J. reine angew. Math. **663** (2012), 91—126
DOI 10.1515/CRELLE.2011.095

Journal für die reine und angewandte Mathematik

Representations up to homotopy of Lie algebroids

By *Camilo Arias Abad* at Zürich and *Marius Crainic* at Utrecht

Abstract. We introduce and study the notion of representation up to homotopy of a Lie algebroid, paying special attention to examples. We use representations up to homotopy to define the adjoint representation of a Lie algebroid and show that the resulting cohomology controls the deformations of the structure. The Weil algebra of a Lie algebroid is defined and shown to coincide with Kalkman's BRST model for equivariant cohomology in the case of group actions. The relation of this algebra with the integration of Poisson and Dirac structures is explained in [3].

1. Introduction

Lie algebroids are infinite dimensional Lie algebras which can be thought of as *generalized tangent bundles* associated to various geometric situations. Apart from Lie algebras and tangent bundles, examples of Lie algebroids come from foliation theory, equivariant geometry, Poisson geometry, riemannian foliations, quantization, etc. Lie algebroids are the infinitesimal counterparts of Lie groupoids exactly in the same way in which Lie algebras are related to Lie groups. Generalizing representations of Lie algebras as well as vector bundles endowed with a flat connection, a representation of a Lie algebroid on a vector bundle is an action by derivations on the space of sections.

The aim of this paper is to introduce and study a more general notion of representation: "representations up to homotopy". Our approach is based on Quillen's superconnections [29] and fits into the general theory of structures up to homotopy. The idea is to represent Lie algebroids in cochain complexes of vector bundles, rather than in vector bundles. In a representation up to homotopy the complex is given an action of the Lie algebroid and homotopy operators ω_k for $k \geqq 2$. The action is required to be flat only up to homotopy, that is, the curvature of the action may be non-zero, but it is homotopic to zero via the homotopy ω_2; in turn, ω_2 is required to satisfy the appropriate coherence condition (Bianchi identity) only up to a homotopy given by ω_3, and there are higher order coherence conditions. Quillen's formalism is used to book-keep the equations involved.

The first author supported by NWO Grant "Symmetries and Deformations in Geometry" and by SNF Grant 200020-121640/1. The second author supported by NWO Vidi Grant no. 639.032.712.

The advantage of considering these representations is that they are flexible and general enough to contain interesting examples which are the correct generalization of the corresponding notions for Lie algebras. In the setting of representations up to homotopy one can give a good definition of the adjoint representation of a Lie algebroid. We will show that, as in the case of Lie algebras, the cohomology associated to the adjoint representation of a Lie algebroid controls the deformations of the structure (it coincides with the deformation cohomology of [14]). There are other seemingly ad-hoc equations that arise from various geometric problems and can now be recognized as cocycle equations for the cohomology associated to a representation up to homotopy. This is explained in Proposition 4.6 for the k-differentials of [20].

Our original motivation for considering representations up to homotopy is the study of the cohomology of classifying spaces of Lie groupoids. We will introduce the Weil algebra $W(A)$ associated to a Lie algebroid A. When the Lie algebroid comes from a Lie groupoid G, the Weil algebra serves as a model for the De-Rham algebra of the total space of the universal principal G-bundle $EG \to BG$. As we shall explain, this generalizes not only the usual Weil algebra of a Lie algebra but also Kalkman's BRST algebra for equivariant cohomology. Further applications of the Weil algebra are given in [3], while applications of the notion of representations up to homotopy to the cohomology of classifying spaces—Bott's spectral sequence—are explained in [2].

A few words about the relationship of this paper with other work that we found in the literature. Representations up to homotopy can also be described in the language of differential graded modules over differential graded algebras [19]. We emphasize however that we insist on working with DG-modules which are sections of vector bundles. Our Weil algebra is isomorphic to the one given in Mehta's thesis [28], where, using the language of supermanifolds, it appears as $\big(C^{\infty}\big([-1]T([-1]A)\big), \mathscr{L}_{d_A} + d\big)$. Similar descriptions were communicated to us by D. Roytenberg and P. Severa (unpublished). Some of the equations that appear in the definition of the adjoint representation were considered by Blaom in [4]. Representations up to homotopy provide examples of the Q-bundles studied by Kotov and Strobl in [25].

This paper is organized as follows. Section 2 begins by collecting the definitions of Lie algebroids, representations and the associated cohomology theories. Then, the connections and curvatures underlying the adjoint representation are described.

In Section 3, we give the definition of representations up to homotopy, introduce several examples and explain the relationship with extensions (Proposition 3.9). We define the adjoint representation and point out that the associated cohomology is isomorphic to the deformation cohomology of [14], Theorem 3.11. We also explain the relation between the adjoint representation and the first jet algebroid (Proposition 3.12). In the case of the tangent bundle we describe the associated parallel transport (Proposition 3.13).

In Section 4, we discuss the main operations on the category of representations up to homotopy and then we have a closer look at the resulting derived category. In particular, we prove that our notion has the usual properties that one expects for "structures up to homotopy" (Proposition 4.12 and Theorem 4.13). Various examples are presented.

In Section 5, we introduce the Weil algebra of a Lie algebroid, generalizing the standard Weil algebra of a Lie algebra. We show that, when applying this construction to the

Lie algebroid associated to a Lie group action on a manifold, one obtains Kalkman's BRST algebra for equivariant cohomology (Proposition 5.5).

In order to fix our conventions, we recall some basic properties of graded algebra and complexes of vector bundles in the appendix.

Acknowledgments. We would like to thank the referee for carefully reading a previous version of this manuscript and suggesting many improvements. We would also like to thank Dmitry Roytenberg for several discussions we had and James Stasheff for valuable comments.

2. Preliminaries

2.1. Representations and cohomology. Here we review some standard facts about Lie algebroids, representations and cohomology. We also make a few general comments regarding the notion of adjoint representation for Lie algebroids. Throughout this paper, A denotes a Lie algebroid over a fixed base manifold M. As a general reference for algebroids, we use [26].

Definition 2.1. A Lie algebroid over M is a vector bundle $\pi : A \to M$ together with a bundle map $\rho : A \to TM$, called the anchor map, and a Lie bracket on the space $\Gamma(A)$ of sections of A satisfying the Leibniz identity:

$$[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta,$$

for every $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$.

Given an algebroid A, there is an associated De-Rham complex $\Omega(A) = \Gamma(\Lambda A^*)$, with the De-Rham operator given by the Koszul formula

$$d_A\omega(\alpha_1, \dots, \alpha_{k+1}) = \sum_{i<j}(-1)^{i+j}\omega([\alpha_i, \alpha_j], \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_{k+1}) + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}\omega(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_{k+1}),$$

where $L_X(f) = X(f)$ is the Lie derivative along vector fields. The operator d_A is a differential ($d_A^2 = 0$) and satisfies the derivation rule

$$d_A(\omega\eta) = d_A(\omega)\eta + (-1)^p \omega d_A(\eta),$$

for all $\omega \in \Omega^p(A)$, $\eta \in \Omega^q(A)$.

Definition 2.2. Let A be a Lie algebroid over M. An A-connection on a vector bundle E over M is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(\alpha, S) \mapsto \nabla_\alpha(S)$ such that:

$$\nabla_{f\alpha}(s) = f\nabla_\alpha(s), \quad \nabla_\alpha(fs) = f\nabla_\alpha(s) + L_{\rho(\alpha)}(f)(s)$$

for all $f \in C^\infty(M)$, $s \in \Gamma(E)$ and $\alpha \in \Gamma(A)$. The A-curvature of ∇ is the tensor given by

$$R_\nabla(\alpha, \beta)(s) := \nabla_\alpha\nabla_\beta(s) - \nabla_\beta\nabla_\alpha(s) - \nabla_{[\alpha,\beta]}(s)$$

for $\alpha, \beta \in \Gamma(A)$, $s \in \Gamma(E)$. The A-connection ∇ is called flat if $R_\nabla = 0$. A representation of A is a vector bundle E together with a flat A-connection ∇ on E.

Given any A-connection ∇ on E, the space of E-valued A-differential forms, $\Omega(A; E) = \Gamma(\Lambda A^* \otimes E)$ has an induced operator d_∇ given by the Koszul formula

$$d_\nabla \omega(\alpha_1, \dots, \alpha_{k+1}) = \sum_{i<j} (-1)^{i+j} \omega([\alpha_i, \alpha_j], \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_{k+1})$$
$$+ \sum_i (-1)^{i+1} \nabla_{\alpha_i} \omega(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_{k+1}).$$

In general, d_∇ satisfies the derivation rule

$$d_\nabla(\omega\eta) = d_A(\omega)\eta + (-1)^p \omega d_\nabla(\eta),$$

and squares to zero if and only if ∇ is flat.

Proposition 2.3. *Given a Lie algebroid A and a vector bundle E over M, there is a* 1-1 *correspondence between A-connections ∇ on E and degree* $+1$ *operators d_∇ on $\Omega(A; E)$ which satisfy the derivation rule. Moreover, (E, ∇) is a representation if and only if $d_\nabla^2 = 0$.*

In a more algebraic language, every Lie algebroid A has an associated DG algebra $(\Omega(A), d_A)$, and every representation E of A gives a DG module over this DG algebra.

Definition 2.4. Given a representation $E = (E, \nabla)$ of A, the cohomology of A with coefficients in E, denoted $H^\bullet(A; E)$, is the cohomology of the complex $(\Omega(A; E), d_\nabla)$. When E is the trivial representation (the trivial line bundle with $\nabla_\alpha = L_{\rho(\alpha)}$), we write $H^\bullet(A)$.

Example 2.5. In the extreme case where A is TM, representations are flat vector bundles over M, while the associated cohomology is the usual cohomology of M with local coefficient given by the flat sections of the vector bundle. At the other extreme, when $A = \mathfrak{g}$ is a Lie algebra, one recovers the standard notion of representation of Lie algebras, and Lie algebra cohomology. For a foliation $\mathscr{F}$ on M, viewed as an involutive sub-bundle of TM, the Lie algebroid cohomology becomes the well-known foliated cohomology (see e.g. [23], [1]). Central to foliation theory is the Bott connection [5] on the normal bundle $\nu = TM/\mathscr{F}$,

$$\nabla_V(X \bmod \mathscr{F}) = [V, X] \bmod \mathscr{F}, \quad V \in \Gamma(\mathscr{F}),$$

which is the linearized version of the notion of holonomy. In our language, ν is a representation of $\mathscr{F}$.

More generally, for any regular Lie algebroid A (regular in the sense that $\rho : A \to TM$ has constant rank), A has two canonical representations. They are the kernel of ρ, denoted $\mathfrak{g}(A)$, with the A-connection

$$\nabla^{\mathrm{adj}}_\alpha(\beta) = [\alpha, \beta],$$

and the normal bundle $\nu(A) = TM/\rho(A)$ of the foliation induced by A, with the connection

$$\nabla^{\mathrm{adj}}_{\alpha}(\overline{X}) = \overline{[\rho(\alpha), X]},$$

where $\overline{X} = X \bmod \rho(A)$.

2.2. Deformation cohomology. The deformation cohomology of A arises in the study of the deformations of the Lie algebroid structure [14]. This cohomology cannot, in general, be realized as the cohomology associated to a representation. The deformation complex $\big(C^{\bullet}_{\mathrm{def}}(A), \delta\big)$ is defined as follows. In degree k, it consists of pairs (c, σ_c) where c is an antisymmetric, $\mathbb{R}$-multilinear map

$$c : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{k\text{-times}} \to \Gamma(A),$$

and σ_c is an antisymmetric, $C^{\infty}(M)$-multilinear map

$$\sigma_c : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{(k-1)\text{-times}} \to \Gamma(TM),$$

which is the symbol of c in the sense that:

$$c(\alpha_1, \ldots, f\alpha_k) = fc(\alpha_1, \ldots, \alpha_k) + L_{\sigma_c(\alpha_1, \ldots, \alpha_{k-1})}(f)\alpha_k,$$

for any function $f \in C^{\infty}(M)$ and sections $\alpha_i \in \Gamma(A)$. The differential

$$\delta : C^{k}_{\mathrm{def}}(A) \to C^{k+1}_{\mathrm{def}}(A)$$

associates to (c, σ_c) the pair $\big(\delta(c), \sigma_{\delta(c)}\big)$ where:

$$\begin{aligned}
\delta(c)(\alpha_1, \ldots, \alpha_{k+1}) &= \sum_{i<j} (-1)^{i+j} c([\alpha_i, \alpha_j], \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_{k+1}) \\
&\quad + \sum_{i=1}^{k+1} (-1)^{i+1} [\alpha_i, c(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_{k+1})], \\
\sigma_{\delta(c)} &= \delta(\sigma_c) + (-1)^{k+1} \rho \circ c,
\end{aligned}$$

and

$$\begin{aligned}
\delta(\sigma_c)(\alpha_1, \ldots, \alpha_k) &= \sum_{i<j} (-1)^{i+j} \sigma_c([\alpha_i, \alpha_j], \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_k) \\
&\quad + \sum_{i=1}^{k} (-1)^{i+1} [\rho(\alpha_i), c(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_k)].
\end{aligned}$$

Definition 2.6. The deformation cohomology of the Lie algebroid A, denoted $H^{\bullet}_{\mathrm{def}}(A)$, is defined as the cohomology of the cochain complex $\big(C^{\bullet}_{\mathrm{def}}(A), \delta\big)$.

Example 2.7. When $A = TM$, the deformation cohomology is trivial (cf. [14], Corollary 2). When $A = \mathfrak{g}$ is a Lie algebra, the deformation cohomology is isomorphic to $H^\bullet(\mathfrak{g};\mathfrak{g})$, the cohomology with coefficients in the adjoint representation. This is related to the fact that deformations of the Lie algebra $\mathfrak{g}$ are controlled by $H^2(\mathfrak{g};\mathfrak{g})$.

In the case of a foliation $\mathscr{F}$, $H^\bullet_{\mathrm{def}}(\mathscr{F})$ is isomorphic to $H^{\bullet-1}(\mathscr{F};\nu)$, the cohomology with coefficients in the Bott representation ([14], Proposition 4). This was already explained in the work of Heitsch [18] on deformations of foliations where he shows that such deformations are controlled by $H^1(\mathscr{F};\nu)$. Due to the analogy with Lie algebras, it seems natural to declare $\nu[-1]$—the graded vector bundle which is ν concentrated in degree one—as the adjoint representation of $\mathscr{F}$.

Remark 2.8. The notion of adjoint representation will be properly defined in Subsection 3.2. For now, we would like to explain why it has to be defined in the setting of representations up to homotopy. With the examples of Lie algebras and foliations in mind, it is tempting to consider the natural representations of A (the $\mathfrak{g}(A)$ and $\nu(A)$ of Example 2.5) and to define the adjoint representation of A as

$$\mathrm{ad} = \mathfrak{g}(A) \oplus \nu(A)[-1], \tag{1}$$

a (graded) representation with $\mathfrak{g}(A)$ in degree zero and $\nu(A)$ in degree one. Even under the assumption that A is regular (so that the bundles involved are smooth), the behavior of the deformation cohomology shows that one should be more careful. Indeed, based on the examples of Lie algebras, one expects the cohomology associated to the adjoint representation to coincide with the deformation cohomology. While there is a long exact sequence ([14], Theorem 3)

$$\cdots \to H^n\big(A;\mathfrak{g}(A)\big) \to H^n_{\mathrm{def}}(A) \to H^{n-1}\big(A;\nu(A)\big) \xrightarrow{\delta} H^{n+1}\big(A;\mathfrak{g}(A)\big) \to \cdots, \tag{2}$$

it is not difficult to find examples for which the connecting map δ is non-zero. As we shall see, definition (1) can be made correct provided we endow the right-hand side with the structure of a representation up to homotopy.

The situation is worse in the non-regular case, when the graded direct sum $\mathfrak{g}(A) \oplus \nu(A)[1]$ is non-longer smooth. One can overcome this by interpreting the direct sum as the cohomology of a cochain complex of vector bundles, concentrated in two degrees:

$$A \xrightarrow{\rho} TM. \tag{3}$$

We will call this the *adjoint complex of A*. The idea of using this complex in order to make sense of the adjoint representation appeared already in [15], and is also present in [12]. However, the presence of the extra-structure of a representation up to homotopy had been overlooked.

2.3. Basic connections and the basic curvature. Keeping in mind our discussion on what the adjoint representation should be, we want to *extend* the canonical flat A-connections ∇^{adj} (from $\mathfrak{g}(A)$ and $\nu(A)$) to A and TM or, even better, to the adjoint complex (3). This construction already appeared in the theory of secondary characteristic classes [12] and was also used in [11].

Definition 2.9. Given a Lie algebroid A over M and a connection ∇ on the vector bundle A, we define:

1. The basic A-connection induced by ∇ on A:

$$\nabla^{\text{bas}}_{\alpha}(\beta) = \nabla_{\rho(\beta)}(\alpha) + [\alpha, \beta].$$

2. The basic A-connection induced by ∇ on TM:

$$\nabla^{\text{bas}}_{\alpha}(X) = \rho\big(\nabla_X(\alpha)\big) + [\rho(\alpha), X].$$

Note that $\nabla^{\text{bas}} \circ \rho = \rho \circ \nabla^{\text{bas}}$, i.e. ∇^{bas} is an A-connection on the adjoint complex (3). On the other hand, the existence of a connection ∇ such that ∇^{bas} is flat is a very restrictive condition on A. It turns out that the curvature of ∇^{bas} hides behind a more interesting tensor—and that is what we will call the basic curvature of ∇.

Definition 2.10. Given a Lie algebroid A over M and a connection ∇ on the vector bundle A, we define the basic curvature of ∇, as the tensor

$$R^{\text{bas}}_{\nabla} \in \Omega^2\big(A; \text{Hom}(TM, A)\big)$$

given by

$$R^{\text{bas}}_{\nabla}(\alpha, \beta)(X) := \nabla_X([\alpha, \beta]) - [\nabla_X(\alpha), \beta] - [\alpha, \nabla_X(\beta)] - \nabla_{\nabla^{\text{bas}}_{\beta} X}(\alpha) + \nabla_{\nabla^{\text{bas}}_{\alpha} X}(\beta),$$

where α, β are sections of A and X is a vector field on M.

This tensor appears when one looks at the curvatures of the A-connections ∇^{bas}. One may think of R^{bas}_{∇} as the expression $\nabla_X([\alpha, \beta]) - [\nabla_X(\alpha), \beta] - [\alpha, \nabla_X(\beta)]$ which measures the derivation property of ∇ with respect to $[\cdot\,,\cdot]$, *corrected* so that it becomes $C^{\infty}(M)$-linear on all arguments.

Proposition 2.11. *For any connection ∇ on A, one has:*

1. *The curvature of the A-connection ∇^{bas} on A equals $-\rho \circ R^{\text{bas}}_{\nabla}$, while the curvature of the A-connection ∇^{bas} on TM equals $-R^{\text{bas}}_{\nabla} \circ \rho$.*

2. *R^{bas}_{∇} is closed with respect to ∇^{bas} i.e. $d_{\nabla^{\text{bas}}}(R^{\text{bas}}_{\nabla}) = 0$.*

Proof. For $\alpha, \beta, \gamma \in \Gamma(A)$,

$$\begin{aligned} R^{\text{bas}}_{\nabla}(\alpha, \beta)\rho(\gamma) &= \nabla_{\rho(\gamma)}([\alpha, \beta]) - [\nabla_{\rho(\gamma)}(\alpha), \beta] - [\alpha, \nabla_{\rho(\gamma)}(\beta)] - \nabla_{\nabla_{\beta}\rho(\gamma)}(\alpha) + \nabla_{\nabla_{\alpha}\rho(\gamma)}(\beta) \\ &= \nabla_{[\alpha, \beta]}(\gamma) - \nabla_{\alpha}\big(\nabla_{\beta}(\gamma)\big) + \nabla_{\beta}\big(\nabla_{\alpha}(\gamma)\big) \\ &= -R_{\nabla^{\text{bas}}}(\alpha, \beta)(\gamma). \end{aligned}$$

On the other hand, if we evaluate at a vector field X the computation becomes:

$$\begin{aligned}
\rho\big(R_{\nabla}^{\mathrm{bas}}(\alpha,\beta)X\big) &= \rho\big(\nabla_X([\alpha,\beta]) - [\nabla_X(\alpha),\beta] - [\alpha,\nabla_X(\beta)] - \nabla_{\nabla_\beta X}(\alpha) + \nabla_{\nabla_\alpha X}(\beta)\big)\\
&= \rho\big(\nabla_X([\alpha,\beta])\big) + [\rho([\alpha,\beta]),X]\\
&\quad + [[\rho(\beta),X],\rho(\alpha)] - \rho\big([\alpha,\nabla_X(\beta)]\big) - \rho\big(\nabla_{\nabla_\beta X}(\alpha)\big)\\
&\quad + [\rho(\beta),[\rho(\alpha),X]] + \rho\big(\nabla_{\nabla_\alpha X}(\beta)\big) - \rho\big([\nabla_X(\alpha),\beta]\big)\\
&= \nabla_{[\alpha,\beta]}(X) - \nabla_\alpha\big(\nabla_\beta(X)\big) + \nabla_\beta\big(\nabla_\alpha(X)\big)\\
&= -R_{\nabla^{\mathrm{bas}}}(\alpha,\beta)(X).
\end{aligned}$$

The proof of the second part is a similar computation that we will omit. □

The following results indicate the geometric meaning of the basic curvature $R_{\nabla}^{\mathrm{bas}}$. The first one refers to the characterization of Lie algebroids which arise from Lie algebra actions.

Proposition 2.12. *A Lie algebroid A over a simply connected manifold M is the algebroid associated to a Lie algebra action on M if and only if it admits a flat connection ∇ whose induced basic curvature $R_{\nabla}^{\mathrm{bas}}$ vanishes.*

Proof. If A is associated to a Lie algebra action, one chooses ∇ to be the obvious flat connection. Assume now that there is a connection ∇ as above. Since M is simply connected, the bundle is trivial. Choose a frame of flat sections $\alpha_1, \dots, \alpha_r$ of A, and write

$$[\alpha_i, \alpha_j] = \sum_{k=1}^{r} c_{ij}^k \alpha_k$$

with $c_{ij}^k \in C^\infty(M)$. Since

$$R_{\nabla}^{\mathrm{bas}}(\alpha_i,\alpha_j)(X) = \sum_{k=1}^{r} \nabla_X(c_{ij}^k \alpha_k) = \sum_{k=1}^{r} X(c_{ij}^k)\alpha_k,$$

we deduce that the c_{ij}^k's are constant. The Jacobi identity for the Lie bracket on $\Gamma(A)$ implies that c_{ij}^k's are the structure constants of a Lie algebra, call it $\mathfrak{g}$. The anchor map defines an action of $\mathfrak{g}$ on M, and the trivialization of A induces the desired isomorphism. □

In particular consider $A = TM$ for some compact simply connected manifold M and a flat connection ∇ on A. The condition that the basic curvature vanishes means precisely that the conjugated connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X(Y) = [X, Y] + \nabla_Y(X)$$

is also flat. This implies that M is a Lie group.

The next result refers to the relation between bundles of Lie algebras (viewed as Lie algebroids with zero anchor map) and Lie algebra bundles, for which the fiber Lie algebra is fixed in the local trivializations.

Proposition 2.13. *Let A be a bundle of Lie algebras over M. Then A is a Lie algebra bundle if and only if it admits a connection ∇ whose basic curvature vanishes.*

Proof. If the bundle of Lie algebras is locally trivial then locally one can choose connections with zero A-curvature. Then one can use partitions of unity to construct a global connection with the same property. For the converse, assume there exists a ∇ with $R_{\nabla}^{A} = 0$, and we need to prove that the Lie algebra structure on the fiber is locally trivial. We may assume that $A = R^n \times R^r$ as a vector bundle. The vanishing of the basic curvature means that ∇ acts as derivations of the Lie algebra fibers:

$$\nabla_X([\alpha, \beta]) = [\nabla_X(\alpha), \beta] + [\alpha, \nabla_X(\beta)].$$

Since derivations are infinitesimal automorphisms, we deduce that the parallel transports induced by ∇ are Lie algebra isomorphisms, providing the necessary Lie algebra bundle trivialization. □

3. Representations up to homotopy

3.1. Representations up to homotopy and first examples. In this section, we introduce the notion of representation up to homotopy and the adjoint representation of Lie algebroids. As before, A is a Lie algebroid over M. We start with the shortest, but less intuitive description of representations up to homotopy.

Definition 3.1. A representation up to homotopy of A consists of a graded vector bundle E over M and an operator, called the structure operator,

$$D : \Omega(A; E) \to \Omega(A; E)$$

which increases the total degree by one and satisfies $D^2 = 0$ and the graded derivation rule:

$$D(\omega\eta) = d_A(\omega)\eta + (-1)^k \omega D(\eta)$$

for all $\omega \in \Omega^k(A)$, $\eta \in \Omega(A; E)$. The cohomology of the resulting complex is denoted by $H^\bullet(A; E)$.

Intuitively, a representation up to homotopy of A is a complex endowed with an A-connection which is "flat up to homotopy". We will make this precise in what follows.

Proposition 3.2. *There is a* 1-1 *correspondence between representations up to homotopy (E, D) of A and graded vector bundles E over M endowed with*:

1. *A degree* 1 *operator ∂ on E making (E, ∂) a complex.*

2. *An A-connection ∇ on (E, ∂).*

3. *An* End(E)*-valued* 2*-form* ω_2 *of total degree* 1, *i.e.*

$$\omega_2 \in \Omega^2\big(A; \underline{\mathrm{End}}^{-1}(E)\big)$$

satisfying

$$\partial(\omega_{[2]}) + R_\nabla = 0,$$

where R_∇ *is the curvature of* ∇.

4. *For each* $i > 2$, *an* End(E)*-valued i-form* ω_i *of total degree* 1, *i.e.*

$$\omega_i \in \Omega^i\big(A; \underline{\mathrm{End}}^{1-i}(E)\big)$$

satisfying

$$\partial(\omega_i) + d_\nabla(\omega_{i-1}) + \omega_2 \circ \omega_{i-2} + \omega_3 \circ \omega_{i-3} + \cdots + \omega_{i-2} \circ \omega_2 = 0.$$

The correspondence is characterized by

$$D(\eta) = \partial(\eta) + d_\nabla(\eta) + \omega_2 \wedge \eta + \omega_3 \wedge \eta + \cdots.$$

We also write

$$D = \partial + \nabla + \omega_2 + \omega_3 + \cdots. \tag{4}$$

Proof. Due to the derivation rule and the fact that $\Omega(A; E)$ is generated as an $\Omega(A)$-module by $\Gamma(E)$, the operator D will be uniquely determined by what it does on $\Gamma(E)$. It will send each $\Gamma(E^k)$ into the sum

$$\Gamma(E^{k+1}) \oplus \Omega^1(A; E^k) \oplus \Omega^2(A; E^{k-1}) \oplus \cdots,$$

hence it will also send each $\Omega^p(A; E^k)$ into the sum

$$\Omega^p(A; E^{k+1}) \oplus \Omega^{p+1}(A; E^k) \oplus \Omega^{p+2}(A; E^{k-1}) \oplus \cdots,$$

and we denote by $D_0, D_1, \ldots$ the components of D. From the derivation rule for D, we deduce that each D_i for $i \neq 1$ is a (graded) $\Omega(A)$-linear map and, by Lemma A.1, it is the wedge product with an element in $\Omega\big(A; \underline{\mathrm{End}}(E)\big)$. On the other hand, D_1 satisfies the derivation rule on each of the vector bundles E^k and, by Proposition 2.3, it comes from A-connections on these bundles. The equations in the statement correspond to $D^2 = 0$. □

Next, one can define the notion of morphism between representations up to homotopy.

Definition 3.3. A morphism $\Phi : E \to F$ between two representations up to homotopy of A is a degree zero linear map

$$\Phi : \Omega(A; E) \to \Omega(A; F)$$

which is $\Omega(A)$-linear and commutes with the structure differentials D_E and D_F.

We denote by $\mathbb{R}\mathrm{ep}^{\infty}(A)$ the resulting category, and by $\mathrm{Rep}^{\infty}(A)$ the set of isomorphism classes of representations up to homotopy of A.

By the same arguments as above, one gets the following description of morphisms in $\mathbb{R}\mathrm{ep}^{\infty}(A)$. A morphism is necessarily of type

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots,$$

where Φ_i is a $\mathrm{Hom}(E, F)$-valued i-form on A of total degree zero:

$$\Phi_i \in \Omega^i\big(A; \mathrm{Hom}^{-i}(E, F)\big)$$

satisfying

$$\partial(\Phi_n) + d_{\nabla}(\Phi^{n-1}) + \sum_{i+j=n,\, i \geqq 2} [\omega_i, \Phi_j] = 0.$$

Note that, in particular, Φ_0 must be a map of complexes.

Example 3.4 (Usual representations). Of course, any representation E of A can be seen as a representation up to homotopy concentrated in degree zero. More generally, for any integer k, one can form the representation up to homotopy $E[-k]$, which is E concentrated in degree k.

Example 3.5 (Differential forms). Any closed form $\omega \in \Omega^n(A)$ induces a representation up to homotopy on the complex which is the trivial line bundle in degrees 0 and $n-1$, and zero otherwise. The structure operator is $\nabla^{\mathrm{flat}} + \omega$ where ∇^{flat} is the flat connection on the trivial line bundle. If ω and ω' are cohomologous, then the resulting representations up to homotopy are isomorphic with isomorphism defined by $\Phi_0 = \mathrm{Id}$, $\Phi_{n-1} = \theta \in \Omega^{n-1}(A)$ chosen so that $d(\theta) = \omega - \omega'$. In conclusion, there is a well-defined map $H^{\bullet}(A) \to \mathrm{Rep}^{\infty}(A)$.

Example 3.6 (Conjugation). For any representation up to homotopy E with structure operator D given by (4), one can form a new representation up to homotopy $\bar{E}$, which has the same underlying graded vector bundle as E, but with the structure operator

$$\bar{D} = -\partial + \nabla - \omega_2 + \omega_3 - \omega_4 + \cdots.$$

In general, E and $\bar{E}$ are isomorphic, with isomorphism $\Phi = \Phi_0$ equal to $(-1)^n \,\mathrm{Id}$ on E^n.

Remark 3.7. Let us now be more explicit on the building blocks of representations up to homotopy which are concentrated in two consecutive degrees, say 0 and 1. From Proposition 3.2, we see that such a representation consists of:

1. Two vector bundles E and F, and a vector bundle map $f : E \to F$.

2. A-connections on E and F, both denoted ∇, compatible with ∂ ($\nabla_{\alpha}\partial = \partial\nabla_{\alpha}$).

3. A 2-form $K \in \Omega^2(A; \mathrm{Hom}(F, E))$ such that

$$R_{\nabla^E} = \partial \circ K, \quad R_{\nabla^F} = K \circ \partial,$$

and such that $d_\nabla(K) = 0$.

Example 3.8 (The double of a vector bundle). Let E be a vector bundle over M. For any A-connection ∇ on E with curvature $R_\nabla \in \Omega^2(A; \underline{\mathrm{End}}(E))$, the complex $E \stackrel{\mathrm{Id}}{\to} E$ concentrated in degrees 0 and 1, together with the structure operator

$$D_\nabla := \mathrm{Id} + \partial + R_\nabla$$

defines a representation up to homotopy of A denoted $\mathscr{D}_{E,\nabla}$. The resulting element

$$\mathscr{D}_E \in \mathrm{Rep}^\infty(A)$$

does not depend on the choice of the connection. To see this, remark that if ∇' is another A-connection, then there is an isomorphism

$$\Phi : (\mathscr{D}_E, D_\nabla) \to (\mathscr{D}_E, D_{\nabla'})$$

with two components:

$$\Phi^0 = \mathrm{Id}, \quad \Phi^1(\alpha) = \nabla_\alpha - \nabla'_\alpha.$$

We will now explain how representations up to homotopy of length 1 are related to extensions.

Proposition 3.9. *For any representation up to homotopy of length one with vector bundles E in degree* 0 *and F in degree* 1 *and structure operator $D = \partial + \nabla + K$, there is an extension of Lie algebroids*:

$$\mathfrak{g}_\partial \to \tilde{A} \to A,$$

where:

1. $\mathfrak{g}_\partial = \mathrm{Hom}(F, E)$, *is a bundle of Lie algebras with bracket* $[S, T]_\partial = S\partial T - T\partial S$.

2. $\tilde{A} = \mathfrak{g}_\partial \oplus A$ *with anchor* $(S, \alpha) \mapsto \rho(\alpha)$ *and bracket*

$$[(S, \alpha), (T, \beta)] = \big([S, T] + \nabla_\alpha(T) - \nabla_\beta(S) + K(\alpha, \beta), [\alpha, \beta]\big).$$

Proof. After a careful computation, we find that the Jacobi identity for the bracket of $\tilde{A}$ breaks into the following equations (cf. [26], Theorem 7.3.7):

$$\nabla_\alpha([S, T]) = [\nabla_\alpha(S), T] + [S, \nabla_\alpha(T)], \tag{5}$$

$$\nabla_{[\beta,\gamma]}(T) - \nabla_\beta\nabla_\gamma(T) + \nabla_\gamma\nabla_\beta(T) = [T, K(\beta, \gamma)], \tag{6}$$

$$\begin{aligned} K([\alpha, \beta], \gamma) + K([\beta, \gamma], \alpha) + K([\gamma, \alpha], \beta) \\ = \nabla_\beta\big(K(\gamma, \alpha)\big) + \nabla_\alpha\big(K(\beta, \gamma)\big) + \nabla_\gamma\big(K(\alpha, \beta)\big), \end{aligned} \tag{7}$$

for $\alpha, \beta, \gamma \in \Gamma(A)$ and $S, T \in \Gamma(\mathfrak{g}_\partial)$. These equations are not equivalent to, but they follow from the equations satisfied by ∂, ∇ and K. The first equation follows from the compatibility of ∂ and ∇, the second equation follows from the two equations satisfied by the curvature, while the last equation is precisely $d_\nabla(K) = 0$. □

Example 3.10. When $A = TM$ and E is a vector bundle, the extension associated to the double of E (Example 3.8) is isomorphic to the "Atiyah extension" induced by E:

$$\mathrm{End}(E) \to \mathfrak{gl}(E) \to TM.$$

This extension is discussed e.g. in [24], Section 1 (where $\mathfrak{gl}(E)$ is denoted by $\mathscr{D}(E)$). Recall that $\mathfrak{gl}(E)$ is the vector bundle over M whose sections are the derivations of E, i.e. pairs (D, X) consisting of a linear map $D : \Gamma(E) \to \Gamma(E)$ and a vector field X on M, such that $D(fs) = fD(s) + L_X(f)s$ for all $f \in C^\infty(M)$, $s \in \Gamma(E)$. The Lie bracket of $\mathfrak{gl}(E)$ is the commutator

$$[(D, X), (D', X')] = (D \circ D' - D' \circ D, [X, X']),$$

while the anchor sends (D, X) to X. A connection on E is the same thing as a splitting of the Atiyah extension, and it induces an identification

$$\mathfrak{gl}(E) \cong \mathrm{End}(E) \oplus TM.$$

Computing the bracket (or consulting [24]) we find that, after this identification, the Atiyah extension becomes the extension associated to the double $\mathscr{D}_E$.

3.2. The adjoint representation. It is now clear that the properties of the basic connections and the basic curvature given in Proposition 2.11 give the adjoint complex the structure of a representation up to homotopy. Choosing a connection on the vector bundle A, the adjoint complex

$$A \xrightarrow{\rho} TM,$$

together with the structure operator

$$D_\nabla := \rho + \nabla^{\mathrm{bas}} + R_\nabla^{\mathrm{bas}}$$

becomes a representation up to homotopy of A, denoted ad_∇. The isomorphism class of this representation is called the adjoint representation of A and is denoted

$$\mathrm{ad} \in \mathrm{Rep}^\infty(A).$$

Theorem 3.11. *Given two connections on A, the corresponding adjoint representations are naturally isomorphic. Also, there is an isomorphism*:

$$H^\bullet(A; \mathrm{ad}) \cong H^\bullet_{\mathrm{def}}(A).$$

Proof. Let ∇ and ∇' be two connections on A. Then, $\Phi = \Phi_0 + \Phi_1$ where

$$\Phi_0 = \mathrm{Id}, \quad \Phi_1(\alpha)(X) = \nabla_X(\alpha) - \nabla'_X(\alpha),$$

defines an isomorphism between the corresponding adjoint representations. For the last part, note that there is an exact sequence

$$0 \to \Omega^k(A;A) \to C^k_{\mathrm{def}}(A) \xrightarrow{-\sigma} \Omega^{k-1}(A;TM) \to 0,$$

where σ is the symbol map (see Subsection 2.2). A connection ∇ on A induces a splitting of this sequence, and then an isomorphism

$$\Psi : C^k_{\mathrm{def}}(A) \to \Omega^k(A;A) \oplus \Omega^{k-1}(A;TM) = \Omega(A,\mathrm{ad})^k,$$
$$D \mapsto (c_D, -\sigma_D),$$

where σ_D is the symbol of D and c_D is given by

$$c_D(\alpha_1,\dots,\alpha_k) = D(\alpha_1,\dots,\alpha_k) + (-1)^{k-1}\sum_{i=1}^{k}(-1)^i\nabla_{\sigma(D)(\alpha_1,\dots,\widehat{\alpha_i},\dots,\alpha_k)}(\alpha_i).$$

This map is an isomorphism of $\Omega(A)$-modules. We need to prove that the operators δ and D_∇ coincide. Since these two operators are derivations with respect to the module structures, it is enough to prove that they coincide in low degrees, and this can be checked by inspection. $\square$

We will now explain the relationship between the adjoint representation and the first jet algebroid. We denote by $J^1(A)$ the first jet bundle of A and by $\pi : J^1(A) \to A$ the canonical projection. For a section α of A we denote by $j^1(\alpha) \in \Gamma(J^1A)$ its first jet. It is well known (for a proof, see e.g. [12]) that $J^1(A)$ admits a unique Lie algebroid structure such that for any section α of A,

$$\rho(j^1\alpha) = \rho(\alpha), \tag{8}$$

and for sections $\alpha, \beta \in \Gamma(A)$:

$$[j^1\alpha, j^1\beta] = j^1([\alpha,\beta]). \tag{9}$$

The jet Lie algebroid fits into a short exact sequence of Lie algebroids

$$\mathrm{Hom}(TM,A) \xrightarrow{i} J^1(A) \xrightarrow{\pi} A, \tag{10}$$

where the inclusion i is determined by the condition

$$\mathrm{Hom}(TM,A) \ni df \otimes \alpha \mapsto fj^1(\alpha) - j^1(f\alpha),$$

for all $f \in C^\infty(M)$, $\alpha \in \Gamma(A)$. On the other hand, Proposition 3.9 associates to the adjoint representation an extension of Lie algebroids.

Proposition 3.12. *The extension associated to the adjoint representation is isomorphic to the first-jet extension* $\mathrm{Hom}(TM,A) \xrightarrow{i} J^1(A) \xrightarrow{\pi} A$.

Proof. Note that, taking global sections in (10), we obtain a split short exact sequence with splitting j^1. With respect to the resulting decomposition, the Lie bracket on $\Gamma(J^1A)$ is given by (9), the Lie bracket on $\mathrm{Hom}(TM, A)$:

$$[T, S] = T \circ \rho \circ S - S \circ \rho \circ T,$$

and the Lie bracket $[j^1(\alpha), T] \in \mathrm{Hom}(TM, A)$ between $j^1(\alpha)$ and $T \in \mathrm{Hom}(TM, A)$:

$$[j^1(\alpha), T](X) = T\big([X, \rho(\alpha)]\big) + [\alpha, T(X)].$$

The last two formulas follow from the Leibniz identity for the bracket on $\Gamma\big(J^1(A)\big)$ and writing T and S as a sum of expressions of type $df \otimes \alpha = fj^1(\alpha) - j^1(f\alpha)$.

Next, giving a connection ∇ on A is equivalent to choosing a vector bundle splitting $j^\nabla : A \to J^1(A)$ of π. The relation between the two is given by

$$j^\nabla(\alpha) = j^1(\alpha) + \nabla.(\alpha),$$

for all $\alpha \in \Gamma(A)$. Here $\nabla.(\alpha) \in \mathrm{Hom}(TM, A)$ is given by $X \mapsto \nabla_X(\alpha)$. Using this and the previous formulas, a careful but straightforward computation shows that

$$[j^\nabla(\alpha), T] = \nabla^{\mathrm{bas}}_\alpha \circ T - T \circ \nabla^{\mathrm{bas}}_\alpha = \nabla^{\mathrm{bas}}_\alpha(T),$$
$$j^\nabla([\alpha, \beta]) - [j^\nabla(\alpha), j^\nabla(\beta)] = R^{\mathrm{bas}}_\nabla(\alpha, \beta),$$

for all $\alpha, \beta \in \Gamma(A)$, $T \in \Gamma\big(\mathrm{Hom}(TM, A)\big)$. This shows that the vector bundle isomorphism $J^1(A) \cong A \oplus \mathrm{Hom}(TM, A)$ induced by the splitting j^∇ identifies the Lie algebroid bracket of $J^1(A)$ with the extension bracket of Proposition 3.9 applied to ad_∇. □

We note that there doesn't seem to be any construction which associates to a Lie algebroid extension of A a representation up to homotopy so that, applying it to $J^1(A)$ one recovers the adjoint representation. In other words, $J^1(A)$ with its structure of extension of A does not contain all the information about the structure of the adjoint representation.

3.3. The case of tangent bundles. The representations up to homotopy of TM are connections on complexes of vector bundles which are *flat up to homotopy*. Indeed, at least the first equation in Proposition 3.2 says that the curvature of ∇ is trivial cohomologically (up to homotopy).

On the other hand, a flat connection ∇ on a vector bundle E can be integrated to a representation of the fundamental groupoid of M. This correspondence is induced by parallel transport. To be more precise, given a vector bundle E endowed with a connection ∇, for any path γ in M from x to y, the parallel transport along γ with respect to ∇ induces a linear isomorphism

$$T_\gamma : E_x \to E_y.$$

This construction is compatible with path concatenation. When ∇ is flat, T_γ only depends on the homotopy class of γ, and this defines an action of the homotopy groupoid of M

on E. It is only natural to ask what is the corresponding notion of parallel transport for connections which are flat up to homotopy.

Proposition 3.13. *Let (E, D) be a representation up to homotopy of TM. Then:*

1. *For any path γ in M from x to y, there is an induced chain map*

$$T_\gamma : (E_x, \partial) \to (E_y, \partial),$$

and this construction is compatible with path concatenation. More precisely, T_γ is the parallel transport with respect to the connection underlying D.

2. *If γ_0 and γ_1 are two homotopic paths in M from x to y then T_{γ_0} and T_{γ_1} are chain homotopic. More precisely, for any homotopy h between γ_0 and γ_1 there is an associated map of degree -1, $T_h : E_x \to E_y$, such that*

$$T_{\gamma_1} - T_{\gamma_0} = [\partial, T_h].$$

Proof. The compatibility of ∇ with the grading and ∂ implies that the parallel transport T_γ is a map of chain complexes. We now prove (2). Given a path $u : I \to E$ $(I = [0,1])$, sitting over some base path $\gamma : I \to M$, we denote by

$$\frac{Du}{Dt} = \nabla_{\frac{d\gamma}{dt}}(u)$$

the derivative of u with respect to the connection ∇. Then, for any path γ, and any s, t, the parallel transport

$$T_\gamma^{s,t} : E_{\gamma(s)} \to E_{\gamma(t)}$$

is defined by the equation

$$\frac{D}{Dt} T_\gamma^{s,t}(u) = 0, \quad T_\gamma^{s,s}(u) = u.$$

The global parallel transport along γ, $T_\gamma : E_x \to E_y$ is obtained for $s = 0$, $t = 1$. Note that, for a path in the fiber above $\gamma(s)$, $\phi : I \to E_{\gamma(s)}$, one has

$$\frac{D}{Dt} T_\gamma^{s,t}(\phi(t)) = T_\gamma^{s,t}\left(\frac{d\phi}{dt}(t)\right).$$

This implies that, for a path $v : I \to E$ above γ and $u_0 \in E_x$, the unique solution of the equation

$$\frac{Du}{dt} = v, \quad u(0) = u_0,$$

can be written in terms of the parallel transport as

$$u(t) = T_{\gamma}^{0,t}\left(u_0 + \int_0^t T_{\gamma}^{t',0}\big(v(t')\big)\,dt'\right).$$

For any map $u : I \times I \to E$ sitting above some $h : I \times I \to M$, we have

$$\frac{D^2 u}{DtD\varepsilon} - \frac{D^2 u}{D\varepsilon Dt} = R\left(\frac{dh}{dt}, \frac{dh}{d\varepsilon}\right)\big(u(\varepsilon, t)\big),$$

where $R = R_{\nabla}$ is the curvature of ∇. Let now h be as in the statement, $u_0 \in E_x$, and consider the previous equation applied to

$$u(\varepsilon, t) = T_{\gamma_\varepsilon}^{0,t}(u_0).$$

We find

$$\frac{D}{Dt}\left(\frac{Du}{D\varepsilon}\right) = R\left(\frac{dh}{dt}, \frac{dh}{d\varepsilon}\right)u,$$

where $\gamma_\varepsilon = h(\varepsilon, \cdot)$. Since $\dfrac{Du}{D\varepsilon}(\varepsilon, 0) = 0$, we find

$$\frac{Du}{D\varepsilon} = T_{\gamma_\varepsilon}^{0,t}\int_0^t T_{\gamma_\varepsilon}^{t',0} R\left(\frac{dh}{dt}, \frac{dh}{d\varepsilon}\right)u(\varepsilon, t')\,dt'.$$

Fixing the argument t, since

$$u(0, t) = T_{\gamma_0}^{0,t}(u_0),$$

by the same argument as above, we deduce that

$$u(\varepsilon, t) = T_{h_t}^{0,\varepsilon}\left[T_{\gamma_0}^{0,t}(u_0) + \int_0^{\varepsilon} T_{h_t}^{\varepsilon',0} T_{\gamma_{\varepsilon'}}^{0,t}\int_0^t T_{\gamma_{\varepsilon'}}^{t',0} R\left(\frac{dh}{dt}, \frac{dh}{d\varepsilon}\right)u(\varepsilon', t')\,dt'\,d\varepsilon'\right],$$

where $h_t(\cdot) = h(\cdot, t)$. Taking $\varepsilon = 1$, $t = 1$, we find

$$T_{\gamma_1}(u_0) = T_{\gamma_0}(u_0) + \int_0^1\int_0^1 T_{\gamma_\varepsilon}^{t,1} R\left(\frac{dh}{dt}, \frac{dh}{d\varepsilon}\right)T_{\gamma_\varepsilon}^{0,t}(u_0)\,dt\,d\varepsilon.$$

Using now that $R + \partial(\omega_2) = 0$, we deduce that

$$T_{\gamma_1}(u_0) - T_{\gamma_0}(u_0) = [\partial, T_h]u_0,$$

where $T_h \in \mathrm{Hom}(E_x, E_y)$ is

$$T_h = -\int_0^1\int_0^1 T_{\gamma_\varepsilon}^{t,1}\omega_2\left(\frac{dh}{dt}, \frac{dh}{d\varepsilon}\right)T_{\gamma_\varepsilon}^{0,t}\,dt\,d\varepsilon.$$

Note that the expression under the integral is in the (ε, t)-independent vector space $\mathrm{Hom}(E_x, E_y)$. $\square$

We would like to mention here the interesting recent work of Igusa [21] where, based on Chen's iterated integrals [9], the author constructs a general parallel transport for flat superconnections.

4. Operations, cohomology and the derived category

4.1. Operations and more examples. As explained in the appendix, the standard operations on vector spaces such as

$$E \mapsto E^*, \quad E \mapsto \Lambda(E), \quad E \mapsto S(E),$$
$$(E, F) \mapsto E \oplus F, \quad (E, F) \mapsto E \otimes F, \quad (E, F) \mapsto \underline{\mathrm{Hom}}(E, F)$$

extend to the setting of graded vector bundles, complexes of vector bundles and complexes of vector bundles endowed with a connection. We will see that these operations are also well-defined for representations up to homotopy.

Example 4.1 (Taking duals). For $E \in \mathbb{R}\mathrm{ep}^\infty(A)$ with associated structure operator D, the operator D^* corresponding to the dual E^* is uniquely determined by the condition

$$d_A(\eta \wedge \eta') = D^*(\eta) \wedge \eta' + (-1)^{|\eta|}\eta \wedge D(\eta'),$$

for all $\eta \in \Omega(A; E^*)$ and $\eta' \in \Omega(A; E)$, where $\wedge$ is the operation

$$\Omega(A; E^*) \otimes \Omega(A; E) \to \Omega(A)$$

induced by the pairing between E^* and E (see Appendix A.1). In terms of the components of D, if $D = \partial + \nabla + \sum_{i \geqq 2} \omega_i$, we find $D^* = \partial^* + \nabla^* + \sum_{i \geqq 2} \omega_i^*$, where ∇^* is the connection dual to ∇ and, for $\eta_k \in (E^k)^*$,

$$\partial^*(\eta) = -(-1)^k \eta \circ \partial, \quad \omega_p^*(\alpha_1, \dots, \alpha_p)(\eta_k) = -(-1)^{k(p+1)} \eta_k \circ \omega_p(\alpha_1, \dots, \alpha_p).$$

In particular, if we start with a representation up to homotopy of length one, $D = \partial + \nabla + K$ on $E \xrightarrow{\partial} F$ (E in degree 0 and F in degree 1), the dual complex will be $F^* \xrightarrow{\partial^*} E^*$ (F^* in degree -1 and E^* in degree 0), with $D^* = \partial^* + \nabla^* - K^*$. The fact that some signs appear when taking duals is to be expected since, for any connection ∇, the curvature of ∇^* equals the negative of the dual of the curvature of ∇.

Example 4.2 (Tensor products). For $E, F \in \mathbb{R}\mathrm{ep}^\infty(A)$, with associated structure operators D^E and D^F, the operator D corresponding to $E \otimes F$ is uniquely determined by the condition

$$D(\eta_1 \wedge \eta_2) = D^E(\eta_1) \wedge \eta_2 + (-1)^{|\eta_1|}\eta_1 \wedge D^F(\eta_2),$$

for all $\eta_1 \in \Omega(A; E)$ and $\eta_2 \in \Omega(A; F)$. More explicitly, if $D^E = \partial^E + \nabla^E + \omega_2^E + \cdots$ and similarly for D^F, then $D = \partial + \nabla + \omega_2 + \cdots$, where:

1. ∂ is just the graded tensor product of ∂^E and ∂^F: $\partial = \partial^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial^F$,

$$\partial(u \otimes v) = \partial^E(u) \otimes v + (-1)^{|u|} u \otimes \partial^F(v).$$

2. ∇ is just the tensor product connection of ∇^E and ∇^F: $d_\nabla = d_{\nabla^E} \otimes \mathrm{Id} + \mathrm{Id} \otimes d_{\nabla^F}$,

$$\nabla_\alpha(u \otimes v) = \nabla^E_\alpha(u) \otimes v + u \otimes \nabla^F_\alpha(v).$$

3. $\omega_p = \omega^E_p \otimes \mathrm{Id} + \mathrm{Id} \otimes \omega^F_p$.

Example 4.3 (Pull-back). A Lie algebroid A over M can be pulled-back along a submersion $\tau : N \to M$ or, more generally, along smooth maps τ which satisfy certain transversality condition, as we now explain. Recall [27] that the pull-back algebroid $\tau^! A$ has the fiber at $x \in N$:

$$\tau^!(A)_x = \{(X, \alpha) : X \in T_x N, \alpha \in A_{\tau(x)}, (d\tau)(X) = \rho(\alpha)\}.$$

The transversality condition mentioned above is that this is a smooth vector bundle over N, which certainly happens if τ is a submersion or the inclusion of a leaf of A. The anchor of $\tau^! A$ sends (X, α) to X, while the bracket is uniquely determined by the derivation rule and

$$[(X, \tau^*\alpha), (Y, \tau^*\beta)] = ([X, Y], \tau^*[\alpha, \beta]).$$

In general, there is a pull-back map (functor)

$$\tau^* : \mathbb{R}\mathrm{ep}^\infty(A) \to \mathbb{R}\mathrm{ep}^\infty(\tau^!(A))$$

which sends E with structure operator $D = \partial + \nabla + \sum \omega_i$ to $\tau^*(E)$ endowed with $D = \partial + \tau^*(\nabla) + \sum \tau^*(\omega_i)$ where $\tau^*\nabla$ is the pull-back connection

$$(\tau^*\nabla)_{(X,\alpha)}(\tau^*(s)) = \tau^*(\nabla_\alpha(s)),$$

while

$$\tau^*(\omega_i)((X_1, \alpha_1), \ldots, (X_i, \alpha_i)) = \omega_i(\alpha_1, \ldots, \alpha_i).$$

Example 4.4 (Semidirect products with representations up to homotopy). If $\mathfrak{g}$ is a Lie algebra and $V \in \mathbb{R}\mathrm{ep}^\infty(\mathfrak{g})$, the operator making $\Lambda(V^*)$ a representation up to homotopy of $\mathfrak{g}$ is a derivation on the algebra

$$\Lambda(\mathfrak{g}^*) \otimes \Lambda(V^*) = \Lambda((\mathfrak{g} \oplus V)^*),$$

i.e. defines the structure of L_∞-algebra (see [31]) on the direct sum $\mathfrak{g} \oplus V$. This L_∞-algebra deserves the name "semi-direct product of $\mathfrak{g}$ and V", and is denoted $\mathfrak{g} \ltimes V$ (it is the usual semi-direct product if V is just a usual representation).

Example 4.5 (Exterior powers of the adjoint representation and k-differentials). In the literature one often encounters equations which look like cocycle conditions, but which do not seem to have a cohomology theory behind them. Such equations arise naturally as infinitesimal manifestations of properties of global objects and it is often useful to interpret them as part of cohomology theories. We now point out one such example.

An almost k-differential ([20]) on a Lie algebroid A is a pair of linear maps $\delta : C^{\infty}(M) \to \Gamma(\Lambda^{k-1}A)$, $\delta : \Gamma(A) \to \Gamma(\Lambda^k A)$ satisfying

(i) $\delta(fg) = \delta(f)g + f\delta(g)$,

(ii) $\delta(f\alpha) = \delta(f) \wedge \delta(\alpha) + f\delta(\alpha)$,

for all $f, g \in C^{\infty}(M)$, $\alpha \in \Gamma(A)$. It is called a k-differential if

$$\delta[\alpha, \beta] = [\delta(\alpha), \beta] + [\alpha, \delta(\beta)]$$

for all $\alpha, \beta \in \Gamma(A)$.

We will now explain how k-differentials are related to representations up to homotopy. Applying the exterior powers construction to the adjoint representation, we find new elements:

$$\Lambda^k \operatorname{ad} \in \operatorname{Rep}^{\infty}(A),$$

one for each positive integer k. These are given by the representations up to homotopy $\Lambda^k(\operatorname{ad}_{\nabla})$, where ∇ is an arbitrary connection on A. Generalizing the case of the cohomology of A with coefficients in the adjoint representation ad, we now show that the cohomology with coefficients in $\Lambda^k \operatorname{ad}$ can be computed by a complex which does not require the use of a connection. More precisely, we define $\big(C^{\bullet}(A; \Lambda^k \operatorname{ad}), d\big)$ as follows. An element $c \in C^p(A; \Lambda^k \operatorname{ad})$ is a sequence $c = (c_0, c_1, \ldots)$ where

$$c_i : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{(p-i) \text{ times}} \to \Gamma(\Lambda^{k-i}A \otimes S^i TM),$$

are multilinear, antisymmetric maps related by

$$c_i(\alpha_1, \ldots, f\alpha_{p-i}) = fc_i(\alpha_1, \ldots, f\alpha_{p-i}) + i(df)\big(c_{i+1}(\alpha_1, \ldots, \alpha_{p-i-1}) \wedge \alpha_{p-i}\big),$$

where $i(df) : S^{i+1}(TM) \to S^i(TM)$ is the contraction by df. We think of $c_1, c_2, \ldots$ as the *tail* of c_0, which measures the failure of c_0 to be $C^{\infty}(M)$-linear. For instance, to define the differential dc, we first define its leading term by the Koszul formula

$$\begin{aligned}(dc)_0(\alpha_1, \ldots, \alpha_{p+1}) = {} & \sum_{i<j} (-1)^{i+j} c_0([\alpha_i, \alpha_j], \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_{p+1}) \\ & + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}\big(c_0(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_{p+1})\big),\end{aligned}$$

and then the tail can be computed by applying the principle we have mentioned above. The case $k = 1$ corresponds to the deformation complex of A.

Proposition 4.6. *The cohomology $H^\bullet(A; \Lambda^k \mathrm{ad})$ is naturally isomorphic to the cohomology of the complex $(C^\bullet(A; \Lambda^k \mathrm{ad}), d)$. Moreover, the 1-cocycles of this complex are precisely the k-differentials on A.*

Proof. For the first part, we pick a connection ∇ to realize the adjoint representation and we claim that the complexes $(C^\bullet(A; \Lambda^k \mathrm{ad}), d)$ and $(\Omega^\bullet(A; \Lambda^k \mathrm{ad}), D_\nabla)$ are isomorphic. For $k = 0$, the statement is trivial while the case $k = 1$ follows from Theorem 3.11. The general statement follows from these two cases if one observes that both $\bigoplus_k C^\bullet(A; \Lambda^k \mathrm{ad})$ and $\bigoplus_k \Omega^\bullet(A; \Lambda^k \mathrm{ad})$ are algebras generated in low degree for which the corresponding differentials are derivations. For the second part, remark that an element in $C^1(A; \Lambda^k \mathrm{ad})$ is a pair (c_0, c_1) where $c_0 : \Gamma(A) \to \Gamma(\Lambda^k A)$ and $c_1 \in \Gamma(\Lambda^{k-1} A \otimes TM)$ satisfy the appropriate equation. Viewing c_1 as the map $C^\infty(M) \to \Gamma(\Lambda^{k-1} A)$, $f \mapsto i(df)(c_1)$, we see that the elements of $C^1(A; \Lambda^k \mathrm{ad})$ are precisely the almost k-differentials on A. The fact that the cocycle equation is precisely the k-differential equation follows by a simple computation. □

Example 4.7 (The coadjoint representation). The dual of the adjoint representation of a Lie algebroid A is called the coadjoint representation of A, denoted ad^*. Using a connection ∇ on A, it is given by the representation up to homotopy

$$\mathrm{ad}^* : \underbrace{T^*M}_{\text{degree } -1} \xrightarrow{\rho^*} \underbrace{A^*}_{\text{degree } 0}, \quad D = \rho^* + (\nabla^{\mathrm{bas}})^* - (R_\nabla^{\mathrm{bas}})^*.$$

As in the case of the adjoint representation, the resulting cohomology can be computed by a complex which does not require the choice of a connection. This complex, denoted $C^\bullet(A; \mathrm{ad}^*)$, is defined as follows. An element in $C^p(A; \mathrm{ad}^*)$ is a pair $c = (c_0, c_1)$ where c_0 is a multilinear antisymmetric map:

$$c_0 : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{p \text{ times}} \to \Omega^1(M),$$

and $c_1 \in \Omega^{p-1}(A; A^*)$ is such that

$$c_0(\alpha_1, \dots, \alpha_{p-1}, f\alpha_p) = fc_0(\alpha_1, \dots, \alpha_{p-1}, \alpha_p) - df \wedge c_1(\alpha_1, \dots, \alpha_{p-1})(\alpha_p),$$

for all $f \in C^\infty(M)$, $\alpha_i \in \Gamma(A)$. The differential of c, $d(c) \in C^{p+1}(A; \mathrm{ad}^*)$ is given by the formulas

$$\begin{aligned}(dc)_0(\alpha_1, \dots, \alpha_{p+1}) &= \sum_{i<j} (-1)^{i+j} c_0([\alpha_i, \alpha_j], \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_{p+1}) \\ &\quad + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}(c_0(\alpha_1, \dots, \widehat{\alpha_i}, \dots, \alpha_{p+1})), \\ (dc)_1(\alpha_1, \dots, \alpha_p) &= \sum_{i<j} (-1)^{i+j} c_0([\alpha_i, \alpha_j], \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_p) \\ &\quad + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}(c_0(\alpha_1, \dots, \widehat{\alpha_i}, \dots, \alpha_p)) + (-1)^{p+1} c(\alpha_1, \dots, \alpha_p) \circ \rho.\end{aligned}$$

Proposition 4.8. $(C^\bullet(A;\mathrm{ad}^*), d)$ *is a cochain complex whose cohomology is canonically isomorphic to* $H^\bullet(A;\mathrm{ad}^*)$.

More generally, for any q, the representation up to homotopy $S^q(\mathrm{ad}^*)$ and its cohomology can be treated similarly. This will be made more explicit in our discussion on the Weil algebra.

4.2. Cohomology, the derived category, and some more examples. As we already mentioned several times, one of the reasons we work with complexes is that we want to avoid non-smooth vector bundles. The basic idea was that a complex represents its cohomology bundle (typically a graded non-smooth vector bundle). To complete this idea, we need to allow ourselves more freedom when comparing two complexes so that, morally, if they have the same cohomology bundles, then they become *equivalent*. This will happen in the derived category. For a more general discussion on the derived category of a DG algebra we refer the reader to [30].

Definition 4.9. A morphism Φ between two representations up to homotopy E and F is called a quasi-isomorphism if the first component of Φ, the map of complexes $\Phi_0 : (E, \partial) \to (F, \partial)$, is a quasi-isomorphism. We denote by $\mathbb{D}\mathrm{er}(A)$ the category obtained from $\mathbb{R}\mathrm{ep}^\infty(A)$ by formally inverting the quasi-isomorphisms, and by $\mathrm{Der}(A)$ the set of isomorphism classes of objects of $\mathbb{D}\mathrm{er}(A)$.

Remark 4.10 (Hom in the derived category). Since we work with vector bundles, there is the following simple realization of the derived category. Given two representations up to homotopy E and F of A, there is a notion of homotopy between maps from E to F. To describe it, we remark that morphisms in $\mathbb{R}\mathrm{ep}^\infty(A)$ from E to F correspond to 0-cycles in the complex with coefficients in the induced representation up to homotopy $\underline{\mathrm{Hom}}(E, F)$:

$$\mathrm{Hom}_{\mathbb{R}\mathrm{ep}^\infty(A)}(E, F) = Z^0\big(\Omega\big(A; \underline{\mathrm{Hom}}(E, F)\big)\big).$$

Two maps $\Phi, \Psi : E \to F$ in $\mathbb{R}\mathrm{ep}^\infty(A)$ are called homotopic if there exists a degree -1 map $H : \Omega(A; E) \to \Omega(A; F)$ which is $\Omega(A)$-linear and satisfies $D_E H + H D_F = \Phi - \Psi$, where D_E and D_F are the structure operators of E and F, respectively. We denote by $[E, F]$ the set of homotopy classes of such maps. Hence,

$$[E, F] := H^0\big(\Omega\big(A; \underline{\mathrm{Hom}}(E, F)\big)\big).$$

As in the case of complexes of vector bundles (see part (2) of Lemma A.4), and by the same type of arguments, we see that a map $\Phi : E \to F$ is a quasi-isomorphism if and only if it is a homotopy equivalence. From this, we deduce the following realization of $\mathbb{D}\mathrm{er}(A)$: its objects are the representations up to homotopy of A, while

$$\mathrm{Hom}_{\mathbb{D}\mathrm{er}(A)}(E, F) = [E, F].$$

Note that, in this language, for any $F \in \mathbb{R}\mathrm{ep}^\infty(A)$,

$$H^n(F) = [\mathbb{R}[n], F].$$

Also, the mapping cone construction gives a function

$$\mathrm{Map} : [E, F] \to \mathrm{Rep}^\infty(A)$$

which, when applied to $E = \mathbb{R}[n]$, gives the construction from Example 3.5. Here, the mapping cone $\mathrm{Map}(f)$ of a morphism $f : E \to F$ between representations up to homotopy, is the representation up to homotopy whose associated DGA $\Omega\big(A; \mathrm{Map}(f)\big)$ is the mapping cone (in the DG sense [19]) of the map f, viewed as a DG map from $\Omega(A; E)$ to $\Omega(A; F)$.

Example 4.11. If $A = \mathscr{F} \subset TM$ is a foliation on M, then the projection from the complex $\mathscr{F} \hookrightarrow TM$ underlying the adjoint representation into $\nu[-1]$ (the normal bundle $\nu = TM/\mathscr{F}$ concentrated in degree 1) is clearly a quasi-isomorphism of complexes. It is easy to see that this projection is actually a morphism of representations up to homotopy, when ν is endowed with the Bott connection (see Example 2.5). Hence, as expected,

$$\mathrm{ad}_{\mathscr{F}} \cong \nu[1] \quad (\text{in } \mathrm{Der}(\mathscr{F})).$$

Similarly, for a transitive Lie algebroid A, i.e. one for which the anchor is surjective,

$$\mathrm{ad}_A \cong \mathfrak{g}(A) \quad (\text{in } \mathrm{Der}(A)).$$

Next, we will show that the cohomology $H^\bullet(A; -)$, viewed as a functor from the category of representations up to homotopy, descends to the derived category.

Proposition 4.12. *Any quasi-isomorphism $\Phi : E \to F$ between two representations up to homotopy of A induces an isomorphism in cohomology $\Phi : H(A; E) \to H(A; F)$.*

Proof. If E is a representation up to homotopy of A, one can form a decreasing filtration on $\Omega(A; E)$ induced by the form-degree

$$\cdots \subset F^2\big(\Omega(A; E)\big) \subset F^1\big(\Omega(A, E)\big) \subset F^0\big(\Omega(A; E)\big) = \Omega(A; E)$$

where

$$F^p\big(\Omega(A; E)\big) = \Omega^p(A; E) \oplus \Omega^{p+1}(A; E) \oplus \cdots.$$

This filtration induces a spectral sequence $\mathscr{E}$ with

$$\mathscr{E}_0^{p,q} = \Omega^p(A; E^q) \Rightarrow H^{p+q}(A; E),$$

where the differential $d_0^{p,q} : \mathscr{E}_0^{pq} \to \mathscr{E}_0^{p,q+1}$ is induced by the differential ∂ of E. Given a morphism $\Phi : E \to F$, one has an induced map of spectral sequences which, at the first level, is induced by Φ^0. Hence, the assumption that Φ is a quasi-isomorphism implies that the map induced at the level of spectral sequences is an isomorphism at the second level. We deduce that Φ induces an isomorphism in cohomology. □

We will now look at the case where E is a regular representation up to homotopy of A in the sense that the underlying complex (E, ∂) is regular. In this case, the cohomology $\mathscr{H}^\bullet(E)$ is a graded vector bundle over M (see Appendix A.2). The following theorem is an application of homological perturbation theory (see e.g. [10], [17]).

Theorem 4.13. *Let E be a regular representation up to homotopy of A. Then the equation*

$$\overline{\nabla}_{\alpha}([S]) := [\nabla_{\alpha}(S)]$$

for $\alpha \in \Gamma(A)$ and $[S] \in \Gamma(\mathscr{H}^{\bullet}(E))$ makes $\mathscr{H}^{\bullet}(E)$ a representation of A. Moreover, the complex $(\mathscr{H}^{\bullet}(E), 0)$ with connection $\overline{\nabla}$ can be given the structure of a representation up to homotopy of A which is quasi-isomorphic to E.

Also, there is a spectral sequence

$$\mathscr{E}_2^{pq} = H^p(A; \mathscr{H}^q(E)) \Rightarrow H^{p+q}(A, E).$$

Proof. The fact that $\overline{\nabla}$ is flat follows from the fact that the curvature of ∇ is exact. The spectral sequence is the one appearing in the previous proof. Next, we have to construct the structure of representation up to homotopy on $\mathscr{H}^{\bullet}(E)$. We will use the notations from the proof of Lemma A.4 (part 3). The linear Hodge decomposition

$$E = \ker \Delta + \mathrm{im}(b) + \mathrm{im}(b^*)$$

provides quasi-isomorphisms $p : E \to \mathscr{H}(E)$ and $i : \mathscr{H}(E) \to E$. The restriction of the Laplacian to $\mathrm{im}(\partial) \oplus \mathrm{im}(\partial^*)$, denoted $\diamondsuit$, is an isomorphism. Then

$$h = -\diamondsuit^{-1}\partial^*$$

satisfies the following equations:

1. $p\partial = 0$,
2. $\partial i = 0$,
3. $ip = \mathrm{Id} + h\partial + \partial h$,
4. $h^2 = 0$,
5. $ph = 0$.

We denote by the same letters the maps induced at the level of forms, for instance ∂ goes from $\Omega(A; E)$ to $\Omega(A; E)$. We recall that here we use the standard sign conventions. Note that these maps still satisfy the previous equations. We consider

$$\delta := D - \partial : \Omega(A; E) \to \Omega(A; E),$$

where D is the structure operator of A. With these,

$$D_{\mathscr{H}} := p(1 + (\delta h) + (\delta h)^2 + (\delta h)^3 + \cdots)\delta i : \Omega(A; \mathscr{H}) \to \Omega(A; \mathscr{H})$$

is a degree one operator which squares to zero, and

$$\Phi := p(1 + \delta h + (\delta h)^2 + (\delta h)^3 + \cdots) : (\Omega(A; E), D) \to (\Omega(A; \mathscr{H}), D_{\mathscr{H}})$$

is a cochain map. The assertions about $D_{\mathscr{H}}$ and Φ follow by direct computation, or by applying the homological perturbation lemma (see e.g. [10]), which also explains our conventions. We now observe that the equations $h^2 = 0$ and $ph = 0$ imply that Φ is $\Omega(A)$-linear and that $D_{\mathscr{H}}$ is a derivation. $\square$

Example 4.14. Applying this result to the first quasi-isomorphism discussed in Example 4.11, we find that the deformation cohomology of a foliation $\mathscr{F}$ is isomorphic to the shifted cohomology of $\mathscr{F}$ with coefficients in v—and this is [14], Proposition 4.

Example 4.15 (Serre representations). Any extension of Lie algebras

$$\mathfrak{l} \to \tilde{\mathfrak{g}} \to \mathfrak{g}$$

induces a representation up to homotopy of $\mathfrak{g}$ with underlying complex the Chevalley–Eilenberg complex $\big(C^\bullet(\mathfrak{l}), d_{\mathfrak{l}}\big)$ of $\mathfrak{l}$. To describe this representation, we use a splitting $\sigma : \mathfrak{g} \to \tilde{\mathfrak{g}}$ of the sequence. This induces:

1. For $u \in \mathfrak{g}$, a degree zero operator

$$\nabla^\sigma_u := \mathrm{ad}^*_{\sigma(u)} : C^\bullet(\mathfrak{l}) \to C^\bullet(\mathfrak{l}),$$

hence a $\mathfrak{g}$-connection ∇^σ on $C^\bullet(\mathfrak{l})$.

2. The *curvature* of σ, $R_\sigma \in C^2(\mathfrak{g}; \mathfrak{l})$ given by

$$R^\sigma(u, v) = [\sigma(u), \sigma(v)] - \sigma([u, v]).$$

To produce an $\underline{\mathrm{End}}^{-1}\big(C^\bullet(\mathfrak{l})\big)$-valued cochain, we use the contraction operator $i : \mathfrak{l} \to \underline{\mathrm{End}}^{-1}\big(C^\bullet(\mathfrak{l})\big)$ (but mind the sign conventions!).

It is now straightforward to check that

$$D := d_{\mathfrak{l}} + \nabla^\sigma + i(R^\sigma)$$

makes $C^\bullet(\mathfrak{l})$ a representation up to homotopy of $\mathfrak{g}$, with associated cohomology complex isomorphic to $C^\bullet(\tilde{\mathfrak{g}})$.

Note that Theorem 4.13 implies that the cohomology of $\mathfrak{l}$ is naturally a representation of $\mathfrak{g}$, and there is a spectral sequence with

$$\mathscr{E}_2^{p,q} = H^p\big(\mathfrak{g}; H^q(\mathfrak{l})\big) \Rightarrow H^{p+q}(\tilde{\mathfrak{g}}).$$

This is precisely the content of the Serre spectral sequence (see e.g. [32]).

The same argument applies to any extension of Lie algebroids, giving us the spectral sequence of [26], Theorem 7.4.6.

In the case of representations of length 1, we obtain the following:

Corollary 4.16. *If (E, D) is a representation up to homotopy of A with underlying regular complex $E = E^0 \oplus E^1$ then there is a long exact sequence*

$$\cdots \to H^n(A; \mathscr{H}^0(E)) \to H^n(A; E) \to H^{n-1}(A; \mathscr{H}^1(E)) \to H^{n+1}(A; \mathscr{H}^0(E)) \to \cdots .$$

Proof. The map

$$H^{n-1}(A; \mathscr{H}^1(E)) \to H^{n+1}(A; \mathscr{H}^0(E))$$

is $d_2^{pq} : E_2^{p,1} \to E_2^{p+2,0}$ in the spectral sequence and is given by the wedge product with ω_2. The map

$$H^n(A; \mathscr{H}^0(E)) \to H^n(A; E)$$

is determined by the natural inclusion at the level of cochains.

The morphism

$$H^n(A; E) \to H^{n-1}(A; \mathscr{H}^1(E))$$

is given at the level of complexes by

$$(\omega_0 + \omega_1) \mapsto \overline{\omega_1}$$

where $\omega_0 \in \Omega^n(A; E^0)$ and $\omega_1 \in \Omega^{n-1}(A, E^1)$. Since the spectral sequence collapses at the third stage, we conclude that the sequence is exact. □

Example 4.17. Assume that A is a regular Lie algebroid. We apply Theorem 4.13 to the adjoint representation. The resulting representations in the cohomology are $\mathfrak{g}(A)$ in degree zero and $\nu(A)$ in degree one (see Example 2.5). Hence, ad_∇ is quasi-isomorphic to a representation up to homotopy on $\mathfrak{g}(A) \oplus \nu(A)[-1]$ where each term is a representation and the differential vanishes, but the ω_2-term (which depends on the connection) does not vanish in general. On the other hand, the previous corollary gives us the long exact sequence (2) ([14], Theorem 3) and we see that the boundary operator δ is induced by ω_2.

5. The Weil algebra and the BRST model for equivariant cohomology

In this section, we will make use of representations up to homotopy to introduce the Weil algebra associated to a Lie algebroid A, generalizing the Weil algebra of a Lie algebra and Kalkman's BRST algebra for equivariant cohomology. Here we give an explicit description of the Weil algebra which makes use of a connection. An intrinsic description (without the choice of a connection) is given in [3]; the price to pay is that the Weil algebra has to be defined as a certain algebra of differential operators instead of sections of a vector bundle.

Let A be a Lie algebroid over the manifold M and let ∇ be a connection on the vector bundle A. Define the algebra

$$W(A,\nabla) = \bigoplus_{u,v,w} \Gamma\big(\Lambda^u T^*M \otimes S^v(A^*) \otimes \Lambda^w(A^*)\big).$$

It is graded by the total degree $u + 2v + w$, and has an underlying bi-grading $p = v + w$, $q = u + v$, so that $W^{p,q}(A,\nabla)$ is the sum over all u, v and w satisfying these equations. The connection ∇ will be used to define a differential on $W(A,\nabla)$. This will be the sum of two differentials

$$d_\nabla = d_\nabla^{\text{hor}} + d_\nabla^{\text{ver}},$$

where, to define d_∇^{hor} and d_∇^{ver} we look at $W(A,\nabla)$ from two different points of view.

First of all recall that, according to our conventions, the symmetric powers of the dual of the graded vector bundle $\mathscr{D} = A \oplus A$ (concentrated in degrees 0 and 1) is

$$S^p\mathscr{D}^* = \underbrace{(\Lambda^p A^*)}_{\text{degree } -p} \oplus \underbrace{(A^* \otimes \Lambda^{p-1} A^*)}_{\text{degree } -p+1} \oplus \cdots \oplus \underbrace{(S^{p-1}A^* \otimes A^*)}_{\text{degree } -1} \oplus \underbrace{(S^p A^*)}_{\text{degree } 0},$$

hence,

$$W^{p,q}(A,\nabla) = \Omega(M; S^p\mathscr{D}^*)^{q-p}.$$

We now use the representation up to homotopy structure induced by ∇ on the double $\mathscr{D}$ of A (see Example 3.8), which we dualize and extend to $S\mathscr{D}^*$. The resulting structure operator will be our vertical differential

$$d_\nabla^{\text{ver}} : W^{p,q}(A,\nabla) \to W^{p,q+1}(A,\nabla).$$

Similarly, for the coadjoint complex ad^* one has

$$S^q \,\mathrm{ad}^* = \underbrace{(\Lambda^q T^*M)}_{\text{degree } -q} \oplus \underbrace{(A^* \otimes \Lambda^{q-1} T^*M)}_{\text{degree } -q+1} \oplus \cdots \oplus \underbrace{(T^*M \otimes S^{q-1}A^*)}_{\text{degree } -1} \oplus \underbrace{(S^q A^*)}_{\text{degree } 0},$$

hence,

$$W^{p,q}(A,\nabla) = \Omega(A; S^q \,\mathrm{ad}^*)^{p-q}.$$

We now use the connection ∇ to form the coadjoint representation ad_∇^*. To obtain a horizontal operator which commutes with the vertical one, we consider the conjugation of the coadjoint representation, i.e. ad^* with the structure operator $-\rho^* + (\nabla^{\text{bas}})^* + (R_\nabla^{\text{bas}})^*$. The symmetric powers will inherit a structure operator, and this will be our horizontal differential

$$d_\nabla^{\text{hor}} : W^{p,q}(A,\nabla) \to W^{p+1,q}(A,\nabla).$$

Proposition 5.1. *Endowed with d_∇^{hor} and d_∇^{ver}, $W(A,\nabla)$ becomes a differential bi-graded algebra whose cohomology is isomorphic to the cohomology of M. Moreover, up to isomorphisms of differential bi-graded algebras, $W(A,\nabla)$ does not depend on the choice of the connection ∇.*

Proof. To prove that the two differentials commute, one first observes that it is enough to check the commutation relation on functions and on sections of T^*M, $\Lambda^1 A^*$ and $S^1 A^*$, which generate the entire algebra. This follows by direct computation (one can also use the local formulas below, but the computations are much more involved). The independence of ∇ can be deduced from the fact that, up to isomorphisms, the adjoint representation and the double of a vector bundle do not depend on the choice of a connection. Alternatively, it follows from the intrinsic description of the Weil algebra [3] and an argument similar to the one in the proof of Proposition 4.6 (see Proposition 3.9 in *loc.cit*). We need to prove that the cohomology equals that of M. Note that for $p > 0$ the column $(W^{p,\bullet}, d_\nabla^{\mathrm{ver}})$ is acyclic, since it corresponds to an acyclic representation up to homotopy of TM. Next, we note that the first column $(W^{0,\bullet}, d_\nabla^{\mathrm{ver}})$ is the De-Rham complex of M and the usual spectral sequence argument provides the desired result. □

Remark 5.2 (Local coordinates). Since the operators are local, it is possible to look at their expressions in coordinates. Let us assume that we are over a chart (x_a) of M on which we have a trivialization (e_i) of A. Over this chart, the Weil algebra will be the bi-graded commutative algebra over the space of smooth functions, generated by elements ∂^a of bidegree $(0,1)$ (1-forms), elements θ^i of bi-degree $(1,0)$ (the dual basis of (e_i), viewed in $\Lambda^1 A^*$), and elements μ^i of bi-degree $(1,1)$ (the dual basis of (e_i), viewed in $S^1 A^*$). A careful but straightforward computation shows that, on these elements:

$$\begin{aligned}
d_\nabla^{\mathrm{ver}}(\partial^a) &= 0,\\
d_\nabla^{\mathrm{ver}}(\theta^i) &= \mu^i - \Gamma_{aj}^i \partial^a \theta^j,\\
d_\nabla^{\mathrm{ver}}(\mu^i) &= -\Gamma_{aj}^i \partial^a \mu^j + \frac{1}{2} r_{abj}^i \partial^a \partial^b \theta^j,\\
d_\nabla^{\mathrm{hor}}(\partial^a) &= -\rho_i^a \mu^i + \left(\frac{\partial \rho_i^a}{\partial x_b} - \Gamma_{bi}^j \rho_j^a\right) \theta^i \partial^b,\\
d_\nabla^{\mathrm{hor}}(\theta^i) &= -\frac{1}{2} c_{jk}^i \theta^j \theta^k,\\
d_\nabla^{\mathrm{hor}}(\mu^i) &= -\left(c_{jk}^i + \sum_a \rho_k^a \Gamma_{aj}^i\right) \theta^j \mu^k + \frac{1}{2} R_{jka}^i \theta^j \theta^k \partial^a.
\end{aligned}$$

Here we use Einstein's summation convention. The functions ρ_i^a are the coefficients of ρ, c_{jk}^i are the structure functions of A, r_{abj}^i are the coefficients of the curvature of ∇ and R_{jka}^i are the coefficients of the basic curvature:

$$\begin{aligned}
\rho(e_i) = \sum \rho_i^a \partial_a, &\quad [e_j, e_k] = \sum c_{jk}^i e_i,\\
R_\nabla(\partial_a, \partial_b) e_j = r_{abj}^i e_i, &\quad R_\nabla^{\mathrm{bas}}(e_j, e_k)\partial_a = R_{jka}^l e_l.
\end{aligned}$$

Note that for a smooth function f,

$$d_\nabla^{\mathrm{ver}}(f) = \partial_a(f)\partial^a, \quad d_\nabla^{\mathrm{hor}}(f) = \partial_a(f)\rho_i^a \omega^i.$$

Also, in case that the connection ∇ is flat, all the Γ and r-terms above vanish, while the R-terms are given by the partial derivatives of the structure functions c^i_{jk}.

Example 5.3 (The standard Weil algebra). In the case of Lie algebras $\mathfrak{g}$, one can immediately see that we recover the Weil algebra $W(\mathfrak{g})$. In particular, the local coordinates description becomes

$$d^{\mathrm{ver}}(\theta^i) = \mu^i,$$

$$d^{\mathrm{ver}}(\mu^i) = 0,$$

$$d^{\mathrm{hor}}(\theta^i) = -\frac{1}{2}\sum_{j,k} c^i_{jk}\theta^j\theta^k,$$

$$d^{\mathrm{hor}}(\mu^i) = -\sum_{j,k} c^i_{jk}\theta^j\mu^k,$$

which is the usual Weil algebra [8]. In this case, d^{ver} is usually called the Koszul differential, denoted d_K, d^{hor} is called the Cartan differential, denoted d_C, and the total differential is denoted by d_W.

Example 5.4 (The BRST algebra). Recall Kalkman's BRST algebra [22] associated to a $\mathfrak{g}$-manifold M. It is $W(\mathfrak{g}, M) := W(\mathfrak{g}) \otimes \Omega(M)$ with differential:

$$\delta = d_W \otimes 1 + 1 \otimes d_{\mathrm{DR}} + \sum_{a=1}^{n} \theta^a \otimes \mathscr{L}_a - \sum_{b=1}^{n} \mu^b \otimes \iota_b.$$

Proposition 5.5. *Let $A = \mathfrak{g} \ltimes M$ be the action algebroid associated to a $\mathfrak{g}$-manifold M. Then*

$$W(\mathfrak{g}, M) = W(A; \nabla^{\mathrm{flat}}),$$

where ∇^{flat} is the canonical flat connection on A.

Proof. Follows immediately from the local coordinates description of the differentials of the Weil algebra. □

A. Appendix

A.1. The graded setting. Here we collect some general conventions and constructions of graded algebras. As a general rule, we will be constantly using the *standard sign convention*: whenever two graded objects x and y, say of degrees p and q, are interchanged, one introduces the sign $(-1)^{pq}$. For instance, the standard commutator $xy - yx$ is replaced by the graded commutator

$$[x, y] = xy - (-1)^{pq} yx.$$

Throughout the appendix, M is a fixed manifold and all vector bundles are over M.

1. Graded vector bundles. By a graded vector bundle over M we mean a vector bundle E together with a direct sum decomposition indexed by integers:

$$E = \bigoplus_n E^n.$$

An element $v \in E^n$ is called a homogeneous element of degree n and we write $|v| = n$. Most of the constructions on graded vector bundles follow by applying point-wise the standard constructions with graded vector spaces. Here are some of them.

1. Given two graded vector bundles E and F, their direct sum and their tensor product have natural gradings. On $E \otimes F$ we always use the total grading:

$$\deg(e \otimes f) = \deg(e) + \deg(f).$$

2. Given two graded vector bundles E and F one can also form the new graded space $\underline{\mathrm{Hom}}(E, F)$. Its degree k part, denoted $\underline{\mathrm{Hom}}^k(E, F)$, consists of vector bundle maps $T : E \to F$ which increase the degree by k. When $E = F$, we use the notation $\underline{\mathrm{End}}(E)$.

3. For any graded vector bundle, the associated tensor algebra bundle $T(E)$ is graded by the total degree

$$\deg(v_1 \otimes \cdots \otimes v_n) = \deg(v_1) + \cdots + \deg(v_n).$$

The associated symmetric algebra bundle $S(E)$ is defined (fiber-wise) as the quotient of $T(E)$ by forcing $[v, w] = 0$ for all $v, w \in E$, while for the exterior algebra bundle $\Lambda(E)$ one forces the relations $vw = -(-1)^{pq} wv$ where p and q are the degrees of v and w, respectively.

4. The dual E^* of a graded vector bundle is graded by

$$(E^*)^n = (E^{-n})^*.$$

2. Wedge products. We now discuss wedge products in the graded context. First of all, given a Lie algebroid A and a graded vector bundle E, the space of E-valued A-differential forms, $\Omega(A; E)$, is graded by the total degree:

$$\Omega(A; E)^p = \bigoplus_{i+j=p} \Omega^i(A; E^j).$$

Wedge products arise in the following general situation. Assume that E, F and G are graded vector bundles and

$$h : E \otimes F \to G$$

is a degree preserving vector bundle map. Then there is an induced wedge-product operation

$$\Omega(A; E) \times \Omega(A; F) \to \Omega(A; G), \quad (\omega, \eta) \mapsto \omega \wedge_h \eta.$$

Explicitly, for $\omega \in \Omega^p(A; E^i)$, $\eta \in \Omega^q(A; F^j)$, $\omega \wedge_h \eta \in \Omega^{p+q}(A; G^{i+j})$ is given by

$$(\alpha_1, \dots, \alpha_{p+q}) \mapsto \sum (-1)^{qi} \operatorname{sgn}(\sigma) h\big(\omega(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}), \eta(\alpha_{\sigma(p+1)} \dots, \alpha_{\sigma(p+q)})\big),$$

where the sum is over all $(p-q)$-shuffles. Here is a list of the wedge products that will appear in this paper:

1. If h is the identity we get:

$$\cdot \wedge \cdot : \Omega(A; E) \otimes \Omega(A; F) \to \Omega(A; E \otimes F).$$

In particular, we get two operations

$$\Omega(A) \otimes \Omega(A; E) \to \Omega(A; E), \quad \Omega(A; E) \otimes \Omega(A) \to \Omega(A; E) \tag{11}$$

which make $\Omega(A; E)$ into a (graded) $\Omega(A)$-bimodule. Note that, while the first one coincides with the wedge product applied to E viewed as a (ungraded) vector bundle, the second one involves a sign.

2. If h is the composition of endomorphisms of E we get an operation

$$\cdot \circ \cdot : \Omega\big(A, \underline{\mathrm{End}}(E)\big) \otimes \Omega\big(A; \underline{\mathrm{End}}(E)\big) \to \Omega\big(A, \underline{\mathrm{End}}(E)\big) \tag{12}$$

which gives $\Omega\big(A, \underline{\mathrm{End}}(E)\big)$ the structure of a graded algebra. Of course, this operation makes sense for general $\underline{\mathrm{Hom}}$'s instead of $\underline{\mathrm{End}}$.

3. If h is the evaluation map $\mathrm{ev} : \underline{\mathrm{End}}(E) \otimes E \to E$, $(T, v) \mapsto T(v)$, we get:

$$\cdot \wedge \cdot : \Omega\big(A; \underline{\mathrm{End}}(E)\big) \otimes \Omega(A; E) \to \Omega(A; E), \tag{13}$$

while when h is the twisted evaluation map $\overline{\mathrm{ev}} : E \otimes \underline{\mathrm{End}}(E) \to E$, $(v, T) \mapsto (-1)^{|v|\,|T|} T(v)$, we get:

$$\cdot \wedge \cdot : \Omega(A; E) \otimes \Omega\big(A; \underline{\mathrm{End}}(E)\big) \to \Omega(A; E). \tag{14}$$

These operations make $\Omega(A; E)$ a graded $\Omega\big(A; \underline{\mathrm{End}}(E)\big)$-bimodule.

4. If $h : \Lambda^\bullet E \otimes \Lambda^\bullet E \to \Lambda^\bullet E$ is the multiplication, we get

$$\cdot \wedge \cdot : \Omega(A; \Lambda^\bullet E) \otimes \Omega(A; \Lambda^\bullet E) \to \Omega(A; \Lambda^\bullet E)$$

which makes $\Omega(A; \Lambda^\bullet E)$ a graded algebra.

Note that the ring $\Omega\big(A; \underline{\mathrm{End}}(E)\big)$ can be identified with the space of endomorphisms of the left $\Omega(A)$-module $\Omega(A; E)$ (in the graded sense). More precisely, we have the following simple lemma:

Lemma A.1. *There is a* 1-1 *correspondence between degree n elements of $\Omega(A;\underline{\mathrm{End}}(E))$ and operators F on $\Omega(A;E)$ which increase the degree by n and which are $\Omega(A)$-linear in the graded sense*:

$$F(\omega\wedge\eta) = (-1)^{n|\omega|}\omega\wedge F(\eta) \quad \forall\omega\in\Omega(A),\ \eta\in\Omega(A;E).$$

Explicitly, $T\in\Omega(A;\underline{\mathrm{End}}(E))$ *induces the operator* $\hat{T}$ *given by*

$$\hat{T}(\eta) = T\wedge\eta.$$

There is one more interesting operation of type $\wedge_h$, namely the one where h is the graded commutator

$$h : \underline{\mathrm{End}}(E)\otimes\underline{\mathrm{End}}(E)\to\underline{\mathrm{End}}(E), \quad h(T,S) = T\circ S - (-1)^{|S|\,|T|}S\circ T.$$

The resulting operation

$$\Omega(A,\underline{\mathrm{End}}(E))\otimes\Omega(A;\underline{\mathrm{End}}(E))\to\Omega(A;\underline{\mathrm{End}}(E))$$

will be denoted by $[-,-]$. Note that

$$[T,S] = T\wedge S - (-1)^{|T|\,|S|}S\wedge T.$$

A.2. Complexes of vector bundles. Here we bring together some rather standard constructions and facts about complexes of vector bundles.

Complexes. By a complex over M we mean a cochain complex of vector bundles over M, i.e. a graded vector bundle E endowed with a degree one endomorphism ∂ satisfying $\partial^2 = 0$:

$$(E,\partial) : \cdots \xrightarrow{\partial} E^0 \xrightarrow{\partial} E^1 \xrightarrow{\partial} E^2 \xrightarrow{\partial} \cdots.$$

We drop ∂ from the notation whenever there is no danger of confusion. A morphism between two complexes E and F over M is a vector bundle map $f : E\to F$ which preserves the degree and is compatible with the differentials. We denote by $\mathrm{Hom}(E,F)$ the space of all such maps. We denote by $\underline{\mathrm{Ch}}(M)$ the resulting category of complexes over M.

Definition A.2. We say that a complex (E,∂) over M is regular if ∂ has constant rank. In this case one defines the cohomology of E as the graded vector bundle over M:

$$\mathscr{H}^\bullet(E) := \mathrm{Ker}(\partial)/\mathrm{Im}(\partial).$$

Remark A.3. Note that $\mathscr{H}^\bullet(E)$ only makes sense (as a vector bundle) when E is regular. On the other hand, one can always take the point-wise cohomology: for each $x\in M$, there is a cochain complex of vector spaces (E_x,∂_x) and one can take its cohomology $H^\bullet(E_x,\partial_x)$. The dimension of these spaces may vary as x varies, and it is constant if and only if E is regular, in which case they fit into a graded vector bundle over M—and that is $\mathscr{H}^\bullet(E)$.

As for cochain complexes of vector spaces, we have the following terminology:

1. Given two complexes of vector bundles E and F and morphisms $f_1, f_2 : E \to F$, a homotopy between f and g is a degree -1 map $h : E \to F$ satisfying

$$h\partial + \partial h = f_1 - f_2.$$

If such an h exists, we say that f_1 and f_2 are homotopic.

2. A morphism $f : E \to F$ between two complexes of vector bundles E and F is called a homotopy equivalence if there exists a morphism $g : F \to E$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps. If such an f exists, we say that E and F are homotopy equivalent. We say that E is contractible if it is homotopy equivalent to the zero-complex or, equivalently, if there exists a homotopy between Id_E and the zero map.

3. A morphism $f : E \to F$ between two complexes of vector bundles is called a quasi-isomorphism if it induces isomorphisms in the point-wise cohomologies. We say that E is acyclic if it is point-wise acyclic.

Lemma A.4. *For complexes of vector bundles over M:*

(1) *If $f : E \to F$ is a quasi-isomorphism at $x \in M$, then it is a quasi-isomorphism in a neighborhood of x. In particular, if a complex E is exact at $x \in M$, then it is exact in a neighborhood of x.*

(2) *A morphism $f : E \to F$ is a quasi-isomorphism if and only if it is a homotopy equivalence. In particular, a complex E is acyclic if and only if it is contractible.*

(3) *If a complex E is regular, then it is homotopy equivalent to its cohomology $\mathscr{H}^\bullet(E)$ endowed with the zero differential.*

Proof. For (1) and (2), it suffices to prove the apparently weaker statements in the lemma coming after "in particular". This follows from the standard mapping complex argument: given a morphism f, one builds a double complex with E as 0-th row, F as 1-st row, and f as vertical differential. The resulting double complex $M(f)$, has the property that it is acyclic, or contractible, if and only if f is a quasi-isomorphism, or a homotopy equivalence, respectively [32].

To prove the weaker statements of (1) and (2), we fix a complex (E, ∂) and we choose a metric in each vector bundle E^i. Denote ∂^* the adjoint of ∂ with respect to the chosen metric and

$$\Delta = \partial\partial^* + \partial^*\partial$$

the correspondent "Laplacian". It is not difficult to see that the complex $(E_x^\bullet, \partial)$ is exact if and only if Δ_x is an isomorphism. Since the isomorphisms form an open set in the space of linear transformations, we get (1). When (E, ∂) is exact, a simple computation shows that $h := \Delta^{-1}\partial^*$ is a contracting homotopy for E, proving (2).

For (3) a linear version of Hodge decomposition gives us

$$E = \mathrm{Ker}(\Delta) \oplus \mathrm{Im}(\partial) \oplus \mathrm{Im}(\partial^*)$$

and an identification $\mathscr{H}^\bullet(E) = \mathrm{Ker}(\Delta)$. The resulting projection $E \to \mathrm{Ker}(\Delta)$ is a quasi-isomorphism. □

Operations. The operations with graded vector bundles discussed in the previous section extend to the setting of complexes. In other words, if E and F are complexes over M, then all the associated graded vector bundles $S(E)$, $\Lambda(E)$, E^*, $\underline{\mathrm{Hom}}(E,F)$, $E \otimes F$, inherit an operator ∂ making them into complexes over M. The induced differentials are defined by requiring that they satisfy the (graded) derivation rule, written formally as

$$\partial(xy) = \partial(x)y + (-1)^{|x|}x\partial(y).$$

For instance, for $E \otimes F$,

$$\partial(v \otimes w) = \partial(v) \otimes w + (-1)^{|v|} v \otimes \partial(w).$$

Also, for $T \in \underline{\mathrm{Hom}}(E,F)$,

$$\partial\big(T(v)\big) = \partial(T)(v) + (-1)^{|T|}T\big(\partial(v)\big),$$

in terms of graded commutators:

$$\partial(T) = \partial \circ T - (-1)^{|T|} T \circ \partial = [\partial, T].$$

If E is a complex over M, its differential ∂ induces a differential ∂ on $\Omega(A;E)$ defined by

$$\partial(\eta) = \partial \wedge \eta.$$

Explicitly, for $\eta \in \Omega^p(A;E^k)$, $\partial(\eta) \in \Omega^p(A;E^{k+1})$ is given by

$$(\alpha_1, \ldots, \alpha_p) \mapsto (-1)^p \partial\big(\eta(\alpha_1, \ldots, \alpha_p)\big).$$

The following simple lemma shows that the various differentials induced on $\Omega\big(A;\underline{\mathrm{End}}(E)\big)$ coincide.

Lemma A.5. *For any* $T \in \Omega\big(A;\underline{\mathrm{End}}(E)\big)$,

$$\partial(T) = [\partial, T] = \partial \wedge T - (-1)^{|T|} T \wedge \partial.$$

Connections. Let A be a Lie algebroid. An A-connection on a graded vector bundle E is just an A-connection on the underlying vector bundle E which preserves the grading. Equivalently, it is a family of A-connections, one on each E^n. If (E, ∂) is a complex over M, an A-connection on (E, ∂) is a graded connection ∇ which is compatible with ∂ (i.e. $\nabla_\alpha \partial = \partial \nabla_\alpha$). Note that, in terms of the operators d_∇ and ∂ induced on $\Omega(A;E)$, the compatibility of ∇ and ∂ is equivalent to $[d_\nabla, \partial] = 0$.

Connections on E and F naturally induce connections on the associated bundles $S(E)$, $\underline{\mathrm{End}}(E)$, $E \otimes F$, etc. The basic principle is, as before, the graded derivation rule. For instance, one has

$$d_\nabla(\eta_1 \wedge \eta_2) = d_\nabla(\eta_1) \wedge \eta_2 + (-1)^{|\eta_1|}\eta_1 \wedge d_\nabla(\eta_2),$$

for all $\eta_1 \in \Omega(A; E)$, $\eta_2 \in \Omega(A; F)$. Also, for $T \in \Omega\bigl(A; \underline{\mathrm{End}}(E)\bigr)$, $d_\nabla(T)$ is uniquely determined by

$$d_\nabla(T \wedge \eta) = d_\nabla(T) \wedge \eta + (-1)^{|T|} T \wedge d_\nabla(\eta),$$

for all $\eta \in \Omega(A; E)$. More explicitly,

$$d_\nabla(T) = [d_\nabla, T].$$

Lemma A.6. *If a complex (E, ∂) admits an A-connection then, for any leaf $L \subset M$ of A, $E|_L$ is regular.*

Proof. When $A = TM$ there is only one leaf $L = M$, and we have to prove that E is regular. Since ∇ is compatible with ∂, it follows that the parallel transport with respect to ∇ commutes with ∂ and therefore induces isomorphisms between the point-wise cohomologies. The same argument applied to parallel transport along A-paths, as explained in [11], implies the general case. $\square$

References

[1] *J. A. Alvarez López*, A decomposition theorem for the spectral sequence of Lie foliations, Trans. Amer. Math. Soc. **329** (1992), no. 1, 173–184.

[2] *C. Arias Abad* and *M. Crainic*, Representations up to homotopy of groupoids and the Bott spectral sequence, to appear.

[3] *C. Arias Abad* and *M. Crainic*, The Weil algebra and Van Est isomorphisms, Ann. Inst. Fourier, to appear.

[4] *A. Blaom*, Lie algebroids and Cartan's method of equivalence, arXiv:math/0509071.

[5] *R. Bott*, Lectures on characteristic classes and foliations, in: Lectures on algebraic and differential topology, Notes by L. Conlon, with two appendices by J. Stasheff (Second Latin American School in Math., Mexico City 1971), Lect. Notes Math. **279**, Springer, Berlin (1972), 1–94.

[6] *R. Bott*, On the Chern–Weil homomorphism and the continuous cohomology of Lie-groups, Adv. Math. **11** (1973), 289–303.

[7] *H. Bursztyn*, *M. Crainic*, *A. Weinstein* and *C. Zhu*, Integration of twisted Dirac brackets, Duke Math. J. **123** (2004), no. 3, 549–607.

[8] *H. Cartan*, Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie, in: Colloque de topologie (espaces fibrés) (Bruxelles 1950), Georges Thone, Liège (1951), 15–27.

[9] *K. T. Chen*, Iterated path integrals, Bull. Amer. Math. Soc. **83** (1997), 831–879.

[10] *M. Crainic*, On the perturbation lemma and deformations, arXiv:math/0403266.

[11] *M. Crainic* and *R. L. Fernandes*, Integrability of Lie brackets, Ann. Math. (2) **157** (2003), no. 2, 575–620.

[12] *M. Crainic* and *R. L. Fernandes*, Secondary characteristic classes of Lie algebroids, in: Quantum field theory and non-commutative geometry, Lect. Notes Phys. **662**, Springer, Berlin (2005), 157–176.

[13] *M. Crainic* and *R. L. Fernandes*, Stability of symplectic leaves, Invent. Math. **180** (2010), no. 3, 481–533.

[14] *M. Crainic* and *I. Moerdijk*, Deformations of Lie brackets: cohomological aspects, J. Eur. Math. Soc. **10** (2008), 1037–1059.

[15] *S. Evens*, *J.-H. Lu* and *A. Weinstein*, Transverse measures, the modular class and a cohomology pairing for Lie algebroids, Quart. J. Math. Oxford Ser. (2) **50** (1999), no. 200, 417–436.

[16] *E. Getzler*, The equivariant Chern character for non-compact Lie groups, Adv. Math. **109** (1994), no. 1, 88–107.

[17] *V. K. A. M. Gugenheim*, *L. Lambe* and *J. Stasheff*, Perturbation theory in differential homological algebra II, Ill. J. Math. **35** (1991), 357–373.

[18] *J. L. Heitsch*, A cohomology for foliated manifolds, Comment. Math. Helv. **50** (1975), 197–218.

[19] *D. Husemoller*, *J. C. Moore* and *J. Stasheff*, Differential homological algebra and homogeneous spaces, J. Pure Appl. Alg. **5** (1974), 113–185.

[20] *D. Iglesias-Ponte*, *C. Laurent-Gengoux* and *P. Xu*, Universal lifting theorem and quasi-Poisson groupoids, arXiv:math/0507396.

[21] *K. Igusa*, Iterated integrals of superconnections, arXiv:math/0912.0249.

[22] *J. Kalkman*, BRST model for equivariant cohomology and representatives for the equivariant Thom class, Comm. Math. Phys. **153** (1993), no. 3, 447–463.

[23] *F. W. Kamber* and *P. Tondeur*, Foliations and metrics, in: Differential geometry (College Park, Md. 1981/1982), Progr. Math. **32**, Birkhäuser Boston, Boston, MA (1983), 103–152.

[24] *Y. Kosmann-Schwarzbach* and *K. C. H. Mackenzie*, Differential operators and actions of Lie algebroids, in: Quantization, Poisson brackets and beyond (Manchester 2001), Contemp. Math. **315**, Amer. Math. Soc., Providence, RI (2002), 213–233.

[25] *A. Kotov* and *T. Strobl*, Characteristic classes associated to Q-bundles, arXiv:math/0711.4106v1.

[26] *K. Mackenzie*, Lie groupoids and Lie algebroids in differential geometry, Lond. Math. Soc. Lect. Note Ser. **124**, Cambridge Univ. Press, Cambridge 1987.

[27] *K. Mackenzie* and *P. Higgins*, Algebraic constructions in the category of Lie algebroids, J. Alg. **129** (1990), 194–230.

[28] *R. Mehta*, Q-algebroids and their cohomology, J. Symplectic Geom. **7** (2009), no. 3, 263–293.

[29] *D. Quillen*, Superconnections and the Chern character, Topology **24** (1985), no. 1, 89–95.

[30] *S. Schwede*, Morita theory in abelian, derived and stable model categories, in: Structured ring spectra, Lond. Math. Soc. Lect. Note Ser. **315**, Cambridge Univ. Press, Cambridge (2004), 33–86.

[31] *J. Stasheff*, Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras, in: Quantum groups (Leningrad 1990), Lect. Notes Math. **1510**, Springer, Berlin (1992), 120–137.

[32] *C. A. Weibel*, An introduction to homological algebra, Cambridge Stud. Adv. Math. **38**, Cambridge Univ. Press, Cambridge 1994.

[33] *A. Weinstein*, Symplectic groupoids and Poisson manifolds, Bull. Amer. Math. Soc. **16** (1987), 101–104.

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland
e-mail: camilo.arias.abad@math.uzh.ch

Mathematics Institute, Utrecht University, 3508 TA Utrecht, The Netherlands
e-mail: m.crainic@uu.nl

Eingegangen 9. Februar 2009, in revidierter Fassung 10. Juni 2010