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Research Article

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On torsion in finitely presented groups

Abstract: We describe an algorithm that, on input of a recursive presentation P of a group, outputs a recursive presentation of a torsion-free quotient of P, isomorphic to P whenever P is itself torsion-free. Using this, we show the existence of a universal finitely presented torsion-free group; one into which all finitely presented torsion-free groups embed (first proved by Belegradek). We apply our techniques to show that recognising embeddability of finitely presented groups is Π_2^0-hard, Σ_2^0-hard, and lies in Σ_3^0. We also show that the sets of orders of torsion elements of finitely presented groups are precisely the Σ_2^0 sets which are closed under taking factors.

Keywords: Higman's embedding theorem, universal finitely presented group, embeddings, arithmetic hierarchy, torsion

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Dedicated to the memory of Greg Hjorth

1 Introduction

By a finite presentation $\langle X|R\rangle$ of a group we mean, as usual, a finite collection of generators X together with a finite set R of defining relations. A recursive (resp. countably generated recursive) presentation $\langle X|R\rangle$ of a group is a finite (resp. countable) collection of generators X together with a recursive enumeration of a possibly infinite set R of defining relations. We use $\overline{P}$ to denote the group presented by a presentation P.

The Higman embedding theorem [7] shows that every recursively presented group embeds into a finitely presented group. Moreover, this embedding can be made *uniform*; there is an algorithm that takes any recursive presentation P and outputs a finite presentation Q and an explicit embedding $\phi : \overline{P} \hookrightarrow \overline{Q}$. This embedding theorem was used by Higman to show the existence of a *universal* finitely presented group; one into which all finitely presented groups embed. By analysing Higman's embedding theorem, we prove:

Theorem 3.10. *There is a universal finitely presented torsion-free group G. That is, G is torsion-free and, for any finitely presented group H, we have that $H \hookrightarrow G$ if (and only if) H is torsion-free.*

Theorem 3.10 first appeared (as far as we are aware) as Theorem A.1 in the appendix by Oleg V. Belegradek of [1]. He gives a proof different to ours, making use of arguments from model theory. Moreover, in [1, Remark A.2] he points out that Theorem 3.10 can also be proved along the lines we follow in the present paper.

Key to many of the important results in this work is the technical observation that the Higman embedding theorem can preserve the set of orders of torsion elements; we state this as Theorem 2.2. Every group G has a unique torsion-free quotient through which all other torsion-free quotients factor (see Corollary 3.4); we call this the *torsion-free universal quotient* G^{tf}. By standard techniques in combinatorial group theory, we show in Proposition 3.8 the existence of an algorithm that takes any finite presentation P and outputs a recursive presentation P^{tf} of the torsion-free universal quotient of $\overline{P}$. Then Theorem 3.10 follows by combining Theorem 2.2 and Proposition 3.8, in a similar way to Higman's original construction of a universal finitely presented group.

In [9] it was shown by Lempp that the problem of recognising torsion-freeness for finitely presented groups is Π_2^0-complete in Kleene's arithmetic hierarchy (see [11] or the introduction to [9] for a description of Σ_n^0 sets, Π_n^0 sets, and Kleene's arithmetic hierarchy). Therefore the set of finitely presented subgroups of any universal torsion-free finitely presented group is Π_2^0-complete, and, in particular, not recursively enumerable. In [4] we gave another proof of the existence of a finitely presented group whose set of finitely presented subgroups is not recursively enumerable, without the use of the results of Lempp [9] or Oleg Belegradek [1]. Building on Theorem 3.10, we show the following.

Theorem 4.5. *For any recursive enumeration $P_1, P_2, \dots$ of all finite presentations of groups, the set $K = \{(i, j) \in \mathbb{N}^2 \mid \overline{P}_i \hookrightarrow \overline{P}_j\}$ is Σ_2^0-hard, Π_2^0-hard, and has a Σ_3^0 description.*

We write $\mathrm{Tord}(G)$ to denote the orders of non-trivial torsion elements of a group G, and say a set $A \subseteq \mathbb{N}$ is *factor-complete* if it is closed under taking multiplicative factors (excluding 1). Applying Theorem 2.2 to an idea of Dorais in the comments to [8, user 1463], we give the following complete characterisation of sets which can occur as $\mathrm{Tord}(G)$ for a finitely (or recursively) presented group G:

Theorem 5.2. *For a set of natural numbers A, the following conditions are equivalent:*
(1) $A = \mathrm{Tord}(G)$ *for some finitely presented group* G.
(2) $A = \mathrm{Tord}(G)$ *for some countably generated recursively presented group* G.
(3) A *is a factor-complete* Σ_2^0 *set.*

It follows (Corollary 5.5) that we can realise any Σ_2^0 set, up to one-one equivalence, as $\mathrm{Tord}(G)$ for some finitely presented group G.

2 Preliminaries

2.1 Notation

With the convention that $\mathbb{N}$ contains 0, we denote by φ_m the m-th *partial recursive function* $\varphi_m : \mathbb{N} \to \mathbb{N}$. The domain of φ_m, W_m, denotes the m-th *partial recursive set* (also known as a *recursively enumerable set*, abbreviated to r.e. set). A presentation $P = \langle X|R\rangle$ is said to be a *countably generated recursive presentation* if X is a recursive enumeration of generators and R is a recursive enumeration of relators. If P, Q are group presentations then we denote their free product presentation by $P * Q$, which is given by taking the disjoint union of their generators and relators; this extends to the free product of arbitrary collections of presentations. If X is a set, we write X^* for the set of finite words on $X \cup X^{-1}$, including the empty word $\emptyset$. If $\phi : X \to Y^*$ is a set map, then we write $\overline{\phi} : X^* \to Y^*$ for the extension of ϕ to X^*. If $g_1, \dots, g_n$ are elements of a group G, then we write $\langle g_1, \dots, g_n\rangle^G$ for the subgroup in G generated by these elements, and $\langle\langle g_1, \dots, g_n\rangle\rangle^G$ for the normal closure of these elements in G. *Cantor's pairing function* is defined by $\langle \cdot, \cdot\rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $\langle x, y\rangle := \frac{1}{2}(x + y)(x + y + 1) + y$, which gives a computable bijection.

2.2 Embedding theorems

Definition 2.1. Let G be a group. We let $\mathrm{o}(g)$ denote the order of a group element g, and say g is *torsion* if $1 \le \mathrm{o}(g) < \infty$. We set

$$\mathrm{Tor}(G) := \{g \in G \mid g \text{ is torsion}\}, \quad \mathrm{Tord}(G) := \{n \in \mathbb{N} \mid \exists g \in \mathrm{Tor}(G) \text{ with } \mathrm{o}(g) = n \ge 2\}.$$

Thus, $\mathrm{Tord}(G)$ is the set of orders of non-trivial torsion elements of G.

As detailed in [4, Lemma 6.9 and Theorem 6.10], the following theorem is implicit in Rotman's proof [12, Theorem 12.18] of the Higman embedding theorem.

Theorem 2.2. *There is a uniform algorithm that, on input of a countably generated recursive presentation $P = \langle X|R\rangle$, constructs a finite presentation $\mathrm{T}(P)$ such that $\overline{P} \hookrightarrow \overline{\mathrm{T}(P)}$ and $\mathrm{Tord}(\overline{P}) = \mathrm{Tord}(\overline{\mathrm{T}(P)})$, along with an explicit embedding $\overline{\phi} : \overline{P} \hookrightarrow \overline{\mathrm{T}(P)}$.*

We will also use the following consequence of Theorem 2.2.

Theorem 2.3 ([4, Lemma 6.11]). *There is a uniform algorithm that, on input of any $n \in \mathbb{N}$, constructs a finite presentation Q_n such that $\mathrm{Tord}(\overline{Q}_n)$ is one-one equivalent to $\mathbb{N} \setminus W_n$. Thus, taking n' with $W_{n'}$ non-recursive gives that $\mathrm{Tord}(\overline{Q}_{n'})$ is not recursively enumerable; thus, the set of finitely presented subgroups of $\overline{Q}_{n'}$ is not recursively enumerable.*

3 Universal finitely presented torsion-free groups

Let G, H be groups with H torsion-free. A surjective homomorphism $h : G \twoheadrightarrow H$ is *universal* if, for any torsion-free K and any homomorphism $f : G \to K$, there is a homomorphism $\phi : H \to K$ such that $f = \phi \circ h : G \to K$, i.e., the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ & \searrow^{f} & \downarrow{\scriptstyle \phi} \\ & & K \end{array}$$

Note that if ϕ exists then it will be unique. Indeed, if ϕ' also satisfies $f = \phi' \circ h$, then $\phi \circ h = \phi' \circ h$, and hence $\phi = \phi'$ as h is a surjection and thus is right-cancellative. Moreover, any such H is unique, up to isomorphism. Such an H is called the *universal torsion-free quotient* for G, denoted by G^{tf}. Observe that if G is itself torsion-free, then G^{tf} exists and $G^{\mathrm{tf}} \cong G$, as the identity map $\mathrm{id}_G : G \to G$ has the universal property above.

A standard construction, showing that G^{tf} exists for every group G, is done via taking the quotient of G by its *torsion-free radical* $\rho(G)$, where $\rho(G)$ is the intersection of all normal subgroups $N \lhd G$ with G/N torsion-free (see [3]). It follows immediately that $G/\rho(G)$ has all the properties of a torsion-free universal quotient for G.

Here we present an alternative construction for G^{tf} which, though isomorphic to $G/\rho(G)$, lends itself more easily to an effective procedure for finitely (or recursively) presented groups, as shown in Proposition 3.8.

Definition 3.1. Given a group G, we inductively define $\mathrm{Tor}_i(G)$ as follows:

$$\mathrm{Tor}_0(G) := \{e\}, \quad \mathrm{Tor}_{i+1}(G) := \langle\langle \{ g \in G \mid g\,\mathrm{Tor}_i(G) \in \mathrm{Tor}\left(G/\mathrm{Tor}_i(G)\right)\} \rangle\rangle^G, \quad \mathrm{Tor}_\infty(G) := \bigcup_{i \in \mathbb{N}} \mathrm{Tor}_i(G).$$

Thus, $\mathrm{Tor}_i(G)$ is the set of elements of G which are annihilated upon taking i successive quotients of G by the normal closure of all torsion elements, and $\mathrm{Tor}_\infty(G)$ is the union of all these.

By construction, we have $\mathrm{Tor}_i(G) \leq \mathrm{Tor}_j(G)$ whenever $i \leq j$. It follows immediately that $\mathrm{Tor}_\infty(G) \lhd G$. The finite presentation $P := \langle x, y, z \mid x^2, y^3, xy = z^6 \rangle$ defines a group for which $\mathrm{Tor}_1(\overline{P}) \neq \mathrm{Tor}_\infty(\overline{P})$, as shown in [5, Proposition 4.1].

Lemma 3.2. *If G is a group, then $G/\mathrm{Tor}_\infty(G)$ is torsion-free.*

Proof. Suppose $g\,\mathrm{Tor}_\infty(G) \in \mathrm{Tor}\left(G/\mathrm{Tor}_\infty(G)\right)$. Then $g^n\,\mathrm{Tor}_\infty(G) = e$ in $G/\mathrm{Tor}_\infty(G)$ for some $n > 1$, so $g^n \in \mathrm{Tor}_\infty(G)$. Thus there is some $i \in \mathbb{N}$ such that $g^n \in \mathrm{Tor}_i(G)$, and hence $g\,\mathrm{Tor}_i(G) \in \mathrm{Tor}\left(G/\mathrm{Tor}_i(G)\right)$. Thus $g \in \mathrm{Tor}_{i+1}(G) \subseteq \mathrm{Tor}_\infty(G)$, and so $g\,\mathrm{Tor}_\infty(G) = e$ in $G/\mathrm{Tor}_\infty(G)$. □

Proposition 3.3. *If G is a group, then $\rho(G) = \mathrm{Tor}_\infty(G)$.*

Proof. Clearly, $\rho(G) \subseteq \mathrm{Tor}_\infty(G)$ by definition of $\rho(G)$ and the fact that $G/\mathrm{Tor}_\infty(G)$ is torsion-free (Lemma 3.2). It remains to show that $\mathrm{Tor}_\infty(G) \subseteq \rho(G)$. We proceed by contradiction, so assume $\mathrm{Tor}_\infty(G) \nsubseteq \rho(G)$. Then there is some $N \lhd G$ with G/N torsion-free, along with some minimal i such that $\mathrm{Tor}_i(G) \nsubseteq N$ (clearly, $i > 0$, as $\mathrm{Tor}_0(G) = \{e\}$). Then, by definition of $\mathrm{Tor}_i(G)$ and the fact that N is normal, there exists some $e \neq g \in \mathrm{Tor}_i(G)$ such that $g\,\mathrm{Tor}_{i-1}(G) \in \mathrm{Tor}\left(G/\mathrm{Tor}_{i-1}(G)\right)$ and $g \notin N$ (or else $\mathrm{Tor}_i(G) \subseteq N$). But then $g^n \in \mathrm{Tor}_{i-1}(G)$ for some $n > 1$. Since $\mathrm{Tor}_{i-1}(G) \subseteq N$ by minimality of i, we have that gN is a (non-trivial) torsion element of G/N, contradicting the torsion-freeness of G/N. Hence, $\mathrm{Tor}_\infty(G) \subseteq \rho(G)$. □

Corollary 3.4. *If G is a group, then $G/\mathrm{Tor}_\infty(G) = G^{\mathrm{tf}}$, which is the torsion-free universal quotient for G.*

What follows is a standard result, which we state without proof.

Lemma 3.5. *Let $P = \langle X|R\rangle$ be a countably generated recursive presentation. Then the set of words $\{w \in X^* \mid w = e \text{ in } \overline{P}\}$ is r.e.*

Lemma 3.6. *Let $P = \langle X|R\rangle$ be a countably generated recursive presentation. Then the set of words $\{w \in X^* \mid w \in \operatorname{Tor}(\overline{P}) \text{ in } \overline{P}\}$ is r.e.*

Proof. Take any recursive enumeration $\{w_1, w_2, \ldots\}$ of X^*. Using Lemma 3.5, start checking if $w_i^n = e$ in $\overline{P}$ for each $w_i \in X^*$ and each $n \in \mathbb{N}$ (by proceeding along finite diagonals). For each w_i we come across which represents an element of finite order, add it to our enumeration. This procedure will enumerate all words in $\operatorname{Tor}(\overline{P})$, and only words in $\operatorname{Tor}(\overline{P})$. Thus, the set of words in X^* representing elements in $\operatorname{Tor}(\overline{P})$ is r.e. □

From this, we deduce the following lemma.

Lemma 3.7. *Given a countably generated recursive presentation $P = \langle X|R\rangle$, the set $T_i := \{w \in X^* \mid w \in \operatorname{Tor}_i(\overline{P}) \text{ in } \overline{P}\}$ is r.e., uniformly over all i and all such presentations P. Moreover, the union $T_\infty := \bigcup T_i$ is r.e., and is precisely the set $\{w \in X^* \mid w \in \operatorname{Tor}_\infty(\overline{P}) \text{ in } \overline{P}\}$.*

Proof. We proceed by induction. Clearly, $\operatorname{Tor}_1(\overline{P})$ is r.e., as it is the normal closure of $\operatorname{Tor}(\overline{P})$, which is r.e. by Lemma 3.6. So assume that $\operatorname{Tor}_i(\overline{P})$ is r.e. for all $i \leq n$. Then $\operatorname{Tor}_{n+1}(\overline{P})$ is the normal closure of $\operatorname{Tor}(\overline{P}/\operatorname{Tor}_n(\overline{P}))$, which again is r.e. by the induction hypothesis and Lemma 3.6. The remaining parts of the lemma follow immediately. □

Proposition 3.8. *There is a uniform algorithm that, on input of a countably generated recursive presentation $P = \langle X|R\rangle$ of a group $\overline{P}$, outputs a countably generated recursive presentation $P^{\text{tf}} = \langle X|R'\rangle$ (on the same generating set X, and with $R \subseteq R'$ as sets) such that $\overline{P^{\text{tf}}}$ is the torsion-free universal quotient of $\overline{P}$, with associated surjection given by extending $\operatorname{id}_X : X \to X$.*

Proof. By Corollary 3.4, $\overline{P}^{\text{tf}}$ is the group $\overline{P}/\operatorname{Tor}_\infty(\overline{P})$. Then, with the notation from Lemma 3.7, it can be seen that $P^{\text{tf}} := \langle X|R \cup T_\infty\rangle$ is a countably generated recursive presentation for $\overline{P}^{\text{tf}}$, uniformly constructed from P. □

Theorem 3.9. *There is a finitely presentable group G which is torsion free and contains an embedded copy of every countably generated recursively presentable torsion-free group.*

Proof. Take an enumeration $P_1, P_2, \ldots$ of all countably generated recursive presentations of groups, and construct the countably generated recursive presentation $Q := P_1^{\text{tf}} * P_2^{\text{tf}} * \cdots$; this is the countably infinite free product of the universal torsion-free quotient of all countably generated recursively presentable groups (with some repetition). As each P_i^{tf} is uniformly constructible from P_i (by Proposition 3.8), we have that our construction of Q is indeed effective, and hence Q is a countably generated recursive presentation. Also, Proposition 3.8 shows that $\overline{Q}$ is a torsion-free group, as we have successfully annihilated all the torsion in the free product factors, and the free product of torsion-free groups is again torsion-free. Moreover, $\overline{Q}$ contains an embedded copy of every torsion-free countably generated recursively presentable group, as the universal torsion-free quotient of a torsion-free group is itself. Now use Theorem 2.2 to embed $\overline{Q}$ into a finitely presentable group $\overline{T(Q)}$. By construction, $\emptyset = \operatorname{Tord}(\overline{Q}) = \operatorname{Tord}(\overline{T(Q)})$, so $\overline{T(Q)}$ is torsion-free. Finally, $\overline{T(Q)}$ has an embedded copy of every countably generated recursively presentable torsion-free group, since $\overline{Q}$ did. Taking G to be $\overline{T(Q)}$ completes the proof. □

From this, we immediately observe the following consequence.

Theorem 3.10. *There is a universal finitely presented torsion-free group G. That is, G is torsion-free, and for any finitely presented group H, we have that $H \hookrightarrow G$ if (and only if) H is torsion-free.*

Note. One may ask why Theorem 3.10 does not follow immediately from Higman's embedding theorem by taking the free product of all finite presentations of torsion-free groups, and using the fact that Higman's theorem preserves orders of torsion elements. This cannot work, as we show later in Theorem 4.2 that the set of finite presentations of torsion-free groups is not recursively enumerable.

Remark. Miller [10, Corollary 3.14], extending a result of Boone and Rogers [2, Theorem 2], showed that there is no universal finitely presented *solvable word problem* group. It can be shown that none of the following group properties admit a universal finitely presented group: *finite*, *abelian*, *solvable*, *nilpotent* (*simple*, however, remains open).

4 Complexity of embeddings

Using the machinery described in Section 2, we can encode the following recursion theory facts into groups.

Lemma 4.1 ([11, §13.2, Theorem VIII]). *The set $\{n \in \mathbb{N} \mid W_n = \mathbb{N}\}$ is Π_2^0-complete; the set $\{n \in \mathbb{N} \mid |W_n| < \infty\}$ is Σ_2^0-complete.*

We thus can recover the following result, first proved in [9, Main Theorem].

Theorem 4.2. *The set of finite presentations of torsion-free groups is Π_2^0-complete.*

Proof. Given $n \in \mathbb{N}$, we use Theorem 2.3 to construct a finite presentation Q_n such that $\mathrm{Tord}(\overline{Q_n})$ is one-one equivalent to $\mathbb{N} \setminus W_n$. Thus, $\overline{Q_n}$ is torsion-free if and only if $W_n = \mathbb{N}$. By Lemma 4.1, $\{n \in \mathbb{N} \mid W_n = \mathbb{N}\}$ is Π_2^0-complete, so the set of torsion-free finite presentations is at least Π_2^0-hard. But this set has the following Π_2^0 description (taken from [9]):

$$G \text{ is torsion-free if and only if } (\forall w \in G)(\forall n > 0)(w^n \neq_G e \text{ or } w =_G e)$$

and hence is Π_2^0-complete. □

Applying this theorem to the universal torsion-free group from Theorem 3.10, we get the following immediate corollary, which extends Theorem 2.3.

Corollary 4.3. *There is a finitely presented group whose finitely presentable subgroups form a Π_2^0-complete set.*

A construction similar to the proof of Theorem 2.3 (as found in [4, Lemma 6.11]) gives us the following:

Proposition 4.4. *For any fixed prime p, the set of finite presentations into which C_p embeds is Σ_2^0-complete.*

Proof. Given $n \in \mathbb{N}$, we form the countably generated recursive presentation P_n as follows:

$$P_n := \left\langle x_0, x_1, \dots \mid \{x_i^p \mid i \in \mathbb{N}\} \cup \{x_0, \dots, x_j \mid j \in W_n\} \right\rangle$$

If $|W_n| < \infty$ then $\overline{P}_n \cong C_p * C_p * \cdots$. Conversely, if $|W_n| = \infty$ then $\overline{P}_n \cong \{e\}$. So

$$\mathrm{Tord}(\overline{P}_n) = \begin{cases} \{p\} & \text{if } |W_n| < \infty, \\ \emptyset & \text{if } |W_n| = \infty. \end{cases}$$

That is, we have $C_p \hookrightarrow \overline{P}_n$ if and only if $|W_n| < \infty$. Now use Theorem 2.2 to construct a finite presentation $\mathrm{T}(P_n)$ such that $\overline{P}_n \hookrightarrow \overline{\mathrm{T}(P_n)}$ with $\mathrm{Tord}(\overline{P}_n) = \mathrm{Tord}(\overline{\mathrm{T}(P_n)})$. Hence $C_p \hookrightarrow \overline{\mathrm{T}(P_n)}$ if and only if $|W_n| < \infty$. So by Lemma 4.1 the set of finite presentations into which C_p embeds is Σ_2^0-hard. But this set has the following straightforward Σ_2^0 description:

$$C_p \hookrightarrow G \text{ if and only if } (\exists w \in G)(w \neq_G e \text{ and } w^p =_G e)$$

and hence is Σ_2^0-complete. □

Now we can prove the following theorem.

Theorem 4.5. *Take an enumeration $P_1, P_2, \dots$ of all finite presentations of groups, where $P_i = \langle X_i | R_i \rangle$. Then the set $K = \{(i,j) \in \mathbb{N}^2 \mid \overline{P}_i \hookrightarrow \overline{P}_j\}$ is Σ_2^0-hard, Π_2^0-hard, and has a Σ_3^0 description.*

Proof. Corollary 4.3 shows that K is Π_2^0-hard, Proposition 4.4 shows that K is Σ_2^0-hard, and the following is a Σ_3^0 description for the set K:

$$K = \{(i,j) \in \mathbb{N}^2 \mid (\exists \phi : X_i \to X_j^*)(\forall w \in X_i^*)(\overline{\phi}(w) =_{\overline{P}_j} e \text{ if and only if } w =_{\overline{P}_i} e)\}. \qquad \square$$

Note that, with the aid of Cantor's pairing function (a computable bijection between $\mathbb{N}^2$ and $\mathbb{N}$), we can view the set K above as being a subset of $\mathbb{N}$. Hence it makes sense to talk of K being Π_2^0-hard etc.

Based on Theorem 4.5, we conjecture the following:

Conjecture. The set K defined above is Σ_3^0-complete. That is, the problem of deciding for finite presentations P_i, P_j if $\overline{P}_i \hookrightarrow \overline{P}_j$ is Σ_3^0-complete.

5 Complexity of $\mathrm{Tord}(G)$

We now apply our techniques to investigate the complexity of $\mathrm{Tord}(G)$ for G a finitely presented group.

Definition 5.1. Call a set $A \subseteq \mathbb{N}_{\geq 2}$ *factor-complete* if it is closed under taking non-trivial factors. That is, $n \in A$ implies $m \in A$ for all $m > 1$ with $m|n$.

We give a set-theoretic description of the factor-complete sets which can appear as $\mathrm{Tord}(G)$ for G finitely (or recursively) presented. We presented a proof of the following result in [8, user 31415] earlier; what follows is a clearer proof pointed out to us by an anonymous referee.

Theorem 5.2. *For a set of natural numbers A, the following conditions are equivalent:*
(1) *$A = \mathrm{Tord}(G)$ for some finitely presented group G.*
(2) *$A = \mathrm{Tord}(G)$ for some countably generated recursively presented group G.*
(3) *A is a factor-complete Σ_2^0 set.*

Proof. (2) ⇒ (1). By Theorem 2.2, any recursively presented group can be embedded into a finitely presented group with the same Tord.

(1) ⇒ (3). First, observe that $\mathrm{Tord}(G)$ is factor-complete (for any group G), because if $o(g) = mn$ then $o(g^m) = n$, for any $g \in G$. Second, $\mathrm{Tord}(G)$ is a Σ_2^0 set. Indeed, if G has a finite presentation $\langle X|R \rangle$, and S is the set of words in X^* which represent the trivial element in G, then

$$\mathrm{Tord}(G) = \{n \mid (\exists w \in X^*)(n > 1 \wedge w^n \in S \wedge \forall i(0 < i < n \Rightarrow w^i \notin S))\}$$

Since S is r.e. (by Lemma 3.5), it is a Σ_1^0 subset of X^*, and so the result follows.

(3) ⇒ (2). As A is a Σ_2^0 set, it has a description of the form

$$A = \{n \in \mathbb{N} \mid \exists x \forall y R(n,x,y)\}$$

for some ternary recursive relation R on $\mathbb{N}$. Let

$$P := \{(n,m) \in \mathbb{N}^2 \mid (\forall x \leq m)(\exists y) \neg R(n,x,y)\}.$$

Clearly, P is r.e. If $n \notin A$ then $(n,m) \in P$ for all m. Conversely, if $n \in A$ then

$$(n,m) \in P \quad \Leftrightarrow \quad m < m_n := \min\{m \mid (\forall y) R(n,m,y)\}.$$

Let $I := \{(n,m) \in \mathbb{N}^2 \mid n > 1\}$, and let $G := \overline{\langle X|T \rangle}$ where

$$X := \{a_{nm} \mid (n,m) \in I\}, \qquad T := \{a_{nm}^n \mid (n,m) \in I\} \cup \{a_{nm} \mid (n,m) \in I \cap P\}.$$

Clearly, T is r.e., and so G has a countably generated recursive presentation. By the observations above, G can be defined by the generators a_{nm} and relators $a_{nm}^n = e$, where $n \in A$ and $m \geq m_n$. Let K_n denote the free product of countably many cyclic groups C_n of order n. Then G is isomorphic to the free product

$$G \cong *_{n \in A} K_n$$

and therefore

$$\mathrm{Tord}(G) = \bigcup_{n \in A} \mathrm{Tord}(C_n) = \bigcup_{n \in A} \{k \mid k|n \wedge k > 1\} = A;$$

the latter equality holds because A is factor-complete. □

Note. Theorem 5.2 was first proved in the more restricted setting of primes (i.e., considering sets of integers consisting only of primes) by Steinberg and separately by Wilton in response to a question asked by Kohl, see [8]. Moreover, Dorais gave a sketch of an alternate proof of the version for primes in comments to [8, user 1463]. Our original proof was a formalisation of the proof by Dorais, and our result is an extension of this to the more general setting of all factor-complete Σ_2^0 sets. We thank Dorais, Kohl, Steinberg, and Wilton for their online discussion, as well as their insight into key aspects of this result; our work in this section is an extension of their ideas and results.

From the uniformity of the constructions in the proof of Theorem 5.2, we make the following observation.

Proposition 5.3. *The equivalence discussed in Theorem 5.2 is computable, in the following sense:*
(a) *Given a countably generated recursive presentation Q, we can compute from it a finite presentation P with* $\mathrm{Tord}(\overline{P}) = \mathrm{Tord}(\overline{Q})$.
(b) *Given a finite presentation P, we can compute from it a ternary recursive relation R on $\mathbb{N}$ for which* $\mathrm{Tord}(\overline{P}) = \{n \in \mathbb{N} \mid \exists x \forall y R(n, x, y)\}$.
(c) *Given a ternary recursive relation R on $\mathbb{N}$ for which $A := \{n \in \mathbb{N} \mid \exists x \forall y R(n, x, y)\}$ is factor-complete, we can compute from it a countably generated recursive presentation Q with* $\mathrm{Tord}(\overline{Q}) = A$.

We adopt the standard numbering of primes $\{p_i\}_{i \in \mathbb{N}}$, ordered by size; the following lemma is then immediate.

Lemma 5.4. *Let $X \subseteq \mathbb{N}$. Then the set*

$$X_{\mathrm{prime}} := \{p_i \mid i \in X\}$$

is factor-complete and one-one equivalent to X.

Applying Lemma 5.4 to Theorem 5.2, we can conclude the following corollary.

Corollary 5.5. *Given any Σ_2^0 set A, the set A_{prime} is one-one equivalent to A, and can be realised as the set of orders of torsion elements of some finitely presented group G.*

6 Further work

This paper gives rise to several questions. We mention some here.

Problem 1. Given the existence of a universal torsion-free group (Theorem 3.10), and the constructions of explicit finite presentations of universal finitely presented groups by Valiev [13, 14], one could perhaps combine these techniques to produce an explicit finite presentation of a universal torsion-free group.

Problem 2. The positions of the following properties in the arithmetic hierarchy have not been fully determined. Techniques such as those we have covered here may be of use in locating them.
1. *Solvable*: Known to have a Σ_3^0 description.
2. *Residually finite*: Known to have a Π_2^0 description.
3. *Simple*: Known to have a Π_2^0 description.
4. *Orderable*: Known to have a Π_3^0 description (the Ohnishi condition).

Properties 1–3 are mentioned in [10, p. 20], while property 4 appears in [6, Lemma 2.2.1]. We note that it may be very well the case that some of these are neither Π_n^0-complete nor Σ_n^0-complete, for any n.

Problem 3. Considering Theorem 5.5 and the uniformity of such a realisation of a Σ_2^0 set A as one-one equivalent to the torsion orders of a finitely presented group, one could perhaps construct an explicit finite presentation P of a group with $\mathrm{Tord}(\overline{P})$ being Σ_2^0-complete by encoding the set $\{n \in \mathbb{N} \mid |W_n| < \infty\}$, which is Σ_2^0-complete (Lemma 4.1).

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