ON POLYNOMIALS WITH COEFFICIENTS OF MODULUS ONE

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Introduction

As in Littlewood [5], we let $\mathcal{G}_n$ be the class of all polynomials of the form

$$g_n(\theta) = \sum_{k=0}^{n} \exp(\alpha_k i) z^k,$$

where the α_k are arbitrary real constants and $z = \exp(2\pi i\theta)$. Clearly $\|g_n\|_{L^2} = (n+1)^{\frac{1}{2}}$ for all $g_n \in \mathcal{G}_n$, and the question "how close can such a g_n come to satisfying $|g_n| \equiv (n+1)^{\frac{1}{2}}$?" has long been the object of intense study. In [5] Littlewood conjectured that there are positive absolute constants A_1 and A_2 such that, for arbitrarily large n, there exist $g_n \in \mathcal{G}_n$ with $A_1 n^{\frac{1}{2}} \leqslant |g_n(\theta)| \leqslant A_2 n^{\frac{1}{2}}$ for all θ. In [3] Erdős conjectured that there is a universal constant $c > 0$ such that for $n \geqslant 2$, $\|g_n\|_\infty \geqslant (1+c)n^{\frac{1}{2}}$ for all $g_n \in \mathcal{G}_n$.

It was shown by Littlewood [5] that the function

$$g(\theta) = \sum_{m=0}^{n} \exp(\tfrac{1}{2}m(m+1)\pi i(n+1)^{-1}) z^m$$

satisfies: (i) for any $\delta > 0$, $|g|n^{-\frac{1}{2}} \to 1$ uniformly in $n^{-\frac{1}{2}+\delta} \leqslant |\theta| \leqslant \frac{1}{2}$, and (ii) $|g| \leqslant 1{\cdot}4n^{\frac{1}{2}}$ for all θ. In the first part of this paper we strengthen part (i) of the Littlewood result in two ways by producing polynomials g which yield an improved estimate for $|g|n^{-\frac{1}{2}}$ in a larger subset of the unit circle. In the second part we use the methods already developed to construct functions which are "almost" in $\mathcal{G}_n$ and which satisfy the Littlewood conjecture with (within the error) $A_1 = A_2 = 1$.

1. To begin our work, we require two elementary lemmas.

LEMMA 1. *Let*

$$F(x, T) = (1 - e^{2\pi i x}) \sum_{m=0}^{\infty} \frac{e^{2\pi i m x}}{m+T},$$

where $T > 1$ and F is defined by continuity when x is an integer. Then $|F(x, T)| < 3/T$ for all x.

Proof of Lemma 1. Let x be fixed and not an integer. Then

$$F(x, T) = \lim_{R\to\infty} (1 - e^{2\pi i x}) \sum_{m=0}^{R} \frac{e^{2\pi i m x}}{m+T} = \frac{1}{T} - \frac{1}{T} \sum_{m=1}^{\infty} \frac{e^{2\pi i m x}}{(m^2 - m)T^{-1} + 2m + T - 1},$$

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and
$$\sum_{m=1}^{\infty} \frac{1}{(m^2-m)T^{-1}+2m+T-1} = \sum_{1}^{[T]} + \sum_{[T]+1}^{\infty} < 1+T\int_T^{\infty} \frac{dx}{x^2} = 2,$$
completing the proof of Lemma 1.

LEMMA 2. *If r and x are not integers, then*
$$\sum_{m=-\infty}^{\infty} \frac{e^{2\pi imx}}{m+r} = \frac{2\pi i\, e^{2\pi ikr}}{1-e^{-2\pi ir}} e^{-2\pi irx},$$
where $k = [x]$.

Proof of Lemma 2. Simply compute the Fourier Series of the function $F(x)$ of period 1 which, for $0 \leqslant x < 1$, is given by
$$F(x) = \frac{2\pi i}{1-e^{-2\pi ir}} e^{-2\pi irx}.$$

Employing these Lemmas, we are able to prove our basic result.

THEOREM 1. *Let N be a positive integer, and define the function*
$$P \in \mathscr{G}_{N^2-1} \text{ by } P(\theta) = \sum_{k=0}^{N-1}\sum_{j=0}^{N-1} \exp(2\pi ijkN^{-1}) z^{j+kN}, \quad z = \exp(2\pi i\theta).$$
Then

(a) $\left|P\left(\dfrac{j}{N^2}\right)\right| = N$ *for all integers* j,

(b) *For any* ε, $N^{-1} < \varepsilon < \frac{1}{2}$, $|P(\theta)| = N+E$ *for* $-1+\varepsilon \leqslant \theta \leqslant -\varepsilon$, *where* $|E| < 1+2\pi^{-1}+5(\pi\varepsilon)^{-1}$,

(c) *For N odd*, $P\left(\dfrac{1}{2N}\right) = O(1)$, *while for N even*, $P\left(\dfrac{N-1}{2N^2}\right) = O(1)$, *and*

(d) $|P(\theta)| < \left(2+\dfrac{3}{\pi}\right)N+O(1)$ *for all* θ.

Proof of Theorem 1. A straightforward calculation shows that for integers m, r with $0 \leqslant m, r < N$ we have
$$P\left(-\frac{mN+r}{N^2}\right) = N \exp\left(-2\pi i \frac{mN+r}{N^2} r\right),$$
and (a) follows.

To obtain (b) we define the functions
$$G(\theta) = \frac{iN}{2\pi}(1-z^N) \sum_{k=-\infty}^{\infty} \frac{z^{kN}}{k+N\theta}$$
and
$$H(\theta) = \frac{iN}{2\pi}(1-z^N) \sum_{k=0}^{N-1} \frac{z^{kN}}{k+N\theta},$$

and the error functions

$$R(\theta) = P(\theta) - H(\theta) \text{ and } S(\theta) = H(\theta) - G(\theta).$$

To estimate $R(\theta)$ we let

$$\phi(x) = \frac{\sin x}{2(1-\cos x)} - \frac{1}{x} \quad \text{and} \quad \psi(x) = \frac{1}{1-\exp(ix)} - \frac{i}{x} = \frac{1}{2} + i\phi(x).$$

We can now write

$$R(\theta) = (1-z^N)\psi(2\pi\theta) + \sum_{k=0}^{N-1} z^{(k+1)N}\left\{\psi\left(2\pi\left(\frac{k+1}{N}+\theta\right)\right) - \psi\left(2\pi\left(\frac{k}{N}+\theta\right)\right)\right\},$$

which, together with the facts that $\phi'(x) < 0$, $-2\pi < x < 2\pi$, and $-1+\varepsilon \leqslant \theta \leqslant -\varepsilon$, $\varepsilon < \frac{1}{2}$ yields

$$\begin{aligned}
|R(\theta)| &\leqslant 2|\psi(2\pi\theta)| + \sum_{k=0}^{N-1}\left|\psi\left(2\pi\left(\frac{k+1}{N}+\theta\right)\right) - \psi\left(2\pi\left(\frac{k}{N}+\theta\right)\right)\right| \\
&= 2|\psi(2\pi\theta)| + \sum_{k=0}^{N-1}\left\{\phi\left(2\pi\left(\frac{k}{N}+\theta\right)\right) - \phi\left(2\pi\left(\frac{k+1}{N}+\theta\right)\right)\right\} \\
&= 2|\psi(2\pi\theta)| + \frac{1}{2\pi}\left(\frac{1}{1+\theta} - \frac{1}{\theta}\right) \\
&< 2\{\tfrac{1}{2} + \phi(2\pi(-1+\varepsilon))\} + \frac{1}{2\pi\varepsilon(1-\varepsilon)} \\
&< 1 + \frac{2}{\pi} + \frac{2}{\pi\varepsilon}. \qquad (1)
\end{aligned}$$

To estimate $S(\theta)$, we observe that $1-N\theta \geqslant \varepsilon N > 1$ and $N(1+\theta) \geqslant \varepsilon N > 1$, so that we may apply Lemma 1 with $x = \pm N\theta$ and $T = \varepsilon N$ to get

$$\begin{aligned}
|S(\theta)| &= \frac{N}{2\pi}|1-z^N|\left|\sum_{k=-\infty}^{-1}\frac{z^{kN}}{k+N\theta} + \sum_{k=N}^{\infty}\frac{z^{kN}}{k+N\theta}\right| \\
&\leqslant \frac{N}{2\pi}|1-e^{2\pi iN\theta}|\left\{\left|\sum_{k=0}^{\infty}\frac{e^{-2\pi ikN\theta}}{k+1-N\theta}\right| + \left|\sum_{k=0}^{\infty}\frac{e^{2\pi ikN\theta}}{k+N(1+\theta)}\right|\right\} < \frac{3}{\pi\varepsilon}. \qquad (2)
\end{aligned}$$

Next we apply Lemma 2 with $x = r = N\theta$ to conclude that

$$|G(\theta)| = N \text{ for all } \theta. \qquad (3)$$

Finally, combining (1), (2) and (3) with the fact that $P(\theta) = G(\theta)+R(\theta)+S(\theta)$, we obtain (b).

For (c) we assume that $|\theta| < \frac{1}{4}$ and apply the techniques of the proof of (b) to get

$$P(\theta) = \frac{iN}{2\pi}(1-e^{2\pi i N\theta})\left(\sum_{k=0}^{[\frac{1}{2}N]} \frac{e^{2\pi i k N\theta}}{k+N\theta} + \sum_{k=[\frac{1}{2}N]+1}^{N-1} \frac{e^{2\pi i k N\theta}}{k-N+N\theta}\right) + O(1)$$

$$= \frac{iN}{2\pi}(1-e^{2\pi i N\theta})\left(\sum_{k=0}^{\infty} \frac{e^{2\pi i k N\theta}}{k+N\theta} + e^{2\pi i N^2\theta} \sum_{k=-\infty}^{-1} \frac{e^{2\pi i k N\theta}}{k+N\theta}\right) + O(1) \tag{4}$$

If N is odd, (4) immediately implies that

$$P\left(\frac{1}{2N}\right) = O(1).$$

For N even, (4) yields

$$P\left(\frac{N-1}{2N^2}\right) = \frac{2iN}{\pi}\left(\sum_{k=0}^{\infty} \frac{(-1)^k e^{-\pi i(k/N)}}{2k+1} - e^{(\pi i/N)} \sum_{k=0}^{\infty} \frac{(-1)^k e^{\pi i(k/N)}}{2k+1}\right) + O(1)$$

$$= \frac{2iN}{\pi}(1-e^{\pi i/N}) \sum_{k=0}^{\infty} \frac{(-1)^k \cos(k\pi/N)}{2k+1} + \frac{4N}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \sin(k\pi/N)}{2k+1}$$

$$+ O(1) = O(1),$$

where the final estimate follows, for example, from Gradshteyn and Ryzhik [**4**], p. 38, formulas 1.442–3 and 4, and (c) is proven.

To establish (d) we may assume that $|\theta| < \frac{1}{4}$, and we let $N\theta = M+t$, where $M = [N\theta]$. For $M \leqslant -1$ we have, by Lemmas 1 and 2,

$$\left|(1-e^{2\pi i N\theta}) \sum_{k=0}^{\infty} \frac{e^{2\pi i k N\theta}}{k+N\theta}\right| = \left|(1-e^{2\pi i t})\left(\sum_{k=-\infty}^{\infty} \frac{e^{2\pi i k t}}{k+t} - \sum_{k=-\infty}^{M-1} \frac{e^{2\pi i k t}}{k+t}\right)\right| < 2\pi+3. \tag{5}$$

It is equally trivial to obtain an identical estimate for $M \geqslant 0$, and the same method also yields

$$\left|(1-e^{2\pi i N\theta}) \sum_{k=-\infty}^{-1} \frac{e^{2\pi i k N\theta}}{k+N\theta}\right| < 2\pi+3 \quad \text{for} \quad |\theta| < \tfrac{1}{4}. \tag{6}$$

Finally, (d) follows from (4), (5), and (6), and the proof of Theorem 1 is complete.

We point out several immediate consequences of this theorem. First, if α is a fixed real number and if we define Q by $Q(\theta) = P(\theta+\alpha)$, it is obvious that $Q \in \mathscr{G}_{N^2-1}$. Therefore, the bad interval in (b) can be shifted to any interval of length 2ε.

Second, if we are interested in a fixed subinterval of the unit circle, the estimate in (b) becomes quite remarkable. For example, setting $\varepsilon = \frac{1}{4}$ we obtain

COROLLARY 1. *On the unit semicircle $-\frac{3}{4} \leqslant \theta \leqslant -\frac{1}{4}$ we have $|P(\theta)| = N+E$, where $|E| < 1+22\pi^{-1} < 9$.*

Third, by employing, for example, a result of Beller [**1**], we are able to extend Theorem 1 to the case of polynomials of arbitrary degree. We have

Corollary 2. *Let n be a positive integer. Then there is a $g \in \mathscr{G}_n$ satisfying*

(e) *For any* ε, $[n^{1/2}]^{-1} < \varepsilon < \frac{1}{2}$, $|g(\theta)| = n^{1/2}+E$ *for* $-1+\varepsilon \leqslant \theta \leqslant -\varepsilon$, *where* $|E| < 2+2\pi^{-1}+5(\pi\varepsilon)^{-1}+2n^{1/4}$, *and*

(f) $|g(\theta)| < (2+3/\pi)n^{1/2}+2n^{1/4}+O(1)$ *for all* θ.

Proof of Corollary 2. Let $N = [n^{1/2}]$, $m = n-N^2$, and choose $P(\theta)$ as in Theorem 1. By Beller's result [1], we can choose $f \in \mathscr{G}_m$ such that $|f(\theta)| < 1{\cdot}172\mathrm{m}^{1/2} < 2n^{1/4}$ for all θ. If we now let $g(\theta) = P(\theta)+z^{N^2}f(\theta)$, the required estimates follow immediately from the Theorem, and Corollary 2 is proven.

Finally we observe that if we choose ε in Corollary 2 to be, for example, $n^{-1/4}\log n$, we obtain the improvement of Littlewood's result mentioned in the introduction.

2. We now proceed with our construction of functions $G(\theta)$ which are almost in $\mathscr{G}_n$, and which satisfy $|G(\theta)| = n^{1/2}+O(n^{1/4})$ for all θ. Toward this end, we have

Theorem 2. Let n be a positive integer, and let N be the even positive integer satisfying $N^2 \leqslant n < (N+2)^2$. Then there exist functions f and g such that

(A) $z^{N^2}f+g \in \mathscr{G}_n$, *and*

(B) $|f(\theta)+g(\theta)| = n^{1/2}+O(n^{1/4})$, *where the error is uniform in* θ *and* n.

Remark. It will be seen from the following construction that g consists of two parts; a polynomial in $\mathscr{G}_{n-2n^{3/4}+O(n^{1/2})}$, plus z to an integral power multiplied by a polynomial with coefficients of modulus $\frac{1}{2}$ and degree $4n^{3/4}+O(n^{1/2})$ in $z^{1/2}$. Also, f is a function of precisely the same type as the second part of g, just described. Thus we see that, except for a relatively small number (i.e., $O(n^{1/2})$) of terms, $f+g \in \mathscr{G}_n$, and so we can use $f+g$ as the function $G(\theta)$ mentioned above.

Proof of Theorem 2. Define δ by $\delta N = [N^{1/2}]$, let $z = \exp(2\pi i\theta)$, $-\frac{1}{2} \leqslant \theta \leqslant \frac{1}{2}$, let $m = n-N^2$, and choose $G_1 \in \mathscr{G}_m$ such that $|G_1(\theta)| = O(n^{1/4})$ (see the proof of Corollary 2). Define the functions

$$f(\theta) = \tfrac{1}{2}z^{(\frac{1}{2}N-\delta N)N}\sum_{j=-\frac{1}{2}N-\delta N}^{-\frac{1}{2}N+\delta N-1} z^{jN}\sum_{k=0}^{2N-1}\exp k\pi i(jN^{-1}+\theta),$$

$$F(\theta) = z^{(\frac{1}{2}N-\delta N)N}\left\{\tfrac{1}{2}\sum_{j=\frac{1}{2}N-\delta N}^{\frac{1}{2}N+\delta N-1} z^{jN}\sum_{k=0}^{2N-1}\exp k\pi i(jN^{-1}+\theta)\right.$$

$$\left.+\sum_{j=-\frac{1}{2}N+\delta N}^{\frac{1}{2}N-\delta N-1} z^{jN}\sum_{k=0}^{N-1}\exp 2k\pi i(jN^{-1}+\theta)\right\},$$

and

$$g(\theta) = F(\theta)+z^{N^2}G_1(\theta).$$

A straightforward calculation yields (A). To establish (B) we proceed as in the proof of Theorem 1(b), and we obtain

$$f(\theta)+g(\theta) = z^{(\frac{1}{2}N-\delta N)N}(1-z^N)\frac{i}{2\pi}\sum_{j=-\frac{1}{2}N-\delta N}^{\frac{1}{2}N+\delta N-1}\frac{z^{jN}}{(j/N)+\theta}+O(\delta^{-1})+O(n^{\frac{1}{4}})$$

$$= z^{(\frac{1}{2}N-\delta N)N}(1-z^N)\frac{iN}{2\pi}\sum_{j=-\infty}^{\infty}\frac{z^{jN}}{j+N\theta}+O(n^{\frac{1}{4}}).$$

Therefore, by Lemma 2, $|f+g| = N+O(n^{\frac{1}{4}}) = n^{\frac{1}{2}}+O(n^{\frac{1}{4}})$, and the proof of Theorem 2 is complete.

In conclusion we mention a result with a similar flavour to Theorem 2 by Beller and Newman [2], who prove the Littlewood conjecture, with A_1 and A_2 both much smaller than 1, for polynomials whose coefficients are bounded by 1 in modulus.

References

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