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A point is normal for almost all maps $\beta x + \alpha$ mod 1 or generalized β-transformations

B. FALLER and C.-E. PFISTER

EPF-L, SB IACS, Station 8, CH-1015 Lausanne, Switzerland
(e-mail: bastien.faller@a3.epfl.ch, charles.pfister@epfl.ch)

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Abstract. We consider the map $T_{\alpha,\beta}(x) := \beta x + \alpha \bmod 1$, which admits a unique probability measure $\mu_{\alpha,\beta}$ of maximal entropy. For $x \in [0, 1]$, we show that the orbit of x is $\mu_{\alpha,\beta}$-normal for almost all $(\alpha, \beta) \in [0, 1) \times (1, \infty)$ (with respect to Lebesgue measure). Nevertheless, we construct analytic curves in $[0, 1) \times (1, \infty)$ along which the orbit of $x = 0$ is $\mu_{\alpha,\beta}$-normal at no more than one point. These curves are disjoint and fill the set $[0, 1) \times (1, \infty)$. We also study the generalized β-transformations (in particular, the tent map). We show that the critical orbit $x = 1$ is normal with respect to the measure of maximal entropy for almost all β.

1. *Introduction*

In this paper, we consider a dynamical system (X, d, T) where (X, d) is a compact metric space endowed with its Borel σ-algebra $\mathcal{B}$ and $T : X \to X$ is a measurable map. Let $C(X)$ denote the set of all continuous functions from X into $\mathbb{R}$. The set $M(X)$ of all Borel probability measures is equipped with the weak*-topology. $M(X, T) \subset M(X)$ is the subset of all T-invariant probability measures. For $\mu \in M(X, T)$, let $h(\mu)$ denote the measure-theoretic entropy of μ. For all $x \in X$ and $n \geq 1$, the empirical measure of order n at x is

$$\mathcal{E}_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_x \circ T^{-i} \in M(X), \tag{1}$$

where δ_x is the Dirac mass at x. Let $V_T(x) \subset M(X, T)$ denote the set of all cluster points of $\{\mathcal{E}_n(x)\}_{n\geq 1}$ in the weak*-topology.

Definition 1. Let $\mu \in M(X, T)$ be an ergodic measure and take $x \in X$. The orbit of x under T is μ*-normal* if $V_T(x) = \{\mu\}$, i.e. for all continuous $f \in C(X)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int f \, d\mu.$$

By the Birkhoff ergodic theorem, μ-almost all points are μ-normal; however, it is difficult to identify a μ-normal point. This paper is devoted to the study of the normality of orbits for piecewise monotone continuous maps of the interval. We consider a family $\{T_\kappa\}_{\kappa\in K}$ of piecewise monotone continuous maps, parameterized by a parameter $\kappa \in K$, such that for all $\kappa \in K$ there is a unique measure μ_κ of maximal entropy. In our case, K is a subset of $\mathbb{R}$ or $\mathbb{R}^2$. For a given $x \in X$, we estimate the Lebesgue measure of the subset of K for which the orbit of x under T_κ is μ_κ-normal.

For example, let $T_{\alpha,\beta} : [0, 1] \to [0, 1]$ be the piecewise monotone continuous map defined by $T_{\alpha,\beta}(x) = \beta x + \alpha \mod 1$; here $\kappa = (\alpha, \beta) \in [0, 1) \times (1, \infty)$. In [**13**], Parry constructed a $T_{\alpha,\beta}$-invariant probability measure $\mu_{\alpha,\beta}$, absolutely continuous with respect to Lebesgue measure, which is the unique measure of maximal entropy. The main result of §3 is Theorem 3, which shows that for all $x \in [0, 1]$, the set

$$\mathcal{N}(x) := \{(\alpha, \beta) \in [0, 1) \times (1, \infty) \mid \text{the orbit of } x \text{ under } T_{\alpha,\beta} \text{ is } \mu_{\alpha,\beta}\text{-normal}\}$$

has full two-dimensional Lebesgue measure. This is a generalization of a theorem of Schmeling in [**17**], where the case with $\alpha = 0$ and $x = 1$ is studied. For β-transformations, the orbit of 1 plays a particular role, so the restriction to $x = 1$ considered by Schmeling is natural. Similarly, for $T_{\alpha,\beta}$, the orbits of 0 and 1 are very important. In Theorem 4, we show that there exist curves in the (α, β)-plane, defined by $\alpha = \alpha(\beta)$, along which the orbits of 0 or 1 are never $\mu_{\alpha,\beta}$-normal. The curve $\alpha = 0$ is a trivial example of such a curve for the fixed point $x = 0$. In §4, we study the generalized β-transformations introduced by Góra [**7**]. A generalized β-transformation is similar to a β-transformation, but each lap is replaced by an increasing or decreasing lap of constant slope β according to a sequence of signs. For a given class of generalized β-transformations, there exists β_0 such that for all $\beta > \beta_0$ there is a unique measure μ_β of maximal entropy, and the set

$$\{\beta > \beta_0 \mid \text{the orbit of 1 under } T_\beta \text{ is } \mu_\beta\text{-normal}\}$$

has full Lebesgue measure, denoted below by λ. Since the tent maps are generalized β-transformations, we obtain an alternative proof of Bruin's results in [**3**].

2. *Preliminaries*

Let us define properly the coding for a piecewise monotone continuous map of the interval. The classical papers on this subject are [**15**], [**13**] and [**10**]. We consider piecewise monotone continuous maps of the following type. Let $k \geq 2$ and $0 = a_0 < a_1 < \cdots < a_k = 1$. We set $\mathbb{A} := \{0, \ldots, k-1\}$, $I_0 = [a_0, a_1)$, $I_j = (a_j, a_{j+1})$ for $j \in \{1, \ldots, k-2\}$, $I_{k-1} = (a_{k-1}, a_k]$ and $S_0 = \{a_j \mid j = 1, \ldots, k-1\}$. For all $j \in \mathbb{A}$, let $f_j : I_j \to [0, 1]$ be a strictly monotone continuous map. A piecewise monotone continuous map $T : [0, 1]\backslash S_0 \to [0, 1]$ is defined by

$$T(x) = f_j(x) \quad \text{if } x \in I_j.$$

Later we will state, in each specific case, how to define T on S_0. We set $X_0 = [0, 1]$ and, for $n \geq 1$,

$$X_n = X_{n-1}\backslash S_{n-1} \quad \text{and} \quad S_n = \{x \in X_n \mid T^n(x) \in S_0\}, \tag{2}$$

so that T^n is well defined on X_n. Finally, we let $S=\bigcup_{n\geq 0} S_n$ so that $T^n(x)$ is well defined for all $x\in[0,1]\backslash S$ and all $n\geq 0$.

Let $\mathtt{A}$ be endowed with the discrete topology and let $\Sigma_k=\mathtt{A}^{\mathbb{Z}_+}$ be the product space. The elements of Σ_k are denoted by $\underline{x}=x_0x_1\cdots$. A finite string $\underline{w}=w_0\cdots w_{n-1}$ with $w_j\in\mathtt{A}$ is called a *word*. The *length* of $\underline{w}$ is $|\underline{w}|=n$. There is a single word of length 0, the *empty word* ε. The set of all words is $\mathtt{A}^*$. For two words $\underline{w}$ and $\underline{z}$, we write $\underline{w}\,\underline{z}$ for the concatenation of the two words. For $\underline{x}\in\Sigma_k$, let $\underline{x}_{[i,j)}=x_i\cdots x_{j-1}$ denote the word formed by the coordinates i to $j-1$ of $\underline{x}$. For a word $\underline{w}\in\mathtt{A}^*$ of length n, the *cylinder* $[\underline{w}]$ is the set

$$[\underline{w}]:=\{\underline{x}\in\Sigma_k\mid \underline{x}_{[0,n)}=\underline{w}\}.$$

The family $\{[\underline{w}]\mid\underline{w}\in\mathtt{A}^*\}$ is a base for the topology and a semi-algebra generating the Borel σ-algebra. For all $\beta>1$, there exists a metric d_β compatible with the topology which is defined by

$$d_\beta(\underline{x},\underline{x}'):=\begin{cases}0 & \text{if } \underline{x}=\underline{x}',\\ \beta^{-\min\{n\geq 0:\ \underline{x}_n\neq\underline{x}'_n\}} & \text{otherwise.}\end{cases}$$

The left shift map $\sigma:\Sigma_k\to\Sigma_k$ is defined by

$$\sigma(\underline{x})=x_1x_2\cdots .$$

It is a continuous map. We define a total order on Σ_k, denoted by $\prec$. We set

$$\delta(j)=\begin{cases}+1 & \text{if } f_j \text{ is increasing},\\ -1 & \text{if } f_j \text{ is decreasing},\end{cases}$$

and, for word $\underline{w}$,

$$\delta(\underline{w})=\begin{cases}1 & \text{if } \underline{w}=\varepsilon,\\ \delta(w_0)\cdots\delta(w_{n-1}) & \text{if } \underline{w} \text{ has length } n.\end{cases}$$

Let $\underline{x}\neq\underline{x}'\in\Sigma_k$ and define $n=\min\{j\geq 0\mid x_j\neq x'_j\}$; then

$$\underline{x}\prec\underline{x}'\iff\begin{cases}x_n<x'_n & \text{if } \delta(\underline{x}_{[0,n)})=+1,\\ x_n>x'_n & \text{if } \delta(\underline{x}_{[0,n)})=-1.\end{cases}$$

When all maps f_j are increasing, this is the lexicographic order.

We define the coding map $\mathtt{i}:[0,1]\backslash S\to\Sigma_k$ by

$$\mathtt{i}(x):=\mathtt{i}_0(x)\mathtt{i}_1(x)\cdots\quad\text{with } \mathtt{i}_n(x)=j\iff T^n(x)\in I_j.$$

The coding map $\mathtt{i}$ is left undefined on S. Henceforth we suppose that T is such that $\mathtt{i}$ is injective. A sufficient condition for the injectivity of the coding is the existence of $c>1$ such that $|f'_j(x)|\geq c$ for all $x\in I_j$ and all $j\in\mathtt{A}$; see [**13**]. This condition is satisfied in all the cases considered in this paper. The coding map is order-preserving, i.e. for all $x,x'\in[0,1]\backslash S$,

$$x<x'\Rightarrow\mathtt{i}(x)\prec\mathtt{i}(x').$$

Define $\Sigma_T := \overline{\mathtt{i}([0,1]\backslash S)}$. We now introduce the φ-expansion as defined by Parry. For all $j \in \mathtt{A}$, let $\varphi^j : [j, j+1] \to [a_j, a_{j+1}]$ be the unique monotone extension of $f_j^{-1} : (c, d) \to (a_j, a_{j+1})$, where $(c, d) := f_j((a_j, a_{j+1}))$. The map $\varphi : \Sigma_k \to [0, 1]$ is defined by

$$\varphi(\underline{x}) = \lim_{n\to\infty} \varphi^{x_0}(x_0 + \varphi^{x_1}(x_1 + \cdots + \varphi^{x_n}(x_n))).$$

Parry proved that this limit exists if $\mathtt{i}$ is injective. The map φ is order-preserving; moreover, $\varphi|_{\mathtt{i}([0,1]\backslash S)} = \mathtt{i}^{-1}$ and, for all $n \geq 0$ and all $x \in [0, 1]\backslash S$,

$$T^n(x) = \varphi \circ \sigma^n \circ \mathtt{i}(x). \tag{3}$$

If the coding map is injective, one can show that the map φ is continuous (see [**6**, Theorem 2.3]). Using the continuity and monotonicity of φ, we have $\varphi(\Sigma_T) = [0, 1]$. We remark that there is, in general, no extension of $\mathtt{i}$ on $[0, 1]$ such that equation (3) would be valid on $[0, 1]$. For all $j \in \mathtt{A}$, define

$$\underline{u}^j := \lim_{x\downarrow a_j} \mathtt{i}(x) \quad \text{and} \quad \underline{v}^j := \lim_{x\uparrow a_{j+1}} \mathtt{i}(x) \quad \text{with } x \in [0, 1]\backslash S.$$

The strings $\underline{u}^j$ and $\underline{v}^j$ are called critical orbits and (see, for instance, [**10**])

$$\Sigma_T = \{\underline{x} \in \Sigma_k \mid \underline{u}^{x_n} \preceq \sigma^n \underline{x} \preceq \underline{v}^{x_n} \ \forall n \geq 0\}. \tag{4}$$

Moreover, the critical orbits $\underline{u}^j$ and $\underline{v}^j$ satisfy, for all $j \in \mathtt{A}$,

$$\begin{cases} \underline{u}^{u_n^j} \preceq \sigma^n \underline{u}^j \preceq \underline{v}^{u_n^j} \\ \underline{u}^{v_n^j} \preceq \sigma^n \underline{v}^j \preceq \underline{v}^{v_n^j} \end{cases} \quad \text{for all } n \geq 0. \tag{5}$$

Let us recall the construction of the Hausdorff dimension. Let (X, d) be a metric space and consider $E \subset X$. Let $\mathcal{D}_\varepsilon(E)$ be the set of all finite or countable covers of E with sets of diameter smaller than ε. For all $s \geq 0$, define

$$H_\varepsilon(E, s) := \inf\left\{\sum_{B\in\mathcal{C}} (\operatorname{diam} B)^s \,\middle|\, \mathcal{C} \in \mathcal{D}_\varepsilon(E)\right\}$$

and $H(E, s) := \lim_{\varepsilon\to 0} H_\varepsilon(E, s)$, the s-Hausdorff measure of E. The Hausdorff dimension of E is

$$\dim_H E := \inf\{s \geq 0 \mid H(E, s) = 0\}.$$

In [**1**], Bowen introduced a definition of the topological entropy of non-compact sets for a continuous dynamical system on a metric space. We now recall this definition. Let (X, d) be a metric space and $T : X \to X$ a continuous map. For $n \geq 1$, $\varepsilon > 0$ and $x \in X$, let

$$B_n(x, \varepsilon) = \{y \in X \mid d(T^j(x), T^j(y)) < \varepsilon \ \forall j = 0, \ldots, n-1\}.$$

For $E \subset X$ such that $T(E) \subset E$, let $\mathcal{G}_n(E, \varepsilon)$ be the set of all finite or countable covers of E with Bowen's balls $B_m(x, \varepsilon)$ for $m \geq n$. For all $s \geq 0$, define

$$C_n(E, \varepsilon, s) := \inf\left\{\sum_{B_m(x,\varepsilon)\in\mathcal{C}} e^{-ms} \,\middle|\, \mathcal{C} \in \mathcal{G}_n(x, \varepsilon)\right\}$$

and $C(E, \varepsilon, s) := \lim_{n\to\infty} C_n(E, \varepsilon, s)$. Now, let

$$h_{\text{top}}(E, \varepsilon) := \inf\{s \geq 0 \mid C(E, \varepsilon, s) = 0\}$$

and, finally, take $h_{\text{top}}(E) = \lim_{\varepsilon\to 0} h_{\text{top}}(E, \varepsilon)$ (this last quantity increases to the limit $h_{\text{top}}(E)$). There is an evident similarity between this definition and that of the Hausdorff dimension; this similarity is the key to the next lemma.

LEMMA 1. *For $\beta > 1$, consider the dynamical system $(\Sigma_k, d_\beta, \sigma)$. Let $E \subset \Sigma_k$ be such that $\sigma(E) \subset E$; then*

$$\dim_H E \leq \frac{h_{\text{top}}(E)}{\log \beta}.$$

Proof. Let $\varepsilon \in (0, 1)$, $s \geq 0$, $n \geq 0$ and $\mathcal{C} \in \mathcal{G}_n(E, \varepsilon)$. Since $\operatorname{diam} B_m(x, \varepsilon) \leq \varepsilon\beta^{-m+1} \leq \varepsilon\beta^{-n+1}$ for all $B_m(x, \varepsilon) \in \mathcal{C}$, $\mathcal{C}$ is a cover of E with sets of diameter smaller than $\varepsilon\beta^{-n+1}$. Moreover,

$$\sum_{B_m(x,\varepsilon)\in\mathcal{C}} \operatorname{diam}\,(B_m(x, \varepsilon))^{s/\log\beta} \leq (\varepsilon\beta)^{s/\log\beta} \sum_{B_m(x,\varepsilon)\in\mathcal{C}} e^{-ms}.$$

Thus, $H_\delta(E, s/\log\beta) \leq (\varepsilon\beta)^{s/\log\beta} C_n(E, \varepsilon, s)$ with $\delta = \varepsilon\beta^{-n+1}$. Taking the limit $n \to \infty$, we obtain

$$H(E, s/\log\beta) \leq (\varepsilon\beta)^{s/\log\beta} C(E, \varepsilon, s).$$

If $s > h_{\text{top}}(E, \varepsilon)$, then $H(E, s/\log\beta) = 0$ and $s/\log\beta \geq \dim_H E$. This is true for all $s > h_{\text{top}}(E, \varepsilon)$; thus

$$\dim_H E \leq \frac{h_{\text{top}}(E, \varepsilon)}{\log\beta} \leq \frac{h_{\text{top}}(E)}{\log\beta}. \qquad \square$$

The next lemma is a classical result about the Hausdorff dimension; it is [**4**, Proposition 2.3].

LEMMA 2. *Let (X, d), (X', d') be two metric spaces and let $\rho : X \to X'$ be an α-Hölder continuous map with $\alpha \in (0, 1]$. Let $E \in X$; then*

$$\dim_H \rho(E) \leq \frac{\dim_H E}{\alpha}.$$

Finally, we restate [**14**, Theorem 4.1]. This theorem will be used to estimate the topological entropy of the sets we are interested in.

THEOREM 1. *Let (X, d, T) be a continuous dynamical system and let $F \subset M(X, T)$ be a closed subset. Define*

$$G := \{x \in X \mid V_T(x) \cap F \neq \emptyset\}.$$

Then

$$h_{\text{top}}(G) \leq \sup_{\nu\in F} h(\nu).$$

3. *Normality for the maps $\beta x + \alpha$ mod1*

In this section, we study the piecewise monotone continuous maps $T_{\alpha,\beta}$ defined by $T_{\alpha,\beta}(x) = \beta x + \alpha$ mod 1, with $\beta > 1$ and $\alpha \in [0, 1)$. These maps were studied by Parry in [**13**] as a generalization of the β-transformations. In his paper Parry constructed a $T_{\alpha,\beta}$-invariant probability measure $\mu_{\alpha,\beta}$ which is absolutely continuous with respect to Lebesgue measure. Its density is

$$h_{\alpha,\beta}(x) := \frac{d\mu_{\alpha,\beta}}{d\lambda}(x) = \frac{1}{N_{\alpha,\beta}} \frac{\sum_{n\geq 0} 1_{x<T^n_{\alpha,\beta}(1)} - \sum_{n\geq 0} 1_{x<T^n_{\alpha,\beta}(0)}}{\beta^{n+1}}, \tag{6}$$

with $N_{\alpha,\beta}$ being the normalization factor. In [**8**], Halfin proved that $h_{\alpha,\beta}(x)$ is non-negative for all $x \in [0, 1]$. Let $\mathtt{i}^{\alpha,\beta}$ denote the coding map under $T_{\alpha,\beta}$, and $\varphi^{\alpha,\beta}$ the corresponding φ-expansion. Also define $\Sigma_{\alpha,\beta} := \Sigma_{T_{\alpha,\beta}} \subset \Sigma_k$ with $k := \lceil \alpha + \beta \rceil$, $\underline{u}^{\alpha,\beta} := \lim_{x\downarrow 0} \mathtt{i}^{\alpha,\beta}(x)$, and $\underline{v}^{\alpha,\beta} := \lim_{x\uparrow 1} \mathtt{i}^{\alpha,\beta}(x)$. We specify how $T_{\alpha,\beta}$ is defined at the discontinuity points. We choose to define $T_{\alpha,\beta}$ by right-continuity at $a_j \in S_0$. By doing this, we can also extend the definition of the coding map $\mathtt{i}^{\alpha,\beta}$ using the disjoint intervals $[a_j, a_{j+1})$ for $j \in \mathtt{A}$, so that $\mathtt{i}^{\alpha,\beta}$ is now defined for all $x \in [0, 1)$†. We can show that $\underline{u}^{\alpha,\beta} = \mathtt{i}^{\alpha,\beta}(0)$ and

$$\mathtt{i}([0, 1)) = \{\underline{x} \in \Sigma_k \mid \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{x} \prec \underline{v}^{\alpha,\beta} \ \forall n \geq 0\}$$

and that equation (3) is true for all $x \in [0, 1)$. It is easy to check that formula (4) becomes

$$\Sigma_{\alpha,\beta} = \{\underline{x} \in \Sigma_k \mid \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{x} \preceq \underline{v}^{\alpha,\beta} \ \forall n \geq 0\} \tag{7}$$

and that the inequalities (5) become

$$\begin{cases} \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{u}^{\alpha,\beta} \preceq \underline{v}^{\alpha,\beta} \\ \underline{u}^{\alpha,\beta} \preceq \sigma^n \underline{v}^{\alpha,\beta} \preceq \underline{v}^{\alpha,\beta} \end{cases} \quad \text{for all } n \geq 0. \tag{8}$$

It is known that the dynamical system $(\Sigma_{\alpha,\beta}, \sigma)$ has topological entropy $\log \beta$. Moreover, Hofbauer showed in [**11**] that it has a unique measure $\hat{\mu}_{\alpha,\beta}$ of maximal entropy, that $\mu_{\alpha,\beta} = \hat{\mu}_{\alpha,\beta} \circ (\varphi^{\alpha,\beta})^{-1}$ and $\mu_{\alpha,\beta}$ is the unique measure of maximal entropy for $T_{\alpha,\beta}$. In view of (7) and (8), for a pair $(\underline{u}, \underline{v}) \in \Sigma_k^2$ satisfying

$$\begin{cases} \underline{u} \preceq \sigma^n \underline{u} \preceq \underline{v} \\ \underline{u} \preceq \sigma^n \underline{v} \preceq \underline{v} \end{cases} \quad \text{for all } n \geq 0, \tag{9}$$

we define the shift space

$$\Sigma_{\underline{u},\underline{v}} := \{\underline{x} \in \Sigma_k \mid \underline{u} \preceq \sigma^n \underline{x} \preceq \underline{v} \ \forall n \geq 0\}. \tag{10}$$

We now give a lemma and a proposition which will be the keys to the main theorem of this section. The lemma says that for given x and α, there is exponential separation between the orbits of x under the two different dynamical systems T_{α,β_1} and T_{α,β_2}. The proposition asserts that the topological entropy of $\Sigma_{\underline{u},\underline{v}}$ is upper semi-continuous with respect to the critical orbits $\underline{u}$ and $\underline{v}$.

† This convention differs from that made in the previous section; it is, however, the most convenient choice when all the f_j are increasing.

LEMMA 3. *Let $x \in [0, 1)$, $\alpha \in [0, 1)$ and $1 < \beta_1 \le \beta_2$. Define $l = \min\{n \ge 0 \mid \mathtt{i}_n^1(x) \ne \mathtt{i}_n^2(x)\}$ with $\mathtt{i}^j(x) = \mathtt{i}^{\alpha,\beta_j}$ for $j = 1, 2$. If $x \ne 0$, then*

$$\beta_2 - \beta_1 \le \frac{\beta_2}{x}\beta_2^{-l}.$$

If $x = 0$ and $\alpha \ne 0$, then

$$\beta_2 - \beta_1 \le \frac{\beta_2^2}{\alpha}\beta_2^{-l}.$$

Proof. Let $\delta := \beta_2 - \beta_1 \ge 0$. We prove by induction that for all $m \ge 1$, $\mathtt{i}_{[0,m)}^1(x) = \mathtt{i}_{[0,m)}^2(x)$ implies

$$T_2^m(x) - T_1^m(x) \ge \beta_2^{m-1}\delta x,$$

where $T_i = T_{\alpha,\beta_i}$. For $m = 1$,

$$T_2(x) - T_1(x) = \beta_2 x + \alpha - \mathtt{i}_0^2(x) - (\beta_1 x + \alpha - \mathtt{i}_0^1(x)) = \delta x.$$

Now suppose that the statement is true for a certain m; then $\mathtt{i}_{[0,m+1)}^1 = \mathtt{i}_{[0,m+1)}^2$ implies

$$\begin{aligned} T_2^{m+1}(x) - T_1^{m+1}(x) &= \beta_2 T_2^m(x) + \alpha - \mathtt{i}_m^2(x) - (\beta_1 T_1^m(x) + \alpha - \mathtt{i}_m^1(x)) \\ &= \beta_2(T_2^m(x) - T_1^m(x)) + \delta T_1^m(x) \ge \beta_2^m \delta x. \end{aligned}$$

On the other hand, $1 \ge T_2^m(x) - T_1^m(x) \ge \beta_2^{m-1}\delta x$. Thus $\delta \le \beta_2^{-m+1}/x$ for all m such that $\mathtt{i}_{[0,m)}^1 = \mathtt{i}_{[0,m)}^2$. If $x = 0$, then $T_1(x) = T_2(x) = \alpha$ and we can apply the first statement to $y = \alpha > 0$. □

PROPOSITION 1. *Let the pair $(\underline{u}, \underline{v}) \in \Sigma_k^2$ satisfy (9). For all $\delta > 0$, there exists $L(\delta, \underline{u}, \underline{v})$ such that for all $L \ge L(\delta, \underline{u}, \underline{v})$ the following claim is true: let the pair $(\underline{u}', \underline{v}') \in \Sigma_k^2$ satisfy (9), and suppose further that $\underline{u}$, $\underline{u}'$ have a common prefix of length L and that $\underline{v}$, $\underline{v}'$ have a common prefix of length L; then*

$$h_{\text{top}}(\Sigma_{\underline{u}',\underline{v}'}) \le h_{\text{top}}(\Sigma_{\underline{u},\underline{v}}) + \delta.$$

To prove Proposition 1, one associates to the subshift $\Sigma_{\underline{u},\underline{v}}$ a graph $\mathcal{G}(\underline{u}, \underline{v})$, called the Markov diagram [**11**]. One then proves a property equivalent to Proposition 1 for these graphs; see Appendix A.

We now state our first theorem and a corollary about the normality of orbits under $T_{\alpha,\beta}$. The proof of the theorem is inspired by the proof of [**17**, Theorem C], where the case of $x = 1$ and $\alpha = 0$ is considered.

THEOREM 2. *Take any $x \in [0, 1)$ and $\alpha \in [0, 1)$ except for $(x, \alpha) = (0, 0)$. Then the set*

$$\{\beta > 1 \mid \text{the orbit of } \mathtt{i}^{\alpha,\beta}(x) \text{ under } \sigma \text{ is } \hat{\mu}_{\alpha,\beta}\text{-normal}\}$$

has full λ-measure.

COROLLARY 1. *Take any $x \in [0, 1)$ and $\alpha \in [0, 1)$ except for $(x, \alpha) = (0, 0)$. Then the set*

$$\{\beta > 1 \mid \text{the orbit of } x \text{ under } T_{\alpha,\beta} \text{ is } \mu_{\alpha,\beta}\text{-normal}\}$$

has full λ-measure.

We remark that the theorem and its corollary may also be formulated for $x \in (0, 1]$ by using a left-continuous extension of $T_{\alpha,\beta}$ on $(0, 1]$ and a coding $\mathtt{i}^{\alpha,\beta}$ defined using intervals $(a_j, a_{j+1}]$ for $j \in \mathbb{A}$.

Proof of Theorem 2. We briefly sketch the proof. It is sufficient to consider a finite interval $[\underline{\beta}, \overline{\beta}]$. We use the uniqueness of the measure $\hat{\mu}_{\alpha,\beta}$ of maximal entropy: for a $\underline{x} \in \Sigma_{\alpha,\beta}$ which is not $\hat{\mu}_{\alpha,\beta}$-normal, there exists $\nu \in V_\sigma(\underline{x})$ such that $h(\nu) < h(\hat{\mu}_{\alpha,\beta}) = \log \beta$. We therefore cover the set of abnormal β in $[\underline{\beta}, \overline{\beta}]$ by sets Ω_N, $N \in \mathbb{N}$, given by

$$\Omega_N := \{\beta \in [\underline{\beta}, \overline{\beta}] \mid \{\mathcal{E}_n(\mathtt{i}^{\alpha,\beta}(x))\}_n \text{ clusters on } \nu \text{ with } h(\nu) < (1 - 1/N) \log \beta\}.$$

We consider each Ω_N separately and cover them by appropriate intervals, which we denote generically by $[\beta_1, \beta_2]$. The main idea is to imbed $\{\mathtt{i}^{\alpha,\beta}(x) : \beta \in [\beta_1, \beta_2]\}$ in a shift space $\Sigma^* := \Sigma_{\underline{u}^*,\underline{v}^*}$ with $\underline{u}^*$ and $\underline{v}^*$ suitably chosen. Writing $D^* \subset \Sigma^*$ for the range of the imbedding, we estimate the Hausdorff dimension of the subset of D^* corresponding to points $\mathtt{i}^{\alpha,\beta}(x)$ which are not $\hat{\mu}_{\alpha,\beta}$-normal. Then we estimate the coefficient of Hölder continuity of the map ρ_* defined as the inverse of the imbedding. This gives us an estimate of the Hausdorff dimension of the non-$\hat{\mu}_{\alpha,\beta}$-normal points in the interval $[\beta_1, \beta_2]$.

To obtain uniform estimates, we restrict our proof to the interval $[\underline{\beta}, \overline{\beta}]$ with $1 < \underline{\beta} < \overline{\beta} < \infty$. All shift spaces below will be subshifts of Σ_k with $k = \lceil \alpha + \overline{\beta} \rceil$. Let $\Omega := \{\beta \in [\underline{\beta}, \overline{\beta}] \mid \mathtt{i}^{\alpha,\beta}(x) \text{ is not } \hat{\mu}_{\alpha,\beta}\text{-normal}\}$. For $\beta \in \Omega$, we have $V_\sigma(\mathtt{i}^{\alpha,\beta}(x)) \neq \{\hat{\mu}_{\alpha,\beta}\}$. Since $\hat{\mu}_{\alpha,\beta}$ is the unique $T_{\alpha,\beta}$-invariant measure of maximal entropy $\log \beta$, there exist $N \in \mathbb{N}$ and $\nu \in V_\sigma(\mathtt{i}^{\alpha,\beta}(x))$ such that $h(\nu) < (1 - 1/N) \log \beta$. Setting

$$\Omega_N := \{\beta \in [\underline{\beta}, \overline{\beta}] \mid \exists \nu \in V_\sigma(\mathtt{i}^{\alpha,\beta}(x)) \text{ s.t. } h(\nu) < (1 - 1/N) \log \beta\},$$

we have $\Omega = \bigcup_{N \geq 1} \Omega_N$. We will prove that $\dim_H \Omega_N < 1$, so that $\lambda(\Omega_N) = 0$ for all $N \geq 1$.

For $N \in \mathbb{N}$ fixed, define $\varepsilon := \underline{\beta} \log \underline{\beta}/(2N - 1) > 0$ and $\delta := \log(1 + \varepsilon/\overline{\beta})$. Let $\beta \in [\underline{\beta}, \overline{\beta}]$ and define $L_\beta = L(\delta/2, \underline{u}^{\alpha,\beta}, \underline{v}^{\alpha,\beta})$ as in Proposition 1. Choose q_β in $\mathbb{Q}$ such that $\log \beta - \delta/2 \leq \log q_\beta \leq \log \beta$. Let

$$J(\beta, L_\beta, q_\beta) := \{\beta' \in [q_\beta, \overline{\beta}] \mid \underline{u}^{\alpha,\beta'}_{[0,L_\beta)} = \underline{u}^{\alpha,\beta}_{[0,L_\beta)}, \underline{v}^{\alpha,\beta'}_{[0,L_\beta)} = \underline{v}^{\alpha,\beta}_{[0,L_\beta)}\}.$$

This set is an interval: if $\beta' \in J(\beta, L_\beta, q_\beta)$ and $\beta' < \beta'' \in J(\beta, L_\beta, q_\beta)$, then $[\beta', \beta''] \subset J(\beta, L_\beta, q_\beta)$ since the maps $\beta' \mapsto \underline{u}^{\alpha,\beta'}$ and $\beta' \mapsto \underline{v}^{\alpha,\beta'}$ are both monotone increasing. Moreover, $\beta \in J(\beta, L_\beta, q_\beta)$. Notice also that the family $\{J(\beta, L_\beta, q_\beta) \mid \beta \in [\underline{\beta}, \overline{\beta}]\}$ is countable. Indeed, the interval $J(\beta, L_\beta, q_\beta)$ is entirely characterized by $\underline{u}^{\alpha,\beta}_{[0,L_\beta)}$, $\underline{v}^{\alpha,\beta}_{[0,L_\beta)}$ and q_β. But there are only countably many triples in $\mathbb{A}^* \times \mathbb{A}^* \times \mathbb{Q}$. Thus $\{J(\beta, L_\beta, q_\beta) \mid \beta \in [\underline{\beta}, \overline{\beta}]\}$ is a countable cover of $[\underline{\beta}, \overline{\beta}]$. To prove that $\lambda(\Omega_N) = 0$, it is sufficient to prove that $\lambda(\Omega_N \cap J(\beta, L_\beta, q_\beta)) = 0$ for all $\beta \in [\underline{\beta}, \overline{\beta}]$. The interval $J(\beta, L_\beta, q_\beta)$ may be open, closed, or neither open nor closed. We need to work on a closed interval, thus we prove an equivalent result, namely that for any closed interval $[\beta_1, \beta_2] \subset J(\beta, L_\beta, q_\beta)$ we have $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$.

Let $\underline{u}^j = \underline{u}^{\alpha,\beta_j}$ and $\underline{v}^j = \underline{v}^{\alpha,\beta_j}$. Using (8) and the monotonicity of $\beta \mapsto \underline{u}^{\alpha,\beta}$ and $\beta \mapsto \underline{v}^{\alpha,\beta}$, we have

$$\begin{aligned} \underline{u}^1 \preceq \sigma^n \underline{u}^1 \preceq \underline{v}^1 \preceq \underline{v}^2 \\ \underline{u}^1 \preceq \underline{u}^2 \preceq \sigma^n \underline{v}^2 \preceq \underline{v}^2 \end{aligned} \qquad \text{for all } n \geq 0.$$

Hence the pair $(\underline{u}^1, \underline{v}^2)$ satisfies (9), and we set $\Sigma^* = \Sigma_{\underline{u}^1,\underline{v}^2}$ and

$$D^* := \{\underline{z} \in \Sigma^* \mid \exists \beta \in [\beta_1, \beta_2] \text{ s.t. } \underline{z} = \mathtt{i}^{\alpha,\beta}(x)\}.$$

We define an map $\rho_* : D^* \to [\beta_1, \beta_2]$ by $\rho_*(\underline{z}) = \beta \Leftrightarrow \mathtt{i}^{\alpha,\beta}(x) = \underline{z}$. This map is well defined: by definition of D^*, for all $\underline{z} \in D^*$ there exists a β such that $\underline{z} = \mathtt{i}^{\alpha,\beta}(x)$; moreover, this β is unique, since by Lemma 3 $\beta \mapsto \mathtt{i}^{\alpha,\beta}(x)$ is strictly increasing. On the other hand, for all $\beta \in [\beta_1, \beta_2]$, we have, from (7),

$$\underline{u}^1 \preceq \underline{u}^{\alpha,\beta} \preceq \sigma^n \mathtt{i}^{\alpha,\beta}(x) \preceq \underline{v}^{\alpha,\beta} \preceq \underline{v}^2 \quad \text{for all } n \geq 0,$$

whence $\mathtt{i}^{\alpha,\beta}(x) \in \Sigma^*$ and $\rho_* : D^* \to [\beta_1, \beta_2]$ is surjective. Let $\log \beta_* := h_{\text{top}}(\Sigma^*)$; then, by Proposition 1,

$$\log \beta^* = h_{\text{top}}(\Sigma^*) \leq h_{\text{top}}(\Sigma_{\alpha,\beta}) + \delta/2 = \log \beta + \delta/2.$$

By definition of q_β, we have $\log \beta - \delta/2 \leq \log q_\beta \leq \log \beta_1$; thus $\log \beta^* \leq \log \beta_1 + \delta$ and

$$\beta_* - \beta_1 \leq \beta_1(\mathrm{e}^\delta - 1) \leq \varepsilon. \tag{11}$$

Let us compute the coefficient of Hölder continuity of $\rho_* : (D^*, d_{\beta_*}) \to [\beta_1, \beta_2]$. Take $\underline{z} \neq \underline{z}' \in D^*$ and $n = \min\{l \geq 0 \mid z_l \neq z_l'\}$; then $d_{\beta_*}(\underline{z}, \underline{z}') = \beta_*^{-n}$. By Lemma 3, there exists C such that

$$|\rho_*(\underline{z}) - \rho_*(\underline{z}')| \leq C\rho_*(\underline{z})^{-n} \leq C\beta_1^{-n} = C(d_{\beta_*}(\underline{z}, \underline{z}'))^{\log \beta_1/\log \beta_*},$$

where

$$C = \max\left\{\frac{\overline{\beta}}{x}, \frac{\overline{\beta}^2}{\alpha}\right\}.$$

By equation (11) and the choice of ε, we have

$$\begin{aligned} \beta_* - \beta_1 \leq \frac{\underline{\beta} \log \underline{\beta}}{2N-1} &\Rightarrow \beta_* - \beta_1 \leq \frac{\beta_1 \log \beta_1}{2N-1} \\ &\Leftrightarrow 1 + \frac{\beta_* - \beta_1}{\beta_1 \log \beta_1} \leq 1 + \frac{1}{2N-1} \\ &\Leftrightarrow \frac{\log \beta_1 + (\beta_* - \beta_1)/\beta_1}{\log \beta_1} \leq \frac{2N}{2N-1} \\ &\Rightarrow \frac{\log \beta_1}{\log \beta_*} \geq \frac{\log \beta_1}{\log \beta_1 + (\beta_* - \beta_1)/\beta_1} \geq 1 - \frac{1}{2N}. \end{aligned}$$

In the last line, we have used the concavity of the logarithm, so the first-order Taylor development is an upper estimate. Thus ρ_* has Hölder-exponent $1 - 1/(2N)$.

Define

$$G_N^* := \{\underline{z} \in \Sigma^* \mid \exists \nu \in V_\sigma(\underline{z}) \text{ s.t. } h(\nu) < (1 - 1/N) \log \beta_*\}.$$

Let $\beta \in \Omega_N \cap [\beta_1, \beta_2]$. Then there exists $\nu \in V_\sigma(\mathtt{i}^{\alpha,\beta}(x))$ such that

$$h(\nu) < (1 - 1/N) \log \beta \leq (1 - 1/N) \log \beta_*.$$

Since $\mathtt{i}^{\alpha,\beta}(x) \in D^* \subset \Sigma^*$, we have $\mathtt{i}^{\alpha,\beta}(x) \in G_N^*$. Using the surjectivity of ρ_*, we obtain $\Omega_N \cap [\beta_1, \beta_2] \subset \rho_*(G_N^* \cap D^*)$. We claim that $h_{\text{top}}(G_N^*) \leq (1 - 1/N) \log \beta_*$. This implies, using Lemmas 2 and 1, that

$$\dim_H(\Omega_N \cap [\beta_1, \beta_2]) \leq \dim_H \rho_*(G_N^* \cap D^*) \leq \frac{\dim_H G_N^*}{1 - 1/2N} \leq \frac{h_{\text{top}}(G_N^*)}{(1 - 1/2N) \log \beta_*} \leq \frac{1 - 1/N}{1 - 1/2N} < 1.$$

Thus $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$.

It remains to prove that $h_{\text{top}}(G_N^*) \leq (1 - 1/N) \log \beta_*$. Recall that $h(\nu) = \lim_n H_n(\nu)/n$, where $H_n(\nu)$ is the entropy of ν with respect to the algebra $\mathcal{A}_n$ of cylinder sets of length n:

$$H_n(\nu) = -\sum_{[\underline{w}] \in \mathcal{A}_n} \nu([\underline{w}]) \log \nu([\underline{w}]).$$

Since the cylinders are both open and closed, $\nu \mapsto H_n(\nu)$ is continuous in the weak*-topology. Moreover, $H_n(\nu)/n$ is decreasing in n. For all $m \geq 1$, we set

$$F_N^*(m) := \left\{ \nu \in M(\Sigma^*, \sigma) \;\middle|\; \frac{1}{m} H_m(\nu) \leq (1 - 1/N) \log \beta_* \right\}$$
$$G_N^*(m) := \{\underline{z} \in \Sigma^* \mid V_\sigma(\underline{z}) \cap F_N^*(m) \neq \emptyset\}.$$

Let $\underline{z} \in G_N^*$; then there exists $\nu \in V_\sigma(\underline{z})$ such that $h(\nu) < (1 - 1/N) \log \beta_*$. Since $H_m(\nu)/m \downarrow h(\nu)$, there exists $m \geq 1$ such that $H_m(\nu)/m \leq (1 - 1/N) \log \beta_*$, whence $\nu \in F_n^*(m)$ and $\underline{z} \in G_N^*(m)$. This implies that $G_N^* \subset \bigcup_{m \geq 1} G_N^*(m)$. Since $H_m(\cdot)$ is continuous, $F_N^*(m)$ is closed for all $m \geq 1$. Finally, by using Theorem 1 we obtain

$$h_{\text{top}}(G_N^*) = \sup_m h_{\text{top}}(G_N^*(m)) \leq \sup_m \sup_{\nu \in F_N^*(m)} h(\nu) \leq \sup_m \sup_{\nu \in F_N^*(m)} \frac{1}{m} H_m(\nu) \leq (1 - 1/N) \log \beta_*. \qquad \square$$

Proof of Corollary 1. Let $\beta > 1$ be such that the orbit of $\mathtt{i}^{\alpha,\beta}(x)$ under σ is $\hat{\mu}_{\alpha,\beta}$-normal. Let $f \in C([0, 1])$; then $\hat{f} : \Sigma_{\alpha,\beta} \to \mathbb{R}$ defined by $\hat{f} := f \circ \varphi^{\alpha,\beta}$ is continuous, since $\varphi^{\alpha,\beta}$ is continuous. Using $\mu_{\alpha,\beta} := \hat{\mu}_{\alpha,\beta} \circ (\varphi^{\alpha,\beta})^{-1}$, we have

$$\int_{[0,1]} f \, d\mu_{\alpha,\beta} = \int_{\Sigma_{\alpha,\beta}} \hat{f} \, d\hat{\mu}_{\alpha,\beta} = \lim_{n \to \infty} \sum_{i=0}^{n-1} \hat{f}(\sigma^i \mathtt{i}^{\alpha,\beta}(x)) = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(\varphi^{\alpha,\beta}(\sigma^i \mathtt{i}^{\alpha,\beta}(x))) = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(T_{\alpha,\beta}^i(x)).$$

The second equality comes from the $\hat{\mu}_{\alpha,\beta}$-normality of the orbit of $\mathtt{i}^{\alpha,\beta}(x)$ under σ, while the last one follows from (3), which is true for all $x \in [0, 1)$ with our convention for the extension of $T_{\alpha,\beta}$ and $\mathtt{i}^{\alpha,\beta}$ on $[0, 1)$. $\square$

The next step is to consider the question of $\mu_{\alpha,\beta}$-normality in the whole (α, β)-plane instead of working with α fixed. Define $\mathcal{R} := [0, 1) \times (1, \infty)$.

THEOREM 3. *For all* $x \in [0, 1)$, *the set*

$$\mathcal{N}(x) := \{(\alpha, \beta) \in \mathcal{R} \mid \textit{the orbit of } x \textit{ under } T_{\alpha,\beta} \textit{ is } \mu_{\alpha,\beta}\textit{-normal}\}$$

has full two-dimensional Lebesgue measure.

Proof. We need only prove that $\mathcal{N}(x)$ is measurable and apply Fubini's theorem and Corollary 1. The first step is to prove that for all $x \in [0, 1)$ and all $n \geq 0$, the maps $(\alpha, \beta) \mapsto \mathtt{i}^{\alpha,\beta}(x)$ and $(\alpha, \beta) \mapsto T^n_{\alpha,\beta}(x)$ are measurable. First, observe that for all $n \geq 1$,

$$T^n_{\alpha,\beta}(x) = \beta^n x + \alpha \frac{\beta^n - 1}{\beta - 1} - \sum_{j=0}^{n-1} \mathtt{i}^{\alpha,\beta}_j(x)\, \beta^{n-j-1}. \tag{12}$$

The proof by induction is immediate. To prove that $(\alpha, \beta) \mapsto \mathtt{i}^{\alpha,\beta}(x)$ is measurable, it is enough to prove that for all $n \geq 0$ and for all words $\underline{w} \in \mathbb{A}^*$ of length n,

$$\{(\alpha, \beta) \in \mathcal{R} \mid \mathtt{i}^{\alpha,\beta}_{[0,n)}(x) = \underline{w}\}$$

is measurable, since the σ-algebra on Σ_k is generated by the cylinders. This set is the subset of $\mathbb{R}^2$ such that

$$\begin{cases} \beta > 1, \\ 0 \leq \alpha < 1, \\ w_j < \beta T^j_{\alpha,\beta}(x) + \alpha \leq w_j + 1 \quad \text{for } 0 \leq j < n. \end{cases}$$

Using (12), this system of inequalities can be rewritten as

$$\begin{cases} \beta > 1, \\ 0 \leq \alpha < 1, \\ \alpha > \dfrac{\beta - 1}{\beta^{j+1} - 1}\left(\displaystyle\sum_{i=0}^{j} w_i \beta^{j-i} - \beta^{j+1} x\right) & \text{for } 0 \leq j < n, \\ \alpha \leq \dfrac{\beta - 1}{\beta^{j+1} - 1}\left(1 + \displaystyle\sum_{i=0}^{j} w_i \beta^{j-i} - \beta^{j+1} x\right) & \text{for } 0 \leq j < n. \end{cases}$$

From this, the measurability of $\mathtt{i}^{\alpha,\beta}$ follows. If $(\alpha, \beta) \mapsto \mathtt{i}^{\alpha,\beta}(x)$ is measurable, then by formula (12), $(\alpha, \beta) \mapsto T^n_{\alpha,\beta}(x)$ is clearly measurable for all $n \geq 0$. Then, for all $f \in C([0, 1])$ and all $n \geq 1$, the map $(\alpha, \beta) \mapsto S_n(f) := \sum_{i=0}^{n-1} f(T^i_{\alpha,\beta}(x))/n$ is measurable and, consequently,

$$\left\{(\alpha, \beta) \,\middle|\, \lim_{n\to\infty} S_n(f) \text{ exists}\right\}$$

is a measurable set.

On the other hand, if $f \in C([0, 1])$, then $(\alpha, \beta) \mapsto \int f\, d\mu_{\alpha,\beta}$ is measurable. Indeed,

$$\int f\, d\mu_{\alpha,\beta} = \int f h_{\alpha,\beta}\, d\lambda$$

and, in view of equation (6) and the measurability of $(\alpha, \beta) \mapsto T_{\alpha,\beta}(x)$, the map $(\alpha, \beta) \mapsto h_{\alpha,\beta}$ is clearly measurable. Therefore

$$\left\{(\alpha, \beta) \,\middle|\, \lim_{n\to\infty} S_n(f) = \int f \, d\mu_{\alpha,\beta}\right\}$$

is measurable for all $f \in C([0, 1])$. Let $\{f_m\}_{m\in\mathbb{N}} \subset C([0, 1])$ be a countable subset which is dense with respect to uniform convergence. Then, setting

$$D_m := \left\{(\alpha, \beta) \in \mathcal{R} \,\middle|\, \lim_{n\to\infty} S_n(f_m) = \int f_m \, d\mu_{\alpha,\beta}\right\},$$

we have $\mathcal{N}(x) = \bigcap_{m\in\mathbb{N}} D_m$, whence it is a measurable set. □

We have shown that for a given $x \in [0, 1)$, the orbit of x under $T_{\alpha,\beta}$ is $\mu_{\alpha,\beta}$-normal for almost all (α, β). The orbits of 0 and 1 are of particular interest; see equation (6). Now we show that through any point (α_0, β_0), there passes a curve defined by $\alpha = \alpha(\beta)$ such that the orbit of 0 under $T_{\alpha(\beta),\beta}$ is $\mu_{\alpha(\beta),\beta}$-normal for at most one β. A trivial example of such a curve is $\alpha = 0$, since $x = 0$ is a fixed point. The idea is to consider curves along which the coding of 0 is constant, i.e. to define $\alpha(\beta)$ such that $\underline{u}^{\alpha(\beta),\beta}$ is constant. The results below depend on reference [**6**], where we solve the following inverse problem: given $\underline{u}$ and $\underline{v}$ satisfying (9), can we find α, β such that $\underline{u} = \underline{u}^{\alpha,\beta}$ and $\underline{v} = \underline{v}^{\alpha,\beta}$?

Let

$$\mathcal{U} := \{\underline{u} \mid \exists\, (\alpha, \beta) \in \mathcal{R} \text{ s.t. } \underline{u} = \underline{u}^{\alpha,\beta}\}.$$

We define an equivalence relation in $\mathcal{R}$ by

$$(\alpha, \beta) \sim (\alpha', \beta') \iff \underline{u}^{\alpha,\beta} = \underline{u}^{\alpha',\beta'}.$$

An equivalence class is denoted by $[\underline{u}]$. The next lemma describes $[\underline{u}]$.

LEMMA 4. *Let $\underline{u} \in \mathcal{U}$ and set*

$$\alpha(\beta) = (\beta - 1) \sum_{j\geq 0} \frac{u_j}{\beta^{j+1}}.$$

Then there exists $\beta_{\underline{u}} \geq 1$ such that

$$[\underline{u}] = \{(\alpha(\beta), \beta) \mid \beta \in I_{\underline{u}}\}$$

with $I_{\underline{u}} = (\beta_{\underline{u}}, \infty)$ or $I_{\underline{u}} = [\beta_{\underline{u}}, \infty)$.

Proof. If $\underline{u} = 000\ldots$, then the statement is trivially true with $\alpha(\beta) \equiv 0$ and $\beta_{\underline{u}} = 1$. So suppose $\underline{u} \neq 000\ldots$. First, we prove that

$$(\alpha, \beta) \sim (\alpha', \beta) \implies \alpha = \alpha',$$

and then that

$$(\alpha, \beta) \in [\underline{u}] \implies (\alpha(\beta'), \beta') \in [\underline{u}] \quad \text{for all } \beta' \geq \beta.$$

Let $(\alpha, \beta) \in [\underline{u}]$. Using (3), we have $\varphi^{\alpha,\beta}(\sigma\underline{u}) = T_{\alpha,\beta}(0) = \alpha$. Since the map $\alpha \mapsto \varphi^{\alpha,\beta}(\sigma\underline{u}) - \alpha$ is continuous and strictly decreasing [**6**, Lemmas 3.5 and 3.6], the first statement is true. Let $\beta' > \beta$. By [**6**, Corollary 3.1], we have that $\varphi^{\alpha,\beta}(\sigma\underline{u}) > \varphi^{\alpha,\beta'}(\sigma\underline{u})$. Therefore there exists a unique $\alpha' < \alpha$ such that $\varphi^{\alpha',\beta'}(\sigma\underline{u}) = \alpha'$. We claim that $\underline{u}^{\alpha',\beta'} = \underline{u}$. By [**6**, Proposition 2.5(1)], we have $\underline{u} \preceq \underline{u}^{\alpha',\beta'}$. By [**6**, Proposition 3.3], we have

$$h_{\mathrm{top}}(\Sigma_{\underline{u},\underline{v}^{\alpha',\beta'}}) = h_{\mathrm{top}}(\Sigma_{\alpha',\beta'}) = \log \beta'.$$

Since $\Sigma_{\alpha,\beta} = \Sigma_{\underline{u},\underline{v}^{\alpha,\beta}}$ and $\beta' > \beta$, we must have $\underline{v}^{\alpha,\beta} \prec \underline{v}^{\alpha',\beta'}$. Therefore

$$\begin{cases} \underline{u} \preceq \sigma^n \underline{u} \prec \underline{v}^{\alpha,\beta} \prec \underline{v}^{\alpha',\beta'} \\ \underline{u} \preceq \underline{u}^{\alpha',\beta'} \prec \sigma^n \underline{v}^{\alpha',\beta'} \preceq \underline{v}^{\alpha',\beta'} \end{cases} \quad \text{for all } n \geq 0$$

are the inequalities [**6**, (4.1)] for the pair $(\underline{u}, \underline{v}^{\alpha',\beta'})$. We can then apply [**6**, Proposition 3.2 and Theorem 4.1] to this pair and get $\underline{u} = \underline{u}^{\alpha',\beta'}$. It remains to show that $\alpha' = \alpha(\beta')$. Following the definition of the φ-expansion of Rényi, we have, for all $x \in [0, 1)$ and all $n \geq 0$,

$$x = \sum_{j=0}^{n-1} \frac{\mathtt{i}_j^{\alpha,\beta}(x) - \alpha}{\beta^{j+1}} + \frac{T_{\alpha,\beta}^n(x)}{\beta^n}.$$

Since $T_{\alpha,\beta}^n(x) \in [0, 1)$, for all $\beta > 1$ we find an explicit expression for $\varphi^{\alpha,\beta}$ on $\Sigma_{\alpha,\beta}$:

$$x = \sum_{j \geq 0} \frac{\mathtt{i}_j^{\alpha,\beta}(x) - \alpha}{\beta^{j+1}}.$$

In particular, by applying this equation to $x = 0$ we obtain, for all $(\alpha, \beta) \in \mathcal{R}$,

$$\alpha = (\beta - 1) \sum_{j \geq 0} \frac{u_j^{\alpha,\beta}}{\beta^{j+1}}.$$

Since for all $\beta > \beta_{\underline{u}}$ we have $\underline{u} \in \Sigma_{\alpha,\beta}$, this completes the proof. □

For each $\underline{u} \in \mathcal{U}$, the equivalence class $[\underline{u}]$ defines an analytic curve in $\mathcal{R}$ which is strictly monotone decreasing (except for $\underline{u} = 000\cdots$):

$$[\underline{u}] = \left\{ (\alpha, \beta) \,\middle|\, \alpha = (\beta - 1) \sum_{j \geq 0} \frac{u_j}{\beta^{j+1}}, \ \beta \in I_{\underline{u}} \right\}.$$

These curves are pairwise disjoint and their union is $\mathcal{R}$.

THEOREM 4. *Let $(\alpha, \beta) \in \mathcal{R}$, $\underline{u} = \underline{u}^{\alpha,\beta}$, and define $\alpha(\beta)$ and $\beta_{\underline{u}}$ as in Lemma 4. Then, for all $\beta > \beta_{\underline{u}}$, the orbit of $x = 0$ under $T_{\alpha(\beta),\beta}$ is not $\mu_{\alpha(\beta),\beta}$-normal.*

Proof. Let $\hat{\nu} \in M(\Sigma_k, \sigma)$ (with k large enough) be a cluster point of $\{\mathcal{E}_n(\underline{u})\}_{n \geq 1}$ (see (1)). By Lemma 4, $\underline{u}^{\alpha(\beta),\beta} = \underline{u}$ for any $\beta > \beta_{\underline{u}}$. Therefore

$$h(\hat{\nu}) \leq h_{\mathrm{top}}(\Sigma_{\alpha(\beta),\beta}) = \log \beta \quad \text{for all } \beta > \beta_{\underline{u}}$$

and $\hat{\nu}$, as well as $\nu_\beta := \hat{\nu} \circ (\varphi^{\alpha(\beta),\beta})^{-1}$ for all $\beta > \beta_{\underline{u}}$, is not a measure of maximal entropy (see [**10**]). □

Recall that

$$\mathcal{N}(0) = \{(\alpha, \beta) \in \mathcal{R} \mid \text{the orbit of 0 under } T_{\alpha,\beta} \text{ is } \mu_{\alpha,\beta}\text{-normal}\}.$$

By Theorem 3, $\mathcal{N}(0)$ has full Lebesgue measure. On the other hand, by Theorem 4, we can decompose $\mathcal{R}$ into a family of disjoint analytic curves such that each curve meets $\mathcal{N}(0)$ in at most one point. This situation is very similar to the one presented in [**12**] by Milnor, following an idea of Katok.

4. *Normality in generalized β-transformations*

In this section, we consider another class of piecewise monotone continuous maps, the generalized β-transformations. Introduced by Góra in [**7**], these maps have only one critical orbit like β-transformations, but they admit increasing and decreasing laps. A family $\{T_\beta\}_{\beta>1}$ of generalized β-transformations is defined by $k \geq 2$ and a sequence $s = (s_n)_{0\leq n<k}$ with $s_i \in \{-1, 1\}$. For any $\beta \in (k-1, k]$, let $a_j = j/\beta$ for $j = 0, \ldots, k-1$ and $a_k = 1$. Then, for all $j = 0, \ldots, k-1$, the map $f_j = I_j \to [0, 1]$ is defined by

$$f_j(x) := \begin{cases} \beta x \bmod 1 & \text{if } s_j = +1, \\ 1 - (\beta x \bmod 1) & \text{if } s_j = -1. \end{cases}$$

In particular, when $s = (1, -1)$, then T_β is a tent map. Here we leave the map undefined on a_j for $j = 1, \ldots, k-1$.

Góra constructed the unique measure μ_β that is absolutely continuous with respect to Lebesgue measure [**7**, Theorem 6 and Proposition 8]. Using the same argument that Hofbauer employed in [**9**], we deduce that a measure of maximal entropy is always absolutely continuous with respect to Lebesgue measure, and hence that the measure μ_β is the unique measure of maximal entropy. Let $k = \lceil\beta\rceil$ and write $\mathtt{i}^\beta$ for the coding map under T_β, $\varphi^\beta := (\mathtt{i}^\beta)^{-1}$ for the inverse of the coding map, $\Sigma_\beta := \Sigma_{T_\beta}$ and $\underline{\eta}^\beta := \lim_{x\uparrow 1} \mathtt{i}^\beta(x)$. Now it is easy to check that formula (4) becomes

$$\Sigma_\beta = \{\underline{x} \in \Sigma_k \mid \sigma^n \underline{x} \preceq \underline{\eta}^\beta \ \forall n \geq 0\} \tag{13}$$

and that inequalities (5) become

$$\sigma^n \underline{\eta}^\beta \preceq \underline{\eta}^\beta \quad \text{for all } n \geq 0. \tag{14}$$

It is known, in all of the cases treated below, that the dynamical system (Σ_β, σ) has topological entropy $\log\beta$ and, by the general theory of Hofbauer in [**12**], a unique measure of maximal entropy $\hat{\mu}_\beta$ such that $\mu_\beta = \hat{\mu}_\beta \circ (\varphi^\beta)^{-1}$ (see [**5**]).

As in the previous section, we state two lemmas which we shall need for the proof of the main theorem of this section. We study the normality of $x = 1$ only, so these lemmas are formulated specifically for $x = 1$. Let $S_n(\beta) \equiv S_n$ and $S(\beta) \equiv S$ be defined by (2).

LEMMA 5. *For any family of generalized β-transformations defined by $(s_n)_{0\leq n<k}$, the set $\{\beta \in (k-1, k] \mid 1 \in S(\beta)\}$ is countable.*

Proof. For a fixed $n \geq 1$, we study the map $\beta \mapsto T_\beta^n(1)$. This map is well defined everywhere in $(k-1, k]$ except at finitely many points, and it is continuous on each interval where it is well defined. Indeed, this is clearly true for $n = 1$. Suppose it is true for some n; then $T_\beta^{n+1}(1)$ is well defined and continuous wherever $T_\beta^n(1)$ is well defined and continuous, except for $T_\beta^n(1) \in S_0(\beta)$. By the induction hypothesis, there exists a finite family of disjoint open intervals J_i and continuous functions $g_i : J_i \to [0, 1]$ such that $(k-1, k]\backslash(\bigcup_i J_i)$ is finite and

$$T_\beta^n(x) = g_i(\beta) \quad \text{if } \beta \in J_i.$$

Then

$$\{\beta \in (k-1, k] \mid T_\beta^n(1) \text{ is well-defined and } T_\beta^n(1) \in S_0(\beta)\} = \bigcup_{i,j} \left\{\beta \in J_i \;\middle|\; g_i(\beta) = \frac{j}{\beta}\right\}.$$

We claim that $\{\beta \in J_i \mid g_i(\beta) = j/\beta\}$ has finitely many points. From the form of the map T_β, it follows immediately that each $g_i(\beta)$ is a polynomial of degree n. Since $\beta > 1$,

$$g_i(\beta) = \frac{j}{\beta} \quad \Longleftrightarrow \quad \beta g_i(\beta) - j = 0.$$

This polynomial equation has at most $n+1$ roots. In fact, using the monotonicity of the map $\beta \mapsto \underline{\eta}^\beta$, we can prove that this set has at most one point. The lemma then follows, since $S(\beta) = \bigcup_{n \geq 0} S_n(\beta)$. □

LEMMA 6. *Consider a family $\{T_\beta\}_{\beta>1}$ of generalized β-transformations defined by a sequence $s = (s_n)_{0 \leq n < k}$. Let $1 < \beta_1 \leq \beta_2$ and $\underline{\eta}^j := \underline{\eta}^{\beta_j}$ for $j = 1, 2$; define $l :=$* $\min\{n \geq 0 \mid \underline{\eta}_n^1 \neq \underline{\eta}_n^2\}$. *If $k \geq 3$, then for all $\beta_0 > 2$ there exists K such that $\beta_1 \geq \beta_0$ implies*

$$\beta_2 - \beta_1 \leq K\beta_2^{-l}.$$

If $s = (+1, +1)$, then

$$\beta_2 - \beta_1 \leq \beta_2^{-l+1}.$$

If $s = (+1, -1)$ or $(-1, +1)$, then for all $\beta_0 > 1$ there exists K such that $\beta_1 \geq \beta_0$ implies

$$\beta_2 - \beta_1 \leq K\beta_2^{-l}.$$

If $s = (-1, -1)$, then there exists $\beta_0 > 1$ and K such that $\beta_1 \geq \beta_0$ implies

$$\beta_2 - \beta_1 \leq K\beta_2^{-l}.$$

The proof is very similar to the proof of Brucks and Misiurewicz for [**2**, Proposition 1]; see also Sands [**16**, Lemma 23].

Proof. Let $\delta := \beta_2 - \beta_1 \geq 0$ and write $T_j = T_{\beta_j}$ and $\mathtt{i}^j = \mathtt{i}^{\beta_j}$ for $j = 1, 2$. Let $a_1, a_2 \in [0, 1]$ such that $r := \mathtt{i}_0^1(a_1) = \mathtt{i}_0^2(a_2)$. Considering four cases according to the signs of $a_2 - a_1$ and s_r, we have

$$|T_2(a_2) - T_1(a_1)| \geq \beta_2|a_2 - a_1| - \delta.$$

Applying this formula n times, we find that $\mathtt{i}_{[0,n)}^1(a_1) = \mathtt{i}_{[0,n)}^2(a_2)$ implies

$$|T_2^n(a_2) - T_1^n(a_1)| \geq \beta_2^n\left(|a_2 - a_1| - \frac{\delta}{\beta_2 - 1}\right).$$

Consider the case $k \geq 3$. Then $a_i = T_i(1)$ for $i = 1, 2$ are such that

$$|a_2 - a_1| = \delta > \frac{\delta}{\beta_0 - 1} \geq \frac{\delta}{\beta_2 - 1}.$$

Using $|T_2^n(a_2) - T_1^n(a_1)| \leq 1$, we conclude that for all $\beta_0 \leq \beta_1 \leq \beta_2$, if $\underline{\eta}^1_{[0,n)} = \underline{\eta}^2_{[0,n)}$, then

$$\delta \leq \frac{\beta_0 - 1}{\beta_0 - 2}\, \beta_2^{-n+1}.$$

For the case $s = (+1, +1)$, we can apply Lemma 3 with $\alpha = 0$ and $x = 1$.

The case where $s = (+1, -1)$ or $(-1, +1)$ is considered in [**16**, Lemma 23].

Now consider the case $s = (-1, -1)$: for a fixed n, we want to find β_0 such that for all $\beta_0 \leq \beta_1 \leq \beta_2$ we have

$$|T_2^n(1) - T_1^n(1)| > \frac{\delta}{\beta_2 - 1}. \tag{15}$$

Then we can conclude as in the $k \geq 3$ case. Formula (15) holds if $|dT_\beta^n(1)/d\beta| > 1/(\beta - 1)$ for all $\beta \geq \beta_0$. When n increases, β_0 decreases. With $n = 3$, we have $\beta_0 \approx 1.53$. □

In the tent map case, the separation of orbits is proved for $\beta \in (\sqrt{2}, 2]$ and then extended arbitrarily near $\beta_0 = 1$ by using the renormalization. In the $s = (-1, -1)$ case there is no such argument, and we are forced to increase n to obtain a lower bound β_0. With the help of a computer, we obtain $\beta_0 \approx 1.27$ for $n = 12$. For more details, see [**5**].

Now we turn to the question of normality for generalized β-transformations. The structure of the proof is very similar to that of Theorem 2 and Corollary 1.

THEOREM 5. *Consider a family $\{T_\beta\}_{k-1<\beta\leq k}$ of generalized β-transformations defined by a sequence $s = (s_n)_{0\leq n<k}$. Let β_0 be defined as in Lemma 6. Then the set*

$$\{\beta > \beta_0 \mid \textit{the orbit of } \underline{\eta}^\beta \textit{ under } \sigma \textit{ is } \hat{\mu}_\beta\textit{-normal}\}$$

has full λ-measure.

COROLLARY 2. *Consider a family $\{T_\beta\}_{\beta>1}$ of generalized β-transformations defined by a sequence $s = (s_n)_{n\geq 0}$. Let β_0 be defined as in Lemma 6. Then the set*

$$\{\beta > \beta_0 \mid \textit{the orbit of } 1 \textit{ under } T_\beta \textit{ is } \mu_\beta\textit{-normal}\}$$

has full λ-measure.

Proof of Theorem 5. Let

$$B_0 := \{\beta \in (\beta_0, \infty) \mid 1 \notin S(\beta)\}.$$

From Lemma 5, this subset has full Lebesgue measure. To obtain uniform estimates, we restrict our proof to the interval $[\underline{\beta}, \overline{\beta}]$ with $\beta_0 < \underline{\beta} < \overline{\beta} < \infty$. Let $k := \lceil \overline{\beta} \rceil$ and $\Omega := \{\beta \in [\underline{\beta}, \overline{\beta}] \cap B_0 \mid \underline{\eta}^\beta$ is not $\hat{\mu}_\beta$-normal$\}$. As before, setting

$$\Omega_N := \{\beta \in [\underline{\beta}, \overline{\beta}] \cap B_0 \mid \exists \nu \in V_\sigma(\underline{\eta}^\beta) \text{ s.t. } h(\nu) < (1 - 1/N) \log \beta\},$$

we have $\Omega = \bigcup_{N\geq 1} \Omega_N$. We prove that $\dim_H \Omega_N < 1$. For $N \in \mathbb{N}$ fixed, define $\varepsilon := (\underline{\beta} \log \underline{\beta})/(2N-1) > 0$ and L such that $\underline{\eta}^{\beta}_{[0,L)} = \underline{\eta}^{\beta'}_{[0,L)}$ implies $|\beta - \beta'| \leq \varepsilon$ (see Lemma 6). Consider the family of subsets of $[\underline{\beta}, \overline{\beta}]$ of the type

$$J(\underline{w}) = \{\beta \in [\underline{\beta}, \overline{\beta}] \mid \underline{\eta}^{\beta}_{[0,L)} = \underline{w}\},$$

where $\underline{w}$ is a word of length L. $J(\underline{w})$ is either empty or an interval. We cover the non-closed $J(\underline{w})$ with countably many closed intervals if necessary. We shall show that $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$ where $\beta_1 < \beta_2$ are such that $\underline{\eta}^{\beta_1}_{[0,L)} = \underline{\eta}^{\beta_2}_{[0,L)}$.

Let $\underline{\eta}^j = \underline{\eta}^{\beta_j}$, and let

$$D^* := \{\underline{z} \in \Sigma_{\underline{\eta}^2} \mid \exists \beta \in [\beta_1, \beta_2] \cap B_0 \text{ s.t. } \underline{z} = \underline{\eta}^{\beta}\}.$$

Define $\rho_* : D^* \to [\beta_1, \beta_2] \cap B_0$ by $\rho_*(\underline{z}) = \beta \Leftrightarrow \underline{\eta}^{\beta} = \underline{z}$. As before, from formula (13) and the strict monotonicity of $\beta \mapsto \underline{\eta}^{\beta}$, we deduce that ρ_* is well defined and surjective. We compute the coefficient of Hölder continuity of $\rho_* : (D^*, d_{\beta_*}) \to [\beta_1, \beta_2]$. Take $\underline{z} \neq \underline{z}' \in D^*$ and $n = \min\{l \geq 0 : z_l \neq z'_l\}$; then $d_{\beta_*}(\underline{z}, \underline{z}') = \beta_*^{-n}$. By Lemma 6, there exists C such that

$$|\rho_*(\underline{z}) - \rho_*(\underline{z}')| \leq C\rho_*(\underline{z})^{-n} \leq C\beta_1^{-n} = C(d_{\beta_*}(\underline{z}, \underline{z}'))^{\log \beta_1/\log \beta_*}.$$

By the choice of L and ε, we have

$$\frac{\log \beta_1}{\log \beta_*} \geq 1 - \frac{1}{2N},$$

and thus ρ_* has Hölder exponent of continuity $1 - 1/(2N)$. Define

$$G^*_N := \{\underline{z} \in \Sigma^* \mid \exists \nu \in V_{\sigma}(\underline{z}) \text{ s.t. } h(\nu) < (1 - 1/N) \log \beta_*\}.$$

As before, we have $\Omega_N \cap [\beta_1, \beta_2] \subset \rho_*(G^*_N \cap D^*)$ and $h_{\text{top}}(G^*_N) \leq (1 - 1/N) \log \beta_*$. Finally, $\dim_H(\Omega_N \cap [\beta_1, \beta_2]) < 1$ and $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$. □

Proof of Corollary 2. The proof is similar to that of Corollary 1. Equation (3) holds since we work on B_0. □

When we consider the tent map ($s = (1, -1)$) in particular, we recover the main theorem of Bruin in [**3**]. We do not state this theorem for all $x \in [0, 1]$ as we did for the map $T_{\alpha,\beta}$, because we do not have an equivalent of Lemma 3 for all $x \in [0, 1]$. This is the one missing step of the proof.

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A. *Appendix*

Let $\mathcal{G}$ be an oriented labeled right-resolving graph, and denote by $\mathbb{V}$ the set of vertices of $\mathcal{G}$. We assume that $\mathcal{G}$ has a root $\mathrm{v}_0 \in \mathbb{V}$. Let $\mathrm{v} \in \mathbb{V}$; the *level* of v is the length of the shortest path on $\mathcal{G}$ from v_0 to v. For $K \in \mathbb{N}$, the graph $\mathcal{G}_K$ is the subgraph of $\mathcal{G}$ whose set of vertices is

$$\mathbb{V}_K := \{\mathrm{v} \in \mathbb{V} \mid \text{the level of } \mathrm{v} \text{ is at most } K\}.$$

We set

$$\ell(n, \mathcal{G}) := \mathrm{card}\{\text{paths of length } n \text{ in } \mathcal{G} \text{ starting at } \mathrm{v}_0\}.$$

Since the graph is right-resolving, a path in $\mathcal{G}$ is uniquely prescribed by the initial vertex of the path and the (ordered) set of labels of its edges. The right-resolving rooted graph $\mathcal{G}$ has the property $\mathcal{P}$ if *for any path starting at* v *there is a unique path starting at the root* v_0 *with the same set of labels*. If $\mathcal{G}$ has the property $\mathcal{P}$, then

$$\ell(n+m, \mathcal{G}) \leq \ell(n, \mathcal{G})\ell(m, \mathcal{G}).$$

It follows that

$$h(\mathcal{G}) := \lim_{n\to\infty} \frac{1}{n} \log \ell(n, \mathcal{G}) = \inf_n \frac{1}{n} \log \ell(n, \mathcal{G}). \tag{16}$$

The quantity $h(\mathcal{G})$ is the *entropy* of $\mathcal{G}$.

LEMMA 7. *Let $\mathcal{G}$ be a right-resolving rooted graph which has the property $\mathcal{P}$. For all $\delta > 0$, there exists $L(\mathcal{G}, \delta)$ such that for all $L \geq L(\mathcal{G}, \delta)$ and all right-resolving rooted graphs $\mathcal{G}'$ satisfying the property $\mathcal{P}$, we have that $\mathcal{G}_L = \mathcal{G}'_L$ implies*

$$h(\mathcal{G}') \leq h(\mathcal{G}) + \delta.$$

Proof. Given $\mathcal{G}$ and $\delta > 0$, choose $L(\mathcal{G}, \delta)$ such that for all $L \geq L(\mathcal{G}, \delta)$ we have

$$\frac{1}{L} \log \ell(L, \mathcal{G}) \leq h(\mathcal{G}) + \delta.$$

Let $\mathcal{G}'$ be a right-resolving rooted graph which has the property $\mathcal{P}$ and is such that $\mathcal{G}'_L = \mathcal{G}_L$. Then, using (16) and the fact that a path of length L in $\mathcal{G}$ (or in $\mathcal{G}'$) remains in $\mathcal{G}_L$ (or in $\mathcal{G}'_L$), we get

$$\begin{aligned} h(\mathcal{G}') &\leq \frac{1}{L} \log \ell(L, \mathcal{G}') = \frac{1}{L} \log \ell(L, \mathcal{G}'_L) \\ &= \frac{1}{L} \log \ell(L, \mathcal{G}_L) = \frac{1}{L} \log \ell(L, \mathcal{G}) \leq h(\mathcal{G}) + \delta. \end{aligned}$$ □

Let $(\underline{u}, \underline{v})$ satisfy (9); we define a labeled graph $\mathcal{G} = \mathcal{G}(\underline{u}, \underline{v})$. A vertex v of the graph is a pair $(p, q) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. We define the out-going labeled edges from $\mathrm{v} = (p, q)$ to $\mathrm{v}' = (p', q')$, the successors of v.

(1) If $u_p = v_q$, then there is a unique out-going edge labeled by u_p from v to $\mathrm{v}' = (p+1, q+1)$.

(2) If $u_p < v_q$, then there is an out-going edge labeled by u_p from v to $\mathrm{v}' = (p+1, 0)$ and an out-going edge labeled by v_q from v to $\mathrm{v}' = (0, q+1)$. Furthermore, if there exists a with $u_p < a < v_q$, then there is an out-going edge labeled by a from v to $\mathrm{v}' = (0, 0)$.

The graph $\mathcal{G}$ is the minimal graph containing (0, 0), the root of $\mathcal{G}$, such that if v is a vertex of $\mathcal{G}$, then all successors of v are vertices of $\mathcal{G}$. All vertices of $\mathcal{G}$ are of the form (p, q) with $p \neq q$, except for the root. Furthermore, (p, q) is a vertex of $\mathcal{G}$ with $p > q$ if and only if the longest suffix of $u_0 \cdots u_{p-1}$ which is a prefix of $\underline{v}$ has length q. Using the map from the vertices of $\mathcal{G}$ to the subsets of $\Sigma_{\underline{u},\underline{v}}$,

$$(p, q) \mapsto [\sigma^p \underline{u}, \sigma^q \underline{v}] := \{\underline{x} \in \Sigma_{\underline{u},\underline{v}} \mid \sigma^p \underline{u} \preceq \underline{x} \preceq \sigma^q \underline{v}\},$$

together with the results of [**6**, §3.1], one can check that $\mathcal{G}$ has the property $\mathcal{P}$, $h(\mathcal{G}) = h_{\text{top}}(\Sigma_{\underline{u},\underline{v}})$, and the level of $\mathrm{v} = (p, q)$ is $\max\{p, q\}$. This last result implies that for $(\underline{u}', \underline{v}')$ satisfying (9), if $\underline{u}$ and $\underline{u}'$ have a common prefix of length L and $\underline{v}$ and $\underline{v}'$ have a common prefix of length L, then $\mathcal{G}_L = \mathcal{G}'_L$. Therefore Lemma 7 implies Proposition 1.

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