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On group actions on groups and associated series

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1. *Introduction.* In this paper we consider Q-groups; that is, Q is a group and we consider groups N endowed with a Q-action, meaning a homomorphism of Q into the group of automorphisms of N. In (4) a *lower central Q-series* was defined for such a Q-group, N, generalizing the lower central series of a group, and results were obtained relating to the *localization* of such a series. Since the ideas in that paper were inspired by the homotopical localization theory of nilpotent spaces (see (6)), the main body of results in (4) was concerned with the case in which N is nilpotent, and perhaps also the group Q and the action of Q on N (in the sense that the lower central Q-series terminates after a finite number of steps with the trivial group $\{1\}$). We now adopt a broader viewpoint and only restrict ourselves to the nilpotent case when our results appear to require us to do so; thus the spirit of this paper is much more that of general group theory as presented in [(8), especially ch. VI]. Thus, while there is some overlap of results, the methods used are not the same and many results (for example, Theorem 3·1) are far more general than any obtained in (4). Moreover, the methods also appear to us to be more appropriate in that essential appeal was made in (4) to a sophisticated theorem of Norman Blackburn on nilpotent groups, whereas here we merely use homological methods, the construction of the semidirect product and, in section 4, some very classical facts of the commutator calculus.

In section 2 we introduce the *lower* central Q-series and the *upper* central Q-series of a Q-group N (this latter notion was not presented in (4)) and characterize them by means of the semidirect product $G = N \rtimes Q$. We prove a proposition relating them which generalizes the familiar relation between the terms of the lower central series and upper central series of a nilpotent group and which allows us to define the concept of a Q-nilpotent Q-group.

In section 3 we consider the behaviour of the lower and upper central Q-series under P-localization, where P is a set of primes. Here we understand, following (8), by the P-localization of N the process of P-localizing the integral homology groups $H_j N, j \geqslant 1$, which coincides with P-localization in the sense of (2) in the category of nilpotent groups. However, in the generality in which we work in this paper we prefer to adopt the terminology of (8), so that a group of N is said to be HPL (homologically P-local) if $H_j N$ is P-local, $j \geqslant 1$; and N is said to be $UP'R$ (unique P'-roots, where P' is the complement of P) if the function $x \mapsto x^m, x \in N$, is a bijection of N for all P'-numbers m. Thus a $UP'R$-group is P-local, in the sense of (6); and if N is nilpotent then the concepts of HPL and $UP'R$ coincide (together with the concept of P-local as defined in (2)); in that case we will also say that N is P-local. Further we follow (8) in declaring a homomorphism $N \to M$ to P-localize if the induced homology homomorphisms

$H_j N \to H_j M$, $j \geqslant 1$, all P-localize. Here again it is known that this definition agrees with that of (2) in the case in which N, M are nilpotent. We close section 3 by proving a theorem about the upper central Q-series of N, in the case in which Q is nilpotent and acts nilpotently on N, which complements a basic result of (4) relating to the lower central Q-series.

In section 4 we introduce the concept of *Q-trivializing normal series* of a Q-group N and, in section 5, we prove results about the P-localization of the terms of such (descending and ascending) normal series of N. For these results we require that N be nilpotent but, except in Theorem 5·7, put no restriction on Q nor on the Q-action. In a brief appendix it is indicated where a generalization from a hypothesis of nilpotency to a hypothesis of local nilpotency is available, exploiting the observations of (3).

Except insofar as our terminology and notation are based on the dependence of our general formulations on those of chapter VI of (8), the most important departure we make from the conventions of (4) is that we revert to that indexing of the terms of the lower central Q-series which is consistent with the standard indexing of the terms of a lower central series. We achieve the corresponding consistency in indexing the terms of the upper central Q-series, and the terms of the descending and ascending normal series of section 4.

2. *Central Q-series*. Let N be a Q-group, that is, a group N on which the group Q acts as a group of automorphisms. We denote the action by

$$u \mapsto u^x,\ u \in N,\ x \in Q.$$

Define the *lower central Q-series* (4) of N as follows:

$$\Gamma^1_Q N = N,\ \Gamma^{i+1}_Q N = gp\{uv^x u^{-1} v^{-1} | u \in N, v \in \Gamma^i_Q N, x \in Q\}, \quad i \geqslant 1. \tag{2·1}$$

If Q acts trivially on N, the lower central Q-series of N is just the usual lower central series of N. If N is commutative, this definition agrees with that given in (6). In general, the lower central Q-series of N is the fastest *descending* central series of N whose successive quotients are trivial Q-modules. Thus, since $[N, \Gamma^i_Q N]$ is a Q-subgroup of the Q-group $\Gamma^{i+1}_Q N$ such that $\Gamma^i_Q N/[N, \Gamma^i_Q N]$ is a Q-module, it follows that

$$\Gamma^i_Q N/\Gamma^{i+1}_Q N = H_0(Q, \Gamma^i_Q N/[N, \Gamma^i_Q N]) = \mathbb{Z} \otimes_Q \Gamma^i_Q N/[N, \Gamma^i_Q N]. \tag{2·2}$$

An alternative description of the lower central Q-series of N employs the semidirect product $G = N \rtimes Q$ (see Proposition 2·6 of (4)). Embedding N in G in the canonical way (as a normal subgroup with quotient Q), we define a descending series $\{N_i\}$ of N by the rule:

$$N_1 = N,\ N_{i+1} = [G, N_i], \quad i \geqslant 1. \tag{2·3}$$

It is then easy to see that

$$N_i = \Gamma^i_Q \mathrm{N}, \quad i \geqslant 1. \tag{2·4}$$

To simplify notation we shall sometimes write $\Gamma^i N$ or even Γ^i for $\Gamma^i_Q N$.

We now define the *upper central Q-series* of N as follows:

$$\nu^0_Q N = \{1\},\ \nu^1_Q N = \{u \in ZN | u^x = u \text{ for all } x \in Q\},\ \nu^{i+1}_Q N/\nu^i_Q N = \nu^1_Q(N/\nu^i_Q N), \quad i \geqslant 1. \tag{2·5}$$

If Q acts trivially on N, the upper central Q-series of N is just the usual upper central series of N. In general, the upper central Q-series of N is the fastest *ascending* central series of N whose successive quotients are trivial Q-modules. Since $Z(N/\nu_Q^i N)$ is a Q-module containing $\nu_Q^{i+1}N/\nu_Q^i N$, it follows that

$$\nu_Q^{i+1}N/\nu_Q^i N = H^0(Q, Z(N/\nu_Q^i N)) = \mathrm{Hom}_Q(\mathbb{Z}, Z(N/\nu_Q^i N)). \tag{2·6}$$

Again, we have an alternative description of the upper central Q-series of N in terms of the semidirect product $G = N \rtimes Q$. We define an ascending series $\{N^i\}$ of N by the rule

$$N^0 = \{1\}, N^1 = ZG \cap N, N^{i+1}/N^i = Z(G/N^i) \cap N/N^i, \quad i \geqslant 1. \tag{2·7}$$

It is then easy to see that

$$N^i = \nu_Q^i N, \quad i \geqslant 0. \tag{2·8}$$

To simplify notation we shall sometimes write $\nu^i N$ or even ν^i for $\nu_Q^i N$.

The following proposition linking the lower and upper central Q-series of N generalizes the familiar relationship in the case of trivial Q-action.

PROPOSITION 2·1. *Let N* be a *Q-group. Then the following statements are equivalent*: (i) $\nu^n = N$; (ii) $\Gamma^{n+1} = \{1\}$.

Proof. We will prove that (i) implies (ii). Assuming (i), the series $\nu^n, \nu^{n-1}, \ldots$, is a descending central series whose successive quotients are trivial Q-modules. Moreover, we may assume inductively that

$$\Gamma^i \subseteq \nu^{n+1-i}, \tag{2·9}$$

which is true if $i = 1$, and, since the lower central Q-series is the fastest descending central series whose successive quotients are trivial Q-modules, it follows that

$$\Gamma^{i+1} \subseteq \nu^{n-i}.$$

Thus (2·9) is established for all i and we obtain (ii) by setting $i = n+1$. The converse (which also proceeds via (2·9)) may be left to the reader.

Definition 2·2. If the Q-group N satisfies (i) or (ii) of Proposition 2·1 it is said to be *Q-nilpotent of class* $\leqslant n$ and we write $\mathrm{nil}_Q N \leqslant n$.

Note that if $\mathrm{nil}_Q N \leqslant n$, then N is a nilpotent group of class $\leqslant n$; note also that (2·9) holds if $\mathrm{nil}_Q N \leqslant n$.

3. *Q-groups and localization of central series*. Let P be a set of primes and let P' be the complement of P. In accordance with [(8), p. 151] we shall call a group N an *HPL-group* if its integral homology groups $H_j N$ are P-local for $j \geqslant 1$. Also, we shall say that N is a *UP′R-group* (*P-local*, in the terminology of (6)) if it has unique mth roots for all P'-numbers m. It is known (see, e.g. Theorem VI·3·3 of (8)) that a *nilpotent* group N is *HPL* if and only if it is *UP′R*; we will then say that N is *P-local*. The following result generalizes Proposition VI·2·4 of (8).

THEOREM 3·1. *Let the Q-group N be an HPL-group. Then, for $i \geqslant 1$, $\Gamma_Q^i N/\Gamma_Q^{i+1}N$ is P-local and $N/\Gamma_Q^i N$ is HPL* (*and hence UP′R*).

Proof. We argue by induction on i. If $i = 1$, then

$$\Gamma^1/\Gamma^2 = H_0(Q, N_{ab}),$$

and N_{ab} is P-local. Now, for any Q-module A, it is easy to see that

$$\textit{if } A \textit{ is } P\textit{-local then } \mathbb{Z} \otimes_Q A \textit{ is } P\textit{-local.} \tag{3·1}$$

Thus Γ^1/Γ^2 is P-local. Of course N/Γ^1 is trivial and hence HPL.

We now assume that, for a given $i \geqslant 2$, Γ^{i-1}/Γ^i is P-local and N/Γ^{i-1} is HPL, and we consider the extension

$$\Gamma^{i-1}/\Gamma^i \rightarrowtail N/\Gamma^i \twoheadrightarrow N/\Gamma^{i-1}. \tag{3·2}$$

By Proposition VI·2·1 of (8) N/Γ^i is HPL. Now consider the extension

$$\Gamma^i \rightarrowtail N \twoheadrightarrow N/\Gamma^i. \tag{3·3}$$

The 5-term sequence in homology yields

$$H_2 N \to H_2(N/\Gamma^i) \to \Gamma^i/[N, \Gamma^i] \to N_{ab} \to (N/\Gamma^i)_{ab} \to 0, \tag{3·4}$$

and we know all terms in (3·4) except $\Gamma^i/[N, \Gamma^i]$ to be P-local. Thus $\Gamma^i/[N, \Gamma^i]$ is also P-local. Since (2·2) $\Gamma^i/\Gamma^{i+1} = \mathbb{Z} \otimes_Q \Gamma^i/[N, \Gamma^i]$ a second application of (3·1) yields the inductive step.

We call a homomorphism $l: N \to M$ a *P-localization map* if it P-localizes in homology (see (8)). Then we know that a homomorphism $l: N \to M$ of nilpotent groups is a P-localization map if and only if it P-localizes in the category of nilpotent groups ((2), (6), (8)). We now prove

PROPOSITION 3·2. *Let $l: N \to M$ be a P-localization map which is a homomorphism of Q-groups. Then, for $i \geqslant 1$, the induced maps*

$$l_i: N/\Gamma_Q^i N \to M/\Gamma_Q^i M,$$

$$k_i: \Gamma_Q^i N/\Gamma_Q^{i+1} N \to \Gamma_Q^i M/\Gamma_Q^{i+1} M$$

are P-localization maps.

Proof. We argue by induction on i. If $i = 1$, then l_1 P-localizes trivially and k_1 P-localizes since $N_{ab} \to M_{ab}$ P-localizes and, for any map $A \to B$ of Q-modules, it is easy to see that

$$\textit{if } A \to B \textit{ } P\textit{-localizes, so does } \mathbb{Z} \otimes_Q A \to \mathbb{Z} \otimes_Q B. \tag{3·5}$$

Now assume inductively that, for a given $i \geqslant 2$, l_{i-1} and k_{i-1} both P-localize. The diagram

$$\begin{array}{ccccc} \Gamma^{i-1}N/\Gamma^i N & \rightarrowtail & N/\Gamma^i N & \twoheadrightarrow & N/\Gamma^{i-1}N \\ \downarrow{\scriptstyle k_{i-1}} & & \downarrow{\scriptstyle l_i} & & \downarrow{\scriptstyle l_{i-1}} \\ \Gamma^{i-1}M/\Gamma^i M & \rightarrowtail & M/\Gamma^i M & \twoheadrightarrow & M/\Gamma^{i-1}M \end{array} \tag{3·6}$$

then shows that l_i P-localizes. Next the diagram

$$\begin{array}{ccccc} \Gamma^i N & \rightarrowtail & N & \twoheadrightarrow & N/\Gamma^i N \\ \downarrow & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l_i} \\ \Gamma^i M & \rightarrowtail & M & \twoheadrightarrow & M/\Gamma^i M \end{array} \tag{3·7}$$

shows, when we apply the 5-term sequence in homology, that

$$\Gamma^i N/[N, \Gamma^i N] \to \Gamma^i M/[M, \Gamma^i M]$$

P-localizes; and, finally, we invoke again (2·2) and (3·5) to infer that

$$k_i: \Gamma^i N/\Gamma^{i+1} N \to \Gamma^i M/\Gamma^{i+1} M$$

P-localizes, completing the inductive step.

Now suppose that N is a Q-group and that it is nilpotent qua group. If $l: N \to N_P$ is the P-localization map (in the category of nilpotent groups) then N_P admits a unique structure of Q-group such that l is a homomorphism of Q-groups; we suppose N_P endowed with this structure. Then we prove (see Theorem 2·8 of (4)).

COROLLARY 3·3. *Let N be a Q-group which is nilpotent qua group, and let $l: N \to N_P$ P-localize. Then the induced maps*

$$\begin{aligned} &l_i: N/\Gamma^i_Q N \to N_P/\Gamma^i_Q N_P, \\ &k_i: \Gamma^i_Q N/\Gamma^{i+1}_Q N \to \Gamma^i_Q N_P/\Gamma^{i+1}_Q N_P, \\ &h_i: \Gamma^i_Q N \to \Gamma^i_Q N_P \end{aligned}$$

P-localize.

Proof. It is only necessary to prove the statement about h_i. However, from the exactness of localization in the category of nilpotent groups (2), (8) and the diagram

$$\begin{array}{ccccc} \Gamma^i_Q N & \rightarrowtail & N & \twoheadrightarrow & N/\Gamma^i_Q N \\ \downarrow h_i & & \downarrow l & & \downarrow l_i \\ \Gamma^i_Q N_P & \rightarrowtail & N_P & \twoheadrightarrow & N_P/\Gamma^i_Q N_P \end{array}$$

it follows that h_i P-localizes.

Remark. Unless we assume N nilpotent (or locally nilpotent (3)) we have no proof that h_i is a P-localization map, nor even that $\Gamma^i_Q N$ is HPL if N is HPL. This contrasts with the situation in Theorem 3·4 and Proposition 3·5 below.

We now turn to the upper central Q-series of N; now we need to assume N a $UP'R$-group. The following result generalizes Proposition VI·3·2 of (8).

THEOREM 3·4. *Let the Q-group N be a $UP'R$-group. Then, for $i \geqslant 1$, $\nu^i_Q N/\nu^{i-1}_Q N$ is P-local and the groups $\nu^i_Q N$, $N/\nu^i_Q N$ are $UP'R$-groups.*

Proof. It is sufficient to prove that ν^1 and N/ν^1 are $UP'R$-groups, since the general result follows, in the light of (2·5), by an easy induction, using the fact that, given a central extension of groups $G' \rightarrowtail G \twoheadrightarrow G''$, then G'' is $UP'R$ if G', G are $UP'R$. Indeed, this same fact allows us merely to show that ν^1 is $UP'R$.

Now since N is $UP'R$, so is ZN. Suppose $u \in \nu^1$ and let m be a P'-number. Then $u = v^m$, $v \in ZN$, and we must show that $v \in \nu^1$. But if $x \in Q$ then

$$(v^x)^m = (v^m)^x = u^x = u = v^m,$$

so that, m^{th} roots being unique, $v^x = v$, and $v \in \nu^1$. This completes the proof.

We remark that $\nu_Q^i N$, being nilpotent, is also HPL.

To obtain an analogue of Corollary 3·3 we need to impose some extra condition on N beyond nilpotency (see Theorem 5·9 of (2)). In fact, we have

THEOREM 3·5. *Let N be a Q-group which is nilpotent qua group, and let $l: N \to N_P$ P-localize. Then the induced maps*

$$l^i: N/\nu_Q^i N \to N_P/\nu_Q^i N_P,$$
$$k^i: \nu_Q^i N/\nu_Q^{i-1} N \to \nu_Q^i N_P/\nu_Q^{i-1} N_P,$$
$$h^i: \nu_Q^i N \to \nu_Q^i N_P$$

P-localize, provided that Q is finitely generated and that N has bounded P'-torsion (*that is, the P'-torsion of N has bounded exponent*).

Proof. We proceed in several steps, which we present as separate propositions.

PROPOSITION 3·6. *If N has bounded P'-torsion, then so has N/ZN.*

Proof. Consider the upper central series $\{Z_i\}$ of N and suppose that N/ZN does not have bounded P'-torsion. Then there exists a smallest $i \geqslant 1$ with the property that Z_{i+1}/Z_i does not have bounded P'-torsion. Choose this i. Further choose a fixed but arbitrary element $a \in N$. Then the map

$$bZ_i \mapsto [b, a] Z_{i-1} \in Z_i/Z_{i-1}, b \in Z_{i+1},$$

is a homomorphism $Z_{i+1}/Z_i \to Z_i/Z_{i-1}$. Hence, if bZ_i is P'-torsion, then so is $[b, a] Z_{i-1}$ for all $a \in N$. By the minimality of i there exists a P'-number n, independent of b and a, such that $Z_{i-1} = [b, a]^n Z_{i-1} = [b^n, a] Z_{i-1}$. Hence $b^n \in Z_i$ for all P'-torsion elements bZ_i in Z_{i+1}/Z_i. Hence the P'-torsion in Z_{i+1}/Z_i is bounded; but this contradicts the choice of i.

PROPOSITION 3·7. *Let A be a commutative Q-group. If A has bounded P'-torsion then so has $A/\nu_Q^1 A$.*

Proof. Let $a\nu_Q^1 A$ be a P'-torsion element in $A/\nu_Q^1 A$. Then there exists a P'-number n with $(a^n)^x = a^n$ for all $x \in Q$. Thus $(a^x a^{-1})^n = 1$ and $a^x a^{-1}$ is a P'-torsion element of A. Hence there exists a P'-number n independent of a such that $a^n \in \nu_Q^1 A$. It follows that $A/\nu_Q^1 A$ has bounded P'-torsion.

Combining Propositions 3·6 and 3·7 and applying an obvious induction with respect to i, we obtain the following generalization of Proposition 3·7.

PROPOSITION 3·8. *Let N be a Q-group which is nilpotent qua group. If N has bounded P'-torsion, then so has $N/\nu_Q^i N$.*

We return now to the proof of Theorem 3·5. We know from (4) that, if M is a Q-group which is nilpotent qua group and if M^Q is the fixpoint set for the Q-action, then $l: M \to M_P$ induces $M^Q \to (M_P)^Q$ which also P-localizes, provided that Q is finitely generated. Thus the conclusion of the theorem follows, by an evident induction, from the following theorem, which is an improved version of Theorem 5·9 of (2).

THEOREM 3·9. *Let M be a nilpotent group. Then the P-localizing map $l: M \to M_P$ induces $l_i: Z_i M \to Z_i(M_P)$ by restriction*; and l_i *P-localizes provided that the P'-torsion of M has bounded exponent.*

Proof. An obvious induction together with Proposition 3·6 shows that it is enough to prove Theorem 3·9 for $i = 1$. From arguments in (2) we know that we have only to show that l_1 is P-surjective. Thus let $x \in Z(M_P)$; since l is P-surjective, there exists a P'-number m with $x^m = ly$, $y \in M$. Then, for any $a \in M$,

$$aya^{-1} = yu,$$

with $u \in \ker l$. Since l is P-injective, $u \in P'$-torsion subgroup of M. Since the P'-torsion of M has bounded exponent, there exists a P'-number n, independent of a, such that $u^n = 1$. It then follows from (2) that

$$ay^{n^c}a^{-1} = (aya^{-1})^{n^c} = y^{n^c},$$

where M is nilpotent of class $\leqslant c$. Thus $y^{n^c} \in Z(M)$ and $l_1(y^{n^c}) = x^{mn^c}$, with mn^c a P'-number. We have proved that l_1 is P-surjective, so that Theorem 3·9 is proved.

Remark. Theorem 3·6 is plainly more general than Theorem 5·9 of (2). For there we required that the nilpotent group M be finitely generated. Of course, if M is finitely generated, so is its P'-torsion subgroup, which is therefore, in fact, finite.

If N is Q-nilpotent then it follows from Corollary 3·3 that N_P is Q-nilpotent and

$$\operatorname{nil}_Q N_P \leqslant \operatorname{nil}_Q N. \tag{3·8}$$

From (3·8) we may deduce that, when N is Q-nilpotent,

$$\operatorname{nil}_Q N = \max_p \operatorname{nil}_Q N_p. \tag{3·9}$$

For, first, it is plain that if M, N are both Q-nilpotent and $N \subseteq M$, then $\operatorname{nil}_Q N \leqslant \operatorname{nil}_Q M$. Second, if $N(i)$ is a family of Q-nilpotent groups with $\operatorname{nil}_Q N(i) \leqslant c$, then $\prod_i N(i)$ is Q-nilpotent with

$$\operatorname{nil}_Q \prod_i N(i) = \max_i \operatorname{nil}_Q N(i) \leqslant c.$$

Now, by (3·8) N_p, for any prime p, is Q-nilpotent with $\operatorname{nil}_Q N_p \leqslant \operatorname{nil}_Q N$. Thus

$$\operatorname{nil}_Q \prod_p N_p = \max_p \operatorname{nil}_Q N_p \leqslant \operatorname{nil}_Q N. \tag{3·10}$$

On the other hand, for each p, $l: N \to N_p$ has a kernel consisting of the elements of N of finite order prime to p (2). Thus N embeds as a Q-subgroup of $\prod_p N_p$, so that

$$\operatorname{nil}_Q N \leqslant \operatorname{nil}_Q \prod_p N_p. \tag{3·11}$$

From (3·10) and (3·11) we immediately deduce (3·9).

If, now, Q is also nilpotent then (Theorem 3·3 of (4)) the P-localizing map $l: Q \to Q_P$ sets up a bijection

$$l^*: A(Q_P, N_P) \cong A(Q, N_P) \tag{3·12}$$

between nilpotent actions of Q_P on N_P and nilpotent actions of Q on N_P, such that (Theorem 3·2 of (4))

$$\Gamma^i_Q N_P = \Gamma^i_{Q_P} N_P \tag{3·13}$$

for any pair of nilpotent actions related by (3·12). We now prove the analogue of (3·13) for the upper central Q-series of N_P.

THEOREM 3·10. *Let Q be nilpotent and let N_P be Q-nilpotent and P-local. Then N_P is Q_P-nilpotent through* (3·12) *and*

$$\nu_Q^i N_P = \nu_{Q_P}^i N_P. \tag{3·14}$$

Proof. It is sufficient to prove this for $i = 1$, since an evident induction then establishes the general case. Let $G = N_P ⋊ Q$ be the semidirect product and consider the extension

$$N_P \rightarrowtail G \twoheadrightarrow Q. \tag{3·15}$$

Then G is nilpotent and we may P-localize (3·15) to obtain the split extension

$$N_P \rightarrowtail G_P \twoheadrightarrow Q_P. \tag{3·16}$$

Moreover the canonical splitting of (3·15) induces a splitting of (3·16) and hence a (nilpotent) action of Q_P on N_P. It was shown in (4) that this is the action paired to the given action of Q on N_P by l^* (3·12).

Now it is obvious from the definition of ν^1 that $\nu_{Q_P}^1 N_P \subseteq \nu_Q^1 N_P$. Thus we must establish the opposite inequality. We apply (2·7) which asserts that $\nu_Q^1 N_P = ZG \cap N_P$, $\nu_{Q_P}^1 N_P = ZG_P \cap N_P$. Now localization respects finite intersections and sends the centre to the centre. Since P-localization is the identity on N_P it therefore follows that

$$ZG \cap N_P \subseteq ZG_P \cap N_P,$$

or

$$\nu_Q^1 N_P \subseteq \nu_{Q_P}^1 N_P,$$

establishing the theorem.

4. *Q-trivializing series*. Let N be a Q-group. We say that a descending normal series of N consisting of Q-subgroups $N(i)$,

$$\ldots \lhd N(2) \lhd N(1) = N$$

Q-trivializes N if the quotients $N(i)/N(i+1)$ are trivial Q-groups for all $i \geqslant 1$. Notice that the series $\{\Gamma_Q^i N\}$ Q-trivializes N; indeed, it even $N ⋊ Q$-trivializes N.

We define the series $\{\Delta_Q^i N\}$ to be the fastest descending normal series of N which Q-trivializes N. Thus, explicitly,

$$\Delta_Q^1 N = N, \Delta_Q^{i+1} N = gp\{v^x v^{-1} | v \in \Delta_Q^i N, x \in Q\}, \quad i \geqslant 1. \tag{4·1}$$

One may prove directly that $\{\Delta_Q^i N\}$ is a normal series of N. However, the following alternative description of $\{\Delta_Q^i N\}$ also achieves this objective. We form the semidirect product $G = N ⋊ Q$ and embed N, Q in G in the canonical way. We then define the series $\{\delta^i N\}$ by the rule

$$\delta^1 N = N, \delta^{i+1} N = [Q, \delta^i N], \quad i \geqslant 1, \tag{4·2}$$

and it is easy to show that $\delta^i N$ is a Q-subgroup of N, that

$$\delta^i N = \Delta_Q^i N, \quad i \geqslant 1, \tag{4·3}$$

and that

$$\delta^{i+1} N \lhd \delta^i N, \quad i \geqslant 1. \tag{4·4}$$

Of course it is obvious, in either description, that $\Delta_Q^i N/\Delta_Q^{i+1}N$ is a trivial Q-group. Where convenient we abbreviate $\Delta_Q^i N$ to $\Delta^i N$ or even Δ^i.

Now we say that an ascending normal series of N consisting of Q-subgroups $N(i)$,

$$\{1\} = N(0) \lhd N(1) \lhd N(2) \lhd \ldots$$

Q-trivializes N if the quotients $N(i)/N(i-1)$ are trivial Q-groups for all $i \geqslant 1$. Notice that the series $\{\nu_Q^i N\}$ Q-trivializes N; indeed it even $N \rtimes Q$-trivializes N.

The following lemma will be needed to define a special ascending Q-trivializing normal series.

LEMMA 4·1. *Let M be a Q-subgroup of the Q-group N and let $\mathsf{N}(M) = \mathsf{N}_N(M)$ be the normalizer of M in N. Then $\mathsf{N}(M)$ is a Q-subgroup of N.*

Proof. Let $a \in M$, $b \in \mathsf{N}(M)$, $x \in Q$. Then $b^x a(b^x)^{-1} = (ba^{x^{-1}}b^{-1})^x \in M$, so that $b^x \in \mathsf{N}(M)$ and $\mathsf{N}(M)$ is closed under the Q-action.

We may now define an ascending normal series $\{\sigma_Q^i N\}$ by the rule

$$\sigma_Q^0 N = \{1\}, \sigma_Q^1 N = N^Q, \sigma_Q^{i+1}N/\sigma_Q^i N = (\mathsf{N}(\sigma_Q^i N)/\sigma_Q^i N)^Q, \quad i \geqslant 1. \tag{4·5}$$

As before, M^Q is the subgroup of the Q-group M consisting of the Q-invariant elements. Plainly, $\{\sigma_Q^i N\}$ is Q-trivializing.

An alternative description of the series $\{\sigma_Q^i N\}$ employs the semidirect product $G = N \rtimes Q$, with N, Q embedded in the canonical way. We define the series $\{\tau^i N\}$ by the rule

$$\tau^0 N = \{1\}, \tau^1 N = N \cap \mathsf{C}(Q), \tau^{i+1}N/\tau^i N = \tau^1(\mathsf{N}(\tau^i N)/\tau^i N), \quad i \geqslant 1, \tag{4·6}$$

where $\mathsf{C}(Q)$ is the centralizer of Q in G; and it is easily verified that $\tau^i N = \sigma_Q^i N, i \geqslant 0$.

We remark that the series $\{\sigma_Q^i N\}$ is the 'fastest ascending' normal series of N which Q-trivializes N in the sense that we start with $\sigma_Q^0 N = \{1\}$, and, given $\sigma_Q^i N$, then $\sigma_Q^{i+1}N$ is the maximal Q-subgroup M of N such that $\sigma_Q^i N \lhd M$ and Q operates trivially on $M/\sigma_Q^i N$. However, it turns out that the term 'fastest ascending' is misleading, since† it may happen that $\sigma_Q^i N \neq N$ is self-normalizing, so that the series $\{\sigma_Q^i N\}$ stabilizes at a proper subgroup of N, whereas N possesses an ascending Q-trivializing normal series $N(i)$ with $N(n) = N$ for some $n \geqslant 1$.

5. *Q-groups and localization of normal series.* Let N be a Q-group. We will discuss the behaviour of the series $\{\Delta_Q^i N\}$ and $\{\sigma_Q^i N\}$ with respect to localization at a family of primes P, in the case when N is a nilpotent group.

THEOREM 5·1. *Let N be a Q-group which is nilpotent qua group. If N is P-local, so is $\Delta_Q^i N$, $i \geqslant 1$.*

Proof. It plainly suffices to consider the case $i = 2$, since the general case then follows by an obvious induction. Thus (see (4·2), (4·3)) we are reduced to showing that $[Q, N]$ is P-local. This was proved in [(3); Theorem 2·2] under the weaker hypothesis

† We are grateful to the referee for pointing out to us that this phenomenon may occur, and also for drawing our attention to the paper (1) in which the authors study the stability groups of descending and ascending normal series of groups.

that N is locally nilpotent. However, we give a different proof here, since the ideas involved will also be useful in proving Theorem 5·2.

We consider the series

$$\mu_1 N = [Q, N], \mu_{i+1} N = [\mu_i N, N], \quad i \geqslant 1. \tag{5·1}$$

Then $\mu_{n+1} N = \{1\}$, for some n, since N is nilpotent, and we argue by induction on n. If $n = 0$, then $[Q, N] = \{1\}$, and so is certainly P-local.

Now let $n \geqslant 1$. We claim that $\mu_n N$ is P-local; of course, $\mu_n N$ is central in N. We must show that $\mu_n N$ has mth roots where m is a P'-number, and it plainly suffices to consider a generator $[a, u]$ of $\mu_n N$, $a \in \mu_{n-1} N$, $u \in N$. Then $u = v^m$, $v \in N$, and

$$[a, u] = [a, v^m] = [a, v]^m,$$

by the centrality of $\mu_n N$.

It is obvious that $\mu_i N \lhd N$; consider $M = N/\mu_n N$. Then M is P-local, moreover M is a Q-group which is nilpotent qua group and $\mu_n M = \{1\}$. Thus our inductive hypothesis permits us to infer that $[Q, M]$ is P-local. However, $[Q, M] = [Q, N]/\mu_n N$, so that finally $[Q, N]$ is P-local.

THEOREM 5·2. *Let N be a Q-group which is nilpotent qua group, and let $l: N \to N_P$ P-localize. Then the induced map*

$$l_i: \Delta_Q^i N \to \Delta_Q^i N_P$$

P-localizes.

Proof. Again it suffices to consider the case $i = 2$. We use the series $\{\mu_i N\}$, $\{\mu_i N_P\}$ from the proof of Theorem 5·1 and we suppose that $\mu_{n+1} N = \{1\}, \mu_{n+1} N_P = \{1\}$. We argue by induction on n, the case $n = 0$ being trivial. We assume $n \geqslant 1$ and show that l induces

$$l_*: \mu_n N \to \mu_n N_P$$

which P-localizes.

Now $\mu_n N_P$ is commutative and is generated by iterated commutators

$$c(x, u_1, u_2, \ldots, u_n) = [\ldots[[x, u_1], u_2], \ldots u_n], x \in Q, u_i \in N_P. \tag{5·2}$$

Moreover the map $(u_1, u_2, \ldots, u_n) \mapsto c(x, u_1, u_2, \ldots, u_n)$ is linear in each argument. We use these remarks to show that l_* is P-surjective; since l_* is plainly P-injective and $\mu_n N_P$ has already been proved to be P-local, it will follow (see (2), (8)) that l_* P-localizes.

It will suffice to show that, for the element c in (5·2), there is a P'-number m with $c^m \in im\, l_*$. Now, for each j, $1 \leqslant j \leqslant n$, there exists a P'-number m_j with $u_j^{m_j} = l v_j$, $v_j \in N$. Then

$$l_* c(x, v_1, v_2, \ldots, v_n) = c(x, u_1^{m_1}, u_2^{m_2}, \ldots, u_n^{m_n}) = c(x, u_1, u_2, \ldots, u_n)^m,$$

with $m = m_1 m_2 \ldots m_n$. Then m is a P'-number and we have shown that l_* P-localizes.

It follows immediately that if l induces $k: N/\mu_n N \to N_P/\mu_n N_P$, then k P-localizes. Writing M, as in the proof of Theorem 4·1, for $N/\mu_n N$, we may write M_P for $N_P/\mu_n N_P$ and we have $\mu_n M = \{1\}$, $\mu_n M_P = \{1\}$, so our inductive hypothesis allows us to infer that k induces

$$k_2: [Q, M] \to [Q, M_P]$$

which P-localizes. Thus we conclude with the diagram

$$\begin{array}{ccccc} \mu_n N & \rightarrowtail & [Q,N] & \twoheadrightarrow & [Q,M] \\ \downarrow{\scriptstyle l_*} & & \downarrow{\scriptstyle l_2} & & \downarrow{\scriptstyle k_2} \\ \mu_n N_P & \rightarrowtail & [Q,N_P] & \twoheadrightarrow & [Q,M_P] \end{array}$$

in which l_* and k_2 P-localize; this implies that l_2 P-localizes and the argument is complete.

Corollary 5·3. *Let N be a Q-group which is nilpotent qua group and let N_Q be obtained from N by killing the action of Q. Then the P-localizing map $l: N \to N_P$ induces $l_Q: N_Q \to (N_P)_Q$ which also P-localizes.*

Proof. It is only necessary to observe that $N_Q = N/\Delta^2$ and exploit the exactness of localization.

We now turn to the series $\{\sigma_Q^i N\}$. We recall from (4) that if N is P-local, so is N^Q. We now prove

Lemma 5·4. *Let M be a Q-subgroup of the Q-group N. Further, let N be nilpotent with both N, M P-local. Then $\mathsf{N}(M)$ is P-local.*

Proof. In fact, we will prove a more general result. Thus initially we do not suppose N, M P-local. Then the diagram $M \lhd \mathsf{N}(M) \subseteq N$ gives rise to

$$\begin{array}{ccccc} M & \lhd & \mathsf{N}(M) & \subseteq & N \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l} \\ M_P & \lhd & \mathsf{N}(M)_P & \subseteq & N_P \end{array} \qquad (5·3)$$

so that

$$\mathsf{N}(M)_P \subseteq \mathsf{N}(M_P). \qquad (5·4)$$

Notice that, in (5·4), the normalizer on the left is taken in N, on the right in N_P. However, this distinction disappears if N is P-local. Thus if we assume both N, M P-local, we obtain $\mathsf{N}(M)_P \subseteq \mathsf{N}(M)$. But, of course, in this case $\mathsf{N}(M) \subseteq \mathsf{N}(M)_P$, so that $\mathsf{N}(M) = \mathsf{N}(M)_P$ and so $\mathsf{N}(M)$ is P-local as claimed.

It follows that, in (5·4), $\mathsf{N}(M_P)$ is P-local. However, we do not know whether (5·4) is, in general, a strict inequality.†

From Lemma 5·4 we infer

Theorem 5·5. *Let N be a Q-group which is nilpotent qua group and let N be P-local. Then the terms of the series $\{\sigma_Q^i N\}$ are P-local.*

We may now prove the following.

Proposition 5·6. *Let N be a Q-group which is nilpotent qua group. If Q is finitely-generated and if* (5·4) *is an equality for all M, then*

$$(\sigma_Q^i N)_P = \sigma_Q^i (N_P), \quad i \geqslant 0.$$

† In fact, (5·4) is an equality if N is finitely generated. See (7).

Proof. We proceed by induction on i, the assertion being trivial if $i = 0$. Assuming then that the assertion is true for $i \geq 0$ and using the hypothesis that (5·4) is an equality for all Q-subgroups M of N, we have

$$\mathsf{N}(\sigma_Q^i N)_P = \mathsf{N}((\sigma_Q^i N)_P) = \mathsf{N}(\sigma_Q^i N_P).$$

Now, for Q finitely generated, and G any nilpotent group,

$$(G^Q)_P = (G_P)^Q,$$

by (4). Thus

$$\begin{aligned}\sigma_Q^{i+1}(N_P)/\sigma_Q^i(N_P) &= (\mathsf{N}(\sigma_Q^i N_P)/\sigma_Q^i N_P)^Q, \quad \text{by (4·5)}\\ &= (\mathsf{N}(\sigma_Q^i N)_P/(\sigma_Q^i N)_P)^Q\\ &= ((\mathsf{N}(\sigma_Q^i N)/\sigma_Q^i N)_P)^Q\\ &= ((\mathsf{N}(\sigma_Q^i N)/\sigma_Q^i N)^Q)_P\\ &= (\sigma_Q^{i+1} N/\sigma_Q^i N)_P\\ &= (\sigma_Q^{i+1} N)_P/\sigma_Q^i(N_P).\end{aligned}$$

Hence

$$\sigma_Q^{i+1}(N_P) = (\sigma_Q^{i+1} N)_P,$$

and the proposition is proved.

Finally, we consider the situation when Q is also nilpotent and prove

THEOREM 5·7. *Let Q be nilpotent and let N_P be Q-nilpotent and P-local. Adopting the action of Q_P on N_P given by* (3·12) *we have*

(i) $\Delta_Q^i N_P = \Delta_{Q_P}^i N_P, \quad i \geq 1$; (ii) $\sigma_Q^i N_P = \sigma_{Q_P}^i N_P, \quad i \geq 0$.

Proof. (i) Plainly we may confine attention to $i = 2$ and so have to show that

$$[Q, N_P] = [Q_P, N_P]. \tag{5·5}$$

But $[Q, N_P]$ is P-local by Theorem 5·1 and its P-localization is $[Q_P, N_P]$ by Theorem 1·2 (iii) of (4). Thus (5·5) follows, since the identity on N_P P-localizes.

(ii) It was proved in (4) that $N_P^Q = N_P^{Q_P}$. Thus $\mathsf{N}(N_P^Q)/N_P^Q = \mathsf{N}(N_P^{Q_P})/N_P^{Q_P}$ and hence, by the same result, $\sigma_Q^2(N_P) = \sigma_{Q_P}^2(N_P)$. Iteration of this simple argument yields the desired result.

Remark. Let N be a nilpotent group, $M \subseteq N$, and let $K(p) = l_p^{-1}\mathsf{N}(M_p)$, where p is a prime and $l_p: N \to N_p$ p-localizes. Then

$$\mathsf{N}(M) = \bigcap_p K(p). \tag{5·6}$$

For certainly $\mathsf{N}(M) \subseteq K(p)$ for each p, so that $\mathsf{N}(M) \subseteq \bigcap_p K(p)$. Conversely, let $y \in \bigcap_p K(p)$ and let $a \in M$. Then $l_p(yay^{-1}) \in M_p$ for each p, so that, by (5), $yay^{-1} \in M$, and $y \in \mathsf{N}(M)$.

6. *Appendix: Locally nilpotent groups.* Various results in this paper have assumed the nilpotency of N or of Q, where N is a Q-group. Here we very briefly discuss which

results can be immediately generalized to an assumption of local nilpotency. Corollary 3·3 remains valid if we merely require that N be locally nilpotent instead of nilpotent. In Theorem 3·10 we may allow Q to be locally nilpotent (this makes the semidirect product locally nilpotent, and any hypothesis having this effect would lead to the conclusion (3·11)). We have already remarked that Theorem 5·1 remains valid if N is merely assumed locally nilpotent, although there is no immediately evident modification of the argument given; on the other hand, we know of no proof of Theorem 5·2 strengthened by replacing the hypothesis on N of nilpotency by one of local nilpotency. Similarly we do not know how to strengthen Corollary 5·3, although we may infer that if N is a P-local, locally nilpotent Q-group, then N_Q is P-local. In Lemma 5·4, Theorem 5·5 and Proposition 5·6 we may merely assume N locally nilpotent (as also in (5·4)). Finally, Theorem 5·7 remains valid if Q is assumed locally nilpotent (see the comment above on Theorem 3·10); and so, too, does the closing remark (5·6) if N is assumed locally nilpotent.

We also remark that the conclusion of Theorem 3·6 remains true if M is locally nilpotent and its P'-torsion is finite.

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