COST-OF-CAPITAL MARGIN FOR A GENERAL INSURANCE LIABILITY RUNOFF

BY

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ABSTRACT

Under new solvency regulations, general insurance companies need to calculate a risk margin to cover possible shortfalls in their liability runoff. A popular approach for the calculation of the risk margin is the so-called cost-of-capital approach. A comprehensive cost-of-capital approach involves the consideration of multiperiod risk measures. Because multiperiod risk measures are rather complex mathematical objects, various proxies are used to estimate this risk margin. Of course, the use of proxies and the study of their quality raises many questions, see IAA position paper [8]. In the present paper we provide a first discourse on multiperiod solvency considerations for a general insurance liability runoff. Within a chain ladder framework, we derive analytic formulas for the risk margin which allow to compare the comprehensive approach to the different proxies used in practice. Moreover, a case study investigates and answers questions raised in [8].

KEYWORDS

Solvency, risk margin, cost-of-capital approach, multiperiod risk measure, risk bearing capital, general insurance runoff, claims reserving, outstanding loss liabilities, claims development result, chain ladder model.

INTRODUCTION

The runoff of general insurance liabilities (outstanding loss liabilities) usually takes several years. Therefore, general insurance companies need to build appropriate reserves (provisions) for the runoff of these outstanding loss liabilities. Such reserves need to be incessantly adjusted according to the latest information available. Under new solvency regulations, general insurance companies have to protect against possible shortfalls in these reserves adjustments with risk bearing capital. In this spirit, this work provides a first comprehensive discourse on multiperiod solvency considerations for a general insurance liability runoff and answers questions raised in the IAA position paper [8]. This

Astin Bulletin 40(2), 415-451. doi: 10.2143/AST.40.2.2061123

discourse involves the description of the cost-of-capital approach in a multiperiod risk measure setting. In a cost-of-capital approach the insurance company needs to prove that it holds sufficient reserves first to pay for the insurance liabilities (claims reserves) and second to pay the costs of risk bearing capital (cost-of-capital margin or risk margin), see Wüthrich et al. [20], Section 5.3. Hence, at time 0, the insurer needs to hold risk-adjusted claims reserves that comprise best-estimate reserves for the outstanding loss liabilities and an additional margin for the coverage of the cashflow generated by the cost-of-capital loadings. Such risk-adjusted claims reserves are often called a market-consistent price for the runoff liabilities (in a marked-to-model approach), see e.g. Wüthrich et al. [20].

Because the multiperiod cost-of-capital approach is rather involved, state-of-the-art solvency models consider a one-period measure together with a proxy for all later periods. In this paper we consider four different approaches denoted by $\mathcal{R}^A, \ldots, \mathcal{R}^D$:

- $\mathcal{R}^A$: "*The Regulatory Solvency Approach*" (currently used in practice). This approach is risk-based with respect to the next accounting year $k=1$, but it is **not** risk-based for all successive accounting years (it uses a proxy for later accounting years $k \geq 2$).
- $\mathcal{R}^B$: "*The Split of the Total Uncertainty Approach*". This approach presents a risk-based adaption of the first approach to all remaining accounting years. Particularly, the risk measures quantify the risk in each accounting year $k \geq 1$ with respect to the initial information available at time 0.
- $\mathcal{R}^C$: "*The Expected Stand-Alone Risk Measure Approach*". This approach incorporates risk measures for each accounting year which are risk-based, i.e. measurable with respect to the previous accounting year. Moreover, the risk-adjusted claims reserves are self-financing in the average but they lack protection against possible shortfalls in the cost-of-capital cashflow.
- $\mathcal{R}^D$: "*The Multiperiod Risk Measure Approach*". This approach gives a complete, methodologically consistent view via multiperiod risk measures, but as a consequence, it is much more technical and complex compared to $\mathcal{R}^A$, $\mathcal{R}^B$, $\mathcal{R}^C$.

The numerical example in Section 5 will support $\mathcal{R}^B$ to be a good approximation for $\mathcal{R}^D$ and it will also show that the risk assessment of the regulatory solvency approach $\mathcal{R}^A$ may not always be conservative.

Note that throughout this paper we only consider nominal values. A first approach to discounted claims reserves for solvency considerations is provided in Wüthrich-Bühlmann [21]. They present a model for the one-year runoff with stochastic discounting which is similar to $\mathcal{R}^A$.

Organisation of the paper. In Section 2 we introduce the claims development result which describes the adjustments and we give a conceptual discussion about the cost-of-capital approach. The underlying claims reserving model for the prediction of the outstanding loss liabilities is discussed in Section 3. In Section 4 we discuss the four different approaches in order to determine the cost-of-capital margins. Finally, Section 5 presents a case study for a general insurance liability runoff portfolio. All the results are proved either in the main body or in the appendix.

2. The Basic Problem

2.1. The Claims Development Result

Let $C_{i,j} > 0$ denote the cumulative payments of *accident year* $i \in \{0, \ldots, I\}$ after *development year* $j \in \{0, \ldots, J\}$ with $J \leq I$. The ultimate claim of accident year i is then given by $C_{i,J}$, and for the information available at time k (for $k = 0, \ldots, J$) we write

$$\mathcal{D}_{I+k} = \{C_{i,j};\ i + j \leq I + k, \quad 0 \leq i \leq I, 0 \leq j \leq J\}.$$

We call the period $(k-1, k]$ *accounting year* k and $\mathcal{D}_{I+k}$ is the information available after accounting year k (or at time k).

Since loss reserving is basically a prediction problem, we are mainly interested in the predictors $\widehat{C}_{i,J}^{(k)}$ of the ultimate claim $C_{i,J}$, given information $\mathcal{D}_{I+k}$. The outstanding loss liabilities for accident year i at time k are defined by $C_{i,J} - C_{i,I-i+k}$ (assume that $I + k \leq i + J$). At time k these are predicted by the claims reserves

$$R_{i,k} = \widehat{C}_{i,J}^{(k)} - C_{i,I-i+k},$$

the index i will always denote accident years and k accounting years. Note that for every further accounting year more data become available and we have to adapt the predictors according to the latest information available. Therefore, we consider the successive predictions of the ultimate claims $C_{i,J}$, i.e.

$$\widehat{C}_{i,J}^{(0)}, \widehat{C}_{i,J}^{(1)}, \ldots, \widehat{C}_{i,J}^{(J+i-I-1)}, \widehat{C}_{i,J}^{(J+i-I)} = C_{i,J}. \tag{2.1}$$

Their increments determine the so-called claims development result (CDR); see Bühlmann et al. [2] and Ohlsson-Lauzeningks [15]. Note that in an early work of De Felice-Moriconi [4], the CDR was called year-end expectation (YEE) view. In new solvency regulations, the CDR is the *central object* of interest for the reserve risk and has to be thoroughly studied.

Definition 2.1. *For accounting year k and accident year i the CDR is defined by*

$$\mathrm{CDR}_{i,k} = \widehat{C}_{i,J}^{(k-1)} - \widehat{C}_{i,J}^{(k)}. \tag{2.2}$$

It refers to the change in the balance sheet in accounting year k so that we always have best-estimate predictions, i.e. the outstanding loss liabilities are covered by claims reserves according to the latest information available. New solvency approaches (see e.g. Solvency II, Swiss Solvency Test [19] and AISAM-ACME [1]) require protection against possible shortfalls in this (one-year) CDR by risk bearing capital. This means that insurance companies give a yearly guarantee that the best-estimate predictions are always covered by a sufficient amount of claims reserves (modulo the chosen risk measure). Therefore, in a business year or solvency view, we need to study the sequence of CDR's, i.e. the uncertainty in this profit & loss statement position, see also Merz-Wüthrich [13]. For a fixed accident year i we consider the time-series

$$\mathrm{CDR}_{i,k}, \ldots, \mathrm{CDR}_{i,J+i-I}, \tag{2.3}$$

which in terms of the claims reserves satisfy

$$\mathrm{CDR}_{i,k} = \widehat{C}_{i,J}^{(k-1)} - \widehat{C}_{i,J}^{(k)} = R_{i,k-1} - \left(X_{i,I-i+k} + R_{i,k}\right),$$

where $X_{i,I-i+k} = C_{i,I-i+k} - C_{i,I-i+k-1}$ are the incremental payments for accident year i in accounting year k.

In conclusion, for accounting year k we need reserves (provisions) that cover both:

1. the expected outstanding liabilities $R_{i,k}$, i.e. the best-estimate reserves at time k,
2. protection against the possible deviations from these best-estimate reserves.

2.2. A Conceptual Discussion of the Cost-of-Capital Loading

As already mentioned, we consider four different approaches $\mathcal{R}^A, \ldots, \mathcal{R}^D$ to cope with this problem. These four different approaches correspond to four different levels of complexity. Three of them serve as approximations to the methodologically consistent approach denoted by $\mathcal{R}^D$. The following presents a conceptual discussion of the four different approaches.

For the time being we choose a fixed accident year i. The update of the available information in each accounting year k then gives a time-series of CDR's as shown in Figure 1, see also (2.3). In order to protect against possible shortfalls in these CDR's (adverse development of the claims reserves) we need to calculate an appropriate risk measure $\rho_{i,k}$ for each accounting year k. In the cost-of-capital approach we do not need to hold the risk measure $\rho_{i,k}$ itself, but rather the price for this risk measure denoted by $\Pi_{i,k}$. The price $\Pi_{i,k}$ for the risk measure $\rho_{i,k}$ can be viewed as a cashflow towards the risk bearer (similar to a re-insurance premium). That is, we need to build reserves for both the claims cashflow $X_{i,I-i+k}$ and the cost-of-capital cashflow $\Pi_{i,k}$, see Wüthrich et al. [20], Section 5.3, and Pelsser [16].

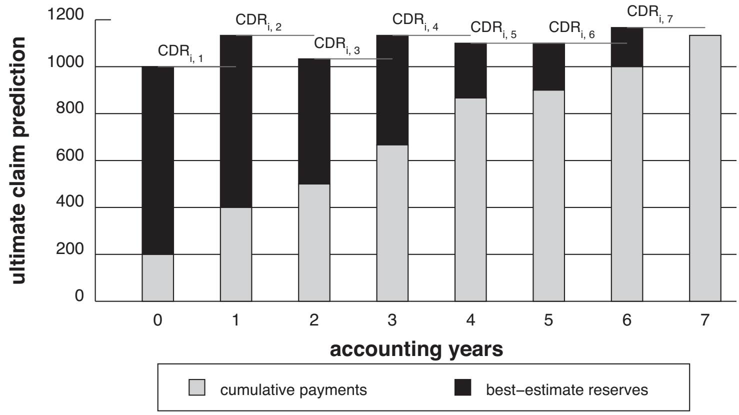

FIGURE 1: Time-series of the claims development results $\mathrm{CDR}_{i,k}$, for a fixed accident year i with $J + i - I = 7$, i.e. $k = 1, \ldots, 7$.

First we consider $\mathcal{R}_i^D$, i.e. the multiperiod risk measure approach, see also Figure 2. This needs to be calculated recursively (see also Ohlsson-Lauzeningks [15]). In the last accounting year $J + i - I$ the risk measure $\rho_{i,J+i-I}$ has to account for the uncertainty in $\mathrm{CDR}_{i,J+i-I}$ which gives the cost-of-capital price $\Pi_{i,J+i-I}$. For the previous accounting year $J + i - I - 1$ we have not only to account for the uncertainty in $\mathrm{CDR}_{i,J+i-I-1}$ but also for the uncertainty of the cost-of-capital cashflow $\Pi_{i,J+i-I}$. Iterating this procedure reveals the recursive structure of the multiperiod risk measure approach and leads to the risk-adjusted claims reserves $\mathcal{R}_i^D$ at time 0. The difference $\mathcal{R}_i^D - R_{i,0}$ measures the risk margin and corresponds to a risk aversion loading in an incomplete market setting. Since such a recursive problem is very complex and soon becomes too

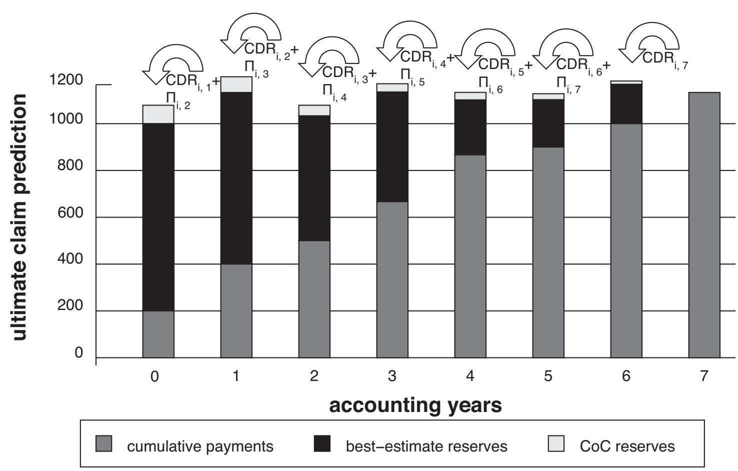

FIGURE 2: Time-series of the multiperiod risk measure approach for $J + i - I = 7$.

time-consuming for simulation (nested simulations), we look for alternative proxy solutions which are either analytically or numerically tractable.

Such a solution is provided by $\mathcal{R}_i^C$, i.e. the expected stand-alone risk measure approach. For each accounting year k, the risk measure $\rho_{i,k}$ accounts for the uncertainty in $\mathrm{CDR}_{i,k}$ with respect to $\mathcal{D}_{I+k-1}$, the latest information available at the beginning of accounting year k. In order to determine the reserves at time 0, we then take the expectation of this $\mathcal{D}_{I+k-1}$-measurable future cost-of-capital with respect to the initial information i.e. $\mathcal{D}_I$. This approach then provides that the risk margin $\mathcal{R}_i^C - R_{i,0}$ is in the average self-financing, but it does not account for possible adverse developments in the cost-of-capital cashflow $\Pi_{i,k}$ itself. That is, we only build reserves for the expected cashflow of $\Pi_{i,k}$.

A further simplified approximation is given by $\mathcal{R}_i^B$, i.e. the split of the total uncertainty approach. Instead of considering the uncertainty of $\mathrm{CDR}_{i,k}$ with respect to the information $\mathcal{D}_{I+k-1}$, we consider the uncertainty in $\mathrm{CDR}_{i,k}$ with respect to the initial information $\mathcal{D}_I$. This approach has the advantage that it can often be calculated analytically and leads to simple formulas.

Finally, approach $\mathcal{R}_i^A$, i.e. the regulatory solvency approach, is the roughest approximation with respect to risk-adjustedness. It calculates a risk measure $\rho_{i,1}$ for the uncertainty of $\mathrm{CDR}_{i,1}$ in the first accounting year. The risk measures $\rho_{i,k}$ for the later accounting years $k \geq 2$ are then simply calculated by volume scaling with the expected runoff of the outstanding loss liabilities. This is the approach used e.g. in the Swiss Solvency Test [19] and it also corresponds to a special case considered in ISVAP [9], Section 1.2.4, called the "flat case".

In the next section we introduce the gamma-gamma Bayes chain ladder model for claims reserving. Restricting our remaining discussion to the consideration of a standard deviation loading as risk measure, this claims reserving model will allow for explicit analytical solutions for all approaches $\mathcal{R}_i^A, \ldots, \mathcal{R}_i^D$ as long as we fix a single accident year i. We will see that the situation becomes more involved as soon as we start to aggregate over accident years i. The explicit analytical solutions then allow for a comparison between the comprehensive approach $\mathcal{R}_i^D$ and its proxies $\mathcal{R}_i^A$, $\mathcal{R}_i^B$, $\mathcal{R}_i^C$. Finally, we would like to mention that for other (more general) risk measures and other claims reserving models one cannot expect explicit closed form solutions and one needs to rely on simulation results. For this reason, we concentrate in the present work on this simple model, where we can directly calculate all terms and study sensitivities of all the parameters involved.

3. Outstanding Loss Liability Model

In the following, the claims reserving is described in a Bayesian chain ladder model framework. This Bayesian framework provides a unified approach for a successive information update in each accounting year, i.e. new information is immediately absorbed by the Bayesian model (see also Bühlmann et al. [2]).

We define the individual claims development factors $F_{i,j} = C_{i,j} / C_{i,j-1}$ for $j = 1, \ldots, J$. Then the cumulative payments $C_{i,j}$ are given by

$$C_{i,j} = C_{i,0} \prod_{m=1}^{j} F_{i,m}.$$

The first payment $C_{i,0}$ plays the role of the initial value of the process $(C_{i,j})_{j=0,\ldots,J}$ and $F_{i,j}$ are the multiplicative changes. In a Bayesian chain ladder framework, we assume that the unknown underlying parameters are described by the random variables $\Theta_1^{-1}, \ldots, \Theta_J^{-1}$. Given these, we further assume that $C_{i,j}$ satisfies a chain ladder model.

Model Assumptions 3.1. (Gamma-Gamma Bayes Chain Ladder Model)

- Conditionally, given $\mathbf{\Theta} = (\Theta_1, \ldots, \Theta_J)$,
  - the cumulative payments $C_{i,j}$ for different accident years i are independent.
  - $C_{i,0}, F_{i,1}, \ldots, F_{i,J}$ are independent with

    $$F_{i,j}|_{\mathbf{\Theta}} \sim \Gamma(\sigma_j^{-2}, \Theta_j \sigma_j^{-2}), \quad \text{for } j = 1, \ldots, J,$$

    where the σ_j's are given positive constants.
- $C_{i,0}$ and $\mathbf{\Theta}$ are independent and $C_{i,0} > 0$, $\mathbb{P}$-a.s.
- $\Theta_1, \ldots, \Theta_J$ are independent with $\Theta_j \sim \Gamma(\gamma_j, f_j(\gamma_j - 1))$ with given prior parameters $f_j > 0$ and $\gamma_j > 1$.

Remarks

- We choose the following parametrisation for the gamma distribution: For $X \sim \Gamma(a,\beta)$ we have that $\mathbb{E}[X] = a/\beta$ and $\text{Var}(X) = a/\beta^2$.
- The gamma-gamma Bayes chain ladder model defines a model that belongs to the exponential dispersion family with associate conjugate priors (see e.g. Bühlmann-Gisler [3]). This family of distributions gives an exact credibility case because the Bayesian estimators coincide with the linear credibility estimators. Hence, it is possible to explicitly calculate the posterior distribution of $\mathbf{\Theta}$, given the observations $F_{i,j}$. We restrict ourselves to the gamma-gamma case because it allows for an explicit calculation of the prediction uncertainties which is essential for multiperiod risk measure decompositions. For most other models, only numerical solutions are available or one considers approximations similar to Theorem 6.5 in Gisler-Wüthrich [7].
- Note that the above model assumes that accident years are conditionally independent, i.e. it does not allow for the modelling of accounting year effects and claims inflation. In general, accounting year effects models can only be solved numerically, see Wüthrich [22].

Conditionally, given $\boldsymbol{\Theta}$, we obtain a chain ladder model with first two moments given by

$$\mathbb{E}[C_{i,j} | \boldsymbol{\Theta}, C_{i,0}, \ldots, C_{i,j-1}] = C_{i,j-1} \mathbb{E}[F_{i,j} | \boldsymbol{\Theta}] = C_{i,j-1} \Theta_j^{-1}, \tag{3.1}$$

$$\operatorname{Var}(C_{i,j} | \boldsymbol{\Theta}, C_{i,0}, \ldots, C_{i,j-1}) = C_{i,j-1}^2 \operatorname{Var}(F_{i,j} | \boldsymbol{\Theta}) = C_{i,j-1}^2 \, \sigma_j^2 \, \Theta_j^{-2}. \tag{3.2}$$

Hence, for the conditional coefficient of variation of $C_{i,j}$, given $\boldsymbol{\Theta}$, we find that

$$\operatorname{Vco}(C_{i,j} | \boldsymbol{\Theta}, C_{i,0}, \ldots, C_{i,j-1}) = \frac{\left[\operatorname{Var}\left(C_{i,j} \middle| \boldsymbol{\Theta}, C_{i,0}, \ldots, C_{i,j-1}\right)\right]^{1/2}}{\mathbb{E}\left[C_{i,j} \middle| \boldsymbol{\Theta}, C_{i,0}, \ldots, C_{i,j-1}\right]} = \sigma_j .$$

Henceforth, Θ_j^{-1} plays the role of the chain ladder factor (see Mack [11]). Note that the variance is proportional to $C_{i,j-1}^2$, this assumption is crucial for the calculation of feasible standard deviation loadings in the multiperiod cost-of-capital approach (but it is different from Mack's [11] classical distribution-free chain ladder model). This modification of the classical chain ladder variance assumption is essential to have a tractable model (otherwise only numerical solutions are available). In the case study below, the numerical differences between these two different models are analysed (see Section 5 below). Moreover, for the prior moments we have

$$\mathbb{E}\left[\Theta_j^{-1}\right] = f_j, \quad \mathbb{E}\left[\Theta_j^{-2}\right] = f_j^2 \frac{\gamma_j - 1}{\gamma_j - 2} \quad \text{and} \quad \operatorname{Var}\left(\Theta_j^{-1}\right) = f_j^2 \frac{1}{\gamma_j - 2}.$$

This shows that we have a prior mean for the chain ladder factor of f_j. In order that the prior second moments exist, we need the additional assumption that $\gamma_j > 2$.

3.1. The Parameter Update Procedure

At time $k \geq 0$, we have information $\mathcal{D}_{I+k}$ and we need to predict the outstanding loss liabilities that correspond to the random variables $C_{i,J} - C_{i,I-i+k}$. This means that we have to update our model according to the information generated by the runoff portfolio for successive accounting years. The following proposition describes the parameter update procedure for the posterior distributions of Θ_j.

Proposition 3.2. *Under Model Assumptions 3.1 we have that for $k \geq 0$ the conditional posterior distributions of Θ_j, given $\mathcal{D}_{I+k}$, are independent gamma distributions $\Gamma(\gamma_{j,k}, c_{j,k})$ with updated parameters*

$$\gamma_{j,k} = \gamma_j + \frac{((I+k-j)\wedge I)+1}{\sigma_j^2} \quad \textit{and} \quad c_{j,k} = f_j(\gamma_j - 1) + \sigma_j^{-2} \sum_{i=0}^{(I+k-j)\wedge I} F_{i,j},$$

where we use the notation $x \wedge y = \min\{x, y\}$.

Proof of Proposition 3.2. We denote the distribution of $C_{i,0}$ by π_i. Then the joint density of $(\mathbf{\Theta}, \mathcal{D}_{I+k})$ is given by

$$\pi(\mathbf{\Theta}, \mathcal{D}_{I+k}) = \prod_{i+j\le I+k} \frac{\left(\Theta_j \sigma_j^{-2}\right)^{\sigma_j^{-2}}}{\Gamma\left(\sigma_j^{-2}\right)} F_{i,j}^{\sigma_j^{-2}-1} \exp\left\{-\Theta_j \sigma_j^{-2} F_{i,j}\right\}$$
$$\times \sum_{j=1}^{J} \frac{\left(f_j(\gamma_j - 1)\right)^{\gamma_j}}{\Gamma(\gamma_j)} \Theta_j^{\gamma_j - 1} \exp\left\{-f_j(\gamma_j - 1)\Theta_j\right\} \prod_{i=0}^{I} \pi_i\left(C_{i,0}\right).$$

This implies that the posterior density of $\mathbf{\Theta}$, given $\mathcal{D}_{I+k}$, satisfies the following proportionality property (only relevant terms are considered)

$$\pi(\mathbf{\Theta} | \mathcal{D}_{I+k}) \propto \prod_{j=1}^{J} \Theta_j^{\gamma_j + \frac{((I+k-j)\wedge I)+1}{\sigma_j^2} - 1} \exp\left\{-\left[f_j(\gamma_j - 1) + \sigma_j^{-2} \sum_{i=0}^{(I+k-j)\wedge I} F_{i,j}\right]\Theta_j\right\}.$$

These are independent gamma densities which proves the proposition. □

The above result implies the following corollary.

Corollary 3.3. *Under the assumptions of Proposition 3.2 we have*

$$\hat{f}_j^{(k)} \overset{def.}{=} \mathbb{E}\left[\Theta_j^{-1} \middle| \mathcal{D}_{I+k}\right] = \frac{c_{j,k}}{\gamma_{j,k} - 1} = \alpha_{j,k} \bar{F}_j^{(k)} + \left(1 - \alpha_{j,k}\right) f_j,$$
$$\mathbb{E}\left[\Theta_j^{-2} \middle| \mathcal{D}_{I+k}\right] = \frac{c_{j,k}^2}{\left(\gamma_{j,k} - 1\right)\left(\gamma_{j,k} - 2\right)} = \left(\hat{f}_j^{(k)}\right)^2 \frac{\gamma_{j,k} - 1}{\gamma_{j,k} - 2},$$

where we have defined

$$\bar{F}_j^{(k)} = \frac{1}{((I+k-j)\wedge I)+1} \sum_{i=0}^{(I+k-j)\wedge I} F_{i,j},$$
$$\alpha_{j,k} = \frac{((I+k-j)\wedge I)+1}{((I+k-j)\wedge I)+1+\sigma_j^2(\gamma_j - 1)}.$$

Remarks

- Note that the $\bar{F}_j^{(k)}$'s differ from the chain ladder estimates resulting from volume weighting in Mack's [11] model. For models where the variance is proportional to the mean squared (see (3.1) and (3.2)) the average of the individual development factors is the optimal estimate of the chain ladder factor as pointed out by Mack [12].
- In order that the second posterior moments exist, we need to assume that $\gamma_{j,k} > 2$, e.g. this is guaranteed by the additional assumption that $\gamma_j > 2$.

It is well-known that the parameter updating procedure can also be done recursively (see Gerber-Jones [6], Kremer [10], Sundt [18] and Bühlmann-Gisler [3], Theorem 9.6). In our case this leads to the helpful representation:

Corollary 3.4. *For $k \geq 1$ and $j \geq k$ we have*

$$\hat{f}_j^{(k)} = a_{j,k} F_{I+k-j,j} + \left(1 - a_{j,k}\right) \hat{f}_j^{(k-1)},$$

where the credibility weight $a_{j,k}$ is given by

$$a_{j,k} = \left(I + k - j + 1 + \sigma_j^2 (\gamma_j - 1)\right)^{-1}.$$

Proof. The proof follows from Corollary 3.3. □

3.2. Ultimate Claim Prediction

The following proposition determines the best-estimate prediction of the ultimate $C_{i,J}$ in our Bayesian chain ladder framework.

Proposition 3.5. *Suppose the assumptions of Proposition 3.2 to hold. The predictor for the ultimate $C_{i,J}$ that has minimal conditional variance, given $\mathcal{D}_{I+k}$, is given by $\widehat{C}_{i,J}^{(k)} = \mathbb{E}[C_{i,J} \mid \mathcal{D}_{I+k}]$. For $I + k < i + J$ we obtain*

$$\widehat{C}_{i,J}^{(k)} = \mathbb{E}\left[C_{i,J} \mid \mathcal{D}_{I+k}\right] = C_{i,I-i+k} \prod_{j=I-i+k+1}^{J} \hat{f}_j^{(k)}.$$

Proof. The proof easily follows from the fact that conditional expectations minimise L^2-distances and from the posterior independence of the Θ_j's (see Proposition 3.2). □

Remark. Note that this is a so-called Bayes chain ladder model (see Bühlmann et al. [2]) where the chain ladder factors $\hat{f}_j^{(k)}$ are credibility weighted averages between the prior estimates f_j and the observations $\bar{F}_j^{(k)}$. For uninformative priors, i.e. $\gamma_j \to 1$ and therefore $a_{j,k} \to 1$, we obtain a frequentist's chain ladder factor estimator that is only based on the observations. Note that for the second posterior moment to exist, we need to have $\gamma_{j,k} > 2$. This may exclude the consideration of the asymptotic uninformative case (see Proposition 3.2).

In the remainder of this paper we are going to characterise the uncertainties in the CDR (Definition 2.1). Our first Corollary states that best-estimate predictions (2.1) form a martingale, see also ISVAP [9], footnote 5 in Section 1.2.1.

Corollary 3.6. *Under Proposition 3.5 we have*

$$\mathbb{E}\left[\mathrm{CDR}_{i,k} \,|\, \mathcal{D}_{I+k-1}\right] = 0.$$

Moreover, the CDR's are uncorrelated, i.e. for $k \geq 1$, $l \leq J$ and $m < max\{k,l\}$ it holds that

$$\mathbb{E}\left[\mathrm{CDR}_{i,k}\,\mathrm{CDR}_{i,l} \,|\, \mathcal{D}_{I+m}\right] = 0.$$

Proof. Note that best-estimate predictions from Proposition 3.5 form a martingale (this follows from the tower property of conditional expectations). Hence, it easily follows that the expected CDR is zero. The second claim then follows from the fact that martingales have uncorrelated increments. □

Remark. At this stage it turns out to be crucial that we have an exact credibility model. Otherwise one obtains a bias term in the CDR (as e.g. in Proposition 3.1 in Bühlmann et al. [2]) that is difficult to control.

4. Cost-of-Capital Margin

For the time being we consider a fixed accident year i. According to Section 2, we need to build reserves (provisions) that cover both, the present best-estimate reserves and a protection against possible adverse developments in the CDR's, i.e. a buffer to be able to balance the profit & loss statement in each remaining accounting year k. Therefore, in accounting year k we choose a risk measure $\rho_{i,k}$ that quantifies the amount needed for such regulatory induced protection. Moreover, we choose a constant $c > 0$, the so-called cost-of-capital rate, i.e. the annual rate required by the risk bearer for providing the risk measure $\rho_{i,k}$. With this, the price of the above risk measure is defined by $\Pi_{i,k} = c\,\rho_{i,k}$ and is called the cost-of-capital margin for accounting year k. Note that $\Pi_{i,k} = c\,\rho_{i,k}$ gives the price of risk but it does not tell us anything about the organisation of the risk bearing, i.e. in addition to the cost-of-capital margin $\Pi_{i,k}$ the regulator also needs to make sure that this capital is used for organising the risk bearing (and not for other purposes), see also Section 5.3 in Wüthrich et al. [20].

Consequently, in addition to the claims reserves $R_{i,k}$, the insurance company needs to build reserves for the cost-of-capital cashflow

$$c\,\rho_{i,k+1},\ \dots,\ c\,\rho_{i,J+i-I}.$$

Assume that $\mathrm{CoC}_{i,k}$ are the reserves (cost-of-capital margin) at time k that cover the aggregated cost-of-capital cashflow $c\,\rho_{i,k+1}, \dots, c\,\rho_{i,J+i-I}$. Hence, the risk-adjusted claims reserves at time k (based on the information $\mathcal{D}_{I+k}$) are given by

$$\mathcal{R}^*_{i,k} = R_{i,k} + \mathrm{CoC}_{i,k}.$$

The $\mathcal{R}^*_{i,k}$'s can be interpreted as a risk-adjusted price for the runoff liabilities in an incomplete market setting, i.e. the provisions for the outstanding loss liabilities and a price for the risk at which the outstanding loss liabilities can be transferred to a third party at time k (in a marked-to-model view). In the following we consider the four different approaches $\mathcal{R}^A, \dots, \mathcal{R}^D$ for $\mathcal{R}^*$ with different versions of standard deviation loadings as risk measures. Note that such loadings can be viewed as measures of preferences, e.g. in Møller [14] the actuarial standard deviation loading is related to the financial standard deviation principle as a concept for pricing in an incomplete market setting using utility theory, see also Pelsser [16].

4.1. Regulatory Solvency Proxy Approach

For $I + k \le J + i$ and $i,k \ge 1$ we define the constant

$$\beta_{i,k} = \left(\sigma^2_{I+k-i} + 1\right)\frac{\gamma_{I+k-i,k-1} - 1}{\gamma_{I+k-i,k-1} - 2}\prod_{j=I+k-i+1}^{J}\left[a^2_{j,k}\left(\left(\sigma^2_j + 1\right)\frac{\gamma_{j,k-1} - 1}{\gamma_{j,k-1} - 2} - 1\right) + 1\right].$$

Note that we have $\beta_{i,k} > 1$ and since the $a_{j,k}$'s and $\gamma_{j,k}$'s do not depend on the observations (see Proposition 3.2 and Corollary 3.4), $\beta_{i,k}$ is also unaffected. This is a crucial property of Model Assumptions 3.1 that we are going to use in the derivations below.

Proposition 4.1. *Under Model Assumptions 3.1 we have, for $I + 1 \le J + i$,*

$$\mathrm{Var}(\mathrm{CDR}_{i,1}\,|\,\mathcal{D}_I) = \mathrm{Var}(\widehat{C}^{(1)}_{i,J}\,|\,\mathcal{D}_I) = (\widehat{C}^{(0)}_{i,J})^2(\beta_{i,1} - 1).$$

Proof. The proposition easily follows from Theorem 4.2 below. □

Many regulators use an approach that is similar to the following proxy for the estimation of the cost-of-capital charge (see e.g. Swiss Solvency Test [19], Sandström [17], Section 6.8, appendix C3 in the IAA position paper [8] or Section 1.3.2 in ISVAP [9]). The expected outstanding loss liabilities at time k viewed from time 0 are given by

$$r_{i,k} = \mathbb{E}\left[R_{i,k} \mid \mathcal{D}_I\right] = \mathbb{E}\left[\widehat{C}_{i,J}^{(k)} - C_{i,I-i+k} \mid \mathcal{D}_I\right] = \widehat{C}_{i,J}^{(0)} - \widehat{C}_{i,I-i+k}^{(0)}.$$

Hence, $r_{i,0}, \ldots, r_{i,J+i-I-1}, r_{i,J+i-I} = 0$ describes the expected runoff of the outstanding loss liabilities viewed from time 0. In the first cost-of-capital approach the risk measure in accounting year k is chosen to be (the upper index in the risk measure notation labels the approaches)

$$\rho_{i,k}^{A} = \frac{r_{i,k-1}}{r_{i,0}}\, \phi \left[\mathrm{Var}\left(\mathrm{CDR}_{i,1} \mid \mathcal{D}_I\right)\right]^{1/2} = \frac{r_{i,k-1}}{r_{i,0}}\, \phi\, \widehat{C}_{i,J}^{(0)} \left(\beta_{i,1} - 1\right)^{1/2},$$

where ϕ is a fixed positive constant determining the security level. In this setup, the appropriate risk measure for accounting year 1 is given by

$$\rho_{i,1}^{A} = \phi \left[\mathrm{Var}\left(\mathrm{CDR}_{i,1} \mid \mathcal{D}_I\right)\right]^{1/2} = \phi\, \widehat{C}_{i,J}^{(0)} \left(\beta_{i,1} - 1\right)^{1/2}.$$

That is, we choose an appropriate risk measure $\rho_{i,1}^{A}$ for the first accounting year which is determined by a standard deviation loading. The risk measures $\rho_{i,k}^{A}$ for later accounting years $k \geq 2$ are then obtained by the $\mathcal{D}_I$-measurable volume scaling $r_{i,k}$ describing the expected runoff of the outstanding loss liabilities. The underlying assumption is that this volume measure is a good proxy for the runoff of the CDR uncertainty. The risk-adjusted claims reserves at time 0 are then given by

$$\mathcal{R}_{i,0}^{A} = R_{i,0} + c\, \phi\, \widehat{C}_{i,J}^{(0)} \left(\beta_{i,1} - 1\right)^{1/2} \sum_{k=1}^{J+i-I} \frac{r_{i,k-1}}{r_{i,0}}. \tag{4.1}$$

We refer to (4.1) as the regulatory solvency proxy approach. We see that the calculation of the risk-adjusted claims reserves is very simple. It only requires the study of the CDR for the first accounting year and all the remaining uncertainties are proportional to the uncertainty in the first accounting year. Hence, this approach meets the simplicity requirements often wanted in practice. However, since this approach is only risk-based for accounting year $k = 1$, the risk-adjustedness for later accounting years is rather questionable.

4.2. Split of the Total Uncertainty Approach

Corollary 3.6 implies that the total uncertainty viewed from time 0 can be split into the single one-year uncertainties for different accounting years as follows

$$
\begin{aligned}
\operatorname{Var}(C_{i,J} \mid \mathcal{D}_I) &= \operatorname{Var}\left(\sum_{k=1}^{J+i-I} \mathrm{CDR}_{i,k} \mid \mathcal{D}_I \right) = \sum_{k=1}^{J+i-I} \operatorname{Var}\left(\mathrm{CDR}_{i,k} \mid \mathcal{D}_I \right) \\
&= \sum_{k=1}^{J+i-I} \mathbb{E}\left[\mathrm{CDR}_{i,k}^2 \mid \mathcal{D}_I \right] = \sum_{k=1}^{J+i-I} \mathbb{E}\left[\operatorname{Var}\left(\widehat{C}_{i,J}^{(k)} \mid \mathcal{D}_{I+k-1} \right) \mid \mathcal{D}_I \right].
\end{aligned}
$$

For the second equality to hold, we have used the uncorrelatedness of the CDR's.

Theorem 4.2. *Under Model Assumptions 3.1 we have, for $I + k \le J + i$,*

$$
\begin{aligned}
\operatorname{Var}\left(\widehat{C}_{i,J}^{(k)} \mid \mathcal{D}_{I+k-1} \right) &= \left(\widehat{C}_{i,J}^{(k-1)} \right)^2 \left(\beta_{i,k} - 1 \right), \\
\operatorname{Var}\left(\mathrm{CDR}_{i,k} \mid \mathcal{D}_I \right) &= \left(\widehat{C}_{i,J}^{(0)} \right)^2 \prod_{j=1}^{k-1} \beta_{i,j} \left(\beta_{i,k} - 1 \right).
\end{aligned}
$$

(An empty product is set equal to 1).

The proof is provided in the appendix. Theorem 4.2 immediately implies:

Corollary 4.3. (Aggregated One-Year Risks) *Under Model Assumptions 3.1 we have*

$$
\begin{aligned}
\operatorname{Var}\left(C_{i,J} \mid \mathcal{D}_I \right) &= \operatorname{Var}\left(\sum_{k=1}^{J+i-I} \mathrm{CDR}_{i,k} \mid \mathcal{D}_I \right) \\
&= \left(\widehat{C}_{i,J}^{(0)} \right)^2 \left[\prod_{k=1}^{J+i-I} \beta_{i,k} - 1 \right] = \left(\widehat{C}_{i,J}^{(0)} \right)^2 \left[\prod_{j=I-i+1}^{J} \left(\left(\sigma_j^2 + 1 \right) \frac{\gamma_{j,0} - 1}{\gamma_{j,0} - 2} \right) - 1 \right].
\end{aligned}
$$

The proof is provided in the appendix.

Remark. Corollary 4.3 gives the prediction uncertainty for the total runoff of the outstanding loss liabilities. This is similar to the famous Mack formula (Mack [11]) in the classical chain ladder model and to the formula in the Bayesian chain ladder model considered in Gisler-Wüthrich [7]. Theorem 4.2 then states how this total uncertainty factorises across the single accounting years. We define the risk measures $\rho_{i,k}^B$ in this second approach by

$$
\rho_{i,k}^B = \phi \left[\operatorname{Var}\left(\mathrm{CDR}_{i,k} \mid \mathcal{D}_I \right) \right]^{1/2} = \phi \, \widehat{C}_{i,J}^{(0)} \prod_{j=1}^{k-1} \beta_{i,j}^{1/2} \left(\beta_{i,k} - 1 \right)^{1/2}.
$$

This means that we analyse the CDR uncertainty for each accounting year viewed from time 0. Since all the underlying terms are $\mathcal{D}_I$-measurable, we can define the risk-adjusted claims reserves at time 0 for the risk measure $\rho_{i,k}^B$ by

$$\mathcal{R}_{i,0}^{B} = R_{i,0} + c\,\phi\,\widehat{C}_{i,J}^{(0)} \sum_{k=1}^{J+i-I} \prod_{j=1}^{k-1} \beta_{i,j}^{1/2} \left(\beta_{i,k} - 1\right)^{1/2}. \tag{4.2}$$

We refer to (4.2) as the split of the total uncertainty approach.

4.3. Expected Stand-Alone Risk Measure Approach

Note that the risk measures $\rho_{i,k}^{A}$ and $\rho_{i,k}^{B}$ are both $\mathcal{D}_I$-measurable. However, one would expect that the risk measure $\rho_{i,k}$ should be $\mathcal{D}_{I+k-1}$-measurable which reflects the claims development up to accounting year k. Note that due to Theorem 4.2 we have

$$\mathrm{Var}\left(\mathrm{CDR}_{i,k} \mid \mathcal{D}_{I+k-1}\right) = \mathrm{Var}\left(\widehat{C}_{i,J}^{(k)} \mid \mathcal{D}_{I+k-1}\right) = \left(\widehat{C}_{i,J}^{(k-1)}\right)^2 \left(\beta_{i,k} - 1\right).$$

Hence, we define the risk measure by

$$\rho_{i,k}^{C} = \phi\left[\mathrm{Var}\left(\mathrm{CDR}_{i,k} \mid \mathcal{D}_{I+k-1}\right)\right]^{1/2} = \phi\,\widehat{C}_{i,J}^{(k-1)}\left(\beta_{i,k} - 1\right)^{1/2}.$$

This corresponds exactly to the regulatory risk bearing capital (modulo the chosen risk measure) that the insurance company needs to hold in order to run its business in accounting year k (and having claims experience $\mathcal{D}_{I+k-1}$). Note that the cost-of-capital margin $c\,\rho_{i,k}^{C}$ is a $\mathcal{D}_{I+k-1}$-measurable cashflow for which we need to put reserves aside at time 0. Therefore, we define the risk-adjusted claims reserves for $\rho_{i,k}^{C}$ by

$$\mathcal{R}_{i,0}^{C} = R_{i,0} + c \sum_{k=1}^{J+i-I} \mathbb{E}\left[\rho_{i,k}^{C} \mid \mathcal{D}_I\right] = R_{i,0} + c\,\phi\,\widehat{C}_{i,J}^{(0)} \sum_{k=1}^{J+i-I} \left(\beta_{i,k} - 1\right)^{1/2}. \tag{4.3}$$

We refer to (4.3) as the expected stand-alone risk measure approach. Note that these risk-adjusted claims reserves are self-financing in the average which means that we have exactly reserved for the expected value of the cashflow

$$X_{i,I-i+1} + c\,\rho_{i,1}^{C}, \; \ldots, \; X_{i,J} + c\,\rho_{i,J+i-I}^{C}.$$

Because $\beta_{i,j} > 1$ or due to Jensen's inequality we can easily see that the following corollary holds true:

Corollary 4.4. *We have*

$$\mathcal{R}_{i,0}^{C} \leq \mathcal{R}_{i,0}^{B}.$$

4.4. Multiperiod Risk Measure Approach

Reserves that are self-financing in the average as above (see formula (4.3)) do not account for the risk inherent in the cost-of-capital cashflow itself. In a multiperiod (dynamic) risk measure approach (see e.g. Föllmer-Penner [5]) we additionally quantify the uncertainty in the cost-of-capital cashflow $c\,\rho_{i,k}$. This then needs a recursive calculation of the necessary risk measures (see also Ohlsson-Lauzeningks [15]) which is explained in this subsection, see also Figure 2. We start with a schematic illustration based on backward induction. Fix accident year $i > I - J$, then the last accounting year for this accident year is given by $J + i - I$. Hence, we initialise the reserves for the cost-of-capital cashflow by $\mathrm{CoC}_{i,J+i-I} = 0$. For $k = 0, \ldots, J + i - I - 1$ the risk-adjusted claims reserves are defined by

$$\mathcal{R}^D_{i,k} = R_{i,k} + \mathbb{E}\left[\mathrm{CoC}_{i,k+1} \mid \mathcal{D}_{I+k}\right] + c\,\rho^D_{i,k+1} \overset{def.}{=} R_{i,k} + \mathrm{CoC}_{i,k},$$

where

$$\begin{aligned}\rho^D_{i,k+1} &= \phi\left[\mathrm{Var}\left(R_{i,k+1} + X_{i,I-i+k+1} + \mathrm{CoC}_{i,k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2} \\ &= \phi\left[\mathrm{Var}\left(\mathrm{CDR}_{i,k+1} + \left(\mathbb{E}\left[\mathrm{CoC}_{i,k+1} \mid \mathcal{D}_{I+k}\right] - \mathrm{CoC}_{i,k+1}\right) \mid \mathcal{D}_{I+k}\right)\right]^{1/2}.\end{aligned}$$

Note that the risk measure $\rho^D_{i,k+1}$ quantifies the CDR uncertainty in the claims cashflow $X_{i,I-i+j}$ *and* in the cost-of-capital cashflow $c\rho^D_{i,j+1}$, $j \geq k + 1$ as well. The total reserves at time 0 for the cost-of-capital cashflow are given by $\mathrm{CoC}_{i,0}$. We define for $k = 1, \ldots, J + i - I$

$$b_{i,k} = 1 + c\,\phi\left(\beta_{i,k} - 1\right)^{1/2}.$$

Proposition 4.5. *The cost-of-capital reserves at time* $k = 0, \ldots, J + i - I - 1$ *are given by*

$$\mathrm{CoC}_{i,k} = \widehat{C}^{(k)}_{i,J}\left(\prod_{m=k+1}^{J+i-I} b_{i,m} - 1\right).$$

The proof goes by induction and is provided in the appendix.

Remark. As a consequence of our model assumptions, the cost-of-capital cashflow turns out to be linear in $\widehat{C}^{(k)}_{i,J}$. This fact allows for an analytic calculation in the multiperiod risk measure approach for single accident years i.

Henceforth, in the multiperiod risk measure approach we have risk-adjusted claims reserves at time 0 given by

$$\begin{aligned}\mathcal{R}_{i,0}^{D} &= R_{i,0} + \mathbb{E}\left[\mathrm{CoC}_{i,1} \mid \mathcal{D}_I\right] + c\,\rho_{i,1}^{D}\\ &= R_{i,0} + \mathrm{CoC}_{i,0} \\ &= R_{i,0} + \widehat{C}_{i,J}^{(0)}\left(\prod_{k=1}^{J+i-I} b_{i,k} - 1\right),\end{aligned} \tag{4.4}$$

referred to as the multiperiod risk measure approach.

Corollary 4.6. *We have*

$$\mathcal{R}_{i,0}^{C} \leq \mathcal{R}_{i,0}^{D}.$$

The deeper reason for Corollary 4.6 to hold is that in addition to the expected cost-of-capital cashflow the risk-adjusted claims reserves in the multiperiod risk measure approach incorporate a margin against possible shortfalls in this cashflow.

The ordering of $\mathcal{R}_{i,0}^{D}$ and $\mathcal{R}_{i,0}^{B}$ depends on the choice of the cost-of-capital rate c and the choice of the security level ϕ. We need to compare

$$\left\{\prod_{k=1}^{J+i-I} b_{i,k} - 1 = \prod_{k=1}^{J+i-I}\left(1 + c\phi\left(\beta_{i,k} - 1\right)^{1/2}\right) - 1\right\} \text{ with}$$

$$\left\{c\phi \sum_{k=1}^{J+i-I} \prod_{j=1}^{k-1} \beta_{i,j}^{1/2}\left(\beta_{i,k} - 1\right)^{1/2}\right\}.$$

Lemma 4.7. *For $g_1, \ldots, g_i \in \mathbb{R}$, we have*

$$\prod_{k=1}^{i}\left(1 + g_k\right) - 1 = \sum_{k=1}^{i} \prod_{j=1}^{k-1}\left(1 + g_j\right) g_k.$$

The lemma is proved in the appendix. Therefore, the question simplifies to comparing

$$\begin{aligned}&\left\{\sum_{k=1}^{J+i-I} \prod_{j=1}^{k-1}\left(1 + c\phi\left(\beta_{i,j} - 1\right)^{1/2}\right)\left(\beta_{i,k} - 1\right)^{1/2}\right\} \text{ with}\\ &\left\{\sum_{k=1}^{J+i-I} \prod_{j=1}^{k-1} \beta_{i,j}^{1/2}\left(\beta_{i,k} - 1\right)^{1/2}\right\}.\end{aligned} \tag{4.5}$$

Corollary 4.8. *Assume for $i = I - J + 2, \ldots, I$*

$$c\phi \geq \max_{j=1,\ldots,J+i-I-1}\left(\beta_{i,j}^{1/2} - 1\right)^{1/2} / \left(\beta_{i,j}^{1/2} + 1\right)^{1/2}.$$

Then we have

$$\mathcal{R}_{i,0}^{B} \le \mathcal{R}_{i,0}^{D}.$$

The proof is provided in the appendix.

Remarks

- Note that for $c\phi$ being smaller than $\min_j (\beta_{i,j}^{1/2} - 1)^{1/2} / (\beta_{i,j}^{1/2} + 1)^{1/2}$ we obtain that the split of total uncertainty approach (4.2) gives higher risk-adjusted claims reserves than the multiperiod risk measure approach (4.4). However, in every other case we cannot say which risk-adjusted claims reserves are more conservative.
- In the practical examples we have considered, the assumptions of Corollary 4.8 were always fulfilled. This means that the multiperiod risk measure approach turned out to result in the most conservative risk-adjusted claims reserves.
- Furthermore, we have noticed that $\beta_{i,j} \approx 1$ which implies that $(\beta_{i,j} - 1)^{1/2} \ll 1$. Moreover, the security level and the cost-of-capital margin typically are such that $c\phi \le 0.3$. This immediately implies that the two terms in (4.5) are almost equal and hence, very often in practical situations we observe that $\mathcal{R}_{i,0}^{B} \approx \mathcal{R}_{i,0}^{D}$.

4.5. Aggregation of Accident Years

In the previous subsections we have studied the cost-of-capital margin for one single accident year i only. Finally, of course, we would like to measure the uncertainty over all accident years $i \in \{I - J + 1, \ldots, I\}$. Hence, the total CDR in accounting year $k = 1, \ldots, J$ is defined by

$$\mathrm{CDR}_k = \sum_{i = I + k - J}^{I} \mathrm{CDR}_{i,k}.$$

Note that the statement of Corollary 3.6 also holds true for CDR_k. Therefore, we prove a similar result like Theorem 4.2 for the aggregated CDR. For $i, m \ge I + k - J$ we have the following covariance decompositions

$$\begin{aligned}&\mathrm{Var}\left(\widehat{C}_{i,J}^{(k)} + \widehat{C}_{m,J}^{(k)} \,|\, \mathcal{D}_{I+k-1}\right)\\ &\quad = \mathrm{Var}\left(\widehat{C}_{i,J}^{(k)} \,|\, \mathcal{D}_{I+k-1}\right) + \mathrm{Var}\left(\widehat{C}_{m,J}^{(k)} \,|\, \mathcal{D}_{I+k-1}\right) + 2\,\mathrm{Cov}\left(\widehat{C}_{i,J}^{(k)}, \widehat{C}_{m,J}^{(k)} \,|\, \mathcal{D}_{I+k-1}\right),\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Var}\left(\mathrm{CDR}_{i,k} + \mathrm{CDR}_{m,k} \,|\, \mathcal{D}_I\right)\\ &\quad = \mathrm{Var}\left(\mathrm{CDR}_{i,k} \,|\, \mathcal{D}_I\right) + \mathrm{Var}\left(\mathrm{CDR}_{m,k} \,|\, \mathcal{D}_I\right) + 2\,\mathrm{Cov}\left(\mathrm{CDR}_{i,k}, \mathrm{CDR}_{m,k} \,|\, \mathcal{D}_I\right).\end{aligned}$$

Thus, it remains to study the covariance terms; the other terms are already considered in Theorem 4.2. For $I + k \leq J + i$ we define

$$\delta_{i,k} = \beta_{i,k}\left[a_{I+k-i,k} + \left(1 - a_{I+k-i,k}\right)\left(\sigma^2_{I+k-i} + 1\right)^{-1} \frac{\gamma_{I+k-i,k-1} - 2}{\gamma_{I+k-i,k-1} - 1}\right] > 1.$$

Theorem 4.9. *Under Model Assumptions 3.1 we have, for $m > i \geq I + k - J$,*

$$\mathrm{Cov}\left(\widehat{C}^{(k)}_{i,J}, \widehat{C}^{(k)}_{m,J} \,|\, \mathcal{D}_{I+k-1}\right) = \widehat{C}^{(k-1)}_{i,J}\, \widehat{C}^{(k-1)}_{m,J}\left(\delta_{i,k} - 1\right),$$

$$\mathrm{Cov}\left(\mathrm{CDR}_{i,k}, \mathrm{CDR}_{m,k} \,|\, \mathcal{D}_I\right) = \widehat{C}^{(0)}_{i,J}\, \widehat{C}^{(0)}_{m,J} \prod_{j=1}^{k-1} \delta_{i,j}\left(\delta_{i,k} - 1\right).$$

The proof is provided in the appendix.

4.5.1. Regulatory Solvency Proxy Approach (4.1)

We define the risk measure for the first accounting year by

$$\begin{aligned}
\rho_1^A &= \phi\left[\mathrm{Var}\left(\mathrm{CDR}_1 \,|\, \mathcal{D}_I\right)\right]^{1/2} = \phi\left[\mathrm{Var}\left(\sum_{i=I+1-J}^{I} \mathrm{CDR}_{i,1} \,|\, \mathcal{D}_I\right)\right]^{1/2} \\
&= \phi\left[\sum_{i=I+1-J}^{I} \mathrm{Var}\left(\mathrm{CDR}_{i,1} \,|\, \mathcal{D}_I\right) + 2 \sum_{I+1-J \leq i < m \leq I} \mathrm{Cov}\left(\mathrm{CDR}_{i,1}, \mathrm{CDR}_{m,1} \,|\, \mathcal{D}_I\right)\right]^{1/2} \\
&= \phi\left[\sum_{i=I+1-J}^{I} \left(\widehat{C}^{(0)}_{i,J}\right)^2\left(\beta_{i,1} - 1\right) + 2 \sum_{I+1-J \leq i < m \leq I} \widehat{C}^{(0)}_{i,J}\, \widehat{C}^{(0)}_{m,J}\left(\delta_{i,1} - 1\right)\right]^{1/2}.
\end{aligned}$$

The aggregated risk-adjusted claims reserves in the regulatory solvency proxy approach (4.1) at time 0 are given by

$$\mathcal{R}_0^A = \sum_{i=I+1-J}^{I} R_{i,0} + c\,\rho_1^A \sum_{k=1}^{J} \frac{\sum_{i=I+k-J}^{I} r_{i,k-1}}{\sum_{i=I+1-J}^{I} r_{i,0}}. \tag{4.6}$$

The last term describes the expected runoff pattern of the outstanding loss liabilities over all accident years. Note that the risk margin term of (4.6) is of the same nature as the flat case risk margin given in ISVAP [9], Section 1.3.2, (1.73).

4.5.2. Split of Total Uncertainty Approach (4.2)

We define the risk measure for accounting year $k \geq 1$ measurable at time $I + t$, $t < k$, by

$$\rho_k^{\mathcal{D}_{I+t}} = \phi\left[\mathrm{Var}\left(\mathrm{CDR}_k \,|\, \mathcal{D}_{I+t}\right)\right]^{1/2} = \phi\left[\mathrm{Var}\left(\sum_{i=I+k-J}^{I} \mathrm{CDR}_{i,k} \,|\, \mathcal{D}_{I+t}\right)\right]^{1/2}$$

$$= \phi\left[\sum_{i=I+k-J}^{I} \mathrm{Var}\left(\mathrm{CDR}_{i,k} | \mathcal{D}_{I+t}\right) + 2\sum_{i<m} \mathrm{Cov}\left(\mathrm{CDR}_{i,k}, \mathrm{CDR}_{m,k} \,|\, \mathcal{D}_{I+t}\right)\right]^{1/2}$$

$$= \phi\left[\sum_{i=I+k-J}^{I} \left(\widehat{C}_{i,J}^{(t)}\right)^2 \prod_{j=t+1}^{k-1} \beta_{i,j}\left(\beta_{i,k} - 1\right) + 2\sum_{i<m} \widehat{C}_{i,J}^{(t)}\, \widehat{C}_{m,J}^{(t)} \prod_{j=t+1}^{k-1} \delta_{i,j}\left(\delta_{i,k} - 1\right)\right]^{1/2}.$$

For $t = 0$ we define $\rho_k^B = \rho_k^{\mathcal{D}_I}$. Then the aggregated risk-adjusted claims reserves at time 0 in the split of total uncertainty approach (4.2) are given by

$$\mathcal{R}_0^B = \sum_{i=I+1-J}^{I} R_{i,0} + c\sum_{k=1}^{J} \rho_k^B. \tag{4.7}$$

4.5.3. Expected Stand-Alone Risk Measure Approach (4.3)

We define the risk measure for accounting year k by $\rho_k^C = \rho_k^{\mathcal{D}_{I+k-1}}$. Then the aggregated risk-adjusted claims reserves in the expected stand-alone risk measure approach (4.3) are given by

$$\mathcal{R}_0^C = \sum_{i=I+1-J}^{I} R_{i,0} + c\,\mathbb{E}\left[\sum_{k=1}^{J} \rho_k^C \,\middle|\, \mathcal{D}_I\right]. \tag{4.8}$$

The term on the right-hand side of (4.8) cannot be calculated in closed form. If we use Jensen's inequality as follows $\mathbb{E}[X] \leq \mathbb{E}[X^2]^{1/2}$ we obtain that $\mathcal{R}_0^C \leq \mathcal{R}_0^B$.

An important remark is that the approaches (4.7) and (4.8) allow for diversification between accident years:

Corollary 4.10. *Let * be B or C and* $k \geq 0$, *then we have*

$$\rho_k^* \leq \sum_{i=I+k-J}^{I} \rho_{i,k}^*.$$

This means that we have subadditive risk measures.

Proof. The proof easily follows from the fact that for any random variables $X_1, \ldots, X_m$ with finite second moment we have $[\mathrm{Var}(\sum_{i=1}^m X_i)]^{1/2} \leq \sum_{i=1}^m [\mathrm{Var}(X_i)]^{1/2}$. □

4.5.4. Multiperiod Risk Measure Approach (4.4)

The aggregation in the multiperiod risk measure approach is more involved and unfortunately does not allow for an analytic solution. Moreover, simulation results are often too time-consuming because the dimensionality of the problem is rather large (nested simulations). Formally, the multiperiod risk measure approach is given recursively. The reserves of the cost-of-capital cashflow are initially given by $\mathrm{CoC}_J = 0$ and for $k = 0, \ldots, J-1$

$$\mathrm{CoC}_k \stackrel{def.}{=} \mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + c\,\rho^D_{k+1},$$

where

$$\begin{aligned}\rho^D_{k+1} &= \phi\left[\mathrm{Var}\left(\mathrm{CDR}_{k+1} + \left(\mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] - \mathrm{CoC}_{k+1}\right) \mid \mathcal{D}_{I+k}\right)\right]^{1/2} \\ &= \phi\left[\mathrm{Var}\left(\sum_{i=I+k+1-J}^{I} \widehat{C}^{(k+1)}_{i,J} + \mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2}.\end{aligned}$$

Thus, the aggregated risk-adjusted claims reserves in the multiperiod risk measure approach (4.4) are given by

$$\mathcal{R}^D_0 = \sum_{i=I+1-J}^{I} R_{i,0} + \mathrm{CoC}_0. \tag{4.9}$$

Let us analyse this expression. If we start the backward induction at accounting year J we see that only accident year I is still active in this accounting year. Therefore, $\rho^D_J = \rho^D_{I,J}$ and for the reserves of the cost-of-capital cashflow in accounting year $J-1$ it follows that

$$\mathrm{CoC}_{J-1} = \mathrm{CoC}_{I,J-1} = c\,\phi\,\widehat{C}^{(J-1)}_{i,J}\left(\beta_{I,J} - 1\right)^{1/2}.$$

One period before, we then obtain

$$\mathrm{CoC}_{J-2} = \mathbb{E}\left[\mathrm{CoC}_{J-1} \mid \mathcal{D}_{I+J-2}\right] + c\,\rho^D_{J-1} = c\left(\phi\,\widehat{C}^{(J-2)}_{i,J}\left(\beta_{I,J} - 1\right)^{1/2} + \rho^D_{J-1}\right),$$

where (using Theorems 4.2 and 4.9)

$$\begin{aligned}\rho^D_{J-1} &= \phi\left[\mathrm{Var}\left(\mathrm{CDR}_{I,J-1}\left(1 + c\phi\left(\beta_{I,J} - 1\right)^{1/2}\right) + \mathrm{CDR}_{I-1,J-1} \mid \mathcal{D}_{I+J-2}\right)\right]^{1/2} \\ &= \phi\left[\left(\widehat{C}^{(J-2)}_{i,J}\right)^2\left(1 + c\phi\left(\beta_{I,J} - 1\right)^{1/2}\right)^2\left(\beta_{I,J-1} - 1\right) + \left(\widehat{C}^{(J-2)}_{I-1,J}\right)^2\left(\beta_{I-1,J-1} - 1\right)\right. \\ &\qquad \left. + 2\,\widehat{C}^{(J-2)}_{i,J}\,\widehat{C}^{(J-2)}_{I-1,J}\left(1 + c\phi\left(\beta_{I,J} - 1\right)^{1/2}\right)\left(\delta_{I-1,J-1} - 1\right)\right]^{1/2}.\end{aligned}$$

At this stage we lose the linearity property in the volume measures $\widehat{C}_{i,J}^{(J-2)}$ and $\widehat{C}_{I-1,J}^{(J-2)}$. For this reason we cannot further expand the calculation analytically, that is, we can neither calculate the conditionally expected value of CoC_{J-2}, given $\mathcal{D}_{I+J-3}$, nor is it possible to calculate the risk measure ρ_{J-2}^{D}. Therefore, we cannot calculate $\mathcal{R}_0^D$ in closed form. Similar difficulties occurred in the expected stand-alone risk measure approach (4.8).

By neglecting diversification effects between accident years, we easily find an upper bound for the aggregated risk-adjusted claims reserves as follows:

$$\mathcal{R}_0^D \leq \sum_{i=I+1-J}^{I} \mathcal{R}_{i,0}^D. \tag{4.10}$$

Another more sophisticated upper bound that considers diversification between accident years within accounting years is given by:

Proposition 4.11. *For $c\phi < 1$ we have*

$$\mathrm{CoC}_0 \leq \widetilde{\mathrm{CoC}_0} \stackrel{def.}{=} \sum_{k=1}^{J} \left(1 + \left(\sqrt{2} - 1\right) c\,\phi\right)^{k-1} c\,\rho_k^B.$$

The proof is provided in the appendix. This proposition motivates the following risk-adjusted claims reserves

$$\widetilde{\mathcal{R}_0^D} = \sum_{i=I+1-J}^{I} R_{i,0} + \widetilde{\mathrm{CoC}_0}. \tag{4.11}$$

Note that for $c\phi < 1$ we have the order

$$\widetilde{\mathcal{R}_0^D} \geq \max\left\{\mathcal{R}_0^B, \mathcal{R}_0^C, \mathcal{R}_0^D\right\}.$$

For the remainder we will refer to (4.1)-(4.4) as Approaches A-D, respectively.

5. Case Study

We present a case study for the different cost-of-capital approaches on a real dataset from practice. The claims data are given by a loss triangle (see Table 1) representing the observed historical cumulative claims payments $C_{i,j}$. Moreover, the data also include prior values for the development factors f_j and γ_j and the corresponding standard deviation parameter σ_j as well as the observed chain ladder factors $\bar{F}_j^{(0)}$. The standard deviation parameter σ_j is obtained from similar business. In a full Bayesian approach this parameter should also be modelled with the help of a prior distribution. But then the model is no longer

TABLE 1

OBSERVED HISTORICAL CUMULATIVE CLAIMS PAYMENTS $C_{i,j}$, AVERAGES OVER THE INDIVIDUAL CLAIMS DEVELOPMENT FACTORS $\bar{F}_j^{(0)}$, PRIOR DEVELOPMENT FACTORS f_j, PRIOR PARAMETERS γ_j, STANDARD DEVIATION PARAMETERS σ_j, CREDIBILITY WEIGHTS $\alpha_{j,0}$ FROM COROLLARY 3.3.

	Cumulative claims payments $C_{i,j}$									
$i \setminus j$	0	1	2	3	4	5	6	7	8	9
0	122'058	183'153	201'673	214'337	227'477	237'968	261'275	276'592	286'337	298'238
1	132'099	193'304	213'733	230'413	243'926	258'877	269'139	284'618	295'745	
2	132'130	186'839	207'919	222'818	237'617	253'623	267'766	284'800		
3	127'767	187'494	207'759	222'644	237'671	256'521	271'515			
4	127'648	179'633	196'260	213'636	229'660	245'968				
5	125'739	181'082	203'281	219'793	237'129					
6	117'470	172'967	190'535	204'086						
7	117'926	172'606	191'108							
8	118'274	171'248								
9	119'932									
$\bar{F}_j^{(0)}$		1.4530	1.1065	1.0750	1.0680	1.0650	1.0629	1.0599	1.0372	1.0416
f_j		1.4500	1.1100	1.0750	1.0700	1.0650	1.0630	1.0600	1.0500	1.0400
γ_j		2.1	3.0	4.1	4.3	4.7	4.8	5.1	6.4	8.8
σ_j		0.0202	0.0080	0.0078	0.0073	0.0117	0.0233	0.0031	0.0026	0.0022
$\alpha_{j,0}$		100.00%	100.00%	100.00%	100.00%	99.99%	99.95%	100.00%	100.00%	100.00%

analytically tractable. Therefore, we use an empirical Bayesian viewpoint using a plug-in estimate from similar business. Furthermore, note that we work with vague priors for Θ_j, i.e. γ_j is close to 2 which results in high credibility weights $\alpha_{j,0}$ (see Table 1).

According to the previous section, we compute the cost-of-capital margins for this runoff portfolio for all the different approaches. Table 2 presents an overview of the numerical results whereas Figure 3 summarises the results for each single accident year. The cost-of-capital rate and the security level are chosen to be $c = 6\%$ and $\phi = 3$. The choice of c corresponds to the rates also used in the IAA position paper [8], page 79, and the Swiss Solvency Test. A cost-of-capital rate of $c = 6\%$ is higher than average returns on major government bonds. This reflects the urge of risk bearers to ask for a higher than risk-free return for the compensation of the risk transfer. A more sophisticated discussion on the choice of an appropriate cost-of-capital rate involves stochastic modelling of c depending on financial markets which goes beyond the scope of this paper.

Discussion of the results

- As expected, we observe that the regulatory solvency approach (Approach A) essentially differs from the other approaches. This is because it is not risk-based for later accounting years. In particular, the regulatory solvency approach

TABLE 2

SUMMARY OF NUMERICAL RESULTS OF THE COST-OF-CAPITAL MARGINS FOR THE APPROACHES A-D FOR SINGLE ACCIDENT YEARS AND FOR THE AGGREGATED CASE. THE BRACKETS IN COLUMN C INDICATE THAT THIS VALUE IS GENERATED BY NUMERICAL SIMULATION AND FOR D IT MEANS THAT WE CALCULATED THE UPPER BOUND GIVEN IN PROPOSITION 4.11. ALL THE OTHER VALUES CAN BE CALCULATED EXACTLY.

i	reserves $R_{i,0}$	ultimate claim predictor $\widehat{C}_{i,J}^{(0)}$	cost-of-capital margins $\mathrm{CoC}_{i,0}$				% of reserves			
			A	B	C	D	A	B	C	D
1	12’292	308’037	173	173	173	173	1.4%	1.4%	1.4%	1.4%
2	22’861	307’661	302	346	346	246	1.3%	1.5%	1.5%	1.5%
3	39’369	310’884	427	543	543	543	1.1%	1.4%	1.4%	1.4%
4	53’394	299’362	3’309	1’897	1’897	1’899	6.2%	3.6%	3.6%	3.6%
5	70’239	307’368	2’188	2’672	2’671	2’678	3.1%	3.8%	3.8%	3.8%
6	78’429	282’515	1’675	2’900	2’900	2’911	2.1%	3.7%	3.7%	3.7%
7	93’284	284’392	2’015	3’372	3’371	3’387	2.2%	3.6%	3.6%	3.6%
8	110’718	281’966	2’232	3’791	3’791	3’811	2.0%	3.4%	3.4%	3.4%
9	166’991	286’923	4’390	4’913	4’912	4’947	2.6%	2.9%	2.9%	3.0%
Total	647’577		16’710	20’606	20’603	20’695	2.6%	3.2%	3.2%	3.2%
Aggregated case			11’693	13’647	(13’646)	(16’082)	1.8%	2.1%	2.1%	2.5%
Diversification effect			30%	34%	34%	22%				

may not provide sufficient protection (in our example accident years 5-9) or on the contrary superfluous protection against shortfalls (accident year 4).

- Note that Approaches B and C serve as good approximations to Approach D. From a mathematical point of view, Approach D provides a methodological comprehensive model in order to quantify the uncertainty for this multi-period risk consideration. On the other hand, in the practical examples that

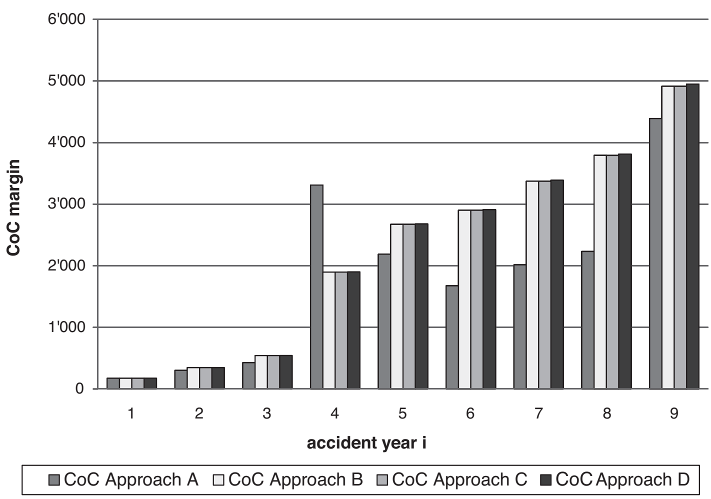

FIGURE 3: Cost-of-capital margin for single accident years.

we have looked at, the differences between Approaches B and C and Approach D turned out to be marginal. Hence, due to its simplicity, Approach B is probably preferable from a practitioners point of view.

- We see that Approach D is more conservative than Approach C. This confirms the result of Corollary 4.6 and corresponds to our intuition since Approach D additionally quantifies the risk in the cost-of-capital cashflow.
- Due to the choice of the cost-of-capital rate c and the security level ϕ, our computation shows that the assumption of Corollary 4.8 is fulfilled. Therefore, we find that in this example Approach B is slightly less conservative than Approach D.
- Intuitively, the further an accident year is developed the less uncertainty there is in the prediction. For Approaches B-D the cost-of-capital margin is growing for consecutive accident years. This agrees with the requirements formulated in the IAA position paper [8], Section 6.2. On the other hand, the regulatory solvency approach still shows a trend but with more fluctuation which might lead to counterintuitive situations (see Figure 3, observe the decrease in the cost-of-capital margin going from accident year 4 to accident year 5).
- **Important observation for premium calculation:** for the last accident year, we compute that the cost-of-capital expenses with respect to Approaches B-D account for approximately 2% of the ultimate claims prediction $\widehat{C}_{i,J}^{(0)}$. This can be read from Table 2, e.g. for accident year 9, one divides the cost-of-capital margin 4'947 of Approach D by the prediction for the ultimate claim 286'923. This implies that the cost-of-capital loadings for the runoff liabilities result in a substantial premium calculation element that is of almost 2% of the total premium! We point out that the cost-of-capital margins strongly depend on the choice of c and ϕ. This means that the percentage calculated in our example may not be representative. However, neglecting this element in premium calculations will lead to a substantial reduction of the P&L result by regulatory capital costs.

Below we extend the above results from single accident years to the study of the cost-of-capital charge for all accident years simultaneously. Figure 4 presents the results in percentage of the claims reserves. Note that in the aggregated case (see Figure 4), only Approach A and B are analytically tractable and allow for direct computation. Since Approach C lacks a closed form calculation, evaluation has been done by Monte Carlo simulation. Since numerical computation is too time-consuming, we have no viable algorithm for Approach D. Therefore, we only computed the bound given in Proposition 4.11.

- As before, the regulatory solvency approach for aggregated accident years turns out to be the less conservative one. This means that also the cost-of-capital charge for the uncertainty over all accident years may not be sufficient compared to Approaches B-D.

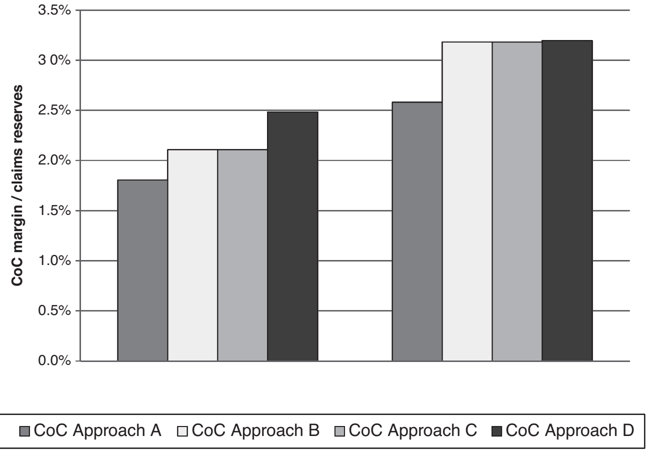

FIGURE 4: Cost-of-capital margins in percentage of the claims reserves for the aggregated case (left panel with diversification) and the total over single accident years (right panel without diversification according to, e.g., the right-hand side of formula (4.10)) for the cost-of-capital margins only.

- If we sum the cost-of-capital margins over single accident years, we immediately get upper bounds for Approaches B-D (see Figure 4). Diversification effects between accident years account for substantial releases of over 34% for Approach B and C and 22% for the upper bound of Approach D. Since our model does not allow for accounting (calendar) year effects modelling, these results may overstate the releases actually feasible in practice.
- We observe that for this example, the upper bound of Approach D turns out to be an upper bound for all the other Approaches. For Approach C, this is immediately clear and, since $c\phi < 1$, the result holds true for Approach B. Note that in general, Approach D is not necessarily more conservative than Approach A.
- The example further confirms that Approach B is more conservative than Approach C but just by a small margin.

The fact that Approach B is analytically tractable for aggregated accident years makes it a preferable approximation to the multiperiod risk measure approach. The computed upper bound for Approach D in the aggregated case only allows for a rough statement about the precision of this approximation. If we compare the results for single accident years, we observe that the uncertainty in the cost-of-capital cashflow accounts just for a marginal proportion. Figure 4 indicates that the upper bound for Approach D is conservative.

Finally, we would like to compare the gamma-gamma Bayes chain ladder model used in this discussion with the classical distribution-free chain ladder model presented in Mack [11]. A main deviation lies in the fact that the chain ladder factors are calculated differently, we use an average over the observed individual claims development factors $F_{i,j}$ (see Mack [12]), whereas the classical chain ladder model takes a volume weighted average thereof.

We denote by

$$\mathrm{msep}_{C_{i,J}|\mathcal{D}_I}\left(\widehat{C}_{i,J}^{(0)}\right) = \mathrm{Var}\left(C_{i,J} \,|\, \mathcal{D}_I\right),$$

the mean square error of prediction (MSEP) of $\widehat{C}_{i,J}^{(0)}$ and by

$$\mathrm{msep}_{\mathrm{CDR}_{i,1}|\mathcal{D}_I}(0) = \mathrm{Var}\left(\mathrm{CDR}_1 \,|\, \mathcal{D}_I\right),$$

the MSEP of the CDR of the first accounting year for the gamma-gamma Bayes chain ladder model (see Corollary 4.3 and Subsection 4.5.1). For the classical chain ladder model, the estimators of the MSEP are denoted by

$$\widehat{\mathrm{msep}}^{\mathrm{Mack}}_{C_{i,J}|\mathcal{D}_I}\left(\widehat{C}_{i,J}^{\mathrm{CL}}\right) \quad \text{and} \quad \widehat{\mathrm{msep}}^{\mathrm{BDGMV}}_{\mathrm{CDR}_{i,1}^{\mathrm{CL}}|\mathcal{D}_I}(0).$$

The first is estimated with the classical Mack formula (Mack [11]), the latter is calculated according to Remark 4.11 in Bühlmann et al. [2].

We only observe marginal deviations, that is, our model choice does not significantly change risk assessment compared to a classical chain ladder model.

Limitations

Our choice of the model, the security level ϕ, a constant for c and the standard deviation as risk measure was mainly motivated by the fact that it leads to a model that is analytically tractable. As a matter of fact, our model does

TABLE 3

Gamma-gamma Bayes chain ladder, the last row provides the aggregated case.

Gamma-gamma Bayes chain ladder					
i	reserves	$\mathrm{msep}_{C_{i,J}\|\mathcal{D}_I}\left(\widehat{C}_{i,J}^{(0)}\right)^{1/2}$	% of reserves	$\mathrm{msep}_{\mathrm{CDR}_{i,1}\|\mathcal{D}_I}(0)^{1/2}$	% of reserves
1	12’292	961	8%	961	8%
2	22’861	1’372	6%	1’091	5%
3	39’369	1’770	4%	1’247	3%
4	53’394	7’981	15%	7’822	15%
5	70’239	9’087	13%	4’288	6%
6	78’429	8’642	11%	2’791	4%
7	93’284	9’014	10%	2’929	3%
8	110’718	9’251	8%	2’958	3%
9	166’991	11’226	7%	6’371	4%
Total	**647’577**	**31’317**	5%	**19’402**	3%

TABLE 4

CLASSICAL CHAIN LADDER MODEL PRESENTED IN MACK [11], THE MSEP FOR THE ONE-YEAR CDR IS CALCULATED ACCORDING TO BÜHLMANN ET AL. [2] AND THE LAST ROW PROVIDES THE AGGREGATED CASE.

Mack model [11]					
i	reserves	$\widehat{\mathrm{msep}}^{\mathrm{Mack}}_{C_{i,J}\mid\mathcal{D}_I}\left(\widehat{C}^{\mathrm{CL}}_{i,J}\right)^{1/2}$	% of reserves	$\widehat{\mathrm{msep}}^{\mathrm{BDGMV}}_{\mathrm{CDR}^{\mathrm{CL}}_{i,1}\mid\mathcal{D}_I}(0)^{1/2}$	% of reserves
1	12'292	965	8%	965	8%
2	22'869	1'380	6%	1'102	5%
3	39'379	1'770	4%	1'248	3%
4	53'212	7'946	15%	7'783	15%
5	70'083	8'957	13%	4'232	6%
6	78'263	8'822	11%	2'840	4%
7	93'112	9'177	10%	2'946	3%
8	110'561	9'454	9%	2'993	3%
9	166'722	11'406	7%	6'482	4%
Total	**646'494**	**31'345**	5%	**19'300**	3%

not account for all the idiosyncrasies one has to deal with in practice. However, it allows for a first investigation of the CDR as a main risk driver in multiperiod solvency considerations of general insurance companies. Moreover, by the means of our specific example we presented a representative comparison of the different proxies used in practice and highlighted their shortcomings.

CONCLUSION AND OUTLOOK

We have studied the CDR for a multiperiod general insurance liability runoff portfolio. Our paper directly addresses open questions discussed in the IAA position paper [8] concerning the calculation of an appropriate cost-of-capital margin. For the four different approaches discussed in this paper, the numerical example indicated that the "Split of the Total Uncertainty Approach" (Approach B) provides a good approximation to the mathematically comprehensive "Multiperiod Risk Measure Approach" (Approach D). Moreover, the example confirms that the cost-of-capital margins may have a substantial implication on premiums which should be accounted for in premium calculations. Note that the cost-of-capital margins strongly depend on the chosen constants. In this context we point out that further research should also consider stochastically modelled cost-of-capital rate c.

We performed our calculation in a Bayesian model framework which represents a canonical way to account for the parameter uncertainty and allows for the immediate absorption of recent information by the model. The underlying distributions and model parameters are chosen in such a way that Approaches A-D for single accident years as well as Approach A and B in the aggregated case have analytic solutions. The study of other stochastic models and claims reserving methods goes beyond the scope of the present paper but is an important topic for further research.

So far, our discussion only considers nominal values. Undiscounted values then incorporate hidden reserves which contribute substantially to the financial strength of general insurance companies. A next step is to extent the results to discounted claims reserves similar to Wüthrich-Bühlmann [21]. This then provides a cost-of-capital margin for discounted claims reserves.

Further research should also investigate the role of other risk measures, dependency modelling along accounting years, and the incorporation of other information available. We emphasise that the analytical tractability was essentially based on the standard deviation loading and our choice of the model.

Acknowledgements

We would like to kindly thank the anonymous referees whose expert comments have helped to appreciably improve the outline of this paper. Furthermore, we are grateful to Prof. Embrechts for providing valuable remarks related to this paper.

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A. PROOFS

Proof of Theorem 4.2. For $I + k \leq J + i$ we obtain

$$\mathrm{Var}\left(\widehat{C}_{i,J}^{(k)} \mid \mathcal{D}_{I+k-1}\right) = \mathrm{Var}\left(C_{i,I+k-i} \prod_{j=I+k-i+1}^{J} \hat{f}_j^{(k)} \mid \mathcal{D}_{I+k-1}\right).$$

Using Corollary 3.4 we rewrite the chain ladder factor estimators as follows

$$\hat{f}_j^{(k)} = \alpha_{j,k} \bar{F}_j^{(k)} + \left(1 - \alpha_{j,k}\right) f_j = a_{j,k} F_{I+k-j,j} + \left(1 - a_{j,k}\right) \hat{f}_j^{(k-1)}.$$

From this we see that conditionally, given $\mathcal{D}_{I+k-1}$, $\hat{f}_j^{(k)}$ is only random in $F_{I+k-j,j}$. Using the posterior independence of $\Theta_1, \ldots, \Theta_J$, given $\mathcal{D}_{I+k-1}$, and that fact that all random variables involved only depend on different accident years and development years, we see that $C_{i,I+k-i}, \hat{f}_{I+k-i+1}^{(k)}, \ldots, \hat{f}_J^{(k)}$ are independent, given $\mathcal{D}_{I+k-1}$. This implies that

$$\begin{aligned}
&\mathrm{Var}\left(C_{i,I+k-i} \prod_{j=I+k-i+1}^{J} \hat{f}_j^{(k)} \mid \mathcal{D}_{I+k-1}\right) \\
&= \mathbb{E}\left[C_{i,I+k-i}^2 \mid \mathcal{D}_{I+k-1}\right] \prod_{j=I+k-i+1}^{J} \mathbb{E}\left[\left(\hat{f}_j^{(k)}\right)^2 \mid \mathcal{D}_{I+k-1}\right] \\
&\quad - \mathbb{E}\left[C_{i,I+k-i} \mid \mathcal{D}_{I+k-1}\right]^2 \prod_{j=I+k-i+1}^{J} \mathbb{E}\left[\hat{f}_j^{(k)} \mid \mathcal{D}_{I+k-1}\right]^2 \\
&= C_{i,I+k-i-1}^2 \left(\mathbb{E}\left[F_{i,I+k-i}^2 \mid \mathcal{D}_{I+k-1}\right] \prod_{j=I+k-i+1}^{J} \mathbb{E}\left[\left(\hat{f}_j^{(k)}\right)^2 \mid \mathcal{D}_{I+k-1}\right] - \prod_{j=I+k-i}^{J} \left(\hat{f}_j^{(k-1)}\right)^2\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left[F_{i,I+k-i}^2 \mid \mathcal{D}_{I+k-1}\right] &= \mathbb{E}\left[\mathbb{E}\left[F_{i,I+k-i}^2 \mid \boldsymbol{\Theta}, \mathcal{D}_{I+k-1}\right] \mid \mathcal{D}_{I+k-1}\right] \\
&= \mathbb{E}\left[\operatorname{Var}\left(F_{i,I+k-i} \mid \boldsymbol{\Theta}\right) + \mathbb{E}\left[F_{i,I+k-i} \mid \boldsymbol{\Theta}\right]^2 \mid \mathcal{D}_{I+k-1}\right] \\
&= \left(\sigma_{I+k-i}^2 + 1\right) \mathbb{E}\left[\Theta_{I+k-i}^{-2} \mid \mathcal{D}_{I+k-1}\right] \\
&= \left(\sigma_{I+k-i}^2 + 1\right) \left(\hat{f}_{I+k-i}^{(k-1)}\right)^2 \frac{\gamma_{I+k-i,\,k-1} - 1}{\gamma_{I+k-i,\,k-1} - 2},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\left(\hat{f}_j^{(k)}\right)^2 \mid \mathcal{D}_{I+k-1}\right] &= \operatorname{Var}\left(\hat{f}_j^{(k)} \mid \mathcal{D}_{I+k-1}\right) + \left(\hat{f}_j^{(k-1)}\right)^2 \\
&= a_{j,k}^2 \operatorname{Var}\left(F_{I+k-j,j} \mid \mathcal{D}_{I+k-1}\right) + \left(\hat{f}_j^{(k-1)}\right)^2 \\
&= \left(\hat{f}_j^{(k-1)}\right)^2 \left[a_{j,k}^2 \left(\left(\sigma_j^2 + 1\right) \frac{\gamma_{j,k-1} - 1}{\gamma_{j,k-1} - 2} - 1\right) + 1\right].
\end{aligned}$$

This proves the first claim of the theorem. Moreover, we have

$$\operatorname{Var}\left(\mathrm{CDR}_{i,k} \mid \mathcal{D}_I\right) = \mathbb{E}\left[\mathrm{CDR}_{i,k}^2 \mid \mathcal{D}_I\right] = \mathbb{E}\left[\operatorname{Var}\left(\widehat{C}_{i,J}^{(k)} \mid \mathcal{D}_{I+k-1}\right) \mid \mathcal{D}_I\right].$$

This implies (with the first statement of the theorem) that

$$\begin{aligned}
\mathbb{E}\left[\operatorname{Var}\left(\widehat{C}_{i,J}^{(k)} \mid \mathcal{D}_{I+k-1}\right) \mid \mathcal{D}_I\right] &= \left(\beta_{i,k} - 1\right)\ \mathbb{E}\left[\left(\widehat{C}_{i,J}^{(k-1)}\right)^2 \mid \mathcal{D}_I\right] \\
&= \left(\beta_{i,k} - 1\right) \mathbb{E}\left(\operatorname{Var}\left(\widehat{C}_{i,J}^{(k-1)} \mid \mathcal{D}_{I+k-2}\right) + \left(\widehat{C}_{i,J}^{(k-2)}\right)^2 \mid \mathcal{D}_I\right).
\end{aligned}$$

Iterating this procedure completes the proof.

□

Proof of Corollary 4.3. From Theorem 4.2 we obtain

$$\operatorname{Var}\left(\sum_{k=1}^{J-i+I} \mathrm{CDR}_{i,k} \mid \mathcal{D}_I\right) = \left(\widehat{C}_{i,J}^{(0)}\right)^2 \sum_{k=1}^{J-i+I} \prod_{j=1}^{k-1} \beta_{i,j}\left(\beta_{i,k} - 1\right).$$

Calculating the last sum gives the first claim. Because of

$$\operatorname{Var}\left(\sum_{k=1}^{J-i+I} \mathrm{CDR}_{i,k} \mid \mathcal{D}_I\right) = \operatorname{Var}\left(C_{i,J} \mid \mathcal{D}_I\right),$$

a straightforward calculation gives

$$\begin{aligned}
\mathrm{Var}\left(C_{i,J}\,|\,\mathcal{D}_I\right) &= \mathbb{E}\left[C_{i,J}^2\,|\,\mathcal{D}_I\right] - \mathbb{E}\left[C_{i,J}\,|\,\mathcal{D}_I\right]^2 \\
&= \mathbb{E}\left[\mathbb{E}\left[C_{i,J}^2\,|\,\mathbf{\Theta},\mathcal{D}_{I+J-1}\right]|\,\mathcal{D}_I\right] - \left(\widehat{C}_{i,J}^{(0)}\right)^2 \\
&= \mathbb{E}\left[\mathrm{Var}\left(C_{i,J}\,|\,\mathbf{\Theta},\mathcal{D}_{I+J-1}\right) + \mathbb{E}\left[C_{i,J}\,|\,\mathbf{\Theta},\mathcal{D}_{I+J-1}\right]^2|\,\mathcal{D}_I\right] - \left(\widehat{C}_{i,J}^{(0)}\right)^2 \\
&= \left(\sigma_J^2+1\right)\mathbb{E}\left[\Theta_J^{-2}\,C_{i,J-1}^2\,|\,\mathcal{D}_I\right] - \left(\widehat{C}_{i,J}^{(0)}\right)^2.
\end{aligned}$$

Iterating this procedure and using posterior independence of Θ_j, given $\mathcal{D}_I$, we obtain

$$\begin{aligned}
\mathrm{Var}(C_{i,J}\,|\,\mathcal{D}_I) &= C_{i,I-i}^2 \prod_{j=I-i+1}^{J}\left(\sigma_j^2+1\right)\mathbb{E}\left[\Theta_j^{-2}\,|\,\mathcal{D}_I\right] - \left(\widehat{C}_{i,J}^{(0)}\right)^2 \\
&= \left(\widehat{C}_{i,J}^{(0)}\right)^2\left[\prod_{j=I-i+1}^{J}\left(\left(\sigma_j^2+1\right)\frac{\gamma_{j,0}-1}{\gamma_{j,0}-2}\right)-1\right].
\end{aligned}$$

This proves the corollary. □

Proof of Proposition 4.5. We calculate the cost-of-capital reserves inductively. For $k = J + i - I - 1$ we obtain (see Theorem 4.2)

$$\begin{aligned}
\mathrm{CoC}_{i,J+i-I-1} &= 0 + c\,\rho_{i,J+i-I}^D = c\,\phi\left[\mathrm{Var}\left(\mathrm{CDR}_{i,J+i-I}\,|\,\mathcal{D}_{J+i-1}\right)\right]^{1/2} \\
&= c\,\phi\,\widehat{C}_{i,J}^{(J+i-I-1)}\left(\beta_{i,J+i-I}-1\right)^{1/2}.
\end{aligned}$$

This proves the claim for $k = J + i - I - 1$.
Induction step: Assume the claim holds true for $k + 1$. We then have that

$$\mathrm{CoC}_{i,k} = \mathbb{E}\left[\mathrm{CoC}_{i,k+1}\,|\,\mathcal{D}_{I+k}\right] + c\,\rho_{i,k+1}^D = \widehat{C}_{i,J}^{(k)}\left(\prod_{m=k+2}^{J+i-I} b_{i,m} - 1\right) + c\,\rho_{i,k+1}^D,$$

where

$$\begin{aligned}
\rho_{i,k+1}^D &= \phi\left[\mathrm{Var}\left(\mathrm{CDR}_{i,k+1} + \left(\mathbb{E}\left[\mathrm{CoC}_{i,k+1}\,|\,\mathcal{D}_{I+k}\right] - \mathrm{CoC}_{i,k+1}\right)|\,\mathcal{D}_{I+k}\right)\right]^{1/2} \\
&= \phi\left[\mathrm{Var}\left(\widehat{C}_{i,J}^{(k+1)} + \widehat{C}_{i,J}^{(k+1)}\left(\prod_{m=k+2}^{J+i-I} b_{i,m} - 1\right)\middle|\,\mathcal{D}_{I+k}\right)\right]^{1/2} \\
&= \phi\prod_{m=k+2}^{J+i-I} b_{i,m}\left[\mathrm{Var}\left(\widehat{C}_{i,J}^{(k+1)}|\,\mathcal{D}_{I+k}\right)\right]^{1/2} = \widehat{C}_{i,J}^{(k)}\prod_{m=k+2}^{J+i-I} b_{i,m}\,\phi\left(\beta_{i,k+1}-1\right)^{1/2},
\end{aligned}$$

where we have used Theorem 4.2 in the last step. This completes the proof. $\square$

Proof of Lemma 4.7. The proof goes by induction. It is obvious that the result holds for $i = 1, 2$. Hence we do the induction step $i \to i + 1$. Using the induction assumption we get

$$\begin{aligned}\prod_{k=1}^{i+1}(1+g_k)-1 &= (1+g_{i+1})\prod_{k=1}^{i}(1+g_k)-1\\ &= \left(\sum_{k=1}^{i}\prod_{j=1}^{k-1}(1+g_j)\,g_k+1\right)+g_{i+1}\left(\prod_{k=1}^{i}(1+g_k)\right)-1\\ &= \sum_{k=1}^{i}\prod_{j=1}^{k-1}(1+g_j)\,g_k+g_{i+1}\prod_{k=1}^{i}(1+g_k).\end{aligned}$$

This proves the result. $\square$

Proof of Corollary 4.8. In view of (4.5) it suffices to prove that for all, $j = 1, \ldots, J + i - I - 1$,

$$1 + c\phi\left(\beta_{i,j}-1\right)^{1/2} \geq \beta_{i,j}^{1/2},$$

or equivalently

$$c\phi\left(\beta_{i,j}-1\right)^{1/2} \geq \beta_{i,j}^{1/2}-1,$$

If we rewrite the left-hand side $(\beta_{i,j} - 1)^{1/2} = (\beta_{i,j}^{1/2} - 1)^{1/2}\,(\beta_{i,j}^{1/2} + 1)^{1/2}$ we obtain that this is equivalent to the assumption

$$c\phi \geq \left(\frac{\beta_{i,j}^{1/2}-1}{\beta_{i,j}^{1/2}+1}\right)^{1/2},$$

which completes the proof. $\square$

Proof of Theorem 4.9. For $m > i \geq I + k - J$ we obtain

$$\begin{aligned}&\mathrm{Cov}\left(\widehat{C}_{i,J}^{(k)},\, \widehat{C}_{m,J}^{(k)} \mid \mathcal{D}_{I+k-1}\right)\\ &= \mathrm{Cov}\left(C_{i,I+k-i}\prod_{j=I+k-i+1}^{J}\hat{f}_j^{(k)}, C_{m,I+k-m}\prod_{j=I+k-m+1}^{I+k-i-1}\hat{f}_j^{(k)}\prod_{j=I+k-i}^{J}\hat{f}_j^{(k)} \mid \mathcal{D}_{I+k-1}\right).\end{aligned}$$

Using Corollary 3.4 we decouple the problem into independent problems similar to Theorem 4.2. This implies

$$\begin{aligned}
&\mathrm{Cov}\Big(\widehat{C}_{i,J}^{(k)}, \widehat{C}_{m,J}^{(k)} \,|\, \mathcal{D}_{I+k-1}\Big) \\
&= \mathbb{E}\Big[C_{m,I+k-m} \,|\, \mathcal{D}_{I+k-1}\Big] \prod_{j=I+k-m+1}^{I+k-i-1} \mathbb{E}\Big[\hat{f}_j^{(k)} \,|\, \mathcal{D}_{I+k-1}\Big] \mathbb{E}\Big[C_{i,I+k-i} \hat{f}_{I+k-i}^{(k)} \,|\, \mathcal{D}_{I+k-1}\Big] \\
&\quad \times \prod_{j=I+k-i+1}^{J} \mathbb{E}\Big[\Big(\hat{f}_j^{(k)}\Big)^2 \,|\, \mathcal{D}_{I+k-1}\Big] - \widehat{C}_{i,J}^{(k-1)} \widehat{C}_{m,J}^{(k-1)}.
\end{aligned}$$

The only difference to Theorem 4.2 is that the calculation of $\mathbb{E}[F_{i,I+k-i}^2 \,|\, \mathcal{D}_{I+k-1}]$ is now replaced by

$$\begin{aligned}
&\mathbb{E}\Big[F_{i,I+k-i} \hat{f}_{I+k-i}^{(k)} \,|\, \mathcal{D}_{I+k-1}\Big] \\
&\quad = \mathbb{E}\Big[F_{i,I+k-i}\Big(a_{I+k-i,k}\, F_{i,I+k-i} + \big(1 - a_{I+k-i,k}\big) \hat{f}_{I+k-i}^{(k-1)}\Big) \,|\, \mathcal{D}_{I+k-1}\Big] \\
&\quad = a_{I+k-i,k}\, \mathbb{E}\Big[F_{i,I+k-i}^2 \,|\, \mathcal{D}_{I+k-1}\Big] + \big(1 - a_{I+k-i,k}\big) \Big(\hat{f}_{I+k-i}^{(k-1)}\Big)^2 \\
&\quad = \Big(\hat{f}_{I+k-i}^{(k-1)}\Big)^2 \left[a_{I+k-i,k}\big(\sigma_{I+k-i}^2 + 1\big) \frac{\gamma_{I+k-i,k-1} - 1}{\gamma_{I+k-i,k-1} - 2} + 1 - a_{I+k-i,k}\right].
\end{aligned}$$

This proves the first claim of the theorem. Moreover, we have

$$\mathrm{Cov}\big(\mathrm{CDR}_{i,k}, \mathrm{CDR}_{m,k} \,|\, \mathcal{D}_I\big) = \mathbb{E}\Big[\mathrm{Cov}\Big(\widehat{C}_{i,J}^{(k)}, \widehat{C}_{m,J}^{(k)} \,|\, \mathcal{D}_{I+k-1}\Big) \,|\, \mathcal{D}_I\Big].$$

This implies (with the first statement of this theorem) that

$$\begin{aligned}
\mathbb{E}\Big[\mathrm{Cov}\Big(\widehat{C}_{i,J}^{(k)}, \widehat{C}_{m,J}^{(k)} \,|\, \mathcal{D}_{I+k-1}\Big) \,|\, \mathcal{D}_I\Big] &= \big(\delta_{i,k} - 1\big)\, \mathbb{E}\Big[\widehat{C}_{i,J}^{(k-1)}\, \widehat{C}_{m,J}^{(k-1)} \,|\, \mathcal{D}_I\Big] \\
&= \big(\delta_{i,k} - 1\big)\, \mathbb{E}\Big[\mathrm{Cov}\Big(\widehat{C}_{i,J}^{(k-1)}, \widehat{C}_{m,J}^{(k-1)} \,|\, \mathcal{D}_{I+k-2}\Big) + \widehat{C}_{i,J}^{(k-2)}\, \widehat{C}_{m,J}^{(k-2)} \,|\, \mathcal{D}_I\Big].
\end{aligned}$$

Iterating this procedure completes the proof.

□

In order to prove Proposition 4.11 we need the following lemma.

Lemma A.1. *Choose $y \geq x \geq 0$ then we have, for $p \in (0,1)$,*

$$p\,x + (1-p)\big(y^2 - x^2\big)^{1/2} \leq \big((1-p)\, y^2 + (2p-1)x^2\big)^{1/2}.$$

Proof of Lemma A.1. We define the discrete random variable Y by $P[Y = \sqrt{2x}] = p$ and $P[Y = y] = 1 - p$ for $p \in (0,1)$. Hence we have

$$E\left[\left(Y^2 - x^2\right)^{1/2}\right] = px + (1-p)\left(y^2 - x^2\right)^{1/2},$$

and on the other hand using Jensen's inequality

$$\begin{aligned} E\left[\left(Y^2 - x^2\right)^{1/2}\right] \le E\left[Y^2 - x^2\right]^{1/2} &= \left(px^2 + (1-p)\left(y^2 - x^2\right)\right)^{1/2} \\ &= \left((1-p)\,y^2 + (2p-1)\,x^2\right)^{1/2}. \end{aligned}$$

This proves the lemma.

□

Proof of Proposition 4.11. For $k \in \{0, 1, \ldots, J-2\}$ fixed, we have that

$$\begin{aligned} \mathrm{CoC}_k &= \mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + c\,\phi\left[\mathrm{Var}\left(\mathrm{CDR}(k+1) - \mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2} \\ &\le c\,\rho_{k+1}^{\mathcal{D}_{I+k}} + \mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + c\,\phi\left[\mathrm{Var}\left(\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2}. \end{aligned}$$

We rewrite the above expression in such a way that it is more suitable for iteration. For this, let $p \in (0,1)$ and

$$\begin{aligned} &\mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + c\,\phi\left[\mathrm{Var}\left(\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2} \\ &\quad = \left(1 - c\,\phi\,\frac{p}{1-p}\right) \mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] \\ &\qquad + \frac{c\,\phi}{1-p}\left(p\,\mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + (1-p)\left[\mathrm{Var}\left(\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2}\right). \end{aligned}$$

Now we apply Lemma A.1 to the last term which provides the following upper bound

$$\begin{aligned} &\mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + c\,\phi\left[\mathrm{Var}\left(\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2} \\ &\quad \le \left(1 - c\,\phi\,\frac{p}{1-p}\right) \mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] \\ &\qquad + \frac{c\,\phi}{1-p}\left((1-p)\,\mathbb{E}\left[\mathrm{CoC}_{k+1}^2 \mid \mathcal{D}_{I+k}\right] + (2p-1)\,\mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right]^2\right)^{1/2}. \end{aligned}$$

Note that the term $2p - 1$ is positive for $p \geq 1/2$ and with Jensen's inequality applied to the last term for $p \in [1/2, 1)$ we obtain

$$\mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + c\,\phi\left[\mathrm{Var}\left(\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2}$$
$$\leq \left(1 - c\,\phi\,\frac{p}{1-p}\right)\mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + c\,\phi\,\frac{p^{1/2}}{1-p}\;\mathbb{E}\left[\mathrm{CoC}_{k+1}^2 \mid \mathcal{D}_{I+k}\right]^{1/2}.$$

Note that $c\phi < 1$ implies $(c\phi + 1)^{-1} > 1/2$. Hence, for $p \in [1/2, (c\phi + 1)^{-1})$ we see that $c\phi\, p / (1 - p) < 1$ and consequently with Jensen's inequality applied to the first term on the right-hand side of the above inequality we obtain

$$\mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + c\,\phi\left[\mathrm{Var}\left(\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2}$$
$$\leq \left(1 + c\,\phi\,\frac{p^{1/2} - p}{1-p}\right)\;\mathbb{E}\left[\mathrm{CoC}_{k+1}^2 \mid \mathcal{D}_{I+k}\right]^{1/2}.$$

For $p = 1/2$, the right-hand side is minimal in p. Define $\kappa = 1 + (\sqrt{2} - 1)\, c\phi$, hence

$$\mathbb{E}\left[\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right] + c\,\phi\left[\mathrm{Var}\left(\mathrm{CoC}_{k+1} \mid \mathcal{D}_{I+k}\right)\right]^{1/2} \leq \kappa\,\mathbb{E}\left[\mathrm{CoC}_{k+1}^2 \mid \mathcal{D}_{I+k}\right]^{1/2}.$$

By iteration we find that

$$\mathrm{CoC}_k \leq c\,\rho_{k+1}^{\mathcal{D}_{I+k}} + \kappa\,\mathbb{E}\left[\mathrm{CoC}_{k+1}^2 \mid \mathcal{D}_{I+k}\right]^{1/2}$$
$$\leq c\,\rho_{k+1}^{\mathcal{D}_{I+k}} + \kappa\,\mathbb{E}\left[\left(c\,\rho_{k+2}^{\mathcal{D}_{I+k+1}} + \kappa\,\mathbb{E}\left[\mathrm{CoC}_{k+2}^2 \mid \mathcal{D}_{I+k+1}\right]^{1/2}\right)^2 \middle| \mathcal{D}_{I+k}\right]^{1/2}.$$

By Minkowski's inequality we obtain

$$\mathrm{CoC}_k \leq c\,\rho_{k+1}^{\mathcal{D}_{I+k}} + \kappa\, c\,\mathbb{E}\left[\left(\rho_{k+2}^{\mathcal{D}_{I+k+1}}\right)^2 \mid \mathcal{D}_{I+k}\right]^{1/2} + \kappa^2\,\mathbb{E}\left[\mathrm{CoC}_{k+2}^2 \mid \mathcal{D}_{I+k}\right]^{1/2}.$$

By iterating this procedure we obtain

$$\mathrm{CoC}_k \leq \sum_{j=0}^{J-k-1} \kappa^j\, c\;\mathbb{E}\left[\left(\rho_{k+j+1}^{\mathcal{D}_{I+k+j}}\right)^2 \mid \mathcal{D}_{I+k}\right]^{1/2}.$$

Note that we have

$$\mathbb{E}\left[\left(\rho_{j+1}^{\mathcal{D}_{I+j}}\right)^2 | \mathcal{D}_I\right]^{1/2} = \rho_{j+1}^{\mathcal{D}_I} = \rho_{j+1}^{B}.$$

This proves the result.

$\square$

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