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Computable Finite Element Error Bounds for Poisson's Equation

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New explicit finite element error bounds are presented for approximation by

(1) piecewise linear elements over triangles and
(2) piecewise bilinear elements over squares and rectangles.

By this the error bounds given in Barnhill, Brown & Mitchell (1981) are improved.

1. Introduction

LET $G \subset \mathbb{R}^2$ be an open and bounded region with polygonal boundary ∂G and let $f \in L_2(G)$. We consider the elliptic boundary value problem

$$\int_G [\nabla u(x, y) \cdot \nabla v(x, y) - f(x, y) v(x, y)] \, dx \, dy = 0, \qquad \forall\, v \in H_0^1(G), \quad u \in H_0^1(G), \tag{1.1}$$

which is the weak form of the classical boundary value problem

$$\begin{aligned} \Delta u(x, y) + f(x, y) &= 0 \quad (x, y) \in G \\ u(x, y) &= 0 \quad (x, y) \in \partial G. \end{aligned} \tag{1.2}$$

The solution u of (1.1) is known to be in $H_0^1(G)$ (Ciarlet, 1978). If G is convex, Barnhill & Wilcox (1977) have shown that $u \in H^2(G)$ and that the inequality

$$|u|_{2,G} \leqslant |f|_{0,G} := \|f\|_{L_2(G)} \tag{1.3}$$

holds, where

$$|u|_{2,G}^2 := \int_G [\partial_{xx} u(x, y)^2 + 2\partial_{xy} u(x, y)^2 + \partial_{yy} u(x, y)^2] \, dx \, dy. \tag{1.4}$$

We will solve the problem (1.1) by the finite element method (Schwarz, 1980): the region G is subdivided in triangles or in squares (if possible). The finite element space $V_h \subset H_0^1(G)$ consists of continuous functions which are piecewise linear or bilinear polynomials in the triangles or squares, respectively. (The index h in V_h

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indicates a measure of the fineness of the subdivisions of G.) The problem (1.1) is solved in V_h giving a finite element solution U which satisfies (Strang & Fix, 1973)

$$|u-U|_{1,G}^2 := \int_G |\nabla(u-U)(x,y)|^2\,dx\,dy \leqslant |u-\tilde{U}|_{1,G}^2, \quad \forall\ \tilde{U} \in V_h. \tag{1.5}$$

In this paper new error bounds are derived for $|u-U|_{1,G}$ and for $|u-U|_{0,G}$ improving the bounds given in Barnhill *et al.* (1981).

2. Error Bound for Triangular Elements

The linear interpolant on the triangle T with vertices (0, 0), (1, 0) and (0, 1) is

$$Q_T u(x,y) := (1-x-y)u(0,0) + xu(1,0) + yu(0,1), \quad u \in C(\bar{T}). \tag{2.1}$$

We define the interpolation error function by

$$e_u(x,y) := u(x,y) - Q_T u(x,y). \tag{2.2}$$

Barnhill *et al.* (1981) gave the H^1-bound

$$|e_u|_{1,T} \leqslant 1{\cdot}207|u|_{2,T}, \quad u \in H^2(T). \tag{2.3}$$

This bound can be improved. Consider an arbitrary $u \in H^2(T)$. By Sobolev's imbedding theorem (Ciarlet, 1978) $u \in C(\bar{T})$ so that $Q_T u$ is defined. As all second derivatives of $Q_T u$ vanish, we have

$$|e_u|_{2,T} = |u - Q_T u|_{2,T} = |u|_{2,T}. \tag{2.4}$$

$|e|_1$ and $|e|_2$ are norms on the error functions space

$$E = \{e \in H^2(T) | e(0,0) = e(1,0) = e(0,1) = 0\}.$$

With (2.3) we find that

$$R[e] = \frac{|e|_{2,T}^2}{|e|_{1,T}^2}, \quad e \in E\backslash\{0\} \tag{2.5}$$

is bounded below by a positive constant (i.e. $1{\cdot}207^{-2}$). We consider (2.5) as the Rayleigh quotient of the eigenvalue problem whose variational form is:

find $(\lambda, v) \in \mathbb{R} \times (E\backslash\{0\})$ such that

$$\int_T [\partial_{xx}v\,\partial_{xx}w + 2\partial_{xy}v\,\partial_{xy}w + \partial_{yy}v\,\partial_{yy}w]\,dx\,dy - \lambda\int_T [\partial_x v\,\partial_x w + \partial_y v\,\partial_y w]\,dx\,dy = 0, \quad \forall\ w \in E. \tag{2.6}$$

Applying Green's formula, we obtain from (2.6) for sufficiently smooth v and w

$$\int_T (\Delta^2 v + \lambda\Delta v)w\,dx\,dy - \int_{\partial T} (\partial_n \Delta v + \lambda\,\partial_n v)w\,ds + \int_{\partial T} \partial_{nn}v\,\partial_n w\,ds + \int_{\partial T} \partial_{nt}v\,\partial_t w\,ds = 0. \tag{2.7}$$

On the edges of the triangle $T\,dt = ds$, thus

$$\int_{\partial T} \partial_{nt} v \, \partial_t w \, ds = -\int_{\partial T} \partial_{ntt} v w \, ds, \tag{2.8}$$

because w vanishes in the corners of T.

Putting (2.8) into (2.7) we obtain the classical form of (2.6)

$$\begin{aligned} \Delta^2 v(x, y) &= -\lambda \Delta v(x, y) \quad (x, y) \in T \\ v(0, 0) = v(1, 0) &= v(0, 1) = 0 \\ \partial_{nn} v(x, y) &= 0 \quad (x, y) \in \partial T \\ \partial_n \Delta v(x, y) + \partial_{ntt} v(x, y) + \lambda \, \partial_n v(x, y) &= 0 \quad (x, y) \in \partial T. \end{aligned} \tag{2.9}$$

With the first eigenvalue μ_1 of (2.9) we have

$$|e|_{1,T} \leqslant \mu_1^{-\frac{1}{2}} |e|_{2,T}, \quad \forall\, e \in E. \tag{2.10}$$

Equality holds if $e = v_1$, v_1 being the first eigenfunction of (2.9). In this sense (2.7) is optimal. μ_1 was computed to a high precision using C^1-finite elements. We could achieve the inclusion

$$4{\cdot}18673 \leqslant \mu_1 \leqslant 4{\cdot}18674.$$

With this (2.10) becomes

$$|u - Q_T u|_{1,T} \leqslant 0{\cdot}4888 |u|_{2,T}. \tag{2.11}$$

Transformation on the triangle T_h with vertices $(0, 0)$, $(h, 0)$, $(0, h)$ gives

$$|u - Q_{T_h} u|_{1,T_h} \leqslant 0{\cdot}4888 h |u|_{2,T_h}. \tag{2.12}$$

3. Error Bound for Square Elements

The bilinear interpolant on the square S with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ is defined by

$$Q_S u(x, y) = (1-x)(1-y)u(0, 0) + (1-x)yu(1, 0) + x(1-y)u(0, 1) + xyu(1, 1), \quad u \in C(\bar{S}). \tag{3.1}$$

We define the interpolation error function by

$$e_u(x, y) = u(x, y) - Q_S u(x, y) \tag{3.2}$$

and the space of all possible error functions obtained by interpolating functions in $H^2(S)$ by

$$E = \{v \in H^2(S) | v(0, 0) = v(1, 0) = v(0, 1) = v(1, 1) = 0\}.$$

Barnhill *et al.* (1981) gave the error bound

$$|e_u|_{1,S} \leqslant 0{\cdot}625 |u|_{2,S}, \quad u \in H^2(S). \tag{3.3}$$

As in the former section this bound can be improved. Consider an arbitrary

$u \in H^2(S)$. $Q_S u$ is again defined. Here $|Q_S u|_{2,S} \neq 0$ due to the xy-terms in (3.1)! But $\partial_{xy} Q_S u(x, y)$ is a constant and because

$$\int_0^1 \int_0^1 \partial_{xy} e(x, y)\, dy\, dx = \int_0^1 [\partial_x e(x, 1) - \partial_x e(x, 0)]\, dx$$
$$= e(1, 1) - e(0, 1) - e(1, 0) + e(0, 0) = 0, \quad \forall\, e \in E, \tag{3.4}$$

it follows that

$$\begin{aligned} |u|^2_{2,S} &= |e_u + Q_S u|^2_{2,S} \\ &= \int_0^1 \int_0^1 [\partial_{xx} e_u^2 + 2(\partial_{xy} e_u + \partial_{xy} Q_S u)^2 + \partial_{yy} e_u^2]\, dx\, dy \\ &= |e_u|^2_{2,S} + 4 \int_0^1 \int_0^1 \partial_{xy} e_u \partial_{xy} Q_S u\, dx\, dy + 2 \int_0^1 \int_0^1 \partial_{xy} Q_S u^2\, dx\, dy \\ &= |e_u|^2_{2,S} + |Q_S u|^2_{2,S} \geqslant |e_u|^2_{2,S}, \quad \forall\, u \in H^2(S). \end{aligned} \tag{3.5}$$

Therefore we can argue as in Section 2 and compute a new bound by minimizing the Rayleigh quotient

$$R[e] := \frac{|e|^2_{2,S}}{|e|^2_{1,S}}$$

in $E \backslash \{0\}$. The minimum $\mu_1 = \pi^2$ is attained for functions in the two-dimensional subspace of E spanned by $\sin \pi x$ and $\sin \pi y$. With this and the use of (3.5) we obtain the inequality

$$\begin{aligned} |u - Q_S u|_{1,S} &\leqslant 0{\cdot}3184 |u - Q_S u|_{2,S} \\ &\leqslant 0{\cdot}3184 |u|_{2,S}, \quad \forall\, u \in H^2(S). \end{aligned} \tag{3.6}$$

Transformation on the square $S_h = (0, h) \times (0, h)$ gives

$$|u - Q_{S_h} u|_{1,S_h} \leqslant 0{\cdot}3184 h |u|_{2,S_h}. \tag{3.7}$$

Remark: For a rectangle with sidelengths h_1 and $h_2 > h_1$ (3.7) holds with $h = h_2$.

4. Numerical Results

The new bounds (2.12) and (3.7) are compared with the bounds in Barnhill *et al.* (1981) and with the numerical results given therein. The three test problems considered there have known solutions. Therefore the theoretical (upper) bounds can be compared with corresponding computable values (Barnhill *et al.*, 1981)

$$K^p_M = \frac{|u - U_M|_{2-p}}{h^p |f|_0}, \qquad p = 1, 2; \quad M = S, T, \tag{4.1}$$

where u is the actual solution and U_M its approximant in a finite element space of mesh width h consisting of triangular ($M = T$) or square ($M = S$) elements. Each test problem was solved with three different mesh sizes h giving nine numbers K^p_M for each pair (p, M). The results are summarized in Table 1.

The bounds

$$|u - U_T|_{0,T} \leqslant 0{\cdot}2389 h^2 |f|_0 \tag{4.2}$$

TABLE 1

Range of computable values K^p_M	Bounds in Barnhill *et al.* (1981)	Our bounds
$0{\cdot}1489 \leqslant K^1_S \leqslant 0{\cdot}2519$	0·7906	0·3184
$0{\cdot}0405 \leqslant K^2_S \leqslant 0{\cdot}0761$	0·6250	0·1014
$0{\cdot}1517 \leqslant K^1_T \leqslant 0{\cdot}3466$	1·207	0·4888
$0{\cdot}0417 \leqslant K^2_T \leqslant 0{\cdot}1333$	1·457	0·2389

and

$$|u-U_S|_{0,S} \leqslant 0{\cdot}1014h^2|f|_0 \tag{4.3}$$

are obtained from (2.12) and (3.7) using the so-called Nitsche trick (Ciarlet, 1978).

5. Concluding Remarks

The error bounds seem to be useful for problems of the form (1.1) on *convex* domains G.

For the three test problems the theoretical bounds are near to the greatest values of K^p_M, $p = 1, 2, M = S, T$. K^1_T and K^1_S are overestimated by factors from 1·2 to 3·2, K_T and K_S by factors from 1·3 to 5·7. So the upper bounds are of the correct order. The bounds are better for the energy norm than for the L_2-norm which is clear because the L_2-norm bounds have been obtained only indirectly.

In another approach to Poisson's equation Kuttler (1979) gives the inequality

$$|u-\psi|_{0,G} \leqslant c_1|f-\Delta\psi|_{0,G}+c_2|\psi|_{0,\partial G}$$

and determines the constants c_1 and c_2 explicitly. Here u is the solution of (1.2), ψ an arbitrary approximation in the domain of Δ. This inequality is not applicable on finite element approximations because piecewise linear or bilinear functions are not elements of the domain of the Laplace operator.

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