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The geometric realization of Wall obstructions by nilpotent and simple spaces

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Introduction. Let π denote a finite group. It is well known that every element of the projective class group $K_0\mathbb{Z}\pi$ may be realized as Wall obstruction of a finitely dominated complex with fundamental group π (cf. (13)). We will study two subgroups $N_0\mathbb{Z}\pi$ and $N\mathbb{Z}\pi$ of $K_0\mathbb{Z}\pi$, which are closely related to the Wall obstruction of nilpotent spaces. If the group π is nilpotent and if S denotes the set of elements $x \in K_0\mathbb{Z}\pi$ which occur as Wall obstructions of nilpotent spaces, then

$$N_0\mathbb{Z}\pi \subset S \subset N\mathbb{Z}\pi.$$

It turns out that in many instances one has $N_0\mathbb{Z}\pi = N\mathbb{Z}\pi$ (cf. Section 3) and one obtains hence new information on S. The main theorem (2·4) provides a systematic way of constructing finitely dominated nilpotent (or even simple) spaces with non-vanishing Wall obstructions.

1. *The groups $T\mathbb{Z}\pi$ and $N\mathbb{Z}\pi$.* If π denotes a finite group then one defines $T\mathbb{Z}\pi \subset K_0\mathbb{Z}\pi$ to be the subgroup consisting of all elements of the form $[(k, N)] - [\mathbb{Z}\pi]$, where $N = \Sigma x$, $x \in \pi$, and (k, N) is the projective ideal in $\mathbb{Z}\pi$ generated by N and an integer k prime to card(π). The group $T\mathbb{Z}\pi$ is known to be trivial if π is cyclic (9). On the other hand $T\mathbb{Z}\pi \neq 0$ if π contains a noncyclic subgroup of odd order (11). $T\mathbb{Z}\pi$ is completely known for π a p-group (10).

It is convenient to think of $K_0\mathbb{Z}\pi$ to be generated by π-modules M of type FP and to write $[M]$ for the element $\Sigma(-1)^i [P_i] \in K_0\mathbb{Z}\pi$, if

$$0 \to P_n \to P_{n-1} \to \dots \to P_0 \to M \to 0$$

is a projective resolution of finite type. For instance, if k is prime to card (π) one has an exact sequence $0 \to \mathbb{Z}\pi \to (k, N) \to \mathbb{Z}/k \to 0$ and hence

$$[(k, N)] - [\mathbb{Z}\pi] = [\mathbb{Z}/k] \in K_0\mathbb{Z}\pi,$$

where $\mathbb{Z}/k$ is considered as a trivial π-module (cf. (4)). If $\pi \neq \{1\}$ then every trivial π-module of type FP is necessarily finite and of order prime to card(π). We can then identify $T\mathbb{Z}\pi$ with the subgroup of $K_0\mathbb{Z}\pi$ consisting of all elements representable in the form $[M]$, where M is a trivial π-module of type FP.

In view of the applications we have in mind, we will define more general subgroups $N_i\mathbb{Z}\pi \subset K_0\mathbb{Z}\pi$ in a similar way.

Definition 1·1. Let π be a finite non-trivial group. Then $N_i\mathbb{Z}\pi \subset K_0\mathbb{Z}\pi$ is the sub-group consisting of all elements of the form $\Sigma(-1)^k\ [P_k]$, where $P = \{P_k\}$ is a

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projective complex of finite type whose homology groups $H_k(P)$ are all nilpotent π-modules and for which $H_j(P) = 0$ for $j > i$. Furthermore, $N\mathbb{Z}\pi = \cup N_i\mathbb{Z}\pi$ and for $\pi = \{1\}$ we define $N\mathbb{Z}\pi = N_i\mathbb{Z}\pi = 0$ for all i.

To see that $N_i\mathbb{Z}\pi$ is indeed a subgroup for $\pi \neq \{1\}$, it suffices to check that all elements of $N_i\mathbb{Z}\pi$ are of finite order ($N_i\mathbb{Z}\pi$ is obviously closed under addition). But this amounts to showing that $\Sigma(-1)^k \operatorname{rank}(P_k) = 0$ for $P = \{P_k\}$ as in 1·1. But this follows immediately from the isomorphism

$$H_k(P) \otimes \mathbb{Q} \xrightarrow{\cong} H_k(P) \otimes_\pi \mathbb{Q}$$

which holds, since $H_k(P)$ is a nilpotent π-module, and from the equalities

$$\Sigma(-1)^k \operatorname{rank}(P_k) = \Sigma(-1)^k \dim_{\mathbb{Q}}(H_k(P) \otimes_\pi \mathbb{Q}) = \frac{1}{|\pi|}\Sigma(-1)^k \dim_{\mathbb{Q}}(H_k(P) \otimes \mathbb{Q}).$$

By definition $N_0\mathbb{Z}\pi$ consists of elements $x = [M]$, where M is a nilpotent π-module of type FP. If P is as in 1·1 and $H_j(P) = 0$ for $j > 0$ then $H_0(P) = M$ is a finite module, since

$$\Sigma(-1)^k \operatorname{rank}(P_k) = \dim_{\mathbb{Q}}(M \otimes_\pi \mathbb{Q}) = 0.$$

Clearly, this M is also cohomologically trivial, since it is of type FP.

COROLLARY 1·2. *The subgroup $N_0\mathbb{Z}\pi \subset K_0\mathbb{Z}\pi$ consists of all elements $x = [M] \in K_0\mathbb{Z}\pi$, where M is a finite, nilpotent, and cohomologically trivial π-module.*

This is clear from the above, since a finite M which is cohomologically trivial is of type FP (and even of projective dimension $\leqslant 1$ by (8)).

In particular we see that $T\mathbb{Z}\pi \subset N_0\mathbb{Z}\pi \subset N\mathbb{Z}\pi$. The following example will illustrate that in general however $T\mathbb{Z}\pi \neq N_0\mathbb{Z}\pi$.

LEMMA 1·3. *If π is cyclic of order* 15, *then $N_0\mathbb{Z}\pi$ is of order two.*

Proof. Choose a map $\pi \to \operatorname{Aut}(\mathbb{Z}/9)$ which maps on to the subgroup of order 3. This defines a nilpotent π-module M with underlying abelian group $\mathbb{Z}/9$. M is a nilpotent, cohomologically trivial π-module and M generates $\tilde{K}_0\mathbb{Z}\pi \cong \mathbb{Z}/2$ (cf. (5), Lemma 2·8). Hence $N_0\mathbb{Z}\pi$ is of order two.

Remark. Let $D\mathbb{Z}\pi$ denote the kernel of the map $K_0\mathbb{Z}\pi \to K_0\overline{\mathbb{Z}\pi}$, induced by including $\mathbb{Z}\pi$ into a maximal $\mathbb{Z}$-order $\overline{\mathbb{Z}\pi}$ in $\mathbb{Q}\pi$. If π is nilpotent, then

$$N\mathbb{Z}\pi \subset D\mathbb{Z}\pi.$$

This is proved in (12) for π cyclic and in (7) for a general nilpotent π. An example is given in (7) to show that in general $N\mathbb{Z}\pi \neq D\mathbb{Z}\pi$, even if π is cyclic.

2. *The realization theorem*. All spaces we consider are supposed to be pointed connected CW-complexes; $\tilde{X}$ denotes the universal covering space of a space X. As usual a homology class is called spherical if it lies in the image of the Hurewicz homomorphism. First, we will describe a particular way of killing certain spherical classes.

LEMMA 2·1. *Let X be an n-dimensional CW-complex and let $P \subset H_n\tilde{X}$ denote a projective $\pi_1 X$-module consisting of spherical classes. Denote by $\phi: L \to H_n\tilde{X}$ a map from a* *free $\pi_1 X$-module L with basis $\{b_\alpha, \alpha \in I\}$, such that $\phi(L) = P$. Then one can form a new complex*

$$X' = X \cup (\coprod_{\alpha \in I} e_\alpha^{n+1})$$

of dimension $n+1$ such that

(1) *there is a commutative diagram*

$$\begin{array}{ccccc} \pi_{n+1}(\tilde{X}', \tilde{X}) & \xrightarrow{\partial} & \pi_n \tilde{X} & \xrightarrow{Hu} & H_n \tilde{X} \\ \phi_L \uparrow \cong & & & \nearrow & \\ L & & \phi & & \end{array}$$

(2)
$$H_i \tilde{X}' \cong \begin{cases} H_i \tilde{X} & \text{if} \quad i \neq n, n+1 \\ (H_i \tilde{X})/P & \text{if} \quad i = n \\ \operatorname{Ker} \phi & \text{if} \quad i = n+1. \end{cases}$$

(3) $H_{n+1}\tilde{X}'$ *is projective and spherical* (*i.e. it consists entirely of spherical classes*).

Proof. Since P is projective, one can find $\bar{P} \subset \pi_n \tilde{X}$ such that $\bar{P}$ is mapped isomorphically onto P by the Hurewicz homomorphism. Hence we can choose $\bar{\phi}: L \to \bar{P}$ to obtain a commutative diagram

$$\begin{array}{ccccc} & & \bar{P} & \subset & \pi_n \tilde{X} \\ & \bar{\phi} \nearrow & & & \\ L & & \cong \downarrow h & & \downarrow Hu \\ & \searrow & & & \\ & \phi & P & \subset & H_n \tilde{X} \end{array}$$

Denote by $\mathrm{pr}: \tilde{X} \to X$ the projection. We attach $(n+1)$-cells to X using the maps $\mathrm{pr}\,(\bar{\phi} b_\alpha)$, $\alpha \in I$, and obtain $X' = X \cup (\coprod e_\alpha^{n+1})$. It is immediate that ϕ lifts to an isomorphism ϕ_L giving rise to diagram (1). We consider now the diagram obtained by mapping the homotopy sequence of $(\tilde{X}', \tilde{X})$ into the homology sequence of this pair:

$$\begin{array}{ccccccc} \pi_{n+1}\tilde{X}' & \xrightarrow{\alpha} & \pi_{n+1}(\tilde{X}', \tilde{X}) & \xrightarrow{\partial_\pi} & \pi_n \tilde{X} & & \\ \downarrow & & \downarrow \cong & & \downarrow & & \\ H_{n+1}\tilde{X}' & \xrightarrow{\beta} & H_{n+1}(\tilde{X}', \tilde{X}) & \xrightarrow{\partial_H} & H_n \tilde{X} & \twoheadrightarrow & H_n \tilde{X}' \end{array}$$

Then $\operatorname{im} \partial_\pi = \bar{P}$ and hence $\operatorname{im} \partial_H = P$. Therefore $H_n \tilde{X}' \cong (H_n \tilde{X})/P$. Note also that $\alpha(\pi_{n+1}\tilde{X}')$ is mapped isomorphically onto $\beta(H_{n+1}\tilde{X}') \cong H_{n+1}\tilde{X}'$, since $\bar{P}$ is mapped isomorphically onto P. Hence $H_{n+1}\tilde{X}' \cong \operatorname{Ker} \partial_\pi \cong \operatorname{Ker} \phi$ and $\pi_{n+1}\tilde{X}' \to H_{n+1}\tilde{X}'$ is onto. Therefore (2) and (3) hold.

THEOREM 2·2. *Let X be a connected CW-complex of dimension $n > 1$ and let M be a $\pi_1 X$-module of cohomological dimension $\leqslant 1$. Then there is a space Y obtained from X by attaching cells of dimension $\geqslant n$, such that*

$$H_i \tilde{Y} = \begin{cases} H_i \tilde{X} & \text{if} \quad i \neq n \\ (H_n \tilde{X}) \oplus M & \text{if} \quad i = n. \end{cases}$$

Furthermore, if X is finitely dominated and M of type FP, then Y can be chosen to be finitely dominated; the reduced Wall obstructions of X and Y are then related by

$$\tilde{w}Y = \tilde{w}X + (-1)^n [M] \in \tilde{K}_0 \mathbb{Z}\pi.$$

Proof. Choose a free resolution

$$\ldots \to L_{n+i} \xrightarrow{\phi_{n+i}} L_{n+i-1} \to \ldots \to L_{n+1} \xrightarrow{\phi_{n+1}} L_n \twoheadrightarrow M.$$

(If M is a type FP, we choose a free resolution of finite type.) Since proj. dim $M \leqslant 1$, im (ϕ_{n+i}) is projective for all $i \geqslant 1$. We construct Y inductively as follows. Let $Y^n = X \vee B$, where B is a bouquet of n-spheres corresponding to a basis of L_n. Then $H_n \tilde{Y}^n \cong (H_n \tilde{X}) \oplus L_n$ and $H_n \tilde{Y}^n$ contains a spherical projective submodule P isomorphic to im (ϕ_{n+1}). Attaching $(n+1)$-cells to Y^n with respect to the map $L_{n+1} \to H_n \tilde{Y}^n$ corresponding to ϕ_{n+1}, we obtain by the previous Lemma a new space Y^{n+1} with

$$H_i \tilde{Y}^{n+1} = \begin{cases} H_i \tilde{X} & \text{if } i \neq n, n+1 \\ (H_n \tilde{X}) \oplus M & \text{if } i = n \\ \operatorname{Ker} \phi_{n+1} & \text{if } i = n+1. \end{cases}$$

Since $\operatorname{Ker} \phi_{n+1} \cong H_{n+1} \tilde{Y}^{n+1}$ is projective and spherical (Lemma 2·1) we can kill this group using $L_{n+2} \to H_{n+1} \tilde{Y}^{n+1}$. By repeating this construction we obtain spaces Y^{n+k}, $k \geqslant 1$, and we can form $Y = \cup Y^{n+k}$. By construction, $\tilde{Y}$ has the homology groups claimed in the theorem. Furthermore, the cellular chain complex of $\tilde{Y}$ is isomorphic to the complex

$$\ldots \to L_{n+1} \to L_{n+i-1} \to \ldots \to L_n \oplus C_n \tilde{X} \to C_{n-1} \tilde{X} \to \ldots,$$

where $C\tilde{X}$ is the cellular chain complex of $\tilde{X}$. Since the complex

$$\ldots \to L_{n+i} \to L_{n+i-1} \to \ldots \to L_{n+1} \to \operatorname{im} \phi_{n+1}$$

is contractible, it follows that Y is a retract of Y^{n+1}. Hence Y is finitely dominated, if X is finitely dominated and M of type FP. From the definition of the Wall obstruction it is immediate that

$$\begin{aligned} \tilde{w}(Y) &= \sum_{i=0}^{n} (-1)^i [\bar{C}_i X] + (-1)^{n+1} [\operatorname{im} \phi_{n+1}] \\ &= \tilde{w}(X) + (-1)^n [M], \end{aligned}$$

where $\bar{C}\tilde{X}$ is a chain complex of type FP, chain homotopy equivalent to $C\tilde{X}$.

Before we apply this Theorem to the construction of certain nilpotent spaces, we need the following elementary lemma.

LEMMA 2·3. *Let π be a finite group. Then there exists a finite complex X with $\pi_1 X \cong \pi$ and Euler characteristic $\chi(X) = 0$, such that all covering transformations $t: \tilde{X} \to \tilde{X}$ are homotopic to the identity.*

Proof. Choose an embedding $\pi \subset SU(k)$. Then $X = SU(k)/\pi$ has the desired properties.

Note that $X = SU(k)/\pi$ is nilpotent, if π is a nilpotent group, and it is a simple space, in case π is abelian.

We can now prove our main theorem.

THEOREM 2·4. *Let π be a finite nilpotent group and let $x \in N_0\mathbb{Z}\pi \subset K_0\mathbb{Z}\pi$. Then there exists a finitely dominated nilpotent space Y with fundamental group π and Wall obstruction $w(Y) = x$. If x lies in $T\mathbb{Z}\pi$ and π is abelian, then Y may be chosen simple.*

Proof. Let $x = [M]$ with M a nilpotent π-module (trivial π-action in case $x \in T\mathbb{Z}\pi$). Choose $X = SU(k)/\pi$ as in the previous lemma (we may assume that $\dim X$ is even). Then, according to Theorem 2·2 we can construct a finitely dominated space Y with $\pi_1 Y \cong \pi_1 X$ and

$$H_i\,\tilde{Y} = \begin{cases} H_i\tilde{X} & \text{if} \quad i \neq \dim X \\ (H_i\tilde{X}) \oplus M & \text{if} \quad i = \dim X. \end{cases}$$

It follows that $wY = [M] = x$, since X is finite with Euler characteristic 0. Moreover, Y is nilpotent since its fundamental group is nilpotent and since $H_i\,\tilde{Y}$ is nilpotent for all i. In order to see that Y is simple in case π is abelian and $x \in T\mathbb{Z}\pi$, we prove the stronger result stating that, if M is a trivial π-module, then all covering transformations $t\colon \tilde{Y} \to \tilde{Y}$ are homotopic to the identity. By the 'Hasse-Principle' for free maps (3) it suffices to show that the localizations $t_p\colon \tilde{Y}_p \to \tilde{Y}_p$ are homotopic to the identity for all primes p. If p does not divide the order of M, then the inclusion $X \subset Y$ induces $H_i\tilde{X}_p \cong H_i\tilde{Y}_p$ and hence $X_p \simeq Y_p$ (the induced map of fundamental groups is certainly an isomorphism). Therefore $t_p \simeq \mathrm{Id}\,\tilde{Y}_p$ since the corresponding result is true for X by construction. If p divides the order of the trivial π-module M, then p is necessarily prime to the order of π, since otherwise M would not be cohomologically trivial. It follows therefore that the projection $\tilde{Y} \to Y$ induces a homotopy equivalence $\tilde{Y}_p \simeq Y_p$ if p divides the order of M; clearly this implies that $t_p \simeq \mathrm{Id}\,\tilde{Y}_p$ and hence the global map t is homotopic to the identity (3).

Remark. If in Theorem 2·4 the assumption that π be nilpotent is dropped, one can still construct the finitely dominated space Y with $w(Y) = x \in N_0\mathbb{Z}\pi$. The space Y will however in general only be *homologically nilpotent* in the sense that $\pi_1 Y$ operates nilpotently on $H_i\tilde{Y}$ for all i.

Theorem 2·4 enables us to construct examples of the following types:

COROLLARY 2·5. (*a*) *There exists a finitely dominated simple space with non-vanishing Wall obstruction.*

(*b*) *There exists a finitely dominated nilpotent space with cyclic fundamental group and non-vanishing Wall obstruction.*

Proof. For (*a*) choose any abelian group π with $T\mathbb{Z}\pi \neq 0$ (e.g. $(\mathbb{Z}/p)\times(\mathbb{Z}/p)\times(\mathbb{Z}/p)$, p any prime) and apply Theorem 2·4. Similarly, for (*b*) we can choose any cyclic group π with $N_0\mathbb{Z}\pi \neq 0$ (e.g. $\mathbb{Z}/15$, cf. Lemma 1·3) and we obtain such an example by Theorem 2·4.

3. *Some computations involving $N\mathbb{Z}\pi$.* One can consider $N_0\mathbb{Z}\pi$ as a lower bound for the elements in $K_0\mathbb{Z}\pi$ which occur as Wall obstructions of finitely dominated homologically nilpotent space. Similarly, $N\mathbb{Z}\pi$ provides an upper bound for this set. The

following examples show that for many groups one has $N_0\mathbb{Z}\pi = N\mathbb{Z}\pi$ and, indeed, we don't know of an example with $N_0\mathbb{Z}\pi \neq N\mathbb{Z}\pi$.

Our main tool will be a homomorphism

$$T\colon U(\mathbb{Z}[1/n]) \to K_0\mathbb{Z}\pi/T\mathbb{Z}\pi$$

which was defined in (5) for groups π with cyclic Sylow subgroups ($U(\mathbb{Z}[1/n])$ the units in $\mathbb{Z}[1/n]$ and $n = \operatorname{card}(\pi)$). For the following computation we will assume that π is of square-free order n (hence T is defined). If $N = \Sigma x$, $x \in \pi$, the projection $\mathbb{Z}\pi \to \mathbb{Z}\pi/N$ induces an injective map

$$\overline{pr}_*\colon K_0\mathbb{Z}\pi/T\mathbb{Z}\pi \rightarrowtail K_0(\mathbb{Z}\pi/N).$$

It is convenient to describe T by considering $K_0(\mathbb{Z}\pi/N)$ as the range of T. If p is a prime dividing $n = \operatorname{card}(\pi)$ then the trivial π-module $\mathbb{Z}/p$ considered as a $\mathbb{Z}\pi/N$-module, is of type FP with respect to the ring $\mathbb{Z}\pi/N$ and

$$\overline{pr}_* T(p) = [\mathbb{Z}/p] \in K_0(\mathbb{Z}\pi/N).$$

Furthermore, $T(-1) = 0$ (for details see (5)).

The connexion with $N\mathbb{Z}\pi$ is given by the next lemma.

LEMMA 3·1. *Let π denote a group of square-free order n and let $x \in N\mathbb{Z}\pi$. Let $P = \{P_i\}$ be a projective π-complex with $x = \Sigma(-1)^i[P_i]$ and H_iP nilpotent for all i. Then*

$$\rho(P) = \operatorname{card} H_{\mathrm{ev}}(\mathbb{Z}\pi/N \otimes_\pi P)/\operatorname{card} H_{\mathrm{odd}}(\mathbb{Z}\pi/N \otimes_\pi P)$$

is a unit in $\mathbb{Z}[1/n]$ and if $\bar{x}$ denotes the image of x in $N\mathbb{Z}\pi/T\mathbb{Z}\pi$ then

$$\bar{x} = T\rho(P) \in K_0\mathbb{Z}\pi/T\mathbb{Z}\pi.$$

In particular one has $N\mathbb{Z}\pi/T\mathbb{Z}\pi \subset \operatorname{im}(T)$.

Proof. This result was proved in ((5), Section 3) in case $x = wX$, the Wall obstruction of a homologically nilpotent space X. The same proof works for an arbitrary $x \in N\mathbb{Z}\pi$.

COROLLARY 3·2. *Let π be of order p or $2p$, p an arbitrary prime. Then $N\mathbb{Z}\pi = 0$.*

Proof. If $\operatorname{card}(\pi) = p$ with p an arbitrary prime or if $\operatorname{card}(\pi) = 2p$, p an odd prime, then $\operatorname{im}(T) = 0$ by ((5), Theorem 2·5). Hence $N\mathbb{Z}\pi = T\mathbb{Z}\pi$ in these cases. But in both cases one has $T\mathbb{Z}\pi = 0$ (cf. (11)). It remains to consider the case $\operatorname{card}(\pi) = 4$. But it is well known that $\tilde{K}_0\mathbb{Z}\pi = 0$ if $\operatorname{card}(\pi) = 4$. Hence the result follows.

THEOREM 3·3. *If π is a cyclic group of square-free order, then*

$$N_0\mathbb{Z}\pi = N\mathbb{Z}\pi = \operatorname{im}(T).$$

Proof. Note that $T\mathbb{Z}\pi = 0$ for π cyclic. Hence $N\mathbb{Z}\pi \subset \operatorname{im}(T)$ by Lemma 3·1 and it suffices therefore to show that $\operatorname{im}(T) \subset N_0\mathbb{Z}\pi$. If p is a prime dividing $\operatorname{card}(\pi)$ then $\overline{pr}_* T(p) = [\mathbb{Z}/p] \in K_0(\mathbb{Z}\pi/N)$. It remains to prove that there exist $x \in N_0\mathbb{Z}\pi$ with $\overline{pr}_* x = [\mathbb{Z}/p]$. If $x = [M]$, M a nilpotent π-module of projective dimension $\leqslant 1$, and if $0 \to P_1 \to P_0 \to M \to 0$ is a resolution of type FP, then by definition

$$pr_* x = [\mathbb{Z}\pi/N \otimes_\pi P_0] - [\mathbb{Z}\pi/N \otimes_\pi P_1].$$

But since M is cohomologically trivial, one has

$$\mathrm{Tor}_1^\pi(\mathbb{Z}\pi/N, M) \cong \mathrm{Ker}\,(M/IM \overset{N}{\to} M) = \hat{H}^{-1}(\pi, M) = 0$$

and therefore the sequence

$$0 \to \mathbb{Z}\pi/N \otimes_\pi P_1 \to \mathbb{Z}\pi/N \otimes_\pi P_0 \to \mathbb{Z}\pi/N \otimes_\pi M \to 0$$

is exact. Hence we can write

$$\overline{pr}_*[M] = [\mathbb{Z}\pi/N \otimes_\pi M].$$

We will now construct for all prime divisors of card(π) nilpotent π-modules of type FP with $\mathbb{Z}\pi/N \otimes_\pi M \cong \mathbb{Z}/p$. First consider the case of an odd prime p. Let $\mathbb{Z}/p$ act in a non-trivial way on $\mathbb{Z}/p^2$ and define a π-action on $\mathbb{Z}/p^2$ using a surjection $\pi \to \mathbb{Z}/p$. One verifies easily that the resulting π-module M is nilpotent and cohomologically trivial. Furthermore, $\overline{pr}_*[M] = [\mathbb{Z}\pi/N \otimes_\pi M] = [M/NM] = [\mathbb{Z}/p]$. If $p = 2$, one can use $\mathbb{Z}/8$ as underlying abelian group for M, which one equips with a $\mathbb{Z}/2$-action by mapping a into $5a$, $a \in \mathbb{Z}/8$, and defining a π-module structure by means of a surjection $\pi \twoheadrightarrow \mathbb{Z}/2$. Again one verifies that $\overline{pr}_*[M] = [\mathbb{Z}/2]$. Hence $\mathrm{im}\,(T) \subset N_0\mathbb{Z}\pi$ and the result follows.

For the groups of Theorem 3·3 we can obtain an upper bound for the order and the exponent of $N\mathbb{Z}\pi$ in terms of the Euler ϕ-function $\phi(n) = \mathrm{card}\ (U(\mathbb{Z}/n))$ and the function $e(n) = $ (exponent of $U(\mathbb{Z}/n)$).

THEOREM 3·4. *If π is a cyclic group of square-free order n, then the order of*

$$N_0\mathbb{Z}\pi = N\mathbb{Z}\pi$$

divides $\phi(n)/e(n)$ and its exponent divides $e(n)$.

Proof. Let p be a prime which divides n and let $\bar{\pi} \subset \pi$ be a subgroup of index p. The $[\mathbb{Z}/p] \in T\mathbb{Z}\bar{\pi} \subset K_0\mathbb{Z}\bar{\pi}$ is mapped to $[\mathbb{Z}\pi \otimes_{\bar{\pi}} \mathbb{Z}/p] = [\mathbb{Z}/p[\pi/\bar{\pi}]] \in K_0\mathbb{Z}\pi$ by the map induced by $\bar{\pi} \subset \pi$ (one uses that $\mathrm{Tor}_{\bar{\pi}}^1(\mathbb{Z}\pi, \mathbb{Z}/p) = 0$). But if $M = \mathbb{Z}/p[\pi/\bar{\pi}]$ then $\mathbb{Z}\pi/N \otimes_\pi M = M/NM$ is a nilpotent π-module of cardinality p^{p-1} and hence $\overline{pr}_*[M] = (p-1)\,\overline{pr}_*\,T(p) \in K_0(\mathbb{Z}\pi/N)$. Since $T\mathbb{Z}\bar{\pi} = 0$ ($\bar{\pi}$ is cyclic), $[M] = 0$ and hence $(p-1)\,T(p) = 0$. We obtain therefore a factorization

$$\begin{array}{ccc} U(\mathbb{Z}[1/n]) & \overset{T}{\to} & K_0\mathbb{Z}\pi \\ \lambda \searrow & & \nearrow \bar{T} \\ & U(\mathbb{Z}/n) & \end{array}$$

where $\lambda: U(\mathbb{Z}[1/n]) \to U(\mathbb{Z}/n) \cong \Pi(\mathbb{Z}/p-1)$ is defined by $\lambda(-1) = 0$ and $\lambda(p) = (0, \ldots, 0, \bar{1}, 0, \ldots 0)$ if p divides n. The diagonal element $\Delta = (\bar{1}, \ldots, \bar{1})$ in $U(\mathbb{Z}/n) \cong \Pi(\mathbb{Z}/p-1)$ is mapped to 0 by $\bar{T}$, since $T(n) = 0$ (cf. Theorem 2·5 of (5)). Moreover, $N_0\mathbb{Z}\pi = N\mathbb{Z}\pi = \mathrm{im}\,(T)$ by Theorem 3·3. Hence the exponent of $N\mathbb{Z}\pi$ divides the exponent of $U(\mathbb{Z}/n)$ and the order of $N\mathbb{Z}\pi$ divides $\phi(n)/e(n)$ which is the order of $U(\mathbb{Z}/n)/\langle\Delta\rangle$.

For example, if π is a cyclic group of order $3p$, p a prime > 3, then card$(N\mathbb{Z}\pi) \leqslant 2$ since $\phi(3p) = 2(p-1)$ and $e(2p) = p-1$.

As a final example, we want to compute $N\mathbb{Z}\pi$ in case $\pi = M(p,q)$, the metacyclic group of square-free order pq, p and q odd primes and $q|p-1$, defined by

$$M(p,q) = \langle x,y | x^p = y^q = 1, y^{-1}xy = x^r \rangle,$$

r a primitive qth root of $1 \bmod p$.

THEOREM 3·5. *Let* $\pi = M(p,q)$. *Then*

$$T\mathbb{Z}\pi = N_0\mathbb{Z}\pi = N\mathbb{Z}\pi \cong \mathbb{Z}/q.$$

Proof. It has been shown in (6) that if $x \in K_0\mathbb{Z}\pi$ is the Wall obstruction of a homologically nilpotent space X with fundamental group $M(p,q)$, then $x \in T\mathbb{Z}\pi$. The same argument shows that for an arbitrary $x \in N\mathbb{Z}\pi$ one has $x \in T\mathbb{Z}\pi$ and hence $N\mathbb{Z}\pi = T\mathbb{Z}\pi$. Furthermore, $T\mathbb{Z}\pi = \mathbb{Z}/q$ by (11).

One can combine the results of this section to obtain the following table for $N\mathbb{Z}\pi$, in case π is a group of small, square-free order.

COROLLARY 3·6. *Let* π *be a group of square-free order* $n < 30$. *Then*

$$N_0\mathbb{Z}\pi = N\mathbb{Z}\pi = \begin{cases} 0 & \textit{if} \quad n \neq 15, 21 \\ \mathbb{Z}/2 & \textit{if} \quad n = 15 \\ \mathbb{Z}/3 & \textit{if} \quad n = 21, \quad \pi \textit{ noncyclic} \\ \mathbb{Z}/2 \text{ or } 0 & \textit{if} \quad n = 21, \quad \pi \textit{ cyclic.} \end{cases}$$

REFERENCES

(1) FRÖHLICH, A. On the class group of integral group rings of finite abelian groups. I. *Mathematika* **16** (1969), 143–152.

(2) FRÖHLICH, A., KEATING, M. E. and WILSON, S. M. J. The class groups of quaternion and dihedral 2-groups. *Mathematika* **21** (1974), 64–71.

(3) HILTON, P., MISLIN, G., ROITBERG, J. and STEINER, R. On free maps and free homotopies into nilpotent spaces. *Springer Lecture Notes in Math. Vol.* 673, 1977.

(4) MISLIN, G. Finitely dominated nilpotent spaces. *Ann. of Math.* **103** (1976), 547–556.

(5) MISLIN, G. Groups with cyclic Sylow subgroups and finiteness conditions for certain complexes. *Comment. Math. Helv.* **52** (1977), 373–391.

(6) MISLIN, G. Finitely dominated complexes with metacyclic fundamental groups. Topology and Algebra, Proceedings of a Colloquium in honour of B. Eckmann, Monographie No. 26, *L'Enseignement Mathématique* 1978, 233–235.

(7) MISLIN, G. and VARADARAJAN, K. The finiteness obstruction for nilpotent spaces lie in D($\mathbb{Z}\pi$). *Inventiones math.* **53** (1979), 185–191.

(8) RIM, D. S. Modules over finite groups. *Ann. of Math.* **63** (1959), 700–712.

(9) SWAN, R. G. Periodic resolutions for finite groups. *Ann. of Math.* **72** (1960), 267–291.

(10) TAYLOR, M. J. Locally free class groups of prime power order. *J. Algebra* **50** (1978), 463–487.

(11) ULLOM, S. V. Nontrivial lower bounds for class groups of integral group rings. *Illinois J. of Math.* **20** (1976), 361–371.

(12) VARADARAJAN, K. Finiteness obstructions for nilpotent spaces. *J. Pure and Appl. Algebra* **12** (1978), 137–146.

(13) WALL, C. T. C. Finiteness conditions for CW-complexes. *Ann. of Math.* **81** (1965), 56–69.