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Cancellation of hyperbolic ε-hermitian forms and of simple knots

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Introduction

An n-knot will be a smooth, oriented submanifold $K^n \subset S^{n+2}$ such that K^n is homeomorphic to S^n. Given two knots K_1^n and K_2^n, we define their connected sum $K_1^n \# K_2^n$ as in [**13**], p. 39. The cancellation problem for n-knots is the following.

Problem. Assume that K_1^n, K_2^n and K^n are n-knots such that $K_1^n \# K^n$ is isotopic to $K_2^n \# K^n$. Does this imply that K_1^n and K_2^n are isotopic?

The cancellation problem has a positive solution if $n = 1$. Indeed, Schubert has shown that every 1-knot has a unique decomposition as a connected sum of finitely many indecomposable 1-knots (cf. [**14**]).

For n odd, counter-examples to the cancellation of n-knots have been given in [**2**]. In the present note we shall show that the cancellation problem also has a negative solution for n even, $n \geqslant 4$. The case $n = 2$ remains open.

The first section is entirely algebraic, and contains counter-examples to the cancellation of hyperbolic ϵ-hermitian forms ($\epsilon = \pm 1$) over orders in algebraic number fields. This is inspired by results of Wiegand concerning the cancellation of torsion-free modules over commutative rings of dimension 1, see [**17**]. We shall apply this in Section 2 to obtain the desired knot-theoretical examples.

1. *Cancellation of hyperbolic ϵ-hermitian forms*

Let F be an algebraic number field with a $\mathbb{Q}$-involution, which we shall denote by $x \to \bar{x}$. Let S be a Dedekind set of primes of F (cf. [**11**], § 21), and let $\tilde{R}$ be the ring of S-integers of F. Let R be an S-order of F, i.e. a subring of finite index of $\tilde{R}$. Assume $\bar{R} = R$. Let M be a torsion free R-module of finite rank, and let $M' = \mathrm{Hom}_R(M, R)$ with the R-module structure given by $rf(x) = f(\bar{r}x)$. We shall say that M is *reflexive* if the evaluation homomorphism $e\colon M \to M''$, defined by $e(m)(f) = \overline{f(m)}$, is an isomorphism.

If M is a reflexive R-module, let us associate to M the hyperbolic ϵ-hermitian ($\epsilon = \pm 1$) form $H(M)$, defined as follows:

$$M \oplus M' \times M \oplus M' \to R$$
$$(x, f) \times (y, g) \to f(y) + \overline{\epsilon g(x)}.$$

Let N be a torsion free R-module, and let $h\colon N \times N \to R$ be an ϵ-hermitian form. We shall say that (N, h) is *unimodular* if the homomorphism induced by h, $\mathrm{ad}(h)\colon N \to N'$, is bijective.

If M is reflexive, then it is easy to check that $H(M)$ is a unimodular form.

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Bak and Scharlau have proved that if R is Dedekind, and if the involution is non-trivial, then cancellation holds for hyperbolic forms, i.e.

$$H(M_1) \boxplus H(M) \cong H(M_2) \boxplus H(M)$$

implies $H(M_1) \cong H(M_2)$, where $\boxplus$ denotes the orthogonal sum, cf. [1], theorem 7·1 (ii). We shall see that this is not necessarily the case when R is not Dedekind. Notice that if the involution is trivial, then there are counter-examples to the cancellation of hyperbolic forms even if R is Dedekind (see Parimala [12]).

Our examples are inspired by some results of Wiegand [17]. We shall begin by recalling these results. Let $c = \{r \in R \text{ such that } r\tilde{R} \subseteq R\}$ be the conductor of R in $\tilde{R}$. For any ring S, we shall denote by S^* the group of units of S. Let

$$E(R) = \operatorname{Coker}(\tilde{R}^* \to (\tilde{R}/c)^*)$$

and let $D(R)$ be the cokernel of the composition $(R/c)^* \to (\tilde{R}/c)^* \to E(R)$. Let M be a torsion-free R-module, and let $a \in E(R)$. Let ϕ be an automorphism of $\tilde{R}M/cM$ such that $\det(\phi)$ maps to a in $E(R)$. We shall define a torsion-free R-module M^a by the pullback

$$\begin{array}{ccccc} M^a & & \longrightarrow & & \tilde{R}M \\ \downarrow & & & & \downarrow \\ M/cM & \rightarrowtail & \tilde{R}M/cM & \xrightarrow{\phi} & \tilde{R}M/cM \end{array}$$

Wiegand shows that the correspondence $(a, M) \to M^a$ is a well-defined action of the group $E(R)$ on the set of isomorphism classes of torsion free R-modules, cf. [17], proposition 2·2.

Let J be an invertible R-ideal, and let $a, b \in E(R)$. Then $J^a J^b = J^{ab}$. In particular, J^a is also invertible.

It is easy to check that $\tilde{R}$ is a reflexive R-module (use the identification of $\tilde{R}' = \operatorname{Hom}_R(\tilde{R}, R)$ with c). Therefore $H(\tilde{R})$ is a unimodular ϵ-hermitian form.

PROPOSITION. *Let $a \in E(R)$, and set $I = R^a$. Then*

(i) $H(I) \boxplus H(\tilde{R}) \cong H(R) \boxplus H(\tilde{R})$.

(ii) *If $\epsilon = +1$ and the involution is trivial, then $H(I) \cong H(R)$ if and only if the class of a is trivial in $D(R)$.*

(iii) *Let us assume that the class of $a\bar{a}^{-1}$ is non-trivial in $D(R)$. Then $I \oplus \bar{I}^{-1} \ncong R \oplus R$. In particular, $H(I) \ncong H(R)$.*

Proof. (i) follows immediately from the fact that $I \oplus \tilde{R} \cong R \oplus \tilde{R}$, see Wiegand [17], theorem 2·3 (iv).

(ii) As the involution is trivial, there are only two isotropic lines in $H(R)$. This implies that if $H(I) \cong H(R)$, then $I \cong R$. Therefore by Wiegand [17], proposition 2·2 (iv), the class of a is trivial in $D(R)$.

(iii) As $I = R^a$, it is easy to check that $\bar{I} = R^{\bar{a}}$. We are assuming that the class of $a\bar{a}^{-1}$ is not trivial in $D(R)$. By Wiegand [17], proposition 2·2 (iv), this implies that $I\bar{I}^{-1} = R^{a\bar{a}^{-1}}$ is not isomorphic to R. Therefore $I \oplus \bar{I}^{-1} \ncong R \oplus R$. But $I \oplus \bar{I}^{-1}$ is the underlying module of $H(I)$, so $H(I) \ncong H(R)$.

Using the above Proposition it is easy to obtain explicit examples of orders R and ideals I such that $H(I) \boxplus H(\tilde{R}) \cong H(R) \boxplus H(\tilde{R})$, but $H(I) \ncong H(R)$. The following examples have been obtained in this way, but they are so simple that we shall check everything by direct computation. These examples have been chosen in such a way that they can be used in Section 2 to construct the desired knot-theoretical examples.

Example 1

Let
$$R = \mathbb{Z}[t,t^{-1}]/(43t^2-85t+43) = \mathbb{Z}[3\alpha][\tfrac{1}{43}],$$
where $\alpha = (1-\sqrt{-19})/2$. Then $\tilde{R} = \mathbb{Z}[\alpha][\frac{1}{43}]$, and $c = 3\tilde{R}$. Let I be the kernel of the $\mathbb{Z}$-homomorphism $\phi\colon R \to \mathbb{F}_5$ defined by $\phi(\sqrt{-19}) = 1$. It is easy to check that $I\bar{I}^{-1}$ is not a principal ideal, so $I \oplus \bar{I}^{-1} \not\cong R \oplus R$. In particular, $H(I) \not\cong H(R)$.

On the other hand, we have $H(I) \boxplus H(\tilde{R}) \simeq H(\tilde{R}) \boxplus H(\tilde{R})$. To see this, it is enough to check that $I \oplus \tilde{R} \simeq R \oplus \tilde{R}$. Let us define an R-homomorphism $f\colon R \oplus \tilde{R} \to I \oplus \tilde{R}$ by the matrix
$$A = \begin{pmatrix} 10 & 1 \\ 9\alpha & \alpha \end{pmatrix}.$$

It is easy to check that $f(R \oplus \tilde{R}) \subseteq I \oplus \tilde{R}$: for instance, $f(\tilde{R}) \subseteq I$ because $3\tilde{R} \subseteq R$ and $3\alpha \in I$. It remains to see that f is an isomorphism. Let us consider the diagram
$$\begin{array}{ccc} R \oplus \tilde{R} & \xrightarrow{f} & I \oplus \tilde{R} \\ \cap & & \cap \\ \tilde{R} \oplus \tilde{R} & \xrightarrow{f} & \tilde{R} \oplus \tilde{R}. \end{array}$$

The index of $R \oplus \tilde{R}$ in $\tilde{R} \oplus \tilde{R}$ is 3. We have $\det A = \alpha$, therefore the index of $f(\tilde{R} \oplus \tilde{R})$ in $\tilde{R} \oplus \tilde{R}$ is 5. On the other hand, the index of $I \oplus \tilde{R}$ in $\tilde{R} \oplus \tilde{R}$ is 15, therefore $f(R \oplus \tilde{R}) = I \oplus \tilde{R}$.

To understand this example from the point of view of Wiegand's results, notice that $D(R) \simeq \mathbb{F}_9^{\cdot}/\mathbb{F}_3^{\cdot}$ and that $I = R^a$, where a is the class of α in $D(R)$. It is straightforward to check that $a \neq \bar{a}$ in $D(R)$.

Example 2

Let $R = \mathbb{Z}[t]/(t^2-t+16) = \mathbb{Z}[3\alpha]$, where $\alpha = (1+\sqrt{-7})/2$. Then $\tilde{R} = \mathbb{Z}[\alpha]$ and $c = 3\tilde{R}$. Let I be the R-ideal, which is generated by 2 and by $3\alpha+1$. As in Example 1, we check that $H(I) \not\cong H(R)$ and that $H(I) \boxplus H(\tilde{R}) \simeq H(R) \boxplus H(\tilde{R})$. An isomorphism $f\colon R \oplus \tilde{R} \to I \oplus \tilde{R}$ is given by the matrix
$$A = \begin{pmatrix} 4 & 1 \\ 3(3\alpha+1) & 2\alpha+1 \end{pmatrix}.$$

Let us consider $S = \mathbb{Z}[t,t^{-1}]/(16t^2-31t+16) = R[\frac{1}{2}]$. Then $S = R[\frac{1}{2}]$, and the conductor of $\tilde{S}$ in S is $3\tilde{S}$. Set $J = IS$. Then $J = S$, so $H(J) = H(S)$.

Notice that $D(R) \simeq \mathbb{F}_9^{\cdot}/\mathbb{F}_3^{\cdot}$, and that $I = R^a$, where a is the class of $1-\alpha$ in $D(R)$. On the other hand, $D(S)$ is trivial.

2. *Cancellation of simple knots*

Let K^n be an n-knot, and let U be a tubular neighbourhood of K^n. Let $X = \overline{S^{n+2}\backslash U}$ be the *complement* of K^n. We shall say that K^n is *simple* if $\pi_i(X) \simeq \pi_i(S')$ for $i < q$, where $n = 2q$ or $2q-1$.

Odd dimensional simple knots

Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ and let L be a $\mathbb{Z}$-torsion free Λ-module such that $1-t\colon L \to L$ is an isomorphism. Let us consider the $\mathbb{Z}$-involution of Λ which sends t to t^{-1}. Assume that there exists a unimodular $(-1)^{q+1}$-hermitian form $h\colon L \times L \to \mathbb{Q}(t)/\Lambda$. Then Kervaire's construction ([**8**], theorem II. 3) can be used to show that for every integer $q \geqslant 3$, there exists a simple $(2q-1)$-knot K such that the Blanchfield form associated to K is (L, h); see Kearton [**6**], addendum 8·3. In particular, $L \simeq H_q(\tilde{X}, \mathbb{Z})$, where X is the complement of K and $\tilde{X}$ is the infinite cyclic cover of X. Moreover, if (L, h) is skew-hermitian and if the signature of the rational quadratic form associated to (L, h) is divisible by 16 (see Trotter [**15**] or [**16**] for the definition of this form), then there exists a simple 3-knot with Blanchfield form isometric to (L, h). Recall that L is called the *knot module* of K.

Example 3

Let $R = \mathbb{Z}[t, t^{-1}]/(43t^2 - 85t + 43)$ and let us consider the ϵ-hermitian forms $H(I)$, $H(R)$ and $H(\tilde{R})$, as in § 1, Example 1. It is easy to see that these forms and their underlying modules have the properties that we have just described. Moreover, if $\epsilon = -1$ then it is straightforward to check that the associated rational quadratic forms have signature zero. Let K_1^{2q-1}, K_2^{2q-1} and K^{2q-1} be the simple knots associated to $H(I)$, $H(R)$ and $H(\tilde{R})$, for $q \geqslant 2$. By results of Levine [**10**], Trotter [**15**] and Kearton [**6**], two simple $(2q-1)$-knots $(q \geqslant 2)$, are isotopic if and only if the associated Blanchfield forms are isometric. Therefore $K_1 \# K$ is isotopic to $K_2 \# K$, but K_1 is not isotopic to K_2. Notice that these knots are not fibred, whereas the counter-examples of [**2**] are fibred.

Example 4

Let $R = \mathbb{Z}[t]/(t^2 - t + 16)$ and $S = \mathbb{Z}[t, t^{-1}]/(16t^2 - 31t + 16) = R[\frac{1}{2}]$ as in § 1, Example 2. Let K_1^{2q-1} and K^{2q-1} be the simple $(2q-1)$-knots $(q \geqslant 3)$, associated to $H(S)$ and $H(\tilde{S})$. The ϵ-hermitian forms $H(I)$ and $H(R)$ correspond to minimal Seifert surfaces M_1 and M_2 of K_1^{2q-1}. Similarly $H(\tilde{R})$ corresponds to a minimal Seifert surface M^{2q} of K^{2q-1}. Then $M_1^{2q} \# M^{2q}$ is isotopic to $M_2^{2q} \# M^{2q}$, but M_1^{2q} and M_2^{2q} are not isotopic (cf. [**3**], § 1).

Even-dimensional simple knots

We shall obtain the counter-examples to the cancellation of even-dimensional knots by spinning: this is an idea of Kearton, cf. [**7**]. If Σ^n is any n-knot, we shall denote by $\sigma(\Sigma^n)$ the $(n+1)$-knot obtained by spinning Σ^n (cf. [**9**]). Let X, σX be the complements of the knots Σ, $\sigma\Sigma$ and let $\tilde{X}$, $\sigma\tilde{X}$ be the infinite cyclic covers. If Σ^{2q-1} is simple, then $\sigma\Sigma^{2q-1}$ is also simple, and $H_q(\sigma\tilde{X}) \simeq H_{q+1}(\sigma\tilde{X}) \simeq H_q(\tilde{X})$, cf. Gordon [**4**], theorem 4·1.

Example 5

Let K_1^{2q-1}, K_2^{2q-1} and K^{2q-1} be the simple knots of Example 3 $(q \geqslant 2)$. Then $K_1^{2q-1} \# K^{2q-1}$ and $K_2^{2q-1} \# K^{2q-1}$ are isotopic. As spinning commutes with connected sums (cf. [**5**]), the $(2q)$-knots $\sigma(K_1^{2q-1})$ and $\sigma(K_2^{2q-1}) \# \sigma(K^{2q-1})$ are also isotopic. Recall that the knot modules of K_1^{2q-1} and K_2^{2q-1} are $I \oplus \bar{I}^{-1}$ and $R \oplus R$, which are not isomorphic. Therefore by the above result of Gordon we see that $\sigma(K^{2q-1})$ and $\sigma(K^{2q-1})$ are not isotopic.

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