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NOTE ON CROSSING CHANGES

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Abstract

For any pair of knots of Gordian distance two, we construct an infinite family of knots which are 'between' these two knots, that is, which differ from the given two knots by one crossing change. In particular, we prove that every knot of unknotting number two can be unknotted via infinitely many different knots of unknotting number one.

1. Introduction and main result

It is an elementary fact that every tame knot in S^3 can be unknotted by a finite sequence of crossing changes. Here, a crossing change is a strand passage operation along an embedded disc (see Fig. 1 for an illustration).

The Gordian unknotting number $u(K)$ of a knot K is the minimal number of crossing changes needed to transform K into the trivial knot [**8**]. More generally, the Gordian distance between two knots is the minimal number of crossing changes needed to transform one knot into the other [**1**, **4**]. However, this process is by far not unique.

THEOREM 1.1 *For every pair of knots K and $\widetilde{K}$ of Gordian distance two, there exist infinitely many non-equivalent knots whose Gordian distance to K and $\widetilde{K}$ is one.*

Theorem 1.1 has an interesting special case.

COROLLARY 1.2 *Every knot of unknotting number two can be unknotted via infinitely many different knots of unknotting number one.*

Proof of Theorem 1.1. If two knots K and $\widetilde{K}$ have Gordian distance two, then there exists a knot K_0 which differs from K and $\widetilde{K}$ by one crossing change. The knot K_0 has a diagram with two distinguished spots where crossing changes should take place to obtain K or $\widetilde{K}$, respectively. This set-up is shown in Fig. 2, where the two spots are labelled A and B; a crossing change at A or B transforms K_0 into K or $\widetilde{K}$, respectively.

Moreover, we may assume that the two spots A and B are neighbouring, as shown in Fig. 3, because we can slide the clasp of one spot, say B, along the knot K_0.

Now we are ready to define a family of knots $\{K_n\}_{n\in\mathbb{N}}$ diagrammatically. A diagram of the knot K_n is shown in Fig. 4. It is understood that it coincides with the diagram of Fig. 3 outside the indicated section.

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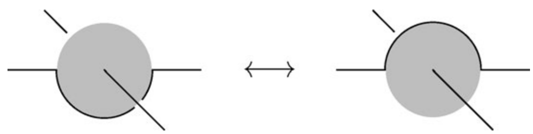

Figure 1. A crossing change along a disc.

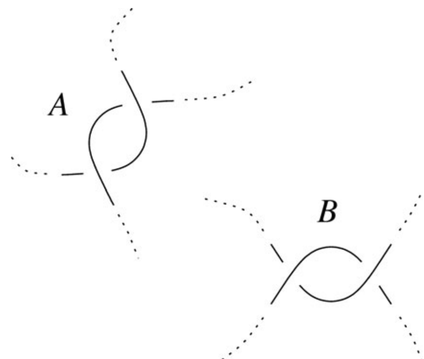

Figure 2. A diagram of K_0.

The horizontal band is twisted $-n$ times, in a manner that gives rise to $2n$ negative crossings. The two vertical bands are twisted n times. We observe that all the knots K_n have the two required properties: changing a crossing at A transforms K_n into K. Similarly, a crossing change at B transforms K_n into $\widetilde{K}$.

However, we have to prove that the family $\{K_n\}_{n\in\mathbb{N}}$ contains infinitely many different knots. For this purpose, we consider the Alexander polynomial written in Conway's notation [**2**]. This polynomial in one variable z is normalized to one for the trivial knot and satisfies the following relation:

$$P_{\times}(z) - P_{\times}(z) = zP_{)(}(z).$$

As we shall apply this skein relation several times, it is convenient to introduce a concise notation for the knots arising from crossing changes and smoothings at different spots.

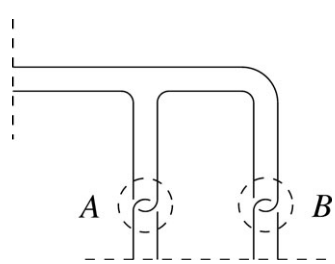

Figure 3. A section of a diagram of K_0.

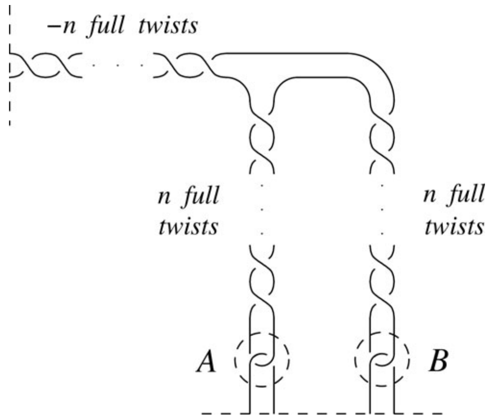

Figure 4. A family of knots $\{K_n\}_{n\in\mathbb{N}}$.

Notation. For $x, y, z \in \mathbb{Z} \cup \{\infty\}$, $K_{x\,y\,z}$ denotes the knot with x full twists in the horizontal band at the top left of the section as shown in Figs 3 and 4, y full twists in the left vertical band and z full twists in the right vertical band. The special case where x (y or z) is ∞ means that we have to smooth the diagram (in an oriented manner) at the corresponding place. As an example, $K_{\infty\,0\,\infty}$ is shown in Fig. 5.

In particular, $K_{-n\,n\,n}$ denotes K_n ($n \in \mathbb{N}$).

LEMMA 1.3

$$\lim_{n\to\infty} \frac{1}{n^2} P(K_n; z) = -z^2 P(K_{\infty 0 \infty}; z).$$

Before we prove Lemma 1.3, let us complete the proof of Theorem 1.1. Suppose the family $\{K_n\}_{n\in\mathbb{N}}$ contained only finitely many different knots, then the limit of Lemma 1.3 would be clearly zero. However, this is not the case because $P(K_{\infty\,0\,\infty}; z)$ is not zero. Indeed, $K_{\infty\,0\,\infty}$ is a connected sum of a knot and two Hopf links. Therefore, its Conway polynomial $P(K_{\infty\,0\,\infty}; z)$ is a product of a Conway polynomial of a knot and $\pm z^2$, which can certainly not be zero.

Proof of Lemma 1.3. Applying the skein relation of the Conway polynomial n times at the crossings of the left vertical band of $K_{-n\,n\,n}$, we get

$$P(K_{-n\,n\,n}; z) = P(K_{-n\,0\,n}; z) + nzP(K_{0\,\infty\,0}; z).$$

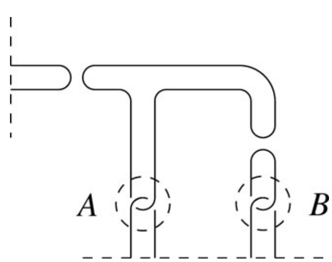

Figure 5. $K_{\infty\,0\,\infty}$.

Continuing at the crossings of the right vertical band of $P(K_{-n\,0\,n}; z)$, we get

$$P(K_{-n\,n\,n}; z) = P(K_{-n\,0\,0}; z) + nzP(K_{-n\,0\,\infty}; z) + nzP(K_{0\,\infty\,0}; z).$$

Finally, we apply the skein relation at the crossings of the horizontal bands of $K_{-n\,0\,0}$ and $K_{-n\,0\,\infty}$ to obtain

$$\begin{aligned} P(K_{-n\,n\,n}; z) = {} & P(K_{0\,0\,0}; z) - nzP(K_{\infty\,0\,0}; z) \\ & + nz(P(K_{0\,0\,\infty}; z) - nzP(K_{\infty\,0\,\infty}; z)) \\ & + nzP(K_{0\,\infty\,0}; z). \end{aligned}$$

We conclude with some remarks and problems about unknotting operations.

Given a knot K of unknotting number two, let $U(K)$ be the set of all knots which lie between K and the trivial knot, that is, knots which can be transformed into K and the trivial knot by one crossing change. According to Theorem 1.1, the set $U(K)$ has infinite (countable) cardinality. In particular, $U(K)$ contains knots of arbitrarily high crossing number. Recently, Uchida found that the set $U(5_1)$ associated with the torus knot 5_1 (in Rolfsen's notation [**7**]) contains knots of arbitrary high bridge number. In contrast, the signature and the four-dimensional genus of all knots in $U(5_1)$ equal one. It might be interesting to study the range of possible values of numerical invariants on the set $U(K)$. We may also ask how much information on a knot K is encoded in the corresponding set $U(K)$.

So far, it seems very hard to compute the unknotting number of knots. A complete list of unknotting numbers exists for knots up to seven crossings only. In the last decade, there has been considerable progress on unknotting numbers of positive knots ([**3**] or [**6**]; see also [**5**] for recent results on unknotting numbers of prime knots). In contrast, very little is known about unknotting numbers of connected sums of positive and negative knots. For example, the unknotting number of the knot $3_1\#!5_1$ is still unknown. Here '#' stands for the connected sum operation and '!' denotes the mirror image operation.

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