

etadata, citation and similar papers at core.ac.uk

Markus Leippold and Liuren Wu*

Abstract

We identify and characterize a class of term structure models where bond yields are quadratic functions of the state vector. We label this class the quadratic class and aim to lay a solid theoretical foundation for its future empirical application. We consider asset pricing in general and derivative pricing in particular under the quadratic class. We provide two general transform methods in pricing a wide variety of fixed income derivatives in closed or semi-closed form. We further illustrate how the quadratic model and the transform methods can be applied to more general settings.

1. Introduction

We identify and characterize a class of term structure models where bond yields are quadratic functions of the Markov process. We label this class the *quadratic class* and aim to lay a solid theoretical foundation for its future empirical application. We identify the necessary and sufficient conditions for the quadratic class and consider the general asset pricing problem under the quadratic framework. In particular, we propose two transform methods to price a wide variety of interest rate derivatives in closed or semi-closed form. We further illustrate how the pricing methods can be applied to more general settings. Examples include option pricing for currencies or stocks with quadratic stochastic volatilities.

Our interests in the quadratic class come mainly from concerns on empirical application. Recent empirical research within the affine framework of Duffie and Kan (1996) indicates an inherent tension between i) delivering good empirical performance in matching salient features of the interest rate data and ii) excluding positive probabilities of having negative interest rates. For example, Backus, Telmer, and Wu (1999) and Dai and Singleton (2000) find that incorporating Gaussian state variables in the affine framework significantly increases the flexibility for model design and greatly improves its empirical performance in capturing the conditional dynamics of interest rates. Dai and Singleton (2001)

*Leippold, leippold@isb.unizh.ch, Swiss Banking Institute, University of Zürich, Plattenstr. 14, 8032 Zürich, Switzerland; and Wu, wu@fordham.edu, Graduate School of Business, Fordham University, 113 West 60th Street, New York, NY 10023. We thank Marco Avellaneda, David Backus, Peter Carr, Pierre Collin, Silverio Foresi, Michael Gallmeyer, Richard Green, Massoud Heidari, Burton Hollifield, Régis Van Steenkiste, Chris Telmer, Stanley Zin, and, in particular, Jon Karpoff (the editor) and Greg Duffee (the referee) for helpful comments.

and Duffee (2001) also find the need for applying an affine market price of risk on Gaussian state variables in explaining the dynamic behavior of expected excess returns to bonds and the anomalies surrounding the various expectations hypotheses. Furthermore, Backus, Foresi, Mozumdar, and Wu (2001) and Backus, Foresi, and Telmer (2001) find that incorporating a negative square-root state variable also helps in explaining the expectation anomalies in both interest rates and currencies. Yet, incorporating either Gaussian variables or negative square-root variables in the affine framework generates negative interest rates with positive probabilities. Therefore, such practices raise concerns, among both practitioners and academics, about potential arbitrage possibilities and their real-time applicability.

In contrast, the quadratic class combines Gaussian state variables with an affine market price of risk into a natural framework and guarantees positive interest rate by a simple parametric restriction. Furthermore, the quadratic relation between interest rates and the normal state variables adds great flexibility for model design. The empirical works of Ahn, Dittmar, and Gallant (2001) and Leippold and Wu (2000) also suggest that quadratic models can outperform affine models in explaining historical bond price behavior in the U.S.

Meanwhile, the analytical tractability of the quadratic class in terms of bond and option pricing is comparable to that of the affine class. We show that, under the quadratic class, the prices of assets, whose future payoffs are exponential-quadratic in the state vector, are exponential-quadratic in the current state. Thus, the price of a zero-coupon bond is merely a degenerating special case. The coefficients for the quadratic functions can be solved analytically for the one-factor case and independent multi-factor cases and are the solutions to a set of ordinary differential equations for general multi-factor cases.

For derivative pricing, we consider the price of a general state-contingent claim and label it as the *state price* in its broadest meaning. A wide variety of fixed income derivatives can be written as an affine function of such a state price. Examples include European options on zero-coupon bonds, interest rate caps and floors, exchange options on zero-coupon bonds, and even Asian options, the payoff of which depends on the path average of bond yields. We define two transforms on the general state price and prove that both can be regarded as an asset with exponential-quadratic payoffs and therefore both can be priced analytically, up to the solution of a series of ordinary differential equations. The state price can then be obtained by a *one-dimensional numerical inversion* of either transform, regardless of the dimension of the state space.

The first transform is similar in nature to the transform defined in Bakshi and Madan (2000) and Duffie, Pan, and Singleton (2000). It regards the state price as an analogue of a cumulative density. The inversion is hence obtained by an extended version of the Lévy inversion formula for cumulative density functions. The second transform is inspired by Carr and Madan (1999) and regards the state price as an analogue of the probability density function. For the second transform to be defined, we need to extend the transform parameter to the complex plane. The transform is hence often referred to as the generalized, or complex, transform. The choice of the imaginary domain depends on the exact structure of the state-contingent payoff. We identify the admissible domain for a wide variety of state-

contingent claims. Given the generalized transform, the inversion can be cast in a way where we can apply the fast Fourier transform (FFT). We can hence rip significant gains in computational efficiency.

The earliest example of quadratic models, to our knowledge, is the double square-root model of Longstaff (1989) and the correction and generalization by Beaglehole and Tenney (1991), (1992). El Karoui, Myneni, and Viswanathan (1992) further develop this quadratic class along the lines of Beaglehole and Tenney (1991). Jamshidian (1996) obtains the ordinary differential equations for bond pricing for the general quadratic class and provides option pricing formulae for a subset of the class (independent Markov process). The SAINTS (squared-autoregressive-independent-variable nominal term structure) model of Constantinides (1992) is also a subset of the quadratic class, where the pricing kernel is exogenously specified as a time-separable quadratic function of the Markov process. Rogers (1997) and Leippold and Wu (1999), starting with modeling the pricing kernel as a potential, also use examples where the pricing kernel is a time-separable quadratic function of the Markov process. Ahn, Dittmar, and Gallant (2001) present a list of assumptions that essentially identify the complete quadratic class. Our paper clarifies the identification problem by proving the necessity and sufficiency of the conditions. Our paper further contributes to the literature by deriving asset pricing implications under the quadratic framework.

Most recently, Filipović (2001) proves, under certain regularity conditions, that if one represents the forward rate as a time-separable polynomial function of the diffusion state vector, the maximal consistent order of the polynomial is *two*. Consistency in this context, as discussed in Björk and Christensen (1999) and Filipović (2000), means that the interest rate model will produce forward rate curves belonging to the parameterized family. Thus, our identification of the quadratic class, together with the identification of the affine class by Duffie and Kan (1996), essentially completes the search for consistent time-separable polynomial term structure models.

The structure of the paper is as follows. The next section identifies the quadratic class, discusses the specification of the pricing kernel, and analyzes the properties of bond yields and forward rates under the quadratic class. Section III considers the general asset pricing problem under the quadratic class and presents our two transform methods of option pricing. Section IV provides numerous examples and other applications. Section V concludes. Additional technical details are provided in the Appendix.

II. Quadratic Term Structure Models

We identify the complete non-degenerating quadratic class of term structure models in terms of the Markov process and the instantaneous interest rate function. We further discuss the specification of the pricing kernel and its impacts on bond pricing, as well as on the pricing of other assets such as currencies and stocks. We then conclude the section by an analysis of the properties of bond yields and forward rates under the quadratic class.

A. Necessary and Sufficient Conditions

We fix a filtered complete probability space $\{\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0 \leq t \leq \mathcal{T}}\}$ satisfying the usual technical conditions¹ with \mathcal{T} being some finite, fixed time. Suppose that X is a Markov process in some state space $\mathcal{D} \subset \mathbb{R}^n$, solving the stochastic differential equation,

$$(1) \quad dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t,$$

where $\mu(X_t)$ is an $n \times 1$ vector defining the drift and $\Sigma(X_t)$ is an $n \times n$ matrix defining the diffusion of the process.

Further assume that for any time $t \in [0, \mathcal{T}]$ and time-of-maturity $T \in [t, \mathcal{T}]$, the market value at time t of a zero-coupon bond with maturity $\tau = T - t$ is fully characterized by $P(X_t, \tau)$ and that the instantaneous interest rate, or the short rate, r , is defined by continuity,

$$r(X_t) \equiv \lim_{\tau \downarrow 0} \frac{-\ln P(X_t, \tau)}{\tau}.$$

Definition 1. In the quadratic class of term structure models, the prices of zero-coupon bonds, $P(X_t, \tau)$, are exponential-quadratic functions of the Markov process X_t ,

$$(2) \quad P(X_t, \tau) = \exp \left[-X_t^\top A(\tau) X_t - \mathbf{b}(\tau)^\top X_t - c(\tau) \right],$$

where $A(\tau)$ is a non-singular $n \times n$ matrix, $\mathbf{b}(\tau)$ is an $n \times 1$ vector, and $c(\tau)$ is a scalar.

$P(X_t, 0) = 1$ for all $X_t \in \mathcal{D}$ implies the boundary conditions: $A(0) = 0$, $\mathbf{b}(0) = 0$, and $c(0) = 0$. By relaxing the non-singularity restriction on $A(\tau)$, we would have the affine class of Duffie and Kan (1996) as a subclass. The affine class is obtained by setting $A(\tau) \equiv 0$ for all τ . A singular $A(\tau)$ matrix would imply a mixture. While we focus on the non-degenerating case to ease deposition and to avoid repetition, relaxing the restriction is straightforward.²

We assume that there exists a so-called risk-neutral measure, or a martingale measure, \mathbb{P}^* , under which the bond price can be written as

$$(3) \quad P(X_t, \tau) = \mathbb{E}^* \left[\exp \left(- \int_t^T r(X_s) ds \right) \middle| \mathcal{F}_t \right],$$

where $\mathbb{E}^*[\cdot]$ denotes expectation under measure \mathbb{P}^* . Under certain regularity conditions, the existence of such a measure is guaranteed by no-arbitrage. The measure is unique when the market is complete. Refer to Duffie (1992) for details. Let $\mu^*(X_t)$ denote the drift function of X_t under measure \mathbb{P}^* . The diffusion function $\Sigma(X_t)$ remains the same under the two measures by virtue of Girsanov’s theorem.

The necessary and sufficient conditions for the quadratic class are identified under measure \mathbb{P}^* .

¹For technical details, see, for example, Jacod and Shiryaev (1987).

²We thank Richard Green, Burton Hollifield, and Stanley Zin for pointing this out.

Proposition 1. The necessary and sufficient conditions for the quadratic class are given by:

i) The instantaneous interest rate $r(X_t)$ is a quadratic function of X_t ,

$$(4) \quad r(X_t) = X_t^\top A_r X_t + \mathbf{b}_r^\top X_t + c_r,$$

with $A_r \in \mathbb{R}^{n \times n}$, $\mathbf{b}_r \in \mathbb{R}^n$ and $c_r \in \mathbb{R}$.

ii) The drift of the Markov process $\mu^*(X_t) = a^* + b^* X_t$, $a^* \in \mathbb{R}$, $b^* \in \mathbb{R}^n$ is affine in X_t .

iii) The diffusion $\Sigma(X_t) \equiv \Sigma \in \mathbb{R}^{n \times n}$ is a constant matrix.

Refer to Part A of the Appendix for the proof. Similar conditions are listed in Ahn, Dittmar, and Gallant (2001), Beaglehole and Tenney (1991), and El Karoui, Myneni, and Viswanathan (1992). We are the first to prove its necessity and sufficiency. Filipović (2001) further proves that the quadratic class represents the highest order of polynomial functions one can apply to consistent time-separable term structure models.

B. The Pricing Kernel

While the conditions are specified under the risk-neutral measure \mathbb{P}^* , for most empirical applications, it is imperative to identify the Markov process under the objective measure \mathbb{P} . To do so, we need to further specify the stochastic process for the *pricing kernel* ξ_t , which relates future cash flows, $K_s, s \in (t, T]$, to today's price, p_t , by

$$p_t = \mathbb{E} \left[\int_t^T \frac{\xi_s K_s ds}{\xi_t} \middle| \mathcal{F}_t \right],$$

where \mathbb{E} is the expectation under the measure \mathbb{P} . Given certain regularity conditions, the conditions for the existence and uniqueness of the martingale measure are equivalent to that for the existence and uniqueness of the pricing kernel. One can perform a multiplicative decomposition on the kernel,

$$\xi_t = \exp \left(- \int_0^t r(X_s) ds \right) M_t,$$

where the variable M_t can be interpreted as the Radon-Nikodým derivative, which takes us from the objective measure \mathbb{P} to the risk-neutral measure \mathbb{P}^* . We can further decompose it into two orthogonal parts,

$$M_t \equiv \frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E} \left(- \int_0^t \gamma(X_s)^\top dW_s \right) \mathcal{E} \left(- \int_0^t \gamma_y^\top dY_s \right),$$

where \mathcal{E} denotes the Doléans exponential³ and $\gamma(X_t)$ is an \mathcal{F}_t -adapted process satisfying the usual regularity conditions and is often referred to as the *market*

³See Jacod and Shiryaev (1987) for a classic reference.

price of risk on the Markov process X . We use Y_t to denote some state vector orthogonal to X_t and γ_y its market price of risk. We leave the vector process Y and its market price of risk unspecified as they do not affect our bond pricing result. Fisher and Gilles (1999) label an independent vector similar to Y as neutrino factors and illustrate how these factors can affect the pricing of other assets such as currencies and stocks, even though they have no effect on bond pricing.

Furthermore, direct application of the Girsanov theorem implies that the drift of the Markov process under measures \mathbb{P} and \mathbb{P}^* are linked by

$$(5) \quad \mu^*(X_t) = \mu(X_t) - \Sigma\gamma(X_t).$$

Hence, for $\mu^*(X_t)$ to be affine in X_t , we require that the affine combination of the drift $\mu(X_t)$ and the market price of risk $\gamma(X_t)$ be affine in X_t . Obviously, there is an infinite number of combinations that result in an affine function in X_t . In principle, one can always augment any arbitrary functions of X_t , say $f(X_t)$, in $\mu(X_t)$ and at the same time add a counterpart in $\gamma(X_t)$, $\Sigma^{-1}f(X_t)$, such that they cancel each other. The pricing of bonds will not be affected. In most empirical applications, one restricts that $\mu(X_t)$ and $\gamma(X_t)$ are non-trivially affine. By non-trivial, we mean that the functions of X_t in $\mu(X_t)$ and $\gamma(X_t)$ do not cancel each other. The canceling function $f(X_t)$ does not affect the term structure as the bond pricing relation only depends on the risk-neutral drift. It does, however, have an impact on the time-series properties of interest rates. For example, if we incorporate a quadratic function into the drift $\mu(X_t)$ and cancel it out through a counterpart in the pricing kernel, interest rates are still quadratic functions of the state vector, yet the time-series properties of the state vector are changed, as they are no longer normally distributed, as implied by an affine drift. While not required, for tractability concerns, one often chooses $\gamma(X_t)$ in such a way that the two functions $\mu(X)$ and $\mu^*(X)$ are of the same type so that the Markov processes are of the same type under the two measures. In addition, the specification of the market price of risk needs to satisfy certain no-arbitrage constraints.

One can also exploit the indeterminacy implied by equation (5) in practical applications. For example, one can specify the objective drift to match the time-series properties of interest rates while the risk-neutral drift matches the cross-sectional property (the term structure) at each day. The difference between the two drifts can then be attributed to the market price of risk. The empirical work by Brandt and Yaron (2001) is analogous in spirit to this philosophy. The outstanding issue, then, is whether the implied market price of risk premium is consistent or supported by any economy.

C. Identification of the Quadratic Class

For tractability, we adopt the “non-canceling” restriction and specify that both the drift $\mu(X_t)$ and the market price of risk $\gamma(X_t)$ are affine in X_t . In particular, under non-degenerating conditions and a possible rescaling and rotation of indices, we can take the Markov process in (1) to have the following simplest possible form,

$$(6) \quad dX_t = -\kappa X_t dt + dW_t,$$

where $\kappa \in \mathbb{R}^{n \times n}$ controls the speed of mean reversion. Such a process is often referred to as a multivariate Ornstein-Uhlenbeck (OU) process. We scale the process to have zero long-run mean and identity instantaneous volatility: $\Sigma(X_t) = I$.⁴

The affine market price of risk is specified as

$$(7) \quad \gamma(X_t) \equiv A_\gamma X_t + \mathbf{b}_\gamma,$$

with $A_\gamma \in \mathbb{R}^{n \times n}$ and $\mathbf{b}_\gamma \in \mathbb{R}^n$. The drift of X under the risk-neutral measure \mathbb{P}^* is hence given by

$$\mu^*(X) = -\mathbf{b}_\gamma - (\kappa + A_\gamma) X.$$

$\kappa^* \equiv \kappa + A_\gamma$ controls the mean-reversion of the Markov process under the risk-neutral measure \mathbb{P}^* . The long run mean of the process under measure \mathbb{P}^* is $-(\kappa + A_\gamma)^{-1} \mathbf{b}_\gamma$.

For identification purposes, we further restrict A_r to be symmetric with no loss of generality as the asymmetric part has zero contribution to the quadratic form. Furthermore, we restrict κ and A_γ to be lower triangular. For the OU process X to be stationary under measure \mathbb{P} , we need all the eigenvalues of κ to be positive, which amounts to a positivity constraint on the diagonal values of the lower triangular matrix. Analogously, for X_t to be stationary under measure \mathbb{P}^* , we need all the eigenvalues of κ^* to be positive.

Under these specifications, the coefficients of bond pricing $\{A(\tau), b(\tau), c(\tau)\}$ are determined by the following ordinary differential equations (ODEs),⁵

$$(8) \quad \begin{aligned} \frac{\partial A(\tau)}{\partial \tau} &= A_r - A(\tau) \kappa^* - (\kappa^*)^\top A(\tau) - 2A(\tau)^2, \\ \frac{\partial \mathbf{b}(\tau)}{\tau} &= \mathbf{b}_r - 2A(\tau) \mathbf{b}_\gamma - (\kappa^*)^\top \mathbf{b}(\tau) - 2A(\tau) \mathbf{b}(\tau), \\ \frac{\partial c(\tau)}{\tau} &= c_r - \mathbf{b}(\tau)^\top \mathbf{b}_\gamma + \text{tr} A(\tau) - \mathbf{b}(\tau)^\top \mathbf{b}(\tau)/2, \end{aligned}$$

subject to the boundary conditions: $A(0) = 0$, $\mathbf{b}(0) = 0$, and $c(0) = 0$. Finite solutions to the ordinary differential equations always exist for $\tau \in [0, \mathcal{T}]$ with some fixed and finite \mathcal{T} . We need further constraint on the parameters to guarantee the existence of a stationary state, i.e., the existence of finite solutions as $\mathcal{T} \rightarrow \infty$. Closed-form solutions exist for one-factor and independent multi-factor cases. See, for example, El Karoui, Myneni, and Viswanathan (1992) and Jamshidian (1996). Solutions for more general cases can readily be computed numerically.

D. Properties of Bond Yields and Forward Rates

Under the non-canceling restriction between the drift $\mu(X)$ and the market price of risk $\gamma(X)$, the Markov process has a constant diffusion matrix and an

⁴Letting \tilde{X} denote a process with a general affine drift $\mu(\tilde{X}) = a + b\tilde{X}$ and a general diffusion matrix $\Sigma(\tilde{X}) = \Sigma$, the process in (6) is obtained by the following linear transformation,

$$X = \Sigma^{-1} \tilde{X} - b^{-1} a,$$

with $\kappa = -\Sigma^{-1} b \Sigma$. Obviously, if the short rate r is quadratic in \tilde{X} , it is also quadratic in its linear transformation X . The transformation is only for identification reasons.

⁵Derivations are available upon request. It follows from equation (19) in the Appendix, Part A.

affine drift under both the objective measure \mathbb{P} and the risk-neutral measure \mathbb{P}^* . Given that the eigenvalues of κ and κ^* are positive, X_t is stationary and distributed multivariate normal both conditionally and unconditionally under both measures,

$$\begin{aligned} \mu &\equiv \mathbb{E}[X_t] = \theta, \\ V &\equiv \mathbb{E}[(X_t - \mu)^2] = \int_0^\infty e^{-s\kappa} e^{-s\kappa^\top} ds, \\ \mu_{t,\tau} &\equiv \mathbb{E}[X_\tau | \mathcal{F}_t] = (I - e^{-\kappa\tau})\theta + e^{-\kappa\tau} X_t, \\ V_\tau &\equiv \mathbb{E}[(X_\tau - \mu_t)^2 | \mathcal{F}_t] = \int_0^\tau e^{-s\kappa} e^{-s\kappa^\top} ds, \end{aligned}$$

where κ is replaced by $\kappa^* \equiv \kappa + A_\gamma$ under measure \mathbb{P}^* and $\theta = 0$ under measure \mathbb{P} and $-(\kappa + A_\gamma)^{-1} \mathbf{b}_\gamma$ under measure \mathbb{P}^* . We drop the subscript t for the conditional variance since it is independent of the current state X_t and is only a function of the time horizon $\tau = T - t$.

Under the quadratic class, the yield $y(X_t, \tau)$ to a zero-coupon bond $P(X_t, \tau)$ is given by

$$y(X_t, \tau) = -\frac{\ln P(X_t, \tau)}{\tau} = \frac{1}{\tau} (X_t^\top A(\tau) X_t + \mathbf{b}(\tau)^\top X_t + c(\tau)).$$

The instantaneous forward rate $f(X_t, \tau)$ with maturity τ is given by

$$f(X_t, \tau) = -\frac{\partial \ln P(X_t, T - t)}{\partial T} = X_t^\top \left[\frac{\partial A(\tau)}{\partial \tau} \right] X_t + \left[\frac{\partial \mathbf{b}(\tau)}{\partial \tau} \right]^\top X_t + \frac{\partial c(\tau)}{\partial \tau}.$$

Therefore, under the quadratic class, both bond yields and forward rates are quadratic forms of normal variates, the properties of which are well-documented in the literature. Reviews of quadratic forms in normal variables may be found, for example, in Holmquist (1996), Johnson and Kotz (1970), Kathri (1980), and Mathai and Provost (1992). In particular, the r th moments and cumulants, as well as their moment-generating functions, are known in closed form.

Property 1. Let \mathbf{x} be an n -dimensional vector having the multivariate normal distribution $N_n(\mu, V)$, let $Q(\mathbf{x}) = \mathbf{x}^\top A \mathbf{x}$, $q_i = \mathbf{x}^\top A_i \mathbf{x}$, and $Q_k = \prod_{i=1}^k \mathbf{x}^\top A_i \mathbf{x} = \prod_{i=1}^k q_i$, where A_i are $n \times n$ matrices. Then

i) The r th cumulant of $Q(\mathbf{x})$ is

$$\kappa_r = 2^{r-1} (r-1)! [\text{tr}((AV)^r) + r \text{tr}((AV)^{r-1} A \mu \mu^\top)], \quad r \geq 1.$$

ii) The moment-generating function of $\mathbf{q} = (q_1, q_2, \dots, q_k)^\top$ is

$$\mathbb{E} \left(e^{\mathbf{s}' \mathbf{q}} \right) = |I_n - C|^{-1/2} \exp \left[-\frac{1}{2} \mu^\top V^{-1} \mu + \frac{1}{2} \mu^\top (I_n - C)^{-1} V^{-1} \mu \right],$$

where $C = \sum_{j=1}^k 2s_j A_j V$.

iii) The expectation of $Q_r = q_1 q_2 \cdots q_r$ is

$$\mathbb{E} [Q_r] = (2r)! \left(\bigotimes_{j=1}^r \mathbf{a}_j \right)^\top S_{n1_{2r}} \sum_{i=0}^r \frac{\mu^{<2r-2i>} \otimes \sigma_2^{<i>}}{i! (2r - 2i)! 2^i},$$

where $\mathbf{a}_i = \text{vec}A_i$ and $\sigma_2 = \text{vec}V$.

The vec operator stacks the columns of the matrix into a vector. \otimes denotes the Kronecker product. $A^{<r>}$ denotes a Kronecker product of the form: $A^{<r>} = \bigotimes_{i=1}^r A = A_1 \otimes A_2 \otimes \cdots \otimes A_r$ with the convention that $A^{<0>} = 1$. The symmetrizer,

$$S_{n1_r} = \sum_{\pi} P_{n1_r}(\pi) / r!,$$

where the summation extends over all $r!$ permutations π in S_r , is a projection operator of the r th tensor power of \mathbb{R}^n onto the r th completely symmetric space over \mathbb{R}^n . Actually, for any matrix A , $\mathbf{x}^\top A \mathbf{x} = \mathbf{x}^\top ((A + A^\top) / 2) \mathbf{x}$ since the asymmetric part $((A - A^\top) / 2)$ of A gives zero contribution: $\mathbf{x}^\top ((A - A^\top) / 2) \mathbf{x} = 0$. In the quadratic class specification, we hence assume A_r being symmetric with no loss of generality. The symmetrizer becomes an identity matrix when the weighting matrix A_i is symmetric.

The fact that moments and cross-moments of all orders for bond yields, forward rates, and bond prices exist in analytical forms illustrates how tractable quadratic models are. Such results on moment conditions not only facilitate our property analysis for the purpose of model design but also simplify estimation, especially when the generalized methods of moments are implemented.

While the analytical tractability of the quadratic model is comparable to that of the affine class, the two classes often imply different behaviors, as discussed below.

1. Nonlinearity in Interest Rate Dynamics: The Role of the Quadratic Term

Numerous studies have documented nonlinearities in the dynamics of interest rates. Examples include Ait-Sahalia (1996), Pfann, Schotman, and Tschernig (1996), Conley, Hansen, Luttmer, and Scheinkman (1997), and Stanton (1997). The quadratic term in bond yields and forward rates provides a direct mechanism to add nonlinearity to the dynamics. As an example, the following proposition illustrates how a one-factor quadratic model can generate rich and nonlinear dynamics in terms of the autocorrelation functions of bond yields.

Proposition 2. A one-factor quadratic model can generate i) a more slowly decaying autocorrelation function than implied by an AR(1) process, and ii) a rich (upward or downward sloping) term structure of persistence for bond yields and forward rates.

Refer to Part B of the Appendix for the proof. Intuitively, a one-factor quadratic model can be thought of as a two-factor affine model with one factor being the original Gaussian factor and the other factor being the square of the Gaussian factor. The autocorrelation function of bond yields is hence a weighted

average of these two factors. Rich dynamics are generated through the interaction between these two factors. For example, to match an autocorrelation, ρ , of an AR(1)-type process, the autocorrelation of the Gaussian factor needs to be set higher due to its weighted average with the square. Yet, the decay of the weighted average is dominated by the Gaussian factor and is hence slower than that of the AR(1) process. Furthermore, the relative weight between the Gaussian factor and its square is determined by their respective coefficients $A(\tau)$ and $\mathbf{b}(\tau)$ and is therefore maturity dependent. Thus, the autocorrelation can be different for yields of different maturities due to the relative weight change although the whole yield curve is driven by merely one Gaussian factor.

The proposition illustrates that even a one-factor quadratic model can generate a rich dynamics for the autocorrelation function and a non-trivial term structure across maturities. Such features cannot be obtained from one-factor AR(1) type models. Within the affine class, one often uses multiple factors to generate the observed nonlinearities in the interest rate dynamics. In contrast, nonlinearity is intrinsically built into the quadratic model.

Most recently, Chapman and Pearson (2000) and Duffee and Stanton (2001) find that econometric problems make even linear models look nonlinear in small samples and thus cast doubt on the robustness of the previous evidence on nonlinearities. Nevertheless, the rich dynamics generated by the quadratic model illustrates its flexibility for model design.

2. Affine Market Price of Risk

Flexible forms for the market price of risk have been proven to be vital in capturing the dynamic behavior of the expected excess returns to bonds. For example, Duffee (2001) finds that the complete affine models of Dai and Singleton (2000) perform poorly in forecasting future changes in Treasury yields in particular because the market price of risk is restricted to be a multiple of the diffusion of the state vector. The performance can be greatly improved by using state variables that have constant diffusions (e.g., the Ornstein-Uhlenbeck process) and applying a general affine market price of risk to such variables. Dai and Singleton (2001) confirm that such a specification adds great ease in capturing both the mean yield curve and the anomalies in the expectation hypothesis. Quadratic models combine Gaussian state variables and market price of risk into a natural framework and hence are poised to perform well in capturing the dynamics behavior of expected excess returns to both bonds and currencies.

Furthermore, if one intends to incorporate an affine market price of risk in the affine class, the diffusion of bond yields is forced to be constant as one is forced to apply the Ornstein-Uhlenbeck process as the state variable,⁶ unless a separate square-root factor is incorporated. In contrast, under the quadratic class, although one is also using the Ornstein-Uhlenbeck process, the quadratic transformation generates state-dependent diffusion for bond yields and forward rates. For example, the diffusion term of the bond yield $y(X_t, \tau)$ is given by $(2A(\tau)X_t + \mathbf{b}(\tau))/\tau$,

⁶As discussed before, this is a must only when one intends to exclude the canceling functions and to retain the affine structure for the drift under both measures.

which is affine in X_t and hence state dependent. Similarly, the diffusion of the forward rate is also affine in X_t .

3. The Virtue of Ornstein-Uhlenbeck Processes

Under the quadratic class, the Markov process follows a multivariate Ornstein-Uhlenbeck (OU) process. It has been found under different circumstances that the OU process renders one with more flexibilities in matching salient features of the interest rate data than the square-root processes also used in affine models. For example, Dai and Singleton (2000) and Backus, Foresi, and Telmer (2001) both observe that the unconditional correlation between two square-root state variables can only be positive, a restriction that runs against evidence. The correlations between OU processes, in contrast, have no such restrictions and can be either negative or positive.

It has been found that flexible correlation structures between state variables significantly improve the model's empirical performance. For example, one often observes a hump shape in the mean term structure of the conditional variance of interest rates. Backus, Telmer, and Wu (1999) find that strong interactions between state variables are necessary to generate such hump dynamics. While a multi-factor correlated square-root process can generate a hump shape, experience indicates that the resulting humps are often not large enough to fit the data. The flexible correlation structure for the multivariate OU process make it a natural choice in capturing such conditional dynamics.

4. Positive Interest Rates

In regard to limits about the square-root process, the OU process has regained its popularity in empirical applications within the affine framework. See the most recent applications in Backus, Telmer, and Wu (1999), Duffee (2001), and Dai and Singleton (2001). In addition, Backus, Foresi, Mozumdar, and Wu (2001) find that some of the limitations of the square-root process can be mitigated by using a negative square-root process as a state variable. However, affine models with either OU processes or negative square-root processes imply positive probabilities of having negative interest rates. While it may be a worthwhile sacrifice if the empirical performance of the model can be significantly improved and if the real probability of having negative interest rates, albeit positive, is small given appropriate choices of parameter estimates, some practitioners and academics alike hold strong opinions against term structure models that imply negative interest rates. Similar concerns also arise when one uses the term structure analogue to model stochastic volatility in currencies and stocks (see Section IV). These concerns can be partially relieved under the quadratic class by restricting A_r to be positive definite and by setting $c_r = \frac{1}{2} \mathbf{b}_r^T A_r^{-1} \mathbf{b}_r$. Under such a restriction, the lower bound for the instantaneous interest rate is zero. Pan (1998) guarantees that the lower bound of all interest rates is zero by further restricting $\mathbf{b}(\tau) = 0$ and $c(\tau) = 0$ for all τ . Such restrictions, however, lead to a degeneration of the quadratic structure such that it is equivalent to a parameterized one-factor affine model.

III. Asset Pricing

In this section, we first extend the bond pricing result to assets with general exponential-quadratic type payoffs and then apply the result to the pricing of state-contingent payoffs.

A. Assets with Exponential-Quadratic Payoffs

Consider an asset which has the following exponential-quadratic payoff structure at time T ,

$$(9) \quad \exp \left(-q_1(X_T) - \int_t^T q_2(X_s) ds \right),$$

where $q_j(X)$ denotes a quadratic function of X , namely $q_j(X) = X^\top A_j X + \mathbf{b}_j^\top X + c_j$. The quadratic form $q_j(X_t)$ can either be regarded as interest rates (bond yield or forward rate) or rates of return on other assets. The integral can be regarded either as an average rate in Asian style payoffs or as a cumulation of continuous payoffs. For example, if we let $q_1(X_T) = \tau y(X_T, \tau)$ and $q_2 = 0$, the payoff would be a τ -year zero-coupon bond; if we set $q_1(X_t) = -\tau y(X_t, \tau)$ and $q_2 = 0$, the payoff would be equivalent to a gross return on a τ -year simple rate,

$$1 + \tau R^\tau = e^{\tau y(X_T, \tau)}.$$

Due to the additivity of quadratic forms, we can also regard q_1 and/or q_2 as linear combinations of many different interest rates (quadratic forms).

We show that assets with such general payoff structures can be priced analytically under the quadratic framework.

Proposition 3. Under the quadratic class, the time- t price of an asset with a payoff function as in (9) is exponential-quadratic in X_t ,

$$(10) \quad \psi \left(q_1 + \int_t^T q_2, \tau \right) \equiv \mathbb{E}^* \left[\exp \left(- \int_t^T r(X_s) ds \right) \times \exp \left(-q_1(X_T) - \int_t^T q_2(X_s) ds \right) \middle| \mathcal{F}_t \right] \\ = \exp \left(-X_t^\top A(\tau) X_t - \mathbf{b}(\tau)^\top X_t - c(\tau) \right).$$

The coefficients $A(\tau)$, $\mathbf{b}(\tau)$, and $c(\tau)$ satisfy the ordinary differential equations in (8) with boundary conditions $A(0) = A_1$, $\mathbf{b}(0) = \mathbf{b}_1$, and $c(0) = c_1$ and with $\{A_r, \mathbf{b}_r, c_r\}$ being replaced by $\{A_r + A_2, \mathbf{b}_r + \mathbf{b}_2, c_r + c_2\}$.

The proof is given in Part C of the Appendix. Note that the price of a zero-coupon bond is just a degenerating special case of the general payoff structure in (9) with $q_1 = q_2 = 0$.

B. State-Contingent Claims

Now consider the time- t price of a contingent claim that pays $\exp(-q_i(X_T))$ at time T in case $q_j(X_T) \leq y$ is true for some fixed number y ,

$$(11) \quad G_{q_i, q_j}(y, \tau) \equiv \mathbb{E}^* \left[\exp \left(- \int_t^T r(X_s) ds \right) e^{-q_i(X_T)} \mathbf{I}_{q_j(X_T) \leq y} \middle| \mathcal{F}_t \right],$$

where y can be regarded as some transform of a strike and \mathbf{I}_x is an indicator function: it equals one when x is true and zero otherwise. As an example, when $y = \infty$, the claim reduces to the asset priced in (10): $G_{q_i, q_j}(\infty, \tau) = \psi(q_i, \tau)$. When we further assume $q_i = 0$, the claim is equivalent to a zero-coupon bond: $G_{0, q_j}(\infty, \tau) = P(X_t, \tau)$. On the other hand, for any fixed number y , if we set $q = 0$, $G_{0, q_j}(y, \tau)$ represents a state price: the price of an asset that pays one dollar if and only if the state event $q_j(X_T) \leq y$ occurs. In what follows, we would refer to $G_{q_i, q_j}(y, \tau)$ as a state price in its broadest meaning. We also relax the notation on quadratic forms and let q_i and q_j denote any quadratic forms, or integral of quadratic forms, or any affine combinations of them. In the next section, we illustrate that many interest rate derivatives such as European options on zero-coupon bonds, interest rate caps and floors, exchange options, and even Asian style options can all be expressed in terms of such a general state price.

In what follows, we define two types of transforms on the state price and prove that both transforms can be regarded as assets with exponential-quadratic payoffs and therefore both can be solved analytically. The state price can then be computed by numerical inversion.

1. Fourier Transform Method

Let $\chi_{q_i, q_j}(z)$ denote the Fourier transform of $G_{q_i, q_j}(y)$ defined as

$$(12) \quad \chi_{q_i, q_j}(z) \equiv \int_{-\infty}^{+\infty} e^{izy} dG_{q_i, q_j}(y), \quad z \in \mathbb{R},$$

where we omit the second argument in τ in the state prices and their transforms in case no confusion occurs. The following proposition derives a closed-form solution for this transform.

Proposition 4. Under the quadratic class, the Fourier transform of the state price $G_{q_i, q_j}(y)$, defined in (12), is equivalent to the price of an asset with exponential-quadratic terminal payoffs,

$$\chi_{q_i, q_j}(z) = \psi(q_i - izq_j).$$

Proof 1. The result is obtained by applying Fubini's theorem and applying the result on the Fourier transform of a Dirac density. \square

The Fourier transform of the state price $G_{q_i, q_j}(y)$ can be regarded as an asset price characterized in (10). Here, of course, the term asset price has to be used with caution since the asset has a complex valued payoff function. But

more importantly, Proposition 4 implies that the Fourier transform of the state-contingent claim retains the exponential-quadratic form and hence the tractability of the quadratic class.

Given the Fourier transform $\chi_{q_i, q_j}(z) = \psi(q_i - izq_j)$, the state price $G_{q_i, q_j}(y)$ can be obtained by an extended version of the Lévy inversion formula, which we prove in the Appendix, Part D.

Proposition 5. The state price $G_{q_i, q_j}(y)$ is given by the following inversion formula,

$$G_{q_i, q_j}(y) = \frac{\chi_{q_i, q_j}(0)}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^{izy} \chi_{q_i, q_j}(-z) - e^{-izy} \chi_{q_i, q_j}(z)}{iz} dz.$$

The above inversion formula involves a numerical integration, similar to the numerical valuation of the normal cumulative densities in the Black-Scholes formula. The prices of many existing fixed income derivatives can be expressed as functions of the general state price $G_{q_i, q_j}(y)$. We can therefore price them through the inversion formula given in Proposition 5. Duffie, Pan, and Singleton (2000) apply a similar approach to asset pricing under an affine jump-diffusion environment.

The key advantage of such a transform method for derivative pricing is its great computational efficiency. In particular, regardless of the dimension of the state space, we only need *one* numerical integration for the inversion. In contrast, methods based on Arrow-Debreu prices⁷ in, for example, Beaglehole and Tenney (1991) and Foresi and Steenkiste (1999), require at least as many numerical integrations (and in general more than) as there are state space dimensions. In these methods, one first prices Arrow-Debreu securities, which are claims with a Dirac function type payoff. Prices of general state-contingent claims are then calculated by integrating the Arrow-Debreu price-weighted cash flows over contingent states. In a general affine or quadratic framework, the Arrow-Debreu price is obtained in a way analogous to our transform method. Further numerical integrations over the state space, and sometimes the time space, need to be performed, finally yielding the state-contingent claim prices.

Obviously, for state-contingent claims of the type in (11), such a procedure is deliberately inefficient. A more direct approach like ours is called for. On the other hand, for claims with more complex payoffs that cannot be represented as a simple function of $G_{q_i, q_j}(y)$, one may need to resort to the Arrow-Debreu price approach or other numerical procedures.

2. Generalized Fourier Transform and FFT

Traditional numerical integration methods for the inversion in Proposition 5, such as the quadrature method used in Singleton (1999) and Duffie, Pan, and Singleton (2000), can be inefficient due to the oscillating nature of the Fourier transform. Instead of working with the inversion formula in Proposition 5, we can also cast the problem in a way such that we can apply the fast Fourier transform

⁷They are also referred to as Green's functions of the state process.

(FFT) and thus take full advantage of its considerable increase in computational efficiency.

For this purpose, let $\varphi_{q_i, q_j}(z)$ denote yet another Fourier transform of $G_{q_i, q_j}(y)$ defined as

$$(13) \quad \varphi_{q_i, q_j}(z) \equiv \int_{-\infty}^{\infty} e^{izy} G_{q_i, q_j}(y) dy, \quad z \in \mathcal{C} \subseteq \mathbb{C}.$$

Comparing the two Fourier transforms defined in (12) and (13), we see that $G_{q_i, q_j}(y)$ is treated as an analogue of a cumulative density function in $\chi_{q_i, q_j}(z)$ while it is treated as a probability density in $\varphi_{q_i, q_j}(z)$. But more importantly, the transform parameter z in (13) is extended to the complex plane with \mathcal{C} being the complex domain of z where $\varphi_{q_i, q_j}(z)$ is well-defined. A Fourier transform that extends to the complex plane is often referred to as the generalized, or complex, Fourier transform. Refer to Titchmarsh (1975) for a comprehensive treatment. As it turns out later, such an extension and the choice of the complex domain are critical for the application of the FFT algorithm.

Proposition 6. Under the quadratic class, the generalized Fourier transform of the state price $G_{q_i, q_j}(y)$ defined in (12), when well-defined, is given by

$$\varphi_{q_i, q_j}(z) = \frac{i}{z} \psi(q_i - izq_j).$$

The result is obtained via integration by parts and Proposition 4,

$$\varphi_{q_i, q_j}(z) = G_{q_i, q_j}(y) \frac{e^{izy}}{iz} \Big|_{-\infty}^{+\infty} - \frac{1}{iz} \int_{-\infty}^{\infty} e^{izy} dG_{q_i, q_j}(y) = \frac{i}{z} \psi(q_i - izq_j).$$

Since $G_{q_i, q_j}(\infty) = \psi(q_i) > 0$, the limit term is well-defined and vanishes only when $\text{Im } z > 0$. In general, the admissible domain \mathcal{C} of z depends on the exact payoff structure of the contingent claim. Table 1 presents the generalized Fourier transforms of various contingent claims and their respective admissible domain for the value of z . Similarly, they are derived via integration by parts and by checking the boundary conditions as $y \rightarrow \pm\infty$. Carr and Madan (1999) consider the special case of pricing a call option on stocks and refer to the imaginary part of z as the *dampening factor* as the call option price needs to be dampened for its transform to be finite.

Let $z = z_r + iz_i$, where z_r and z_i denote, respectively, the real and imaginary part of z . Let $\varphi(z)$ denote the generalized Fourier transform of some state price function $G(y)$, which can be in any of the forms presented in Table 1. Then, given that $\varphi(z)$ is well-defined, the corresponding state price function $G(y)$ is obtained via the inversion formula,

$$G(y) = \frac{1}{2} \int_{iz_i - \infty}^{iz_i + \infty} e^{-izy} \varphi(z) dz.$$

This is an integral along a straight line in the complex z -plane parallel to the real axis. z_i can be chosen to be any real number satisfying the restriction in Table 1 for the corresponding state price function. The integral can also be written as

$$G(y) = \frac{e^{z_i y}}{\pi} \int_0^{\infty} e^{-iz_r y} \varphi(z_r + iz_i) dz_r,$$

TABLE 1
 Generalized Fourier Transforms of Various Contingent Claims
 (α, β, a, b are real constants with $\alpha < \beta$)

Contingent Claim	Generalized Transform $-iz\varphi(z)$	Restrictions on $\text{Im } z$
$G_{q_i, q_j}(y)$	$\psi(q_i - izq_j)$	$(0, \infty)$
$G_{q_i, q_j}(-y)$	$\psi(q_i + izq_j)$	$(-\infty, 0)$
$e^{\alpha y} G_{q_i, q_j}(y)$	$\psi(q_i - (\alpha + iz)q_j)$	(α, ∞)
$e^{\beta y} G_{q_i, q_j}(-y)$	$\psi(q_i + (\beta + iz)q_j)$	$(-\infty, \beta)$
$ae^{\alpha y} G_{q_i, q_j}(y)$	$a\psi(q_i - (\alpha + iz)q_j)$	
$+be^{\beta y} G_{q_i, q_m}(-y)$	$+b\psi(q_i + (\beta + iz)q_m)$	(α, β)

which can be approximated on a finite interval by

$$(14) \quad G(y) \approx \frac{e^{z_i y}}{\pi} \sum_{k=0}^{N-1} e^{-iz_r(k)y} \varphi(z_r(k) + iz_i) \Delta z_r,$$

where $z_r(k)$ are the nodes of z_r and Δz_r the grid of the nodes.

Recall that the FFT is an efficient algorithm for computing the discrete Fourier coefficients. The discrete Fourier transform is a mapping of $\mathbf{f} = (f_0, \dots, f_{N-1})^\top$ on the vector of Fourier coefficients $\mathbf{d} = (d_0, \dots, d_{N-1})^\top$, such that

$$(15) \quad d_j = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-jk \frac{2\pi}{N} i}, \quad j = 0, 1, \dots, N - 1.$$

FFT allows the efficient calculation of \mathbf{d} if N is an even number, say $N = 2^m$, $m \in \mathbb{N}$. The algorithm reduces the number of multiplications in the required N summations from an order of 2^{2m} to that of $m2^{m-1}$, a very considerable reduction. By a suitable choice of Δz_r and a discretization scheme for y , we can cast the approximation in the form of (15) to take advantage of the computing efficiency of FFT. For instance, if we set $z_r(k) = \eta k$ and $y_j = -b + \lambda j$, and require $\eta\lambda = 2\pi/N$, we can cast the state price approximation in (14) into the form of FFT summation in (15),

$$G(y_j) \approx \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-jk \frac{2\pi}{N} i}, \quad j = 0, 1, \dots, N - 1,$$

with

$$f_k = \frac{N}{\pi} e^{z_i y_j + i b z_r(k)} \eta \varphi(z_r(k) + iz_i).$$

Under such a discretization scheme, the effective upper limit for the integration is $a = N\eta$, the range of strike level y is from $-b$ to $N\lambda - b$, with a regular spacing of size λ . The restriction that $\eta\lambda = 2\pi/N$ indicates the trade-off between a fine grid in strike and a fine grid in summation. Thus, if we choose η small to obtain a fine grid for the integration, then we can only compute state prices G at strike spacings

that are relatively large. To increase the accuracy of integration with relative fine grids on strike y , one can also incorporate Simpson's rule into the summation,

$$G(y_j) \approx \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-jk \frac{2\pi}{N} i} \left[1 - \frac{1}{3}(-1)^j - \frac{1}{3}\delta_j \right],$$

where δ_n is the Kronecker delta function that is unity for $n=0$ and zero otherwise. See Carr and Madan (1999) for an application to pricing call options on stocks. Obviously, such an algorithm can also be applied to the affine economy. Application of the FFT algorithm drastically increases the computational efficiency as one can obtain option prices on the whole spectrum of strikes with merely *one* FFT transformation. In particular, in some computing languages (such as MATLAB) that allow for vectorizing the FFT algorithm, option prices on the whole surface of strike and maturity are obtained at one stroke.

IV. Examples and Applications

A. Examples of Fixed Income Derivatives

1. European Options on Bonds

Let C_t denote a European call option at time t on a zero-coupon bond $P(X_t, \tau_p)$ of maturity τ_p . Let τ_c denote the maturity of the call option, and K the strike price. Since the bond price $P(X_T, \tau_p)$ has a quadratic form, we write $P(X_T, \tau_p) \equiv e^{-q(\tau_p)(X_T)}$. Then, from equation (11) we obtain

$$C_t = G_{q(\tau_p), q(\tau_p)}(-\ln K, \tau_c) - KG_{0, q(\tau_p)}(-\ln K, \tau_c).$$

Similarly, the price of a put option on the same bond with the same maturity τ_c and strike price K can be written as

$$P_t = KG_{0, -q(\tau_p)}(\ln K, \tau_c) - G_{q(\tau_p), -q(\tau_p)}(\ln K, \tau_c).$$

The pricing formulae for caps and floors take a similar structure as they can be written as options on bonds.

2. Exchange Options

The payoff of an exchange option on zero-coupon bonds can be written as

$$(m_1 P(X_T, \tau_1) - m_2 P(X_T, \tau_2))^+,$$

which denotes the right to exchange m_2 bonds with maturity τ_2 for m_1 bonds with maturity τ_1 at time $T = t + \tau$. Again, straightforward application of the state price definition yields the price of such an exchange option,

$$M(t, \tau, \tau_1, \tau_2) = m_1 G_{q(\tau_1), q(\tau_1) - q(\tau_2)} \left(\ln \frac{m_1}{m_2} \right) - m_2 G_{q(\tau_2), q(\tau_1) - q(\tau_2)} \left(\ln \frac{m_1}{m_2} \right).$$

Note that given the exponential-quadratic form for the bond prices, the exercise condition $m_1 P(X_T, \tau_1) \geq m_2 P(X_T, \tau_2)$ implies

$$[q(\tau_1) - q(\tau_2)](X_T) \leq \ln \frac{m_1}{m_2}.$$

3. Asian Options

Consider an Asian-style call option whose payoff depends on the path average of the fixed maturity (τ_p) bond yields over the maturity ($\tau_c = T - t$) of the option,

$$\left(\exp \left(-\frac{\tau_p}{\tau_c} \int_t^T y(X_s, \tau_p) ds \right) - K \right)^+.$$

The price of such an option has the representation,

$$C_t^A = G_{\int q(\tau_p), \int q(\tau_p)}(\ln K, \tau_c) - KG_{0, \int q(\tau_p)}(\ln K, \tau_c).$$

An Asian put option is priced in very much the same way,

$$P_t^A = KG_{0, -\int q(\tau_p)}(-\ln K, \tau_c) - G_{\int q(\tau_p), -\int q(\tau_p)}(-\ln K, \tau_c).$$

Given the path dependence of the Asian payoff, the tractability of the pricing relation is remarkable.

B. Option Pricing under Quadratic Stochastic Volatility Models

As the affine jump-diffusion framework has been widely applied to stochastic volatility modeling,⁸ we illustrate how the quadratic framework can also be applied to model stochastic volatility.

Let S denote the price of an asset (stock or exchange rate), which is assumed to satisfy the following stochastic differential equation under measure \mathbb{P}^* ,

$$dS_t/S_t = (r - \delta)dt + \sqrt{v(t)}dZ_t,$$

where Z_t denotes a standard scalar Brownian motion, r the instantaneous interest rate, and δ the continuously compounded dividend yield for stocks and the foreign interest rate for currencies. To be consistent with the quadratic framework, we assume that both r and δ are quadratic functions of Markov process X .

The instantaneous variance rate of the process is denoted by $v(t)$. We allow it to be stochastic and model it by a quadratic function of the Markov process X_t : $v(t) \equiv v(X_t)$,

$$v(X_t) = X_t^\top A_v X_t + b_v^\top X_t + c_v.$$

Similar to instantaneous interest rate, positivity of variance rate can be guaranteed easily by a simple parametric constraint.

Further assume that Z_t is independent of the Brownian motion vector W_t in X_t . The generalized Fourier transform of the log return $s_\tau = \ln S_T/S_t$ over maturity $\tau = T - t$ is given by

$$\begin{aligned} (16) \quad \psi_s(u) &\equiv \mathbb{E}^* \left[e^{ius_\tau} \middle| \mathcal{F}_t \right], \quad u \in \mathbb{C} \\ &= \mathbb{E}^* \left[\exp \left(\int_t^T (iur(X_s) - iu\delta(X_s) - \lambda v(X_s)) ds \right) \middle| \mathcal{F}_t \right], \end{aligned}$$

⁸Prominent examples include, Bakshi, Cao, and Chen (1997), Bates (1996), (2000), and Heston (1993).

where $\lambda = (iu + u^2) / 2$. The last line is obtained by the principle of conditional expectation.

Note that by redefining an interest rate $\tilde{r}(X_t) = -iur(X_t) + iu\delta(X_t) + \lambda v(X_t)$, the transform $\psi_s(u)$ has the same form as the bond pricing formula in (3) and with the same boundary condition: $\psi_s(u) = 1$ at $\tau = 0$. We can thus solve the transform analytically as an exponential-quadratic function of X_t , with coefficients determined by the series of ordinary differential equations in (8).

Under such a setup, interest rates, dividend yields (or foreign interest rates), and stochastic volatility are tightly linked together by the Markov process X_t . Such a tight link, however, can be broken easily, if necessary, by expanding the state vector and assuming that r , δ , and v are each a quadratic form of a subset of the state vector. The subset can be orthogonal or overlapping, depending on the required correlation structure. Alternatively, as is common practice for option pricing on stocks and currencies, we can simply assume constant interest rates and dividend yields and factor out the term $\exp(-(r - \delta)\tau)$. The residual expectation still has the bond pricing form with interest rate redefined as $\tilde{r}(X_t) = \lambda v(X_t)$. An analytical solution is readily obtained. Empirically, Bakshi, Cao, and Chen (1997) have found that incorporating stochastic interest rate does not significantly improve the model's performance in pricing S&P 500 index options.

Given the transform $\psi_s(u)$ on asset returns and analogous to the previous section, we can derive transforms on many state-contingent claims with the asset as the underlying. In particular, consider the time- t price of a contingent claim that pays $\exp(-bs_\tau)$ at time T in case $cs_\tau \leq y$ is true for some fixed number y ,

$$(17) \quad G_{b,c}(y) \equiv e^{-r\tau} \mathbb{E}^* [e^{-bs_\tau} \mathbf{I}_{cs_\tau \leq y} | \mathcal{F}_t],$$

where we assume constant interest rate r for clarity.

The two Fourier transforms of the state price G are given by

$$\begin{aligned} \chi_{b,c}(z) &\equiv \int_{-\infty}^{\infty} e^{izy} dG_{b,c}(y) = e^{-r\tau} \psi_s(zc + bi), \quad z \in \mathbb{R}, \\ \varphi_{b,c}(z) &\equiv \int_{-\infty}^{\infty} e^{izy} G_{b,c}(y) dy = \frac{i}{z} e^{-r\tau} \psi_s(zc + bi), \quad z \in \mathcal{C} \subseteq \mathbb{C}, \end{aligned}$$

where $\psi_s(\cdot)$ is the Fourier transform of the asset return s_τ defined in (16). The proofs are analogous to those for Propositions 4 and 6. If we keep the quadratic interest rate assumption, the interest rate term will be absorbed into a modified transform $\psi_s(\cdot)$. In solving the coefficients for the modified transform $\psi_s(\cdot)$, we need to modify the interest rate term yet again: $\tilde{r}(X_t) = r(X_t) + \tilde{r}(X_t)$.

Given the two transforms, state price $G_{b,c}(y)$ can be solved numerically by either of the two inversion methods proposed in the previous section. Many European style options can be written in terms of $G_{b,c}(y)$. For example, the price of a call option on the asset with strike K can be written as

$$C_t = S_t G_{-1,-1}(\ln(S_t/K)) - K G_{0,-1}(\ln(S_t/K)).$$

The price of a put option with the same strike K is given by

$$P_t = K G_{0,1}(-\ln(S_t/K)) - S_t G_{-1,1}(-\ln(S_t/K)).$$

C. Estimation of Quadratic Models

Regarding $\psi_s(u)$ in (16) as the characteristic function of the return s_τ , we can invert it to obtain the conditional density of the asset return. An analytical form for $\psi_s(u)$ is hence also useful for maximum likelihood calibration of the model. Characteristic functions for bond yields and forward rates can also be obtained from Proposition 10 by setting $r(X_s) = q_2(X_s) = 0$ and letting $q_1(X_T) = -iuy(X_T, \tau_p)$.⁹ For example, Singleton (1999) exploits the knowledge of ψ to derive maximum likelihood estimators for affine models. He obtains the conditional density via inverting the characteristic function. Chacko (1999) and Chacko and Viceira (2000) also propose a spectral generalized methods of moments estimation technique based on the characteristic function.

However, due to the nonlinear relation between yields and the state vector under the quadratic framework, identifying the state variables from the yields becomes a more challenging task. That also limits the application of the maximum likelihood calibration as the conditional densities derived above are conditional on the state vector.

V. Conclusion

We identify and characterize a class of term structure models and price assets with general payoff structures under such a class. In particular, we propose two transform methods for efficiently pricing a wide variety of state-contingent claims. The transform methods can also be applied to econometric estimation and to option pricing on other securities, such as currencies and stocks, with quadratic stochastic volatilities. These results lay a solid foundation for future empirical applications of the quadratic class to term structure modeling, fixed income derivatives pricing, and asset pricing in general.

Appendix

A. Proof of Proposition 1

Assume that the bond pricing formula under the risk-neutral measure yields finite bond prices,

$$P(X_t, \tau) = \mathbb{E}^* \left[\exp \left(- \int_t^T r(X_s) ds \right) \middle| \mathcal{F}_t \right].$$

Applying the Feynman-Kac formula gives

$$(18) \quad r(X_t)P(X_t, \tau) = \frac{\partial P(X_t, \tau)}{\partial t} + \mathcal{A}^*P(X_t, \tau),$$

⁹The unconditional density can also be obtained similarly by letting $T \rightarrow \infty$, given that a stationary state exists. Characteristic functions under the objective measures can also be obtained analogously by replacing $\mu^*(X_t)$ with $\mu(X_t)$ in the partial differential equation in (19) or by setting A_γ and b_γ to zero in the ordinary differential equations in (8).

where A^* denotes the infinitesimal generator on X_t under measure \mathbb{P}^* ,

$$A^*P(X_t, \tau) \equiv \left[\frac{\partial P(X_t, \tau)}{\partial X_t} \right]^\top \mu^*(X_t) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[\frac{\partial^2 P(X_t, \tau)}{\partial X_t \partial X_t^\top} \right]_{ij} \left[\sigma(X_t) \sigma(X_t)^\top \right]_{ij},$$

where the subscript ij denotes the (i, j) th element of the matrix in the bracket. Assume that indeed $P(X_t, \tau)$ has an exponential-quadratic form as in (2), since the instantaneous interest rate $r(X_t)$ is assumed to be well-defined by continuity, the exponential-quadratic form for the bond price implies that $r(X_t)$ is also a quadratic function of X_t . Evaluate the partial derivatives of the bond price $P(X_t, \tau)$ in (2), plug them into the partial differential equation in (18), and rearrange, we have

$$(19) \quad r(X_t) = X_t^\top \left[\frac{\partial A(\tau)}{\partial \tau} \right] X_t + \left[\frac{\partial \mathbf{b}(\tau)}{\partial \tau} \right]^\top X_t + \frac{\partial c(\tau)}{\partial \tau} - [2A(\tau) X_t + \mathbf{b}(\tau)]^\top \mu^*(X_t) - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[2A(\tau) - [2A(\tau) X_t + \mathbf{b}(\tau)] [2A(\tau) X_t + \mathbf{b}(\tau)]^\top \right]_{ij} \times \left[\sigma(X_t) \sigma(X_t)^\top \right]_{ij},$$

for all $\tau \leq T$ and for all $X \in \mathcal{D}$. Under mild non-degeneracy conditions (e.g., $A(\tau)$ being nonsingular), equation (19) and the principle of matching imply that

- i) $\sigma(X_t) \sigma(X_t)^\top$ is a constant matrix, independent of X_t .
- ii) $\mu^*(X_t)$ is affine in X_t .

This provides the necessity part.

Conversely, suppose that $\mu^*(X_t)$ is affine in X_t and $\sigma(X_t)$ is a constant matrix. Consider the candidate exponential-quadratic function for the bond price given in (2) for some $A(\tau)$, $\mathbf{b}(\tau)$, and $c(\tau)$. If we can choose $A(\tau)$, $\mathbf{b}(\tau)$, and $c(\tau)$ so that (19) is satisfied, then the bond price will indeed be exponential-quadratic in X_t . Given that we have a finite solution to the ordinary differential equations in (8), there is indeed a solution for $A(\tau)$, $\mathbf{b}(\tau)$, and $c(\tau)$ satisfying (19), implying that the bond price is exponential-quadratic in X_t , as in (2). This proves the sufficiency part. \square

B. Proof of Proposition 2

Under the specification of a one-factor quadratic term structure model, the variance and auto-covariance of bond yields y_t^τ with maturity τ are given by

$$\begin{aligned} \text{var}(y_t^\tau) &= 2 \left(\frac{A(\tau)}{\tau} V \right)^2 + \left(\frac{\mathbf{b}(\tau)}{\tau} \right)^2 V, \\ \text{cov}(y_{t+n}^\tau, y_t^\tau) &= \phi^{2n} 2 \left(\frac{A(\tau)}{\tau} V \right)^2 + \phi^n \left(\frac{\mathbf{b}(\tau)}{\tau} \right)^2 V, \end{aligned}$$

where $\phi = e^{-\kappa h}$ denotes the autocorrelation of the Markov process X with discrete interval h . The n th order autocorrelation function is defined as

$$\rho(n) = \frac{\text{cov}(y_{t+nh}^\tau, y_t^\tau)}{\text{var}(y_t^\tau)}.$$

Straightforward manipulation yields equation

$$a(\tau)\phi^{2n} + b(\tau)\phi^n = \rho(n),$$

with the weights given by

$$a(\tau) = \frac{2(A(\tau)V)^2}{2(A(\tau)V)^2 + \mathbf{b}(\tau)^2V}, \quad b(\tau) = \frac{\mathbf{b}(\tau)^2V}{2(A(\tau)V)^2 + \mathbf{b}(\tau)^2V}.$$

Note that the weights $a(\tau)$ and $b(\tau)$ are positive and sum to one. In case of the short rate, we replace $A(\tau)/\tau$ and $\mathbf{b}(\tau)/\tau$ with A_r and \mathbf{b}_r .

i) For any AR(1) type process, the n th-order autocorrelation, $\rho_{\text{AR}}(n)$, is equal to the n th-power of its first-order autocorrelation, $\rho_{\text{AR}}(1)^n$:

$$\rho_{\text{AR}}(n) = \rho_{\text{AR}}(1)^n.$$

Letting $\rho(n)$ denote the n th order autocorrelation of the one-factor quadratic model, we claim that, for any order $n \geq 1$, given that $\rho_{\text{AR}}(n) = \rho(n)$, we have $\rho_{\text{AR}}(2n) < \rho(2n)$. To see this, we compute the difference between the two,

$$\begin{aligned} \rho(2n) - \rho_{\text{AR}}(2n) &= \rho(2n) - \rho_{\text{AR}}(n)^2 \\ &= a(\tau)\phi^{4n} + b(\tau)\phi^{2n} - (a(\tau)\phi^{2n} + b(\tau)\phi^n)^2 \\ &= a(\tau)b(\tau) (\phi^{2n} - \phi^n)^2 > 0, \end{aligned}$$

which is always greater than zero since the weights $a(\tau)$ and $b(\tau)$ are positive. Since this result holds for any $n \geq 1$, it implies that the autocorrelation function of an AR(1) specification decays faster than implied by a quadratic one-factor model.

ii) The n th order autocorrelation is determined by the autocorrelation of the state variable X and the relative weight $a(\tau)$ and $b(\tau) = 1 - a(\tau)$, which depends on the maturity of the bond yield. The term structure for the n th order autocorrelation is upward (downward) sloping if $a(\tau)$ decreases (increases) with τ . □

C. Proof of Proposition 3

First, due to the additivity of quadratic forms, we can rewrite the expectation in (10) as

$$\psi(q_1 + \int_t^T q_2, \tau) = \mathbb{E}^* \left[\exp \left(- \int_t^T \tilde{r}(X_s) ds \right) \exp(-q_1(X_T)) \Big| \mathcal{F}_t \right],$$

where $\tilde{r}(X_s) = r(X_s) + q_2(X_s)$ retains the quadratic form of the instantaneous interest rate. Applying the Feynman-Kac formula gives

$$\tilde{r}(X_t)\psi(\cdot, \tau) = \frac{\partial \psi(\cdot, \tau)}{\partial t} + \mathcal{A}^*\psi(\cdot, \tau),$$

an equation analogous to (18). Assume that indeed $\psi(\cdot, \tau)$ has an exponential-quadratic form as in (10), the partial differential equation is reduced to

$$\begin{aligned}
 (20) \quad \tilde{r}(X_t) &= X_t^\top \left[\frac{\partial A(\tau)}{\partial \tau} \right] X_t + \left[\frac{\partial \mathbf{b}(\tau)}{\partial \tau} \right]^\top X_t + \frac{\partial c(\tau)}{\partial \tau} \\
 &+ [2A(\tau)X_t + \mathbf{b}(\tau)]^\top [\kappa^* X_t + \mathbf{b}_\gamma] \\
 &- \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[2A(\tau) - [2A(\tau)X_t + \mathbf{b}(\tau)] [2A(\tau)X_t + \mathbf{b}(\tau)]^\top \right]_{ij},
 \end{aligned}$$

which has the same form as (19), with $r(X_t)$ replaced by $\tilde{r}(X_t)$ and with $\mu^*(X_t) \equiv -b_\gamma + \kappa^* X_t$ and $\sigma(X) \equiv I$ replaced by their respective parametric specifications. Collecting terms, we obtain the same ordinary differential equations as in (8), with only a substitution of $\{A_r, \mathbf{b}_r, c_r\}$ by $\{A_r + A_2, \mathbf{b}_r + \mathbf{b}_2, c_r + c_2\}$ to reflect the instantaneous interest rate adjustment. The boundary conditions also differ to reflect the terminal payoff difference: $A(0) = A_1$, $\mathbf{b}(0) = \mathbf{b}_1$, and $c(0) = c_1$. \square

D. Proof of Proposition 5

To prove the inversion formula for state prices, we follow the proof of the inversion formula for cumulative density functions. See, for example, Chapter 4 of Alan and Ord (1987). The only difference is that the limit of the state prices is given by $\lim_{u \rightarrow +\infty} G_{q_i, q_j}(u) = \psi(q_i)$ while the limit of a cumulative density goes to unity, $\lim_{u \rightarrow +\infty} F(u) = 1$.

We also need the following results,

$$\begin{aligned}
 \frac{1}{\pi} \int_0^\infty \frac{e^{izy} - e^{-izy}}{iz} dz &= \frac{2}{\pi} \int_0^\infty \frac{\sin zy}{z} dz = \operatorname{sgn} y, \\
 \lim_{u \rightarrow -\infty} G_{q_i, q_j}(u) &= 0, \\
 \int_{-\infty}^\infty \operatorname{sgn}(u - y) dG_{q_i, q_j}(u) &= - \int_{-\infty}^y dG_{q_i, q_j}(u) + \int_y^\infty dG_{q_i, q_j}(u) \\
 &= \psi(q_i) - 2G_{q_i, q_j}(y).
 \end{aligned}$$

For a positive number c , the uniformly convergent integral

$$\begin{aligned}
 I_c &\equiv \frac{1}{2\pi} \int_0^c \frac{e^{izy} \chi_{q_i, q_j}(-z) - e^{-izy} \chi_{q_i, q_j}(z)}{iz} dz \\
 &= \frac{1}{2\pi} \int_0^c \frac{e^{izy} \int_{-\infty}^\infty e^{-izu} dG_{q_i, q_j}(u) - e^{-izy} \int_{-\infty}^\infty e^{izu} dG_{q_i, q_j}(u)}{iz} dz \\
 &= \frac{1}{2\pi} \int_0^c \int_{-\infty}^\infty \frac{e^{-iz(u-y)} - e^{iz(u-y)}}{iz} dG_{q_i, q_j}(u) dz \\
 &= \frac{1}{2\pi} \int_0^c \int_{-\infty}^\infty \frac{-2 \sin z(u-y)}{z} dG_{q_i, q_j}(u) dz.
 \end{aligned}$$

Because the integral is uniformly convergent, we may change the order of integration to obtain

$$I_c = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^c \frac{-2 \sin z(u-y)}{z} dz dG_{q_i, q_j}(u).$$

The integral with respect to z is continuous and bounded. We may therefore let c tend to infinity to obtain

$$\begin{aligned} \lim_{c \rightarrow \infty} I_c &= \frac{1}{4} \int_{-\infty}^{\infty} -2 \operatorname{sgn}(u-y) dG_{q_i, q_j}(y) \\ &= -\frac{1}{2} [\psi(q_i) - 2G_{q_i, q_j}(y)]. \end{aligned}$$

We therefore have the result in Proposition 5.

Furthermore, note that $\chi_{q_i, q_j}(z)$ and $\chi_{q_i, q_j}(-z)$ are conjugate quantities and hence, if $\mathcal{R}(z)$ and $\mathcal{I}(z)$ are the real and imaginary parts of $\chi_{q_i, q_j}(z)$, we have

$$G_{q_i, q_j}(y) = \frac{\chi_{q_i, q_j}(0)}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\mathcal{R}(z) \sin yz - \mathcal{I}(z) \cos yz}{z} dz. \quad \square$$

References

- Ahn, D.-H.; R. F. Dittmar; and A. R. Gallant. "Quadratic Term Structure Models: Theory and Evidence." *Review of Financial Studies*, 15 (2001), 243–288.
- Ait-Sahalia, Y. "Testing Continuous-Time Models of the Spot Interest Rate." *Review of Financial Studies*, 9 (1996), 385–426.
- Alan, S., and J. K. Ord. *Kendall's Advanced Theory of Statistics*, 5th ed. New York, NY: Oxford Univ. Press, vol. 1 (1987).
- Backus, D.; S. Foresi; A. Mozumdar; and L. Wu. "Predictable Changes in Yields and Forward Rates." *Journal of Financial Economics*, 59 (2001), 281–311.
- Backus, D.; S. Foresi; and C. Telmer. "Affine Term Structure Models and the Forward Premium Anomaly." *Journal of Finance*, 56 (2001), 279–304.
- Backus, D.; C. Telmer; and L. Wu. "Design and Estimation of Affine Yield Models." Unpubl. Manuscript, New York Univ. (1999).
- Bakshi, G.; C. Cao; and Z. Chen. "Empirical Performance of Alternative Option Pricing Models." *Journal of Finance*, 52 (1997), 2003–2049.
- Bakshi, G., and D. Madan. "Spanning and Derivative-Security Valuation." *Journal of Financial Economics*, 55 (2000), 205–238.
- Bates, D. "Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options." *Review of Financial Studies*, 9 (1996), 69–107.
- _____. "Post-'87 Crash Fears in the S&P 500 Futures Option Market." *Journal of Econometrics*, 94 (2000), 181–238.
- Beaglehole, D. R., and M. Tenney. "General Solution of Some Interest Rate-Contingent Claim Pricing Equations." *Journal of Fixed Income*, 1 (1991), 69–83.
- _____. "A Nonlinear Equilibrium Model of Term Structures of Interest Rates: Corrections and Additions." *Journal of Financial Economics*, 32 (1992), 345–454.
- Björk, T., and B. J. Christensen. "Interest Rate Dynamics and Consistent Forward Rate Curves." *Mathematical Finance*, 9 (1999), 323–348.
- Brandt, M., and A. Yaron. "Time-Consistent No-Arbitrage Models of the Term Structure." Unpubl. Manuscript, Univ. of Pennsylvania (2001).
- Carr, P., and D. B. Madan. "Option Valuation Using the Fast Fourier Transform." *Journal of Computational Finance*, 2 (1999), 61–73.
- Chacko, G. "Continuous-Time Estimation of Exponential Separable Term Structure Models: A General Approach." Unpubl. Manuscript, Harvard Univ. (1999).
- Chacko, G., and L. Viceira. "Spectral GMM Estimation of Continuous-Time Processes." Unpubl. Manuscript, Harvard Univ. (2000).

- Chapman, D. A., and N. D. Pearson. "Is the Short Rate Drift Actually Nonlinear?" *Journal of Finance*, 55 (2000), 355–388.
- Conley, T. G.; L. P. Hansen; E. G. J. Luttmer; and J. A. Scheinkman. "Short-Term Interest Rates as Subordinated Diffusions." *Review of Financial Studies*, 10 (1997), 525–577.
- Constantinides, G. M. "A Theory of the Nominal Term Structure of Interest Rates." *Review of Financial Studies*, 5 (1992), 531–552.
- Dai, Q., and K. Singleton. "Specification Analysis of Affine Term Structure Models." *Journal of Finance*, 55 (2000), 1943–1978.
- _____. "Expectation Puzzles, Time-Varying Risk Premia, and Affine Models of the Term Structure." *Journal of Financial Economics*, (forthcoming 2002).
- Duffee, G. R. "Term Premia and Interest Rate Forecasts in Affine Models." *Journal of Finance*, 57 (2002), 405–443.
- Duffee, G. R., and R. H. Stanton. "Estimation of Dynamic Term Structure Models." Unpubl. Manuscript, Univ. of California, Berkeley (2001).
- Duffie, D. *Dynamic Asset Pricing Theory*, 2nd ed. Princeton, NJ: Princeton Univ. Press (1992).
- Duffie, D., and R. Kan. "A Yield-Factor Model of Interest Rates." *Mathematical Finance*, 6 (1996), 379–406.
- Duffie, D.; J. Pan; and K. Singleton. "Transform Analysis and Asset Pricing for Affine Jump Diffusions." *Econometrica*, 68 (2000), 1343–1376.
- El Karoui, N.; R. Myneni; and R. Viswanathan. "Arbitrage Pricing and Hedging of Interest Rate Claims with State Variables: I Theory." Working Paper, Univ. of Paris (1992).
- Filipović, D. "Exponential-Polynomial Families and the Term Structure of Interest Rates." *Bernoulli*, 6 (2000), 1–27.
- _____. "Separable Term Structures and the Maximal Degree Problems." Unpubl. Manuscript, ETH Zürich, Switzerland (2001).
- Fisher, M., and C. Gilles. "Modeling the State-Price Deflator and the Term Structure of Interest Rates." Unpubl. Manuscript, Federal Reserve Bank of Atlanta/Bear and Stearns & Co. (1999).
- Foresi, S., and R. V. Steenkiste. "Arrow-Debreu Prices for Affine Models." Unpubl. Manuscript, Goldman Sachs and Salomon Smith Barney (1999).
- Heston, S. "Closed-Form Solution for Options with Stochastic Volatility, with Application to Bond and Currency Options." *Review of Financial Studies*, 6 (1993), 327–343.
- Holmquist, B. "Expectations of Products of Quadratic Forms in Normal Variates." *Stochastic Analysis and Applications*, 14 (1996), 149–164.
- Jacod, J., and A. N. Shiryaev. *Limit Theorems for Stochastic Processes*. Berlin: Springer-Verlag (1987).
- Jamshidian, F. "Bond, Futures and Option Valuation in the Quadratic Interest Rate Model." *Applied Mathematical Finance*, 3 (1996), 93–115.
- Johnson, N. L., and S. Kotz. *Distributions in Statistics: Continuous Univariate Distributions 2*. Boston, MA: Houghton Mifflin Co. (1970), ch. 29.
- Kathri, C. G. "Quadratic Forms in Normal Variables." In *Handbook of Statistics*, P. R. Krishnaiah, ed. Amsterdam: North-Holland (1980), vol. 1, 443–469.
- Leippold, M., and L. Wu. "The Potential Approach to Bond and Currency Pricing." Unpubl. Manuscript, Univ. of St. Gallen/Fordham Univ. (1999).
- _____. "Design and Estimation of Quadratic Term Structure Models." Unpubl. Manuscript, Univ. of St. Gallen and Fordham Univ. (2000).
- Longstaff, F. A. "A Nonlinear General Equilibrium Model of the Term Structure of Interest Rates." *Journal of Financial Economics*, 23 (1989), 195–224.
- Mathai, A. M., and S. B. Provost. *Quadratic Forms in Random Variables*. New York, NY: Marcel Dekker (1992).
- Pan, E. "Collapse of Detail." *International Journal of Theoretical and Applied Finance*, 1 (1998), 247–282.
- Pfann, G. A.; P. C. Schotman; and R. Tschernig. "Nonlinear Interest Rate Dynamics and Implications for the Term Structure." *Journal of Econometrics*, 74 (1996), 149–176.
- Rogers, L. C. G. "The Potential Approach to the Term Structure of Interest Rates and Foreign Exchange Rates." *Mathematical Finance*, 7 (1997), 157–176.
- Singleton, K. "Estimation of Affine Asset Pricing Models Using the Empirical Characteristic Function." *Journal of Econometrics*, 102 (2001), 111–141.
- Stanton, R. "A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk." *Journal of Finance*, 52 (1997), 1973–2002.
- Titchmarsh, E. C. *Theory of Fourier Integrals*, 2nd ed. London: Oxford Univ. Press (1975).