilbert space H_0, with norm $\|\,.\,\|_2$.

LEMMA 4. *The topology of $U(\mathfrak{A})$ is that of a complete metric group with metric given by $d(U, V) = \|U-V\|_2$.*

Proof. We denote in this proof by Ω rather than by 1 the trace vector in H_0 determined by the trace τ. Then $\|X\|_2 = \|X\Omega\|_2$ for all X in $\mathfrak{A}(E)$. The vector Ω is cyclic and separating for $\mathfrak{A}(E)$, hence also for the commutant $\mathfrak{A}(E)'$.

Let $(X_\beta)_{\beta \in B}$ be a net in $\mathfrak{A}(E)_1$ converging in the strong operator topology to some X in $\mathfrak{A}(E)_1$. Then $(X_\beta\Omega)$ is a net in H_0 converging to $X\Omega$, so that (X_β) converges to X in the $\|\,.\,\|_2$-topology.

Suppose conversely that (X_β) is a net in $\mathfrak{A}(E)_1$ converging in the $\|\,.\,\|_2$-topology to some X in $\mathfrak{A}(E)_1$. For all Y' in $\mathfrak{A}(E)'$ we have

$$\begin{aligned}\|X_\beta Y'\Omega - XY'\Omega\|_2 = \|Y'(X_\beta - X)\Omega\|_2 &\leqslant \|Y'\|_\infty . \|(X_\beta - X)\Omega\|_2 \\ &= \|Y'\|_\infty \|X_\beta - X\|_2.\end{aligned}$$

Hence the net (X_β) in H_0 converges to $X\xi$ for any vector ξ in $\mathfrak{A}(E)'\Omega$; as the latter lies dense in H_0, this holds also for any ξ in H_0, so that (X_β) converges strongly to X.

We have shown that the strong operator topology and the $\|\,.\,\|_2$-topology coincide on $\mathfrak{A}(E)_1$; they coincide *a fortiori* on $U(\mathfrak{A})$.

The equality $XY - X_0Y_0 = X(Y-Y_0) + (X-X_0)Y_0$ shows that multiplication is strongly continuous on $\mathfrak{A}(E)_1$. The involution $X \mapsto X^*$ is an isometry in the $\|\,.\,\|_2$ norm, since

$$\|X^*\|_2{}^2 = \tau(XX^*) = \tau(X^*X) = \|X\|_2{}^2$$

for all X in $\mathfrak{A}(E)$. Hence $U(\mathfrak{A})$ with the strong operator topology is indeed a topological group. Finally, the metric group $U(\mathfrak{A})$ is complete because it is strongly closed in $\mathfrak{A}(E)_1$, which is strongly complete by von Neumann's density theorem. This ends the proof.

Let $R \in SO(E)_\infty$. The spectral theorem for compact operators implies that there exists a decomposition into orthogonal planes $E = \bigoplus_{n \in N} E_n$ such that the restriction $R|E_n$ is a plane rotation $R(\theta_n)$ through an angle $\theta_n \in (-\pi, \pi]$ for each $n \in N$; as $R-I$ is compact, the sequence (θ_n) converges to 0. It is then easy to compute that

$$\|I-R\|_1 = 2^{3/2} \sum_{n \in N} (1-\cos\theta_n)^{1/2}$$

$$\|I-R\|_2 = 2\{\sum_{n \in N} (1-\cos\theta_n)\}^{1/2}$$

where $\|\,.\,\|_1$ and $\|\,.\,\|_2$ denote respectively the nuclear and the Hilbert–Schmidt norms of operators on E, namely

$$\|T\|_1 = Tr\,|T|,$$

$$\|T\|_2 = \{Tr\,|T|^2\}^{1/2}$$

where $|T| = (T^*T)^{\frac{1}{2}}$. Since $1-\cos\theta \sim \theta^2/2$ as $\theta \to 0$, it follows that

(i) $R \in SO(E)_1$ if and only if (θ_n) is in l^1

(ii) $R \in SO(E)_2$ if and only if (θ_n) is in l^2.

Let (e_n) be an orthonormal basis in E such that $\{e_{2n-1}, e_{2n}\}$ span E_n for each n. Define

$$U_n = e_{2n-1}\{\cos(\theta_n/2)e_{2n-1} - \sin(\theta_n/2)e_{2n}\}$$

$$= \cos(\theta_n/2) - e_{2n-1}e_{2n}\sin(\theta_n/2)$$

$$V_n = U_1 U_2 \dots U_n$$

which are clearly all unitary and even elements in $\mathscr{C}(E)$, and *a fortiori* in $\mathfrak{A}(E)$.

LEMMA 5. *If $m, n \in N$ with $m < n$, then*

(i) $\|V_m - V_n\|_\infty = 2\sin(|\theta_{m+1}| + \dots + |\theta_n|)/4$, *provided that* $|\theta_{m+1}| + \dots + |\theta_n| \leqslant 2\pi$;

(ii) $\|V_m - V_n\|_2{}^2 = 2 - 2\cos\theta_{m+1}/2 \dots \cos\theta_n/2$.

Proof. (i) In order to compute the C*-norm of $V_m - V_n$ we shall work in a Fock representation of $\mathscr{C}(E)$, which must be well adapted to the basis (e_n), namely to R; this device is used by Shale and Stinespring [**20**; Section 3.2]. To be quite explicit, let J be the orthogonal complex structure defined on E by

$$Je_{2n-1} = e_{2n}, \qquad Je_{2n} = -e_{2n-1}$$

for each $n \in N$. The complex Hilbert space E_J defined by E and J has an orthonormal basis (f_n) with $f_n = e_{2n-1}$ for each $n \in N$ (the sets underlying E and E_J coincide).

Let H_J be the *Fock space* defined by J, which is the Hilbert space completion of the exterior algebra $\Lambda(E_J)$ furnished with its standard hermitian form; the space H_J has an orthonormal basis consisting of the vectors

$$\eta_S = f_{n_1} \wedge f_{n_2} \wedge \dots \wedge f_{n_k}$$

where k runs over $\{0\} \cup N$ and where $S = \{n_1, \dots, n_k\}$ runs over strictly increasing sequences of positive integers. To each basis vector e_n of E is associated an operator C_n defined on H_J by

$$C_n(\eta) = f_n \wedge \eta \qquad \text{for all } \eta \text{ in } H_J;$$

it is a so-called *creator*; its adjoint C_n^* is an *annihilator*; they satisfy the *canonical anticommutation relations* (CAR):

$$C_nC_n^* + C_n^*C_n = 1 \qquad n \in N$$

$$C_nC_m + C_mC_n = C_n^*C_m^* + C_m^*C_n^* = 0.$$

It follows from the universal property of Clifford algebras that the assignments

$$\pi_J(e_{2n-1}) = C_n + C_n^* \qquad \pi_J(e_{2n}) = i(C_n - C_n^*)$$

for each $n \in N$ extend uniquely to a morphism of C*-algebras $\pi_J : \mathscr{C}(E) \to L(H_J; H_J)$ which is the *Fock representation* defined by J; it has a cyclic vector Ω_J which is the unit in $\Lambda(E_J) = H_J$ and which is called the *Fock vacuum*. It is easy to check that

$$\pi_J(e_{2n-1}e_{2n}) = i(C_n^*C_n - C_nC_n^*) \quad n \in N$$

$$\pi_J(U_n)\eta_S = \begin{cases} \exp(i\theta_n/2)\eta_S & n \in S \\ \exp(-i\theta_n/2)\eta_S & n \notin S \end{cases}$$

and that

$$\pi_J(U_{m+1}U_{m+2} \dots U_n)\eta_S = \exp\{i(\pm\theta_{m+1} \pm \dots \pm\theta_n)/2\}\eta_S$$

where θ_j has a minus sign if and only if $j \notin S$. Hence

$$\pi_J(V_m - V_n)\eta_S = \omega\{1 - \exp i(\pm\theta_{m+1} \pm \dots \pm\theta_n)/2\}\eta_S$$

where ω is a complex number of modulus 1.

The situation now is that the η_S constitute an orthonormal basis in H_J of eigenvectors for the operator $\pi_J(V_m - V_n)$; the norm of this operator is consequently the maximum modulus of the corresponding eigenvalues, which is precisely

$$2\sin(|\theta_{m+1}| + \dots + |\theta_n|)/4.$$

But $\mathscr{C}(E)$ is simple, π_J is isometric, so the C*-norm of $V_m - V_n$ is indeed given by (i).

Notice incidentally that the notations are consistent: if a, b are vectors in E, then

$$\langle \pi_J(ab)\Omega_J | \Omega_J \rangle = \langle a|b \rangle + i\langle Ja|b \rangle = \langle a|b \rangle_J$$

which is their scalar product in H_J, while

$$\langle \pi_0(ab)\Omega | \Omega \rangle = \langle a|b \rangle$$

which is their scalar product computed in H_0. Notice also that, if E were finite-dimensional, π_J would be the *spin representation* of $\mathrm{Cl}(E)^{\mathbf{C}} \cong \mathscr{C}(E)$, and that the equivalence class of π_J would not depend on J. When, on the contrary, E is infinite-dimensional, this class does depend on J: see, for example **[11]**.

(ii) The proof of the second equality reduces to a short computation:

$$\begin{aligned} \|V_m - V_n\|_2^2 &= \tau\{(V_m - V_n)^*(V_m - V_n)\} \\ &= \tau(1 - V_m^*V_n - V_n^*V_m + 1) \\ &= 2 - \tau(V_m^*V_n + V_n^*V_m) \\ &= 2 - \tau(U_{m+1} \dots U_n + U_n^* \dots U_{m+1}^*) \\ &= 2 - 2\cos\theta_{m+1}/2 \dots \cos\theta_n/2 \end{aligned}$$

since τ annihilates all of $\mathscr{C}(E)$ except the one-dimensional subspace spanned by 1.

The following lemma covers one half of Theorems 1(ii) and 2(ii).

LEMMA 6. (i) *If $R \in SO(E)_1$, then the infinite product ΠU_n is convergent in $U(\mathscr{C})$ in the* C*-*norm and implements $\mathscr{C}(R)$.*

(ii) *If $R \in SO(E)_2$, then the infinite product ΠU_n is convergent in $U(\mathfrak{A})$ in the $\|.\|_2$ norm and implements $\mathfrak{A}(R)$.*

Proof. (i) Let $R \in SO(E)_1$; then (θ_n) lies in l^1. Therefore (V_n) is a Cauchy sequence in $U(\mathscr{C})$ in the C*-norm by Lemma 5(i). But $U(\mathscr{C})$ is a complete metric group, so that (V_n) converges in C*-norm to V, say. Then V implements $\mathscr{C}(R)$ by a straightforward continuity argument.

(ii) Let $R \in SO(E)_2$; then the series with positive terms $\Sigma(1-\cos\theta_n)$ converges. For the proof, let c_n denote $\cos\theta_n$; as $\theta_n \to 0$, there exists m_0 with $c_n > 0$ as soon as $n \geqslant m_0$. The partial products $c_{m_0} \ldots c_n$ converge to some real number $d \neq 0$ (see for example Theorem 14.41 in **[11]**). If $m, n \in N$ with $m_0 < m < n$, it means that

$$c_m \ldots c_n \to 1 \qquad \text{as } m, n \to \infty.$$

By Lemma 5(ii), this implies that (V_n) is a Cauchy sequence in $U(\mathfrak{A})$ in the $\|.\|_2$-topology, which converges by Lemma 4 to V, say. Then V implements $\mathfrak{A}(R)$ by continuity. This ends the proof.

The preceding lemma shows that naive judgment is confirmed and that

$$\mathscr{C}\{R(\theta_1) \oplus R(\theta_2) \oplus \ldots\} = \mathrm{Ad}\,(U_1 U_2 \ldots)$$

$$\mathfrak{A}\{R(\theta_1) \oplus R(\theta_2) \oplus \ldots\} = \mathrm{Ad}\,(U_1 U_2 \ldots)$$

where the infinite product in the first displayed equation converges in the C*-norm, and that in the second converges in the strong operator topology. Recall that each U_n is a product of the vectors e_{2n-1} and $e'_{2n} = \cos(\theta_n/2)e_{2n-1} - \sin(\theta_n/2)e_{2n}$ in E_n, which is a plane in E. The angle between e_{2n-1} and e'_{2n} is $\theta_n/2$ and we have

$$\mathrm{Ad}\,(e_{2n-1}e'_{2n})Z = -e_{2n-1}(-e'_{2n}Ze'_{2n})e_{2n-1}$$

for each Z in the appropriate algebra, so that $\mathrm{Ad}\, U_n$ acts as a product of two reflections.

In this sense, any operator in $SO(E)_1$ (respectively $SO(E)_2$) is a product of countably many reflections in subspaces of codimension 1. This is a satisfactory generalization of the well-known fact that each element of $SO(n)$ is a product of an even number of reflections in hyperplanes of R^n.

5. *Second part of the proofs*

The next move (Lemma 8) is to prove what is essentially the converse of Lemma 6. The following easy technical lemma could be part of the proof of Lemma 8(i), but it seems clearer to present it beforehand: it is essentially Lemma 3 of **[20]**.

LEMMA 7. *Let H, K be complex Hilbert spaces, let A be a unitary operator on H and let B be a unitary operator on K. Then, for any unitary operator C on H, there exists $\lambda \in S^1$ (the complex numbers of modulus* 1) *with*

$$\|C \otimes 1_K - A \otimes B\| \geqslant \|\lambda A \otimes 1_K - A \otimes B\|.$$

Proof. As A and B are unitary, one has

$$\|C \otimes 1_K - A \otimes B\| = \|(C \otimes 1_K - A \otimes B)A^{-1} \otimes B^{-1}\|$$
$$= \|(CA^{-1}) \otimes B^{-1} - 1_H \otimes 1_K\|$$

and similarly

$$\|\lambda A \otimes 1_K - A \otimes B\| = \|1_H \otimes \lambda B^{-1} - 1_H \otimes 1_K\| = \|\lambda B^{-1} - 1_K\|.$$

The lemma results from the following fact: if U (respectively V) is a unitary operator on H (respectively K) then there exists $\lambda \in S^1$ with $\|U \otimes V - 1_H \otimes 1_K\| \geqslant \|\lambda V - 1_K\|$.

Let us prove this fact. We denote by $\sigma(X)$ the spectrum of an operator X. If $\mu \in \sigma(U)$ and $\nu \in \sigma(V)$ then $\mu\nu$ is clearly in $\sigma(U \otimes V)$ (incidentally, the converse holds [3]). Now choose $\lambda \in \sigma(U)$ which minimizes $\|\lambda V - 1_K\|$; then

$$\|U \otimes V - 1_H \otimes 1_K\| = \sup\{|\omega - 1| : \omega \in \sigma(U \otimes V)\}$$
$$\geqslant \sup\{|\lambda\nu - 1| : \nu \in \sigma(V)\}$$
$$= \|\lambda V - 1\|.$$

Note that the lemma is not correct with $\lambda = 1$ (look, for example, at the cases for which $U = -1_H$, $V = -1_K$ and $-C = A = 1_H$, $B = -1_K$).

LEMMA 8. (i) *Let $R \in O(E)$ with $R \in SO(E)_\infty$ or $-R \in SO(E)_\infty$. If $\mathscr{C}(R)$ is inner, then $R \in SO(E)_1$.*

Proof. We work again in a Fock representation π_J with J adapted as above to R. Let E_J be E made complex and let R_J be the unitary operator on E_J corresponding to R. It is easy to check (on the basis vectors η_S) that

$$\pi_J(Re) = \Lambda(R_J)\pi_J(e)\Lambda(R_J)^{-1}$$

for all e in E (see also the lemma in Section 3 of [13]). Since π_J is irreducible, there is at most one unitary operator on H_J which implements $\pi_J(\mathscr{C}(R))$, up to a scalar multiple of modulus 1. Consequently, if $\mathscr{C}(R) = \operatorname{Ad} W$ with W unitary in $\mathscr{C}(E)$, then we may assume that $\pi_J(W) = \Lambda(R_J)$.

For each $n \in N$, let F_n be the complex subspace of E_J spanned by $f_1, \ldots, f_n$. Then

$$H_J = \Lambda(E_J) = \Lambda(F_n) \otimes \Lambda(F_n{}^\perp).$$

Let R_n, R_n' be the restriction to F_n, $F_n{}^\perp$ of R_J. Then

$$\Lambda(R_J) = \Lambda(R_n) \otimes \Lambda(R_n').$$

The * subalgebra of $\mathscr{C}(E)$ generated by the union of the F_n is dense in $\mathscr{C}(E)$. Hence there exists for each $n \in N$ an element W_n in $\mathrm{Cl}\,(F_n)^{\mathrm{C}}$ such that the sequence (W_n) converges to W; we may choose each W_n unitary (if not, replace it by the unitary part in its polar decomposition). For each $n \in N$, the operator $\pi_J(W_n)$ is of the form $Y_n \otimes 1$ on $\Lambda(F_n) \otimes \Lambda(F_n{}^\perp)$, and is arbitrarily near $\Lambda(R_n) \otimes \Lambda(R_n')$ if n is large enough. By Lemma 7, we can choose W_n such that $Y_n = \alpha_n \Lambda(R_n)$ for some α_n in S^1. Now

$$(Y_n \otimes 1)\Omega_J = \alpha_n \Omega_J \to \Lambda(R_J)\Omega_J = \Omega_J$$

so that $\alpha_n \to 1$, and we may assume that each α_n is 1 to start with. We have shown that the hypothesis $\mathscr{C}(R) = \operatorname{Ad} W$ implies that the sequence $\Lambda(R_n) \otimes 1$ converges to $\Lambda(R_J)$ in the C*-norm.

With the notations introduced in the proof of Lemma 5(i), the η_S are the elements of an orthonormal basis of H_J with respect to which $\Lambda(R_n) \otimes 1$ is diagonal for each $n \in N$. It follows readily that

$$\|\Lambda(R_n) \otimes 1 - \Lambda(R_m) \otimes 1\| = 2 \sin\{|\theta_{m+1}| + \ldots + |\theta_n|\}/2$$

as soon as $m < n$. As $\Lambda(R_n) \otimes 1$ is a Cauchy sequence of operators on H_J, the sequence of non-negative numbers $|\theta_n|$ is summable and $R \in SO(E)_1$. This ends the proof, due essentially to Shale and Stinespring [**20**; p. 322].

Let $R \in SO(E)_1$; it is easy to verify that, with V_n as in Lemma 5(i) and V as in Lemma 6(i),

$$(\det(R_n))^{1/2} \pi_J(V_n) = \Lambda(R_n)$$

$$(\det(R_J))^{1/2} \pi_J(V) = \Lambda(R_J).$$

In other words, $\mathscr{C}(R)$ is implemented both by $\pi_J^{-1}(\Lambda(R_J))$ and V, which differ by a scalar of modulus 1: this number is a square root of the determinant of R_J. It may be helpful to recall the definition of $\Lambda(R_J)$: $\Lambda(R_J)$ is the unique unitary on $\Lambda(E_J)$ such that

$$\Lambda(R_J) f_{n_1} \wedge \ldots \wedge f_{n_k} = R_J f_{n_1} \wedge \ldots \wedge R_J f_{n_k}$$

whenever $n_1 < \ldots < n_k$.

LEMMA 8. (ii) *Let $R \in O(E)$ with $R \in SO(E)_\infty$ or $-R \in SO(E)_\infty$. If $\mathfrak{A}(R)$ is inner, then $R \in SO(E)_2$.*

Proof. Let $\mathfrak{A}(R) = \operatorname{Ad} U$. Since the e_S constitute an orthonormal basis of H_0, there exists an index S with $\langle U | e_S \rangle \neq 0$. Let $V = e_S{}^* U$. Then

$$\langle V | 1 \rangle = \langle e_S{}^* U | 1 \rangle = \langle U | e_S \rangle \neq 0$$

so that in particular V is even. Define R' by $R'e = e_S{}^* e e_S$ for each e in E. Then $R' \in O^t(E)_2$ (indeed, either $R' - I$ or $R' + I$ is of finite rank, depending on whether $|S|$ is even or odd). Let $T = R'R$. Then

$$\mathfrak{A}(T) = \mathfrak{A}(R')\mathfrak{A}(R) = \operatorname{Ad} e_S{}^* . \operatorname{Ad} U = \operatorname{Ad} V.$$

But $R \in SO(E)_2$ if and only if $T \in SO(E)_2$. Thus we may assume without loss of generality that U is even and that $\langle U | 1 \rangle \neq 0$.

Let $V_n : H_0 \to H_0$ be given by $V_n(X) = (Re_n) X e_n$ and let $M_n = \{X \in H_0 \colon V_n(X) = X\}$ be the invariant subspace of V_n for each $n \in N$; define M to be the intersection of the M_n. For each $n \in N$, let $P_n = (1 + V_n)/2$ be the projection on M_n and let $Q_n = P_1 \ldots P_n$. Now the V_n mutually commute so the P_n mutually commute; therefore (Q_n) is a monotonically decreasing sequence of projections, and so is a strongly convergent sequence (see [7; p. 255]); the limit is the projection Q on M. Now U, regarded as a vector in H_0, has the property that

$$V_n(U) = (Re_n) U e_n = (U e_n U^*) U e_n = U$$

for each $n \in N$; hence U lies in M and $P \leqslant Q$, where P is the projection on the one-dimensional subspace of H_0 spanned by U. Therefore

$$0 < |\langle U | 1 \rangle|^2 = \langle P(1) | 1 \rangle \leqslant \langle Q(1) | 1 \rangle = \lim_n \langle Q_n(1) | 1 \rangle.$$

It is easy to verify that

$$\langle Q_{2n-1}(1)|1\rangle = \langle Q_{2n}(1)|1\rangle = \cos^2\theta_1/2 \dots \cos^2\theta_n/2.$$

Therefore

$$0 < \prod_{n\in N} \cos^2\theta_n/2 \leqslant 1.$$

By an elementary result concerning infinite products, this implies that $\Sigma\sin^2\theta_n/2$ is convergent. Since $\sin\theta \sim \theta$ as $\theta \to 0$, this implies that $\Sigma\theta_n{}^2$ is convergent and so $R \in SO(E)_2$. The ideas in this proof are again due to Blattner [**2**; p. 671].

Let Aut $(\mathscr{C})$ be the group of * automorphisms of $\mathscr{C}(E)$; we may assume that they are uniformly continuous, though we need not [**18**; p. 162]. Let Inn $(\mathscr{C})$ be the normal subgroup of inner automorphisms. We may identify

$$\mathrm{Aut}_E(\mathscr{C}) = \{\phi \in \mathrm{Aut}\,(\mathscr{C}) : \phi(E) \subset E\}$$

with $O(E)$ and

$$\mathrm{Inn}_E(\mathscr{C}) = \mathrm{Aut}_E(\mathscr{C}) \cap \mathrm{Inn}(\mathscr{C})$$

with a normal subgroup G_1 of $O(E)$.

We shall denote by SG_1 (respectively SG_2) the group of those $R \in O(E)$ for which the inner automorphism $\mathscr{C}(R)$ (respectively $\mathfrak{A}(R)$) is even. It is clear that $G_1 \subset G_2$ and that $SG_1 \subset SG_2$.

LEMMA 9. (i) *SG_1 is of index two in G_1.*

(ii) *SG_2 is of index two in G_2.*

Proof. (i) Let t be a unit vector in E and let $T \in O(E)$ be defined by

$$Te = -e + 2\langle e|t\rangle t$$

for all e in E; one has $T^2 = I$ and $T \in O^b(E)_1$. Then $\mathscr{C}(T) = \mathrm{Ad}\,t$ and $T \in G_1$, $T \notin SG_1$. If R is in $O(E)$ then $\mathscr{C}(R)$ is inner odd if and only if $\mathscr{C}(RT)$ is inner even; hence SG_1 is of index two in G_1. Notice also that R is in $O^b(E)_1$ if and only if RT is in $SO(E)_1$. The proof of (ii) is similar.

Proof of Theorem 1. Let $R \in G_1$. By Lemma 2, one has either R or $-R$ in $O^t(E)_\infty = SO(E)_\infty \cup O^b(E)_\infty$. If one of R, $-R$ is in $SO(E)_\infty$, then $R \in SO(E)_1$ by Lemma 8(i). If one of R, $-R$ is in $O^b(E)_\infty$, then $R \in O^b(E)_1$ by Lemmas 9(i) and 8(i). Hence $G_1 \subset O^t(E)_1$. The opposite inclusion follows from Lemmas 6(i) and 9(i).

By Lemma 6(i) one has $SO(E)_1 \subset SG_1$. As $SO(E)_1$ is of index two in $O^t(E)_1$ and as SG_1 is of index two in G_1, this implies that $SO(E)_1 = SG_1$.

Proof of Theorem 2. The above argument works with the help of Lemmas 6(ii), 8(ii) and 9(ii).

6. *Remarks*

(1). It follows from Theorems 1 and 2 that SG_1 contains the derived group of SG_2. Indeed, let $R_j = I + X_j \in SO(E)_2$, so that $X_j + X_j{}^* + X_jX_j{}^* = 0$ $(j = 1, 2)$; if $R = R_1R_2R_1{}^{-1}R_2{}^{-1}$, then $R - I$ is a sum of products of two and more Hilbert–Schmidt operators, so that $R - I$ is nuclear and $R \in O(E)_1$; a connectedness argument shows that $R \in SO(E)_1$. A simple direct proof of this, in terms of $\mathscr{C}(E)$ and $\mathfrak{A}(E)$, would probably shed light on both theorems.

(2). Stronger versions of Theorems 1 and 2 in terms of universal covering groups of $SO(E)_1$ and $SO(E)_2$ appear in **[9]** and **[15]**.

(3). The group G_1^{rep} of those R in $O(E)$ for which $\pi \circ \mathscr{C}(R)$ and π are unitarily equivalent for any representation π of $\mathscr{C}(E)$ coincides with G_1, namely with $O^t(E)_1$. This is due to Slawny **[23]**. We conjecture that the group G_2^{rep} of those R in $O(E)$ for which $\pi \circ \mathfrak{A}(R)$ and π are unitarily equivalent for any normal representation π of $\mathfrak{A}(E)$ coincides with G_2. Indeed, if $O(E)$ is identified with $\text{Aut}_E(\mathfrak{A})$, then $G_2^{\text{rep}} = O(E) \cap N$ where $N = \{\alpha \in \text{Aut}(\mathfrak{A}) : \pi \circ \alpha$ and π are equivalent for any normal representation π of $\mathfrak{A}(E)\}$ is clearly a normal subgroup of Aut $(\mathfrak{A})$ containing Inn$(\mathfrak{A})$. Now if one could check that $N \neq \text{Aut}(\mathfrak{A})$, it would follow from a result of Connes [5; Corollary 4] that $N = \text{Inn}(\mathfrak{A})$. Consequently $G_2^{\text{rep}} = O(E) \cap \text{Inn}(\mathfrak{A})$ would be equal to $G_2 = O^t(E)_2$.

(4). The problem remains of characterizing those automorphisms of $\mathscr{C}(E)$ or $\mathfrak{A}(E)$ that can be written as $\mathscr{C}(R)$ or as $\mathfrak{A}(R)$ for some R in $SO(E)_\infty$. Results in **[16]** and **[22]** make it hard to guess the answer.

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Section de Mathématiques,
Université de Genève,
Case postale 124, 1211 Geneva 24,
Switzerland.

Department of Mathematics,
The University,
Manchester M13 9PL.