On hyperbolic Coxeter n-polytopes with $n + 2$ facets

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Abstract. A convex polytope admits a *Coxeter decomposition* if it is tiled by finitely many Coxeter polytopes such that any two tiles having a common facet are symmetric with respect to this facet. In this paper, we classify all Coxeter decompositions of compact hyperbolic Coxeter n-polytopes with $n + 2$ facets. Furthermore, going out from Schläfli's reduction formula for simplices we construct in a purely combinatorial way a volume formula for arbitrary polytopes and compute the volumes of all compact Coxeter polytopes in $\mathbb{H}^4$ which are products of simplices.

1 Introduction

1. A polytope in the hyperbolic space $\mathbb{H}^n$ is called a *Coxeter polytope* if all its dihedral angles are integer parts of π. A convex polytope admits a *Coxeter decomposition* if it is tiled by finitely many Coxeter polytopes such that two tiles having a common facet are symmetric with respect to this facet. The tiles of the decomposition are called *fundamental polytopes*. A decomposition containing exactly one fundamental polytope is called *trivial*.

The study of Coxeter decompositions is important for the classification of subgroups generated by reflections in Coxeter groups. Namely, any Coxeter decomposition of a Coxeter polytope corresponds to a finite index reflection subgroup in a Coxeter group. It seems to be impossible to find a general classification of Coxeter decompositions, but there are some partial results. For example, [5] contains the classification of Coxeter decompositions of hyperbolic simplices, [7] contains the classification of decompositions of compact prisms in $\mathbb{H}^3$, in [4] the problem is solved for quadrilaterals in $\mathbb{H}^2$.

In this paper, we classify all Coxeter decompositions of n-dimensional compact hyperbolic Coxeter polytopes with $n + 2$ facets. We also compute the volumes of even-dimensional compact Coxeter polytopes of this type.

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2. For the classification of Coxeter decompositions it is very helpful to know the volumes of the Coxeter polytopes. In general the construction of volume functions for hyperbolic (and spherical) polytopes is difficult and the solution of the volume problem in $\mathbb{H}^n$ (and $\mathbb{S}^n$) requires a strategy adapted to the parity of the dimension. In even dimensions the so-called *Schläfli reduction formula* is valid, which expresses the volume of an even-dimensional simplex in $\mathbb{H}^{2m}$ (and $\mathbb{S}^{2m}$) as a linear function of the odd-dimensional angles. Any polytope can be decomposed into simplices and hence, in principle, the even-dimensional volume problem can be reduced to the problem of the determination of odd-dimensional spherical volumes. In this paper we will work out this idea and we will prove a generalization of Schläfli's reduction formula, which expresses the volume of a polytope in $\mathbb{H}^{2m}$ (and $\mathbb{S}^{2m}$) as a linear function of all angles and with coefficients depending on the combinatorial structure of the corresponding apexes and depending on the combinatorics of the simplicial decomposition of this apex. For 4-dimensional polytopes in $\mathbb{H}^4$ (and $\mathbb{S}^4$) we can choose a simplicial decomposition, such that the constructed formula simplifies to a reduction formula, which is due to L. Schäfli.

For a general survey, especially for the volume problem in odd-dimensional hyperbolic spaces, we recommend the article [12].

3. Compact hyperbolic Coxeter n-dimensional polytopes with $n+2$ facets are classified by I. Kaplinskaja [10] (see also [15]) and F. Esselmann [3]. A polytope of this type is either a simplicial prism or a product of two simplices. More precisely, I. Kaplinskaja classified all Coxeter simplicial prisms of finite volume. There are finitely many compact simplicial prisms of dimension greater than 3. None of these prisms has dimension greater than 5. In $\mathbb{H}^3$ there are several series of compact Coxeter triangular prisms. F. Esselmann classified the remaining compact n-polytopes with $n+2$ facets. It turns out that a compact Coxeter n-polytope with $n+2$ facets different from a simplicial prism is one of seven 4-polytopes combinatorially equivalent to a product of two triangles.

It follows from the definition of a Coxeter decomposition that any two fundamental polytopes of the decomposition are congruent. Let P be a polytope admitting a Coxeter decomposition with a fundamental polytope F. It is proved in [8] that if P is a hyperbolic polytope with k facets and if P admits a Coxeter decomposition with fundamental polytope F then F has at most k facets. This means that if P is an n-polytope with $n+2$ facets then the fundamental polytope is either a polytope with $n+2$ facets or a simplex. Clearly, if P is a compact polytope then F is a compact polytope too. There are finitely many hyperbolic Coxeter simplices of dimension greater than 2, and no compact Coxeter simplex exists in $\mathbb{H}^n$, where $n > 4$ (see [13]). Hence, if P is a compact hyperbolic Coxeter polytope in $\mathbb{H}^n$ with $n+2$ facets, $n > 3$, then $n = 4$ or $n = 5$ and there are finitely many possibilities for P and F. For n equal to 2 and 3 the lists of Coxeter decompositions are contained in [4] and [7] respectively.

In Section 2, we discuss some general properties of Coxeter decompositions. In Section 3, we construct the a volume formula for hyperbolic and spherical polytopes and compute the volumes of compact Coxeter 4-polytopes with 6 facets (see Table 5.1 for the volumes of the simplicial prisms and Table 5.2 for the volumes of the products of two simplices). In Section 4 we use the results of Sections 2 and 3 to classify decompositions of compact n-polytopes with $n+2$ facets in $\mathbb{H}^4$ and $\mathbb{H}^5$.

2 Properties of Coxeter decompositions

Lemma 1 (Volume Property). *Let P be a polytope and $\mathrm{Vol}(P)$ be the volume of P. If P admits a Coxeter decomposition with fundamental polytope F then*

$$\frac{\mathrm{Vol}(P)}{\mathrm{Vol}(F)} \in \mathbb{Z}.$$

The lemma is evident. We will use the Volume Property for 4-dimensional polytopes. The volumes of compact hyperbolic Coxeter 4-dimensional polytopes with 6 facets are computed in Section 3. The volumes of hyperbolic Coxeter simplices are contained in [9].

Notation. Let H_i^- $(i \in I)$ be a family of closed halfspaces in $\mathbb{X}^n$ such that $P = \bigcap_{i \in I} H_i^-$ is a Coxeter n-polytope with the Coxeter diagram $\Sigma(P)$. Denote by $G_{\Sigma(P)}$ the group generated by the reflections with respect to the facets of P. Sometimes we write G_P instead of $G_{\Sigma(P)}$. Furthermore, we define

$$P_J := P \cap \Big(\bigcap_{j \in J} H_j \Big)$$

for all subsets $J \subset I$. H_j denotes the hyperplane corresponding to the halfspace H_j^-. The subdiagram of $\Sigma(P)$ associated to the set J is denoted by $\Sigma(P_J | P)$. If P_J is a face of P then $\Sigma(P_J | P)$ is the diagram of the corresponding face figure.

Lemma 2 (Subdiagram Property). *Let P be a Coxeter polytope admitting a Coxeter decomposition with fundamental polytope F. Let $\Sigma(P)$ and $\Sigma(F)$ be the Coxeter diagrams of P and F. For any elliptic subdiagram $\Sigma'(P)$ of $\Sigma(P)$ there exists an elliptic subdiagram $\Sigma'(F)$ of $\Sigma(F)$ such that $G_{\Sigma'(P)}$ is a subgroup of $G_{\Sigma'(F)}$.*

Proof. In [15], E. B. Vinberg proved that any elliptic subdiagram of a Coxeter diagram corresponds to a face of the polytope. Since any face contains a vertex, there exists a subdiagram $\Sigma''(P)$ of $\Sigma(P)$ such that $\Sigma''(P)$ has n nodes, where n is the dimension of P, and $\Sigma''(P)$ contains $\Sigma'(P)$ as a subdiagram. Let A be a vertex of P determined by $\Sigma''(P)$.

Consider a section of P by a small sphere s centered in A. This section is a spherical Coxeter simplex, denote it by p. The Coxeter decomposition of P with fundamental polytope F induces on s a Coxeter decomposition of p with some fundamental simplex f. Let $\Sigma'(F)$ be a Coxeter diagram of f. Clearly, $\Sigma'(F)$ is an elliptic subdiagram of $\Sigma(F)$. The Coxeter diagram of p coincides with $\Sigma''(P)$, and $G_{\Sigma''(P)}$ is a subgroup of $G_{\Sigma'(F)}$. At the same time, $G_{\Sigma'(P)}$ is a subgroup of $G_{\Sigma''(P)}$ and the lemma is proved. $\square$

Reflection subgroups of spherical Coxeter groups were studied by E. B. Dynkin in [2]. The paper [2] contains the list of root subsystems in root systems. Hence, if P is an indecomposable spherical simplex different from G_2^m, H_3 and H_4 then all subgroups of G_P are listed in [2] (we use the standard notation of spherical Coxeter groups, see for example [16]). Subgroups of $G_{G_2(m)}$, H_3 and H_4 are listed in [6]. Note that if G is a direct product of spherical Coxeter groups then any reflection subgroup of G is a direct product of reflection subgroups of the multiples.

Lemma 3. *Let P be a polytope admitting a Coxeter decomposition with fundamental polytope F. If any two facets of P are adjacent then any two facets of F are adjacent too.*

Proof. Suppose that any two facets of P are adjacent, but the facets a and b of F have no common points. Since F is a Coxeter polytope, the hyperplanes containing a and b have no common point (see [1]). Let $F_0 \in P$ be a copy of F. Consider the sequence F_i, $i \in \mathbb{Z}$, of copies of F, such that F_i and F_{i+1} have a common facet f_i that is a copy of either a or b. Let $\bar{f}_i$ be a hyperplane containing f_i. Clearly, no pair of hyperplanes $\bar{f}_i$ and $\bar{f}_j$ has a common point. Since P contains only finitely many of polytopes F_i, there exist two hyperplanes $\bar{f}_i$ and $\bar{f}_j$ containing facets of P. This contradicts the assumption that any two facets of P are adjacent.   □

In particular, the lemma shows that a product of two triangles admits no decomposition into simplicial prisms.

Let P be a hyperbolic Coxeter simplicial prism of dimension n. A simplicial facet of P will be called a *base*. There exists a hyperplane orthogonal to all facets of P except the bases. The section of P by this hyperplane is a hyperbolic simplex p of dimension $n-1$. The Coxeter diagram $\Sigma(p)$ coincides with $\Sigma(P) \setminus \{b_1, b_2\}$, where b_1 and b_2 are the nodes of $\Sigma(P)$ that correspond to the bases.

Lemma 4. *Let P be a Coxeter simplicial prism admitting a Coxeter decomposition with fundamental polytope F, and F is a simplicial prism. Let $b_1(F)$, $b_2(F)$, $b_1(P)$, $b_2(P)$ be the nodes of $\Sigma(F)$ and $\Sigma(P)$ corresponding to the bases. Let $T(F) = \Sigma(F) \setminus \{b_1(F), b_2(F)\}$ and $T(P) = \Sigma(P) \setminus \{b_1(P), b_2(P)\}$. Then $G_{T(P)}$ is a subgroup of $G_{T(F)}$.*

Proof. Let F_1 be an arbitrary tile of the decomposition and $a_1(F_1), \dots, a_k(F_1)$ be all facets of F_1 different from the bases. The group generated by the reflections with respect to $a_1(F_1), \dots, a_k(F_1)$ is the group $G_{T(F)}$. Let $a_1(P), \dots, a_k(P)$ be all facets of P different from the bases. The proof of the previous lemma shows that no copy of a base of F belongs to $a_1(P), \dots, a_k(P)$. Hence, $G_{T(F)}$ contains a reflection with respect to $a_i(P)$, $1 \le i \le k$. Thus, $G_{T(P)}$ is a subgroup of $G_{T(F)}$.   □

Note that any prism from the list presented in [10] has a base orthogonal to all adjacent facets. Any other compact Coxeter simplicial prism can be composed of two of these prisms (glue two prisms together along congruent bases that are orthogonal to all adjacent facets). Gluing together two congruent prisms we obtain a Coxeter decomposition of the big prism.

3   The volume of $2m$-polytopes

Throughout this section let $\mathbb{X}^n$ be one of the spaces $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ of constant curvature K. Let P be an n-polytope in $\mathbb{X}^n$ and $a^k(P)$ the number of k-dimensional faces of P for $0 \le k \le n$. If $\mathbb{X}^n = \mathbb{H}^n$ then $a^0(P)$ is the number of ordinary vertices and vertices at infinity of P. Let $\mathcal{D} = \mathcal{D}(P)$ be an arbitrary polytopal decomposition of P. Then

we denote by $\Omega^k(\mathcal{D})$ the set of all k-dimensional elements of $\mathcal{D}$ for $1 \leq k \leq n$ and $\Omega^0(\mathcal{D})$ denotes the set of all ordinary vertices of $\mathcal{D}$. If $\mathcal{D}$ is the trivial decomposition then $\Omega^k(\mathcal{D}) = \Omega^k(P)$ is the set of all k-dimensional faces of P.

Let $P^k \in \Omega^k(P)$ for $0 \leq k \leq n-1$. Then we denote by $\alpha_{n-k-1}(P^k)$ the $(n-k-1)$-dimensional (*normalized*) *angle* of P at the apex P^k. The angles are normalized in such a way that the whole sphere is measured as 1. Furthermore, we define $\alpha_{-1}(P) := 1$. A 1-dimensional angle of P is also called a *dihedral angle*.

We have the so-called Schläfli reduction formula for simplices (see [14, page 240]). We remark that Schläfli has normalized the angles in such a way that the generalized octand is measured as one.

Theorem 1 (Schläfli's reduction formula). *Let T be a $2m$-simplex in $\mathbb{X}^{2m}$. Then*

$$2K^m c_{2m}^{-1} \operatorname{Vol}(T) = 2 \sum_{k=0}^{m} (-1)^k a_{2k+1} \sum_{T^{2k} \in \Omega^{2k}(T)} \alpha_{2m-2k-1}(T^{2k}),$$

where c_{2m} denotes the volume of the $2m$-dimensional unit sphere and the constants a_{2k+1} are the tangent numbers defined by the series

$$\tan\left(\frac{z}{2}\right) = \sum_{k=0}^{\infty} \frac{a_{2k+1}}{(2k+1)!} z^{2k+1}.$$

Of course, a polytope can be decomposed into simplices in several ways. Hence we can construct volume formulas for arbitrary polytopes by combining Schläfli's reduction formulas for the decomposition simplices. The problem is to understand the combinatorics of the decomposition. We further have to note that the sum of all decomposition angles with the same apex is an angle in the polytope, but generally not of the same dimension. The change of the dimension depends on the dimension of the face of P, which contains the relative interior of the decomposition apex D in its relative interior $(\operatorname{ri}(D) \subset \operatorname{ri}(P))$. So we can prove in a combinatorial way (compare [17]) the following result.

Theorem 2. *Let P be a $2m$-polytope in $\mathbb{X}^{2m}$, $\mathcal{D} = \mathcal{D}(P)$ a simplicial decomposition of P and for a j-dimensional face $P^j \in \Omega^j(P)$ let $z(i, P^j, \mathcal{D})$ be the number of elements in $\{D \in \Omega^i(\mathcal{D}) : \operatorname{ri}(D) \subset \operatorname{ri}(P^j)\}$ for all $0 \leq i, j \leq 2m$. Then*

$$2K^m c_{2m}^{-1} \operatorname{Vol}(P) = \sum_{\substack{P^j \in \Omega^j(P) \\ j=0,\ldots,2m}} E(P^j, \mathcal{D}) \alpha_{2m-j-1}(P^j)$$

with

$$E(P^j, \mathcal{D}) = 2 \sum_{k=0}^{m} (-1)^k a_{2k+1} z(2k, P^j, \mathcal{D}).$$

In general, this is not a proper reduction formula because the volume is a function of angles of all dimensions, except in special cases (plain polytopes, skillful decompositions). Such a special case is a 4-polytope with the barycentric decomposition, without

decomposing 1-dimensional faces of P. We get a general reduction formula, which is due to L. Schläfli (compare [14, page 276]).

Corollary 1 (Reduction formula for 4-Polytopes). *Let P be a 4-polytope in $\mathbb{X}^4 = \mathbb{S}^4$ or $\mathbb{H}^4$. Then*

$$2c_4^{-1}\operatorname{Vol}(P) = \sum_{P^0 \in \Omega^0(P)} \sigma^0(P^0)\alpha_3(P^0) + \sum_{P^2 \in \Omega^2(P)} \sigma^2(P^2)\alpha_1(P^2) + \sigma^4(P)$$

with $\sigma^0(P^0) = 1$,

$$\sigma^2(P^2) = 1 - \frac{1}{2}a^0(P^2)$$

$$\sigma^4(P) = 1 - \frac{1}{2}(a^0(P) + a^2(P)) + \frac{1}{4}\sum_{P^2 \in \Omega^2(P)} a^0(P^2).$$

The combinatorial invariants $\sigma^{2i}(P)$ $(i \geq 0)$ are called $2i$-dimensional *Schläfli invariants*. L. Schläfli also proved a generalized reduction formula (compare [14, page 280]), but an explicit description of the Schläfli invariants is unknown. The corollary gives such an explicit description in dimensions $0, 2$ and 4. For the Schläfli invariants of simple polytopes compare [16, page 122].

Now we will use the corollary to determine the volumes of the compact 4-dimensional Coxeter polytopes, classified by I. M. Kaplinskaja and F. Esselmann. We remark that the polytopes with a linear graph are so-called orthoschemes and their volumes are already determined by R. Kellerhals (see [11]) by using a reduction formula of analytic type.

From the combinatorial point of view the polytopes P classified by Kaplinskaja are prisms over a 3-simplex. We see that $a^0(P) = 8$, $a^1(P) = 16$, $a^2(P) = 14$, $a^3(P) = 6$ and that the set of 2-dimensional faces consists of 8 triangles and 6 rectangles. So we get $\sigma^4(P) = 2$.

The polytopes P, classified by Esselmann, are products of two 2-simplices. We see that $a^0(P) = 9$, $a^1(P) = 18$, $a^2(P) = 15$ and $a^3(P) = 6$ and that the set of 2-dimensional faces consists of 6 triangles and 9 rectangles. So we get $\sigma^4(P) = 5/2$.

The angles $\alpha_3(P^0)$ of P can be computed as the volumes of the spherical Coxeter 3-polytopes with diagram $\Sigma(P^0|P)$. The angles $\alpha_1(P^2)$ of P can be computed as the volumes of the spherical Coxeter 1-polytopes (segments) with diagram $\Sigma(P^2|P)$. Furthermore, we have to determine the combinatorics of the 2-dimensional faces of P. This is possible by construction of the *complex $\mathcal{F}(P)$* of the polytope P: Let $P = \bigcap_{i \in I} H_i^-$ be a Coxeter n-polytope in $\mathbb{X}^n$. Then $\mathcal{F}(P)$ is the set of all subsets of the form $I(P^k)$ with

$$I(P^k) = \{i \in I : P^k \subset H_i\}$$

for $P^k \in \Omega^k(P)$ and $0 \leq k \leq n - 1$. Under inclusion $\mathcal{F}(P)$ is a partially ordered set, anti-isomorphic to the face lattice of P. Hence $\mathcal{F}(P)$ carries the whole information of the combinatorial structure of P and we can reformulate Theorem 3.1 from [15] in the following way.

Theorem 3. *A set $J \subset I$ is in $\mathcal{F}(P)$ if and only if the diagram $\Sigma(P_J | P)$ is elliptic. Furthermore, let P_{J_1} and P_{J_2} be two faces of P ($J_1, J_2 \in \mathcal{F}(P)$). Then P_{J_1} is a face of P_{J_2} if and only if $J_2 \subset J_1$.*

3.1   Example. We will compute the volume of the Coxeter 4-polytope $P = \bigcap_{i=1}^{6} H_i^-$ with Coxeter diagram $\Sigma(P)$ in Figure 1.
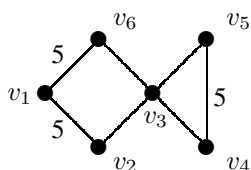

Figure 1. The diagram $\Sigma(P)$.

We number the vertices of the diagram so that every vertex corresponds to a hyperplane $H_i = v_i^\perp$ for all $i = 1, \ldots, 6$. Now we can construct the complex $\mathcal{F}(P)$ of P by studying all subdiagrams of $\Sigma(P)$ and from the complex we can read off the inclusions of the sets $I(P^k)$. Hence we can determine the whole combinatorics of the polytope P. In the following two tables we give the sets $I(P_J)$ for the 0-dimensional ($|J| = 4$) and the 2-dimensional ($|J| = 2$) faces P_J of P.

$i(P_J)$, $|J| = 4$	$\Sigma(P_J \| P)$	$\alpha_{|J|-1}(P_J)$
$(1,2,3,4), (1,2,3,5),$ $(1,3,4,6), (1,3,5,6)$		$1/14400$
$(1,2,4,5), (1,4,5,6)$		$1/100$
$(2,4,5,6)$		$1/40$
$(2,3,4,6), (2,3,5,6)$		$1/192$

$i(P_J)$, $|J| = 2$	$\alpha_{|J|-1}(P_J)$	Type of P_J
$(1,2), (1,6), (4,5)$	$1/10$	3-gone
$(3,4), (3,5)$	$1/6$	3-gone
$(2,3), (3,6)$	$1/6$	4-gone
$(2,6)$	$1/4$	3-gone
$(1,3), (1,4), (1,5),$ $(2,4), (2,5), (4,6), (5,6)$	$1/4$	4-gone

So we see with Corollary 1 that

$$\mathrm{Vol}(P) = \frac{4}{3}\pi^2\Big(\frac{401}{7200} - \frac{101}{40} + \frac{5}{2}\Big) = \frac{221}{5400}\pi^2.$$

4   Classification of decompositions

A compact Coxeter n-polytope with $n + 2$ facets is a polytope in $\mathbb{H}^2$, $\mathbb{H}^3$, $\mathbb{H}^4$ or $\mathbb{H}^5$. Coxeter decompositions of quadrilaterals in $\mathbb{H}^2$ are listed in [4]. Any compact Coxeter polytope in $\mathbb{H}^3$ with 5 facets is a triangular prism. Coxeter decompositions of triangular prisms in $\mathbb{H}^3$ are listed in [7]. In this section, we classify Coxeter decompositions of compact Coxeter n-polytopes with $n + 2$ facets in $\mathbb{H}^4$ and $\mathbb{H}^5$.

4.1   4-dimensional decompositions. A compact Coxeter polytope in $\mathbb{H}^4$ with 6 facets is either a simplicial prism or a product of two triangles. Let P be a compact Coxeter 4-polytope with 6 facets admitting a Coxeter decomposition with fundamental polytope F. As it was mentioned above F has at most 6 facets. Hence, F is either a simplex or a simplicial prism, or a product of two simplices.

Lemma 5. *Let P be a compact Coxeter polytope in $\mathbb{H}^4$ combinatorially equivalent to a product of two triangles. If P admits a Coxeter decomposition then the fundamental polytope F is combinatorially equivalent to a product of two triangles and P contains exactly two copies of the fundamental polytope.*

Proof. Suppose F is a product of two triangles. Applying the Volume Property to the volumes from Table 5.2 we obtain that P contains exactly two copies of F. By Lemma 3 F is not a simplicial prism. Thus, it is sufficient to show that F is not a simplex.

Suppose F is a simplex. Applying the Subdiagram Property we obtain that $P = [(3,5,5,3),(3,5,3)]$ (the lowest polytope from Table 5.2) and either $F = [4,3,3,5]$ or $F = [5,3,3^{1,1}]$ (see Fig. 2 for the notation). Since the simplex $[5,3,3^{1,1}]$ can be decomposed into two copies of $[4,3,3,5]$, it is sufficient to show that $P = [(3,5,5,3),(3,5,3)]$ admits no Coxeter decomposition into several copies of $F = [4,3,3,5]$.
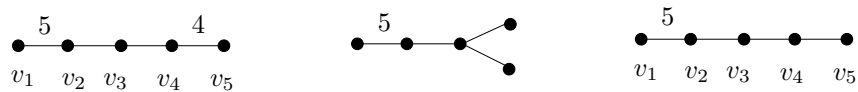

Figure 2. The simplices $[4,3,3,5]$ (left), $[5,3,3^{1,1}]$ (middle) and $[3,3,3,5]$ (right).

Suppose the contrary. Consider a decomposition of $P = [(3,5,5,3),(3,5,3)]$ into several copies of $F = [4,3,3,5]$ in the linear model of $\mathbb{H}^4$. We use coordinates $x_0, x_1, \ldots, x_4$ with bilinear form $(x,x) = -x_0^2 + x_1^2 + \cdots + x_4^2$. Let F_1 be one of the fundamental simplices of the decomposition and $v_1, v_2, \ldots, v_5$ be the unit outward normals to the facets of F_1 (we choose numbering of the facets as it is shown in Fig. 2). We can express

$v_1, v_2, \ldots, v_5$ as follows:

$$
\begin{aligned}
v_1 &= & (0, & \ 0, & -1, & \ 0, & 0) \\
v_2 &= & \left(0, & \ \tfrac{1}{2}, & \tfrac{1+\sqrt5}{4}, & \ \tfrac{1-\sqrt5}{4}, & 0\right) \\
v_3 &= & (0, & -1, & 0, & \ 0, & 0) \\
v_4 &= & \left(0, & \ \tfrac{1}{2}, & 0, & \ \tfrac{1+\sqrt5}{4}, & \tfrac{1-\sqrt5}{4}\right) \\
v_5 &= & \left(\sqrt{2+\sqrt5}, & \ 0, & 0, & \ 0, & \tfrac{\sqrt2(1+\sqrt5)}{2}\right)
\end{aligned}
$$

The simplex $[4,3,3,5]$ contains the subdiagram H_4. No indecomposable spherical Coxeter group contains H_4 as a subgroup. Hence, we can assume that v_1, v_2, v_3 and v_4 are the outward normals to the facets of P. Denote by v_6 and v_7 the remaining outward normals (see Fig. 3).
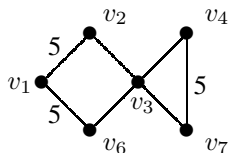
Figure 3. The numbering of vertices of $[(3,5,5,3),(3,5,3)]$.

The vector $\pm v_6$ is a reflection image of one of $v_1, \ldots, v_5$ under a sequence of reflections with respect to $v_1, \ldots, v_5$. The simplex $[4,3,3,5]$ does not have dihedral angles different from $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{5}$. Hence, we can write $v_6 = \sum_{i=1}^5 c_i v_i$, where $c_i \in \mathbb{Q}(\sqrt2, \sqrt5)$ (we use the explicit formula of the reflection $R_e(x) = x - 2\frac{(x,e)}{(e,e)}e$). The direct calculation shows that

$$
v_6 = \left(\sqrt{40+18\sqrt5}, \frac{1}{2}, \frac{1+\sqrt5}{4}, \frac{5+3\sqrt5}{4}, 4+\sqrt5\right)
$$

and $\frac{\sqrt{40+18\sqrt5}}{\sqrt{2+\sqrt5}} \notin \mathbb{Q}(\sqrt2, \sqrt5)$. The contradiction shows that the decomposition does not exist and the lemma is proved. $\square$

Lemma 6. *Let P be a compact Coxeter simplicial prism in $\mathbb{H}^4$. Suppose P admits a Coxeter decomposition with the fundamental polytope F; then F is a simplicial prism and P contains exactly two copies of F.*

Proof. Suppose that F is a simplex. Then the Subdiagram Property and the Volume Property show that $F = [5,3,3,3]$ (see Fig. 2) and P is one of the prisms shown in Fig. 4.
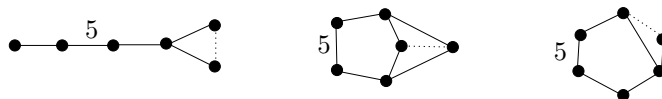

Figure 4. Three possibilities for a prism decomposed into simplices.

Consider $F = [5,3,3,3]$ in the linear model of $\mathbb{H}^4$. Let $v_1, v_2, \ldots, v_5$ be the unit outward normals to the facets of F. Each of the prisms shown in Fig. 4 contains the

subdiagram H_4. No indecomposable spherical Coxeter group contains H_4 as a subgroup. Hence, we can assume that v_1, v_2, v_3 and v_4 are the outward normals to the facets of P. Let v_6 and v_7 be the remaining outward normals of P. Then we can write v_6 and v_7 as a sum $\sum_{i=1}^5 c_i v_i$, where $c_i \in \mathbb{Q}(\sqrt{5})$. The explicit coordinate expression of v_6 and v_7 shows that this is impossible. Hence, F is not a simplex. The Subdiagram Property and the Volume Property show that F is not a product of two triangles. Hence, F is a prism.

Using the Volume Property again, we obtain that either P contains exactly two copies of F or (F, P) is one of the pairs shown in Fig. 5.

PSfrag replacements
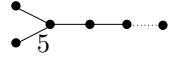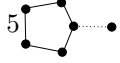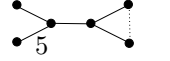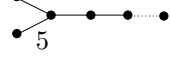
Figure 5. Three possibilities for the pair (F, P).

Looking at the volumes of two tetrahedra shown in Fig. 6 we can see that none of these tetrahedras can be decomposed into copies of the another one.

Figure 6. Volumes of two compact tetrahedra.

Hence, by Lemma 4 (F, P) is not a pair shown in Fig. 5 and the lemma is proved. □

4.2   5-dimensional decompositions. According to [3], any compact Coxeter polytope in $\mathbb{H}^5$ with 7 facets is a simplicial prism. Let P be a simplicial prism admitting a Coxeter decomposition with fundamental polytope F. As it was mentioned above F has at most 7 facets. There are no compact Coxeter simplices in $\mathbb{H}^5$. Hence, F is a simplicial prism, too.

Using the Subgroup Property it is easy to see that P consists of exactly two copies of F; one of the bases of F is orthogonal to all adjacent facets.

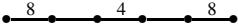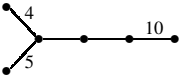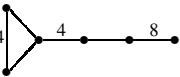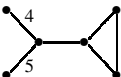We can summarize the result of this section as follows:

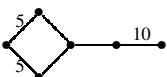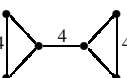*Let P be a compact Coxeter hyperbolic n-polytope with n + 2 facets, where n > 3. If P admits a Coxeter decomposition with fundamental polytope F then P contains exactly two copies of F and both P and F are either simplicial prisms or products of two triangles.*

5   Appendix

5.1   Compact simplicial prisms in $\mathbb{H}^4$.

Diagram	Volume	Value
	$\frac{41}{10800}\pi^2$	0.03746794
	$\frac{17}{4320}\pi^2$	0.03883872
	$\frac{61}{10800}\pi^2$	0.05574499
	$\frac{41}{5400}\pi^2$	0.07493588
	$\frac{11}{1440}\pi^2$	0.07539281
	$\frac{17}{2160}\pi^2$	0.07767744
	$\frac{61}{5400}\pi^2$	0.11148998
	$\frac{61}{5400}\pi^2$	0.11148997
	$\frac{17}{1350}\pi^2$	0.12428391
	$\frac{163}{10800}\pi^2$	0.14895792
	$\frac{11}{720}\pi^2$	0.15078562
	$\frac{61}{2700}\pi^2$	0.22297995
	$\frac{17}{675}\pi^2$	0.24856781

5.2 Products of two simplicies in $\mathbb{H}^4$.

Diagram	Volume	Value
8 — 4 — 8	$\frac{11}{1728}\pi^2$	0.06282734
4, 5, 10	$\frac{221}{21600}\pi^2$	0.10098067
4, 4, 8	$\frac{11}{864}\pi^2$	0.12565468
4, 5, 5	$\frac{221}{10800}\pi^2$	0.20196135
5, 5, 10	$\frac{221}{10800}\pi^2$	0.20196135
4, 4, 4	$\frac{11}{432}\pi^2$	0.25130937
5, 5, 5	$\frac{221}{5400}\pi^2$	0.40392269

References

[1] E. M. Andreev, Intersection of plane boundaries of a polytope with acute angles. *Math. Notes* **8** (1970), 761–764. MR0279680 (43 #5401) Zbl 0209.26506

[2] E. Dynkin, Semisimple subalgebras of semisimple Lie algebras. *Am. Math. Soc. Trans.* (II) **6** (1957), 111–243. MR0047629 (13,904c) Zbl 0077.03404

[3] F. Esselmann, The classification of compact hyperbolic Coxeter d-polytopes with $d+2$ facets. *Comment. Math. Helv.* **71** (1996), 229–242. MR1396674 (97j:52015) Zbl 0856.51016

[4] A. A. Felikson, Coxeter decompositions of hyperbolic polygons. *European J. Combin.* **19** (1998), 801–817. MR1649962 (99k:52018) Zbl 0927.51023

[5] A. A. Felikson, Coxeter decompositions of hyperbolic simplexes. Sb. Math. **193** (2002), 1867–1888. Translated from *Mat. Sb.* **193** (2002), 134–156. Eprint math.MG/0210067 MR1992107 (2004e:52013) Zbl 1063.52005

[6] A. A. Felikson, Coxeter decompositions of spherical simplexes with uncut dihedral angles. *Russian Math. Surveys* **57** (2002), 420–421. MR1918206 Zbl 1053.52014

[7] A. Felikson, Coxeter decompositions of compact hyperbolic pyramids and triangular prisms. Math. Notes **75** (2004), 583–593. Translated from *Mat. Zametki* **75** (2004), 624–636. Eprint math.MG/0212195 MR2068772 (2005e:52021)

[8] A. A. Felikson, P. Tumarkin, Reflection subgroups of reflection groups. *Funct. Anal. Appl.* **38** (2004), 313–314. Zbl 1072.20058

[9] N. W. Johnson, J. G. Ratcliffe, R. Kellerhals, S. T. Tschantz, The size of a hyperbolic Coxeter simplex. *Transform. Groups* **4** (1999), 329–353. MR1726696 (2000j:20070) Zbl 0953.20041

[10] I. Kaplinskaja, Discrete groups generated by reflections in the faces of simplicial prisms in Lobachevskian spaces. *Math. Notes* **15** (1974), 88–91. MR0360858 (50 #13305) Zbl 0288.50014

[11] R. Kellerhals, On Schläfli's reduction formula. *Math. Z.* **206** (1991), 193–210. MR1091935 (92e:52017) Zbl 0717.52011

[12] R. Kellerhals, Shape and size through hyperbolic eyes. *Math. Intelligencer* **17** (1995), 21–30. MR1336070 (96b:51024) Zbl 0849.51009

[13] F. Lannér, On complexes with transitive groups of automorphisms. *Comm. Sém., Math. Univ. Lund* [*Medd. Lunds Univ. Mat. Sem.*] **11** (1950), 71. MR0042129 (13,58c) Zbl 0043.17904

[14] L. Schläfli, Theorie der vielfachen Kontinuität. Denkschrift der schweizerischen naturforschenden Gesellschaft, vol. 38, Zürcher and Furrer, 1901, 1–137 (written 1850–1852). Reprint in: L. Schläfli, *Gesammelte Mathematische Abhandlungen*, vol. 1, Birkhäuser, 1950, 167–387. MR0034587 (11,611b) JFM 32.0083.10

[15] E. Vinberg, Hyperbolic reflection groups. *Russ. Math. Surv.* **40** (1985), 31–75. MR783604 (86m:53059) Zbl 0579.51015

[16] E. Vinberg, editor, *Geometry. II*, volume 29 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin 1993. MR1254931 (94f:53002) Zbl 0786.00008

[17] T. Zehrt, Polytopal complexes in spaces of constant curvature. PhD thesis, University of Basel, 2003.

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