TAUBERIAN RESULTS FOR DENSITIES WITH GAUSSIAN TAILS

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ABSTRACT

We study a class of probability densities with very thin upper tails. These densities generate exponential families which are asymptotically normal. Furthermore the class is closed under convolution. In this paper we shall be concerned with Abelian and strong Tauberian theorems for moment generating functions and Laplace transforms with respect to these densities. We obtain a duality relation between this class of densities and the associated class of moment generating functions which is closely related to the duality relation for convex functions.

0. *Introduction*

Abelian and Tauberian results relate the asymptotic behaviour of a function to the asymptotic behaviour of the transformed function. In this paper we consider an integrable nonnegative function f on the real line with a very thin upper tail and the transform

$$\hat{f}(\tau) = \int_{-\infty}^{\infty} e^{\tau t} f(t)\, dt. \tag{0.1}$$

We are interested in deriving information about the upper tail of f from the upper tail of the transform $\hat{f}$. Note that $\hat{f}(-\tau)$ is the two-sided Laplace transform of the function f. The most famous Tauberian result for Laplace-transforms goes back to Karamata (see, for example, [**12**]) and links regular variation of f with that of $\hat{f}$; we refer to Feller's lucid exposition (see, for example, [**9**, Chapter 13]) for details. Here we assume not regular variation but rather that f has a very thin tail in the sense that

$$f(t) > 0 \quad t > t_0, \tag{0.2}$$

$$f(t)\, e^{nt} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty, \quad \forall n \geqslant 1. \tag{0.3}$$

This ensures that $\hat{f}(\tau)$ is defined for all $\tau \geqslant 0$.

An example of a function with a very thin tail is $f = e^{-\psi}$ where ψ is a convex function whose slope tends to infinity as $t \to \infty$. We are interested in the situation where ψ is asymptotically parabolic. Intuitively this means that in the neighbourhood of $+\infty$ one can approximate ψ locally by a second degree polynomial. This is made precise in the following definitions.

Recall that a function $s\colon \mathbb{R} \to (0, \infty)$ is *self-neglecting* or *Beurling slowly varying* if

$$\frac{s(t + xs(t))}{s(t)} \longrightarrow 1 \quad \text{as } t \longrightarrow \infty, \quad \text{uniformly on bounded } x\text{-intervals}, \tag{0.4}$$

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or equivalently, if it is asymptotic to a function which has a continuous derivative which vanishes in $+\infty$. Note that this implies that $s(t) = o(t)$, as $t \to \infty$. See [5, Section 2.11] for further information.

DEFINITION. A function $\psi : \mathbb{R} \to \mathbb{R}$ *is asymptotically parabolic* if it has a continuous strictly positive second derivative ψ'' such that the function $s = 1/\sqrt{\psi''}$ is self-neglecting. The function s is called the *scale function* of ψ.

The following result gives an alternative definition of 'asymptotically parabolic'. It reflects the more intuitive description above. We write $f \sim g$ if $f(t)/g(t) \to 1$, as $t \to \infty$.

THEOREM A. *Suppose that ψ is asymptotically parabolic with scale function s. Then there exist affine functions $A_t(x) = a_t x + b_t$ such that*

$$\psi(t + xs(t)) - A_t(x) \longrightarrow \tfrac{1}{2}x^2 \quad \textit{as } t \longrightarrow \infty, \quad \textit{uniformly on bounded } x\textit{-intervals.} \tag{0.5}$$

Conversely if ψ is measurable and if there exist a positive function s and affine functions A_t such that (0.5) *holds uniformly on* $[0, q]$ *for some $q > 0$ then there exists an asymptotically parabolic function ψ_0 such that $\psi_0(t) - \psi(t) \to 0$ and $\psi_0''(t)^{-1/2} \sim s(t)$ as $t \to \infty$, so the two scale functions are asymptotically equivalent.*

Proof. For the first part use $b_t = \psi(t)$, $a_t = \psi'(t)s(t)$ and apply Taylor's formula together with the fact that $\psi''(t)^{-1/2}$ is self-neglecting. The second part is contained in a forthcoming paper of the first author [**1**] and needs rather lengthy constructions.

Examples are $\psi(t) = t^p$ with $p > 1$, and $\psi(t) = e^{\alpha t}$ with $\alpha > 0$.

In order to illustrate the kind of results we are interested in, we formulate the Abelian theorem which describes the asymptotic behaviour of the transform $\hat{f}(\tau)$ for $\tau \to \infty$. This Abelian result which is due to Feigin and Yashchin, see [**8**], will be discussed in greater detail in Section 1.

THEOREM B. *Let $f \geqslant 0$ be an integrable function which satisfies* (0.2) *and* (0.3). *Define $\hat{f}$ by* (0.1). *If $f(t) \sim e^{-\psi(t)}$ with ψ asymptotically parabolic, then $\hat{f}(\tau) \sim e^{\phi(\tau)}$ with ϕ asymptotically parabolic. Moreover,*

$$\hat{f}(\tau) \sim \sqrt{(2\pi)}\, s(t)\, e^{\tau t} f(t), \tag{0.6}$$

where $s = (\psi'')^{-1/2}$ is the scale function of ψ and τ is the slope of ψ at t (that is, $\tau = \psi'(t)$).

In particular if f is the Gaussian density $n_{\mu,\sigma}$ then $\tau = (t - \mu)/\sigma^2$ and $s(t) \equiv \sigma$. In this case relation (0.6) becomes the identity

$$\hat{f}(\tau) = \exp\left(\mu\tau + \tfrac{1}{2}\sigma^2\tau^2\right) = \sqrt{(2\pi)}\, \sigma e^{\tau t}\, n_{\mu,\sigma}(t).$$

The relation (0.6) is an application of Laplace's principle for computing the integral of a unimodal positive function. The definition of an asymptotically parabolic function is exactly the condition which allows the principle to work.

One can describe the relation between the functions ψ and ϕ in the exponents of Theorem B more precisely. Therefore recall that the *Legendre transform* or *convex conjugate* of the convex function ψ is the function $\psi^*(\tau) = \sup_t\{\tau t - \psi(t)\}$. If ψ has a continuous strictly positive second derivative the supremum above is achieved in the uniquely determined point t satisfying $\tau = \psi'(t)$. It is not difficult to see (compare [**3**, Theorem 5.3]) that ψ^* is asymptotically parabolic with scale function $s^* = 1/\sqrt{\psi^{*''}}$ and that t and τ are conjugate variables in the sense that

$$\tau = \psi'(t) \quad \text{and} \quad t = \psi^{*\prime}(\tau). \tag{0.7}$$

Then the scale functions are related by

$$s(t) = 1/s^*(\tau) \tag{0.8}$$

and we have the relation $t\tau = \psi(t) + \psi^*(\tau)$.

In Section 1 we shall explain in more detail the essentially geometric argument which yields the asymptotic expression (0.6) for the moment generating function $\hat{f}$ in terms of the convex conjugate of ψ and its second derivative for a more general class of densities. However the main purpose of this paper is to investigate the following question.

Given an integrable non-negative function f whose moment generating function $\hat{f}$ is defined in a neighbourhood of $+\infty$, and satisfies $\hat{f}(\tau) \sim e^{\phi(\tau)}$, as $\tau \to \infty$, with ϕ asymptotically parabolic, what extra Tauberian conditions will ensure that $f(t) \sim e^{-\psi(t)}$, as $t \to \infty$, with ψ asymptotically parabolic? In each of the Sections 2 to 5 we shall discuss different approaches.

The results are related to Tauberian theorems for Laplace transforms of rapidly decreasing functions. Usually weak Tauberian results are given in this context, relating $\log f$ and $\log \hat{f}$ only, see [**2**, **6**, **13**, **14**], whereas we discuss strong Tauberian conclusions relating f and $\hat{f}$. Results of this kind are of interest in Probability Theory for questions where thin tails of distribution functions are crucial as, for instance, in exponential inequalities, large deviations (see, for example, Feller's book [**9**] or [**11**]) and in the stochastic approach to heat equations (see [**10**, **20**]), but also in Analysis in, for example, Summability Theory (see [**19**]). For our investigations one starting point was a paper of Feigin and Yashchin [**8**], who derive a strong Tauberian theorem for the Laplace transform of probability distributions and densities. As a special case they consider densities with Gaussian tails. In the present paper we treat this important case in more detail. The Abelian part was studied in [**3**], which was the second starting point of this paper.

1. *Basic tools and the Abelian result*

The Introduction contained definitions and two basic results. In the present section we present the tools which we shall use in the ensuing sections to analyse the relation between the asymptotic behaviour of the function f and its transform $\hat{f}$ introduced in (0.1). We shall introduce the probabilistic setting of an exponential family of distribution functions. This probabilistic approach is suggested by the positivity of the function f. It will be validated by the crucial role which the Gaussian probability distribution plays in our analysis. As we shall see at the end of this section Tauberian conditions for (0.6) can be regarded as conditions which upgrade weak convergence of distribution functions to locally uniform convergence of densities.

But at first we shall use the exponential family to sketch a proof of a more general form of the Abelian theorem in the Introduction. The exponential family will allow us to reformulate the Abelian result in terms of convergence of densities.

Consider a probability density f on $\mathbb{R}$ which satisfies (0.2) and (0.3). Now we introduce the exponential family of densities f_τ given by

$$f_\tau(t) = e^{\tau t} f(t)/e^{\chi(\tau)} \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

where we choose χ such that f_τ is a density again, that is, we choose

$$\chi(\tau) = \log\left(\int_{-\infty}^{\infty} e^{\tau t} f(t)\, dt\right).$$

The asymptotic behaviour of f_τ as $\tau \to \infty$ yields an elegant proof for the Abelian result (0.6) when f is a logconcave density.

Assume that $f(t) = e^{-\psi(t)}$ where ψ is convex with a continuous strictly positive second derivative. The density f_τ has a similar form with ψ replaced by $\psi_\tau(u) = \psi(u) - \tau u + c$ where $c = \chi(\tau)$. Note that $\psi_\tau'' = \psi''$. Since we are interested in the behaviour of $\psi_\tau(u)$ as a function of u, the value $c = \chi(\tau)$ is not important for the following, so we do not need to know the function $\chi(\tau)$ explicitly. Under rather weak assumptions on ψ we can approximate the function $\psi_\tau(u)$ near its maximum $u = t$ by the parabola $\psi_\tau(u) = \frac{1}{2}\psi_\tau''(t)(u-t)^2 + b$. Observe that t and b depend on τ. Since t and τ are conjugate variables related by (0.7) we can choose t as our independent variable. If this parabolic approximation is valid on u-intervals of length $c\psi''(t)^{-1/2}$ then we have convergence to the standard normal density $n_{0,1}(u) = (2\pi)^{-1/2}\exp(-u^2/2)$. The obvious normalization by a translation over t and a scaling by $s(t) = 1/\sqrt{\psi''(t)}$ gives

$$f_\tau^0(u) := s(t) f_\tau(t + us(t)) \longrightarrow n_{0,1}(u) \quad \text{as } \tau \longrightarrow \infty.$$

Convexity takes care of all convergence problems and – as it is often the case in probability theory – the constants b and c are taken care of automatically. Furthermore we have uniform convergence on $\mathbb{R}$ by convexity. Indeed for any $M > 0$ we have

$$(f_\tau^0(u) - n_{0,1}(u))\, e^{M|u|} \longrightarrow 0 \quad \text{as } \tau \longrightarrow \infty, \quad \text{uniformly in } u \in \mathbb{R}. \tag{1.2}$$

See [3, Proposition 6.5] for details.

Using the convergence $f_\tau^0(u) \to n_{0,1}(u)$ in $u = 0$ and with $s(t) = (\psi''(t))^{-1/2}$ we find that

$$s(t)\exp(\tau t - \psi(t) - \chi(\tau)) = f_\tau^0(0) \longrightarrow \sqrt{(1/2\pi)}.$$

Now observe that $\exp(\chi(\tau)) = \hat{f}(\tau)$. Then the identity $t\tau = \psi(t) + \psi^*(\tau)$ (see below (0.8)) yields (0.6) in a more specific way,

$$\hat{f}(\tau) \sim \sqrt{(2\pi)}\, e^{\psi^*(\tau)}/s^*(\tau),$$

where $s^*(\tau) = 1/s(t) = \sqrt{\psi''(\psi'^{\leftarrow}(\tau))}$ is the scale function of ψ^*, see (0.8).

An essentially geometric argument has yielded a non-trivial asymptotic expression for the moment generating function $\hat{f}$ in terms of the convex conjugate of the function $\psi = -\log f$ and the second derivative of ψ. The condition which ensures that the argument above is valid is both intuitive and simple. The function $s = 1/\sqrt{\psi''}$ should be self-neglecting and hence ψ should be asymptotically parabolic with scale function s.

The derivation above is a mix of potent ingredients: exponential families, asymptotic normality, the Legendre transform, self-neglecting functions. The active part is Laplace's principle for the evaluation of the integral of a unimodal positive function $g(u) = e^{-\psi(u)+\tau u}$. The recipe is well known. We now formulate the result in purely analytic terms after slightly generalizing our situation.

DEFINITION. Let ψ be asymptotically parabolic with scale function s. A function $h:\mathbb{R} \to (0, \infty)$ is *flat* for ψ if

$$\frac{h(t+xs(t))}{h(t)} \longrightarrow 1 \quad \text{as } t \longrightarrow \infty, \quad \text{uniformly on bounded } x \text{ intervals.} \tag{1.3}$$

REMARK 1.1. A function h which is flat for ψ behaves like a positive constant in asymptotic expressions involving $e^{-\psi}$ or e^{ψ}. The notation of flat functions is convenient but not indispensable. If h is flat for ψ then there exists an asymptotically parabolic function ψ_0 with scale function $s_0(t) \sim s(t)$ as $t \to \infty$, such that $h(t)\,e^{-\psi(t)} \sim e^{-\psi_0(t)}$ as $t \to \infty$ (see [3, Proposition 5.10]).

Finally observe (see [3, Proposition 5.6]) that the positive function h is flat for ψ if and only if the function h^* is flat for the convex conjugate ψ^*, where h and h^* are related by the change of variables (0.7), that is,

$$h(t) = h^*(\tau). \tag{1.4}$$

Examples of flat functions for ψ are $s(t)$, $\psi''(t)$, t and constants α. Sums and products of flat functions are again flat.

Summarizing we obtain the following result; a detailed proof of which can be found in [3].

THEOREM C. *Let $f \geqslant 0$ be an integrable function on $\mathbb{R}$, which satisfies* (0.2) *and* (0.3). *If $f(t) \sim h(t)\,e^{-\psi(t)}$, where ψ is asymptotically parabolic and h is flat for ψ, then $\hat{f}(\tau) \sim \beta(\tau)\,e^{\psi^*(\tau)}$, where ψ^* is the convex conjugate of ψ, β is flat for ψ^* and satisfies*

$$\beta(\tau) = \sqrt{(2\pi)}\,s(t)\,h(t) \tag{1.5}$$

with s the scale function of ψ and τ and t conjugate variables related by (0.7).

EXAMPLE. In general an explicit expression for the Legendre transform of a convex function does not exist. However, if $\psi(t) = (at)^p/p$ with $a > 0$ and $p > 1$ then the Legendre transform $\phi = \psi^*$ has the form $(\alpha\tau)^q/q$ with $\alpha = 1/a$ and $1/q + 1/p = 1$. Moreover the conjugate variables $\tau = \psi'(t) = a^p t^{p-1}$ and t are related by $(at)^p = (\alpha\tau)^q$ and the scale functions s of ψ and σ of ϕ are given by

$$s(t) = \sqrt{((q-1)/a^p)}\,t^{1-p/2}, \quad \sigma(\tau) = \sqrt{((p-1)/\alpha^q)}\,\tau^{1-q/2},$$

hence $\hat{f}(\tau) \sim \sqrt{(2\pi\alpha^q/(p-1))}\,\tau^{q/2-1}\exp((\alpha\tau)^q/q)$.

The probabilistic setup was helpful for deriving the Abelian conclusion and it will be even more helpful for deriving the Tauberian results. In the following we present the details.

Consider a random variable U with distribution function F. We shall again consider the exponential family and now derive asymptotic normality from the tail behaviour of the transform of the distribution function F.

DEFINITION. The distribution function F has a *very thin tail* if the function $1-F$ satisfies (0.2) and (0.3).

The assumption that F has a very thin tail will be made throughout the paper. It implies that the moment generating function $\hat{f}(\tau)$ exists for $\tau \geqslant 0$. Indeed conditions (0.2) and (0.3) on $1-F$ are equivalent to the conditions (0.3) and (0.2) on the function $1/\hat{f}$. The assumption of a very thin tail is not strictly necessary. It is made for the sake of simplicity. The theory (with slight modifications) also holds if the upper endpoint of the distribution function F or of the moment generating function $\hat{f}$ is finite, see [**3**, **8**].

Given the random variable U with distribution function F we define the exponential family of random variables U_τ for $\tau \geqslant 0$, with distribution functions

$$F_\tau(t) = \int_{-\infty}^{t} e^{\tau v}\, dF(v)/e^{\chi(\tau)} \quad \text{for } t \in \mathbb{R}, \tag{1.6}$$

where $\chi = \log \hat{f}$ is the so-called *cumulant generating function* of F. The distribution functions F_τ are also collectively called the *Esscher transform* of F. It plays a fundamental role in large deviation theory, see, for example, Feller [**9**]. The function $\chi(z)$ is analytic on $\Re z > 0$ and the cumulant generating function of U_τ is given by $\chi_\tau(z) = \chi(z+\tau)-\chi(\tau)$. The expectation and variance of the random variable U_τ are obtained by differentiation:

$$\mu(\tau) = \mathscr{E} U_\tau = \chi'(\tau), \quad \sigma^2(\tau) = \operatorname{var} U_\tau = \chi''(\tau). \tag{1.7}$$

The normalized random variables

$$U_\tau^* = \frac{U_\tau - \mathscr{E} U_\tau}{\sqrt{\operatorname{var} U_\tau}} \tag{1.8}$$

have distribution function F_τ^* with density f_τ^* if F is absolutely continuous, moment generating function $\hat{f}_\tau^*$ and cumulant generating function

$$\chi_\tau^*(w) = \chi(\tau + w/\sigma(\tau)) - \chi(\tau) - \frac{\chi'(\tau)\, w}{\sigma(\tau)} = \frac{\chi''(\tau+\theta w/\sigma(\tau))}{\sigma^2(\tau)}\frac{w^2}{2} \tag{1.9}$$

for some $\theta = \theta(w,\tau)$ with $|\theta| \leqslant 1$.

At this point we may apply the classical theory of weak convergence for probability measures and their moment generating functions. From (1.9) it is apparent that $\hat{f}_\tau^*(w) \to e^{w^2/2}$ uniformly on bounded w-intervals in $\mathbb{R}$ if the function $1/\sigma$ is self-neglecting, and then U_τ is asymptotically normal as $\tau \to \infty$.

THEOREM 1.2. *Suppose that the distribution function F has a very thin tail and the moment generating function $\hat{f}$ has the form $\hat{f} \sim \beta e^{\phi}$, where ϕ is asymptotically parabolic and β is flat for ϕ. Let $N_{0,1}$ denote the standard normal distribution function. Then the distribution functions of the normalized random variables U_τ^* in* (1.8) *satisfy*

$$F_\tau^* \xrightarrow{\ w\ } N_{0,1} \quad \text{as } \tau \longrightarrow \infty. \tag{1.10}$$

Proof. By Remark 1.1 we have $\beta e^{\phi} \sim e^{\phi_0}$, where ϕ_0 is asymptotically parabolic with scale function $s_0 \sim s$. Set $U_\tau^0 = (U_\tau - \mu_0)/\sigma_0$ with $\mu_0(\tau) = \phi_0'(\tau)$ and $\sigma_0^2(\tau) = \phi_0''(\tau) = s_0^{-2}(\tau)$. Theorem A yields

$$\phi_0(\tau + us_0) - A_\tau(u) \longrightarrow \tfrac{1}{2}u^2 \quad \text{as } \tau \longrightarrow \infty, \quad \text{uniformly on bounded } u\text{-intervals}$$

for $A_\tau(u) = \phi_0 + u\mu_0 s_0$, and this relation also holds for χ since $\chi - \phi_0 \to 0$. The argument above applies. The moment generating function of U_τ^0 converges to that of the standard Gauss distribution. This gives $U_\tau^0 \xrightarrow{d} N_{0,1}$. Convergence of the moment generating function entails convergence of all moments. Therefore we may use the first two moments to normalize, obtaining (1.10).

REMARK 1.3. (a) The distribution F may be discrete, such as, for instance, the Poisson distribution, or may be absolutely continuous, such as, for example, $f(t) = ce^{-t^p}$ for $t \geqslant 0$ (and $p > 1$) in Theorem 1.2.

(b) If F has a very thin tail, then $U_\tau \xrightarrow{p} \infty$, that is, $P(U_\tau \leqslant M) \to 0$ as $\tau \to \infty$ for all $M > 0$. If (1.10) holds then $\mu(\tau) \to \infty$ and $\sigma(\tau)/\mu(\tau) \to 0$, as $\tau \to \infty$.

To see this let $\tau_n \to \infty$ and assume that $\mu(\tau_n) \leqslant M\sigma(\tau_n)$. Then $F_{\tau_n}(\mu(\tau_n) - M\sigma(\tau_n)) \leqslant F_{\tau_n}(0)$. The left-hand side converges by (1.10) to $N_{0,1}(-M) > 0$ whereas the right-hand side is $P(U_{\tau_n} \leqslant 0) \to 0$ as $n \to \infty$.

REMARK 1.4. From a probabilistic point of view the derivation of the result above can be generalized to the following problem: normalize the family of random variables $\{U_\tau, \tau \geqslant 0\}$ suitably. What are the possible limit distributions? In our case we ended with the normal distribution. Assume now that the random variables $U_\tau^{**} = (\alpha - \tau)\, U_\tau$ for $\tau \in (0, \alpha)$ have densities f_τ^{**} which converge as $\tau \to \alpha$, that is,

$$f_\tau^{**}(u) = e^{-u} f\left(\frac{u}{\alpha - \tau}\right) \exp\left(\frac{\alpha u}{\alpha - \tau}\right) \Big/ (\hat{f}(\tau)\,(\alpha - \tau)) \longrightarrow f^{**}(u).$$

Putting $1/(\alpha - \tau) = \xi$ we find that $f(\xi u)\, e^{\alpha u \xi}/C(\xi) \to e^u f^{**}(u)$ as $\tau \to \alpha$, where the exact form of C is not relevant. This implies that $f(\xi u)\, e^{\alpha u \xi}$ is regularly varying and $e^u f^{**}(u) = cu^{\beta - 1}$ with some $\beta > 0$ (see, for example, [5]). Hence the convergence is uniform on compact subsets and $f^{**}(u) = e^{-u} u^{\beta-1}/\Gamma(\beta)$ for $u > 0$, that is, is a Gamma-density. This is the link between Karamata's theory of regular variation and the theory of Gaussian tails presented in this paper.

It is simple to check that the gamma distributions on $(0, \infty)$ or on $(-\infty, 0)$ generate an exponential family of gamma distributions with constant shape parameter $\beta > 0$; also the Gauss distributions generate an exponential family of Gauss distributions. It is known (see, for example, [15, p. 36]) that these are the only instances of exponential group families.

We shall now investigate the asymptotic normality of the exponential family F_τ from the point of view of densities. Let f_τ^* denote the density of U_τ^* as defined in (1.8), that is, $f_\tau^*(u) = \sigma(\tau) \exp(\tau(\mu(\tau) + u\sigma(\tau))) f(\mu(\tau) + u\sigma(\tau))/\hat{f}(\tau)$.

DEFINITION. A probability density f as a *Gaussian tail* if it satisfies (0.2) and (0.3) and if $f(t) \sim e^{-\psi(t)}$ with some asymptotically parabolic-function ψ.

The notation becomes clear from the next result.

PROPOSITION 1.5. *If the probability density f has a Gaussian tail then*

$$f_\tau^*(u) \longrightarrow n_{0,1}(u) \quad \text{as } \tau \longrightarrow \infty, \quad \text{locally uniformly in } u, \tag{1.11}$$

where f_τ^* is the density of the normalized random variables U_τ^* in (1.8).

Conversely suppose that the distribution function F has a very thin tail and is absolutely continuous. If there exist norming functions $a(\tau) > 0$ and $b(\tau)$ such that the densities of $(U_\tau - b(\tau))/a(\tau)$ converge to the standard Gaussian density uniformly on bounded u-intervals, then f has a Gaussian tail.

Proof. We have sketched the proof for the first statement above; for full details see [3]. The converse conclusion is based on Theorem A. The uniform convergence on bounded u-intervals,

$$af(b+au)\exp((b+au)\tau - \chi(\tau)) \longrightarrow e^{u^2/2}/\sqrt{(2\pi)} \quad \text{as } \tau \longrightarrow \infty,$$

gives the relation $\phi(b(\tau)+a(\tau)u)-(q(\tau)+p(\tau)u) \to \frac{1}{2}u^2$ for the logarithms. Note that b tends to infinity by Remark 1.3(b). If b is continuous we may apply Theorem A. We choose $b(\tau)$ to be the median of U_τ. Then b is strictly increasing. Let $t_n \to \infty$; we can choose τ_n so that $F_{\tau_n}(t_n) = \frac{1}{2}$. (Apply the Mean Value Theorem to the continuous function $e^{-\chi(\tau)}\int_{-\infty}^{t_n} e^{\tau u}\,dF(u)$.) This implies that $f_{\tau_n}(t_n) \sim 1/\sqrt{(2\pi)}\,a(\tau_n) > 0$. Hence (0.2) holds on a neighbourhood of $+\infty$. This makes b eventually continuous. Theorem A now implies that $f \sim e^{-\phi_0}$, where ϕ_0 is asymptotically parabolic. In particular ϕ_0 is convex and the limit $\phi_0'(\infty) \leqslant \infty$ exists. Since the distribution function has a very thin tail this limit is infinite and f satisfies (0.3).

REMARK 1.6. (a) If f has a Gaussian tail, then for any $M > 0$ we have

$$|f_\tau^*(u) - n_{0,1}(u)|\, e^{M|u|} \longrightarrow 0 \quad \text{as } \tau \to \infty, \quad \text{uniformly in } u \in \mathbb{R}. \tag{1.12}$$

See (1.2) above and [3, Proposition 6.5].

(b) In particular formula (1.11) gives a tool for relating the asymptotic behaviour of f and $\hat{f}$, namely, putting $u = 0$ we obtain

$$\sigma(\tau)\,e^{\tau\mu(\tau)} f(\mu(\tau))/\hat{f}(\tau) \longrightarrow \frac{1}{\sqrt{(2\pi)}} \quad \text{as } \tau \longrightarrow \infty, \tag{1.13}$$

where $\mu(\tau) = \chi'(\tau)$. Now we again have a formula relating the asymptotics of f and $\hat{f}$.

REMARK 1.7. If F is a distribution function with $1-F(t) \sim e^{-\psi(t)}$, and ψ is asymptotically parabolic, investigating the right tail of F, we have by partial integration

$$\hat{f}(\tau) = \tau \int_{-\infty}^{\infty} e^{t\tau}(1-F(t))\,dt$$

and we would consider the density

$$\tilde{f}_\tau(u) = \tau(1-F(u))\,e^{\tau u}/\hat{f}(\tau),$$

with expectation $\mu(\tau) - 1/\tau$ and variance $\sigma^2(\tau) + 1/\tau^2$ (compare [8]) and similar considerations would apply.

We now return to the question posed at the end of the Introduction. Start with an integrable function $f \geqslant 0$ on $\mathbb{R}$ and assume that the transform $\hat{f}$ defined in (0.1) has the form $\hat{f} \sim \beta e^{\phi}$, where ϕ is asymptotically parabolic and β is flat for ϕ. Also assume that $1/\hat{f}$ satisfies (0.2) and (0.3). What extra (Tauberian) condition will ensure that the source function f satisfies (0.2) and (0.3) and has the form $f \sim e^{-\psi}$, where ψ is asymptotically parabolic?

On scaling by a suitable function we may assume that f is a probability density. The conditions on $\hat{f}$ imply that the distribution function F has a very thin tail, and by Theorem 1.2 that the associated exponential family U_τ is asymptotically normal in the sense of (1.10). What additional conditions will ensure convergence of the densities (uniformly on bounded intervals)? By Proposition 1.5 such a condition ensures that f has a Gaussian tail in the sense that (0.2) and (0.3) hold and that $f \sim e^{-\psi}$ for some asymptotically parabolic function ψ. This results in the following important observation: any condition which upgrades (1.10) to convergence of densities automatically is a Tauberian condition for the Theorems B and C.

2. *Convexity*

In the theory of regular variation, monotonicity is used as a Tauberian condition. In our situation convexity assumptions can be applied. The following result is essentially due to Feigin and Yashchin [**8**].

THEOREM D. *Let the distribution function F have a very thin tail, that is,* $1-F$ *satisfies* (0.2) *and* (0.3), *and suppose that the cumulant generating function is asymptotically parabolic. If F has a density f which is logconcave on a neighbourhood of infinity, then f has a Gaussian tail.*

The arguments of Section 1 allow us to reduce Theorem D to a simple result on logconcave densities. Since we could not find it in the literature we provide a proof. Compare also [**3**; **8**; **17**, Theorem 25.7].

LEMMA 2.1. *Consider a sequence of distribution functions* $(F_n)_{n\in\mathbb{N}}$ *which have logconcave densities* $(f_n)_{n\in\mathbb{N}}$. *If* $F_n \overset{w}{\to} N_{0,1}$, *then* $\sup_x |f_n(x) - n_{0,1}(x)|\, e^{M|x|} \to 0$ *as* $n \to \infty$ *for any* $M > 0$.

Proof. Write $f_n(x) = e^{-\psi_n(x)}$ for $x \in \mathbb{R}$, with convex functions $\psi_n(x) \leqslant \infty$. Since $F_n \overset{d}{\to} N_{0,1}$, the sequence $\psi_n(x)$ has to be bounded from below and above for any fixed x (otherwise $\lim_{n\to\infty} F_n(x)$ would have a jump or be zero below a certain point). By Helly's selection theorem (see, for example, [**9**]), which applies also to convex functions, we find a subsequence such that $\psi_{n_k}(x) \to \psi(x)$ where ψ is convex and continuous, and so the convergence is uniform on compact subsets. Then a standard argument shows that the whole sequence must converge to $\psi(x)$ and that $\exp(-\psi(x))$ is $n_{0,1}(x)$. Since the one-sided derivatives of $\psi_n(x)$ exist and converge, we get eventually that $\psi_n(x) \geqslant Mx - c$ for any large M with suitable $c = c_M$ uniformly in x by convexity. This yields the desired result.

We shall now weaken the geometric Tauberian condition in Theorem D to obtain a necessary and sufficient condition. This condition will be weakened further in (4.5) below.

THEOREM 2.2. *Let the distribution function F have a very thin tail and a density f and assume that the exponential family* (U_τ) *is asymptotically normal, that is,* (1.10) *holds. Then f has a Gaussian tail if and only if f is strictly positive on a neighbourhood of* $+\infty$ *and*

$$\limsup_{t\to\infty}\sup_{a>0}\sup_{0<\theta<1}\frac{f(t)^{1-\theta}f(t+a)^{\theta}}{f(t+\theta a)}\leqslant 1. \tag{2.1}$$

Proof. By Remark 1.3 one may choose $r(\tau)\to\infty$ so that $t(\tau)=\mu(\tau)-r(\tau)\sigma(\tau)\to\infty$ for $\tau\to\infty$ (μ and σ are defined in (1.7)). Define the norming transformations $A_\tau(u)=\mu(\tau)+\sigma(\tau)u$. If F_τ is the distribution function of U_τ, then $F_\tau(A_\tau(u))=F_\tau^*(u)$ is the distribution function of the normalized variables U_τ^* having density $f_\tau^*(u)$.

Reformulate (2.1) as follows: for any $\varepsilon>0$ there exists $t_0>0$ such that

$$0\leqslant \log f+\psi_t\leqslant\varepsilon \quad \text{on } [t,\infty) \quad \text{for } t\geqslant t_0,$$

where ψ_t is the convex hull of the function which is infinite on $(-\infty,t)$ and equals $-\log f$ on $[t,\infty)$. Now introduce the density

$$g_\tau(x)=c_\tau e^{-\psi_t(x)}1_{[t,\infty)}(x) \quad \text{with } t=t(\tau)=\mu(\tau)-r(\tau)\sigma(\tau).$$

Here $c_\tau>0$ is a norming constant. The corresponding distribution function G_τ satisfies $G_\tau(\mu(\tau)+u\sigma(\tau))\to N_{0,1}(u)$, since $r(\tau)\to\infty$. This also implies that $c_\tau\to 1$. Then we conclude by Theorem E above that $\sigma(\tau)g_\tau(\mu(\tau)+u\sigma(\tau))\to n_{0,1}(u)$ uniformly on $\mathbb{R}$. Hence $f_\tau^*(u)\to n_{0,1}(u)$ uniformly on any half line $[-M,\infty)$, which means that f has a Gaussian tail. For the converse conclusion use the fact that $-\log f\sim\psi$, with ψ asymptotically parabolic, so that, in particular, ψ is convex, and this yields (2.1).

3. *Convergence in one point*

As before we assume that the distribution function F has a density f and that the normalized random variables U_τ^* in (1.8) with densities f_τ^* converge in distribution to $N_{0,1}$. Here we shall prove the following: if $f_\tau^*(a)$ converges in one point $a\in\mathbb{R}$ for $\tau\to\infty$ then f has a Gaussian tail.

The convexity condition of the previous section has a surprising consequence. Convexity is a local property. In order to ensure that a function is convex it suffices to check convexity on a sequence of overlapping intervals which cover $\mathbb{R}$. The same holds for asymptotic convexity provided that the intersection intervals are sufficiently long to establish true convexity. This leads to the following result.

PROPOSITION 3.1. *Suppose that the distribution function F has a very thin tail and a bounded density f and let* (1.10) *hold. If there is an interval* (a,b) *such that*

$$f_\tau^*(u)\longrightarrow n_{0,1}(u) \quad \textit{for } \tau\longrightarrow\infty,$$

uniformly for $u\in(a,b)$*, then f has a Gaussian tail.*

Note that we do not assume convergence in any point x outside the interval (a, b) in Proposition 3.1. We shall now sharpen the result by shrinking the interval (a, b) to a single point.

For notational convenience we introduce two families of operators. If one writes the exponential family of distributions F_τ for $\tau \geqslant 0$ as $F_\tau = S^\tau F$, then it is obvious that the operators S^τ for $\tau \geqslant 0$ form a semigroup (on the set of distribution functions with very thin tail), that is, $S^\tau(S^\rho F) = S^{\tau+\rho}F$. Since we often consider the normalized distribution functions F_τ^* it is convenient to introduce a second family of operators T^τ for $\tau \geqslant 0$, by setting $T^\tau F = F_\tau^*$. Define $\mu(\tau)$ and $\sigma(\tau)$ by (1.7). Then

$$T^\rho T^\tau = T^{\tau+\rho/\sigma(\tau)},$$

since for the associated random variables

$$\begin{aligned} T^\rho T^\tau U &= T^\rho((U_\tau - \mu(\tau))/\sigma(\tau)) \\ &= \{\sigma(\tau)\,((U_\tau - \mu(\tau))/\sigma(\tau))_\rho - \mu(\tau+\rho/\sigma(\tau)) + \mu(\tau)\}/\sigma(\tau+\rho/\sigma(\tau)) \\ &= (U_{\tau+\rho/\sigma(\tau)} - \mu(\tau+\rho/\sigma(\tau))/\sigma(\tau+\rho/\sigma(\tau)) \\ &= T^{\tau+\rho/\sigma(\tau)}U, \end{aligned}$$

observing that

$$\begin{aligned} E((U_\tau - \mu_\tau)/\sigma(\tau))_\rho &= \chi_\tau^{*\prime}(\rho) = (\mu(\tau+\rho/\sigma(\tau)) - \mu(\tau))/\sigma(\tau), \\ \operatorname{Var}((U_\tau - \mu_\tau)/\sigma(\tau))_\rho &= \sigma^2(\tau+\rho/\sigma(\tau))/\sigma^2(\tau). \end{aligned}$$

Furthermore we need the following result from Analysis.

LEMMA 3.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions on $\mathbb{R}$ and assume that the sequence $(f_n(x_n))_{n\in\mathbb{N}}$ converges whenever the sequence $(x_n)_{n\in\mathbb{N}}$ converges. Then the sequence $(f_n)_{n\in\mathbb{N}}$ has a continuous limit and convergence is uniform on compact subsets of $\mathbb{R}$.*

Proof. Pointwise convergence of (f_n) to a limit function f follows from the choice $(x_n) = (x)$ for arbitrary $x \in \mathbb{R}$. Furthermore if $(x_n) \to x$ then we have $f_n(x_n) \to f(x)$ since the sequence (y_n) defined by $y_n = x_n$ for odd n and $y_n = x$ for even n converges to x again. Now assume that the limit function is not continuous, that is, there exist $x \in \mathbb{R}$, $\varepsilon > 0$ and a sequence $(x_n) \to x$ such that $|f(x_n) - f(x)| > \varepsilon$. Define inductively

$\tilde{x}_n = x_1$ for $1 \leqslant n \leqslant n_1$, where n_1 is defined such that $|f_n(x_1) - f(x_1)| \leqslant \frac{1}{2}\varepsilon, \forall n \geqslant n_1$;

$\tilde{x}_n = x_2$ for $n_1 \leqslant n \leqslant n_2$, where $n_2 \geqslant n_1$ is defined such that

$$|f_n(x_2) - f(x_2)| \leqslant \tfrac{1}{2}\varepsilon, \forall n \geqslant n_2;$$

etc.

Then we have $(\tilde{x}_n) \to x$ but $|f_{n_k}(\tilde{x}_{n_k}) - f(x)| \geqslant \frac{1}{2}\varepsilon$, contradicting our assumption.

Now assume that the convergence is not uniform on a compact set K. Then there exist some $\varepsilon > 0$ and a sequence $(x_n) \in K$ such that $|f_n(x_n) - f(x_n)| \geqslant \varepsilon$. Without loss of generality we may assume that (x_n) converges to some $x \in K$. By our assumption and the continuity of f we end with the following contradiction,

$$\varepsilon \leqslant |f_n(x_n) - f(x_n)| \leqslant |f_n(x_n) - f(x)| + |f(x) - f(x_n)| \longrightarrow 0.$$

We can now prove the main result of this section.

THEOREM 3.3. *Suppose that the distribution function F has a very thin tail and density f. Assume furthermore that the cumulant generating function is asymptotically parabolic. If there exist constants a, $b \in \mathbb{R}$ such that $f_\tau^*(a) \to b$ as $\tau \to \infty$, then f has a Gaussian tail.*

Proof. Let $\tau_n \to \infty$ and let χ_n denote the cumulant generating function of $f_n := f_{\tau_n}^*$. As in (1.9) $\chi_n(\tau) \to \frac{1}{2}\tau^2$ and $\chi_n''(\tau) \to 1$. Let $c_n \to c \in \mathbb{R}$. Then

$$T^{\tau_n + c_n/\sigma(\tau_n)} f(a) = T^{c_n} T^{\tau_n} f(a) = T^{c_n} f_n(a) = \sigma_n (S^{c_n} f_n)(\mu_n + \sigma_n a)$$
$$= \sigma_n e^{c_n(\mu_n + \sigma_n a)} e^{-\chi_n(c_n)} f_n(\mu_n + \sigma_n a) = e^{A_n} f_n(x_n)$$

with $\mu_n = \chi_n'(c_n)$ and $\sigma_n^2 = \chi_n''(c_n)$. Now $c_n \to c$ implies that $\mu_n \to c$, $\sigma_n^2 \to 1$ and $e^{-\chi_n(c_n)} \to e^{-c^2/2}$. For any sequence $x_n \to x$ one can choose $c_n \to c$ so that $\mu_n + \sigma_n a = x_n$ by continuity of χ_n' and χ_n''. Hence $c = x - a$ and $A_n \to ca + \frac{1}{2}c^2 = \frac{1}{2}x^2 - \frac{1}{2}a^2$. The left-hand side gives $T^{\tau_n + c_n/\sigma_n} f(a) \to b$, yielding that $f_n(x_n) \to b e^{a^2/2} e^{-x^2/2}$. By Lemma 3.2 convergence is uniform on bounded intervals with limit function $h(x) = b e^{a^2/2} e^{-x^2/2}$. Observing that the cumulant generating function is asymptotically parabolic we find that $U_\tau^* \xrightarrow{d} N_{0,1}$ and therefrom we obtain that h is a density and $h = n_{0,1}$.

4. *Conditions on the oscillation*

We begin with some simple observations about asymptotic equality for moment generating functions. Suppose that the distribution function F has a very thin tail. Let G be tail equivalent to F, that is, $1 - F(t) \sim 1 - G(t)$ as $t \to \infty$. Then G also has a very thin tail and the moment generating functions $\hat{f}$ and $\hat{g}$ are also asymptotically equivalent.

Now consider the function βe^ϕ where ϕ is asymptotically parabolic and β is flat for ϕ. This function need not be the moment generating function of a probability measure. However (see below) there always exists a distribution function G with moment generating function $\hat{g} \sim \beta e^\phi$, and we may even choose G to be absolutely continuous with a bounded and continuous density g with a Gaussian tail. If F is another distribution function with moment generating function $\hat{f} \sim \beta e^\phi$ then by Theorem 1.2 both exponential families F_τ and G_τ are asymptotically normal with the same norming constants. If the distribution function F has a density f this allows us to formulate a Tauberian condition: the density f has a Gaussian tail if and only if $f(t) \sim g(t)$ as $t \to \infty$. In this section we shall work out these ideas in greater detail.

Before doing so let us have another look at the asymptotic relations introduced above,

$$f(t) \sim g(t) \Rightarrow 1 - F(t) \sim 1 - G(t) \Rightarrow \hat{f}(\tau) \sim \hat{g}(\tau).$$

Neither of the two implications can be reversed. For the first one this is obvious, for the second one choose F to be the Poisson-distribution. The cumulant generating function $\lambda(e^\tau - 1)$ is asymptotically parabolic. By the arguments above there exists a density g with a Gaussian tail such that $\hat{g} \sim \hat{f}$. Now $(1 - F(n-))/(1 - F(n)) \sim n/\lambda \to \infty$ and hence F cannot be tail-equivalent to G. The symmetry between the asymptotic behaviour in the source space and the transform space which holds in the theory of regular variation is absent in the theory of Gaussian tails.

LEMMA 4.1. *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous distribution functions with densities $(f_n)_{n\in\mathbb{N}}$ and suppose that $F_n \overset{w}{\to} N_{0,1}$ as $n\to\infty$. Then we have*

$$\int_v^w \frac{f_n(x)}{n_{0,1}(x)}\,dx \longrightarrow w-v \quad \text{as } n\longrightarrow\infty, \quad \text{for any fixed } v<w. \tag{4.1}$$

Proof. The function $1_{[v,w]}/n_{0,1}$ is bounded and almost everywhere continuous with respect to Lebesgue measure or the Gaussian measure. Hence

$$\int_{\mathbb{R}} \frac{1_{[v,w]}(x)}{n_{0,1}(x)}\,dF_n(x) \longrightarrow \int_R \frac{1_{[v,w]}(x)}{n_{0,1}(x)}\,dN_{0,1}(x) = v-w.$$

(see [7, Proposition 8.12]).

PROPOSITION 4.2. *Let the distribution function F have a very thin tail and density f and suppose that the cumulant generating function χ is asymptotically parabolic. Then for any $-\infty<v<w<\infty$ we have*

$$\sqrt{(2\pi)}\int_{t+vs(t)}^{t+ws(t)} f(x)\,e^{\chi^*(x)}\,dx \longrightarrow w-v \quad \text{as } t\longrightarrow\infty, \tag{4.2}$$

where χ^* is the Legendre transform of χ and s the scale function of χ^*.

Proof. Choose a logconcave density $g(t)\sim(1/(s(t)\sqrt{(2\pi)}))\,e^{-\chi^*(t)}$. This is possible since s is flat for χ^* by Remark 1.1. The density g has a Gaussian tail and $\chi^{**}=\chi$ implies that $\hat{g}\sim\hat{f}$ by Theorem C in Section 1. Use the norming constants $\mu=\chi'$ and $\sigma=\sqrt{\chi''}$ to normalize the exponential family F_τ generated by the density f and the exponential family G_τ generated by g. Then $g_\tau^*\to n_{0,1}$ uniformly on $\mathbb{R}$ (see Theorem C and (1.11)), and we obtain by Lemma 4.1

$$\int_{\mu(\tau)+v\sigma(\tau)}^{\mu(\tau)+w\sigma(\tau)} \frac{f(x)/\hat{f}(\tau)}{g(x)/\hat{g}(\tau)}\frac{dx}{\sigma(\tau)} = \int_v^w \frac{f_\tau^*(u)}{g_\tau^*(u)}\,du \longrightarrow w-v \quad \text{as } \tau\longrightarrow\infty.$$

We may delete the factor $\hat{g}(\tau)/\hat{f}(\tau)$ which tends to 1. Introduce the conjugate variable $t=\chi'(\tau)=\mu(\tau)$, see (0.7). Then $s(t)=\sigma(\tau)$ by (0.8). Since the function s is self-neglecting we have $s(x)\sim s(t)$ for $x=t+us(t)$ uniformly in $v\leqslant u\leqslant w$, and hence we may replace the denominator $g(x)\,\sigma(\tau)$ by $(1/\sqrt{(2\pi)})\,e^{-\chi^*(x)}$. This gives (4.2).

REMARK 4.3. If $\hat{f}(\tau)\sim\beta(\tau)\,e^{\phi(\tau)}$ with ϕ asymptotically parabolic and β flat for ϕ one obtains a similar relation:

$$\frac{\sqrt{(2\pi)}}{b(t)}\int_{t+vs(t)}^{t+ws(t)} f(x)\,e^{\phi^*(x)}\,dx \to w-v \quad \text{as } t\longrightarrow\infty \quad \text{for any } v<w, \tag{4.3}$$

where s is the scale function of the Legendre transform ϕ^* and $b(t)=\beta(\tau)$ with $t=\phi'(\tau)$ and τ conjugate variables (see (0.7)).

Now we come to our main results in this section, which are related to results in [19].

THEOREM 4.4. *Let the distribution function F have a very thin tail and density f and assume that the moment generating function satisfies* $\hat{f}(\tau) \sim \beta(\tau)e^{\phi(\tau)}$ *as* $\tau \to \infty$, *with* ϕ *asymptotically parabolic and* β *flat for* ϕ. *Write s for the scale function of the Legendre transform* ϕ^* *of* ϕ *and set* $b(t) = \beta(\tau)$ *with* $t = \phi'(\tau)$ *and* τ *conjugate variables* (*see* (0.7)). *The following are equivalent*:

(1) *the density f has a Gaussian tail*;
(2) *the function* $h(t) = f(t)e^{\phi^*(t)}$ *is flat for* ϕ^*;
(3) $f(t) \sim \dfrac{b(t)}{s(t)\sqrt{(2\pi)}} e^{-\phi^*(t)}$.

Proof. (1) ⇒ (3) Apply Theorem C and use (0.7) and (0.8).

(3) ⇒ (2) Use that s and b are flat for ϕ^* (see [3, Proposition 5.6]).

(3) ⇒ (1) The relation $f(t) \sim h(t)e^{-\phi^*(t)}$ and Theorem C, together with Proposition 1.5, imply that f has a Gaussian tail.

(2) ⇒ (3) Since h is flat for ϕ^* and s is self-neglecting we obtain with Remark 4.3

$$w - v = \frac{(1+o(1))}{b(t)}\sqrt{(2\pi)}\int_{t+vs(t)}^{t+ws(t)} f(x)\,e^{\phi^*(x)}\,dx$$

$$= (1+o(1))\sqrt{(2\pi)}\frac{s(t)}{b(t)}(w-v)f(t)\,e^{\phi^*(t)} \quad \text{as } t \to \infty. \tag{4.4}$$

EXAMPLE. Continuing the example below Theorem C we now assume that the moment generating function has the form

$$\hat{f}(\tau) \sim \beta(\tau)\,e^{\phi(\tau)}, \quad \phi(\tau) = (\alpha\tau)^q/q, \quad \beta(\tau) = c_0\tau^{c_1}\exp(c_2\tau^{\theta q}),$$

with ϕ as above and $c_0 > 0$. Then β is flat for ϕ if $\sigma\beta'/\beta \to 0$ and this will hold if $\theta < \frac{1}{2}$. By Theorem 4.4 the density f has a Gaussian tail if and only if $f \sim he^{-\psi}$ with $\psi = \phi^*$, as above, and $h = b/(s\sqrt{(2\pi)})$. We find that $h(t) = a_0 t^{a_1}\exp(a_2 t^{\theta p})$, where the coefficients $a_0 > 0$, a_1 and a_2 can be simply calculated. See [18] for an application to extreme value theory.

REMARK 4.5. Statement (2) can be considered as an oscillation condition, typically used as a Tauberian condition (see, for example, [19] and the results cited therein), namely

$$\limsup_{t\to\infty} \max_{t \leqslant u \leqslant t+\phi^{*\prime\prime}(t)^{-1/2}} |h(u)-h(t)|/h(t) = 0.$$

Usually a weaker Tauberian condition as, for example,

$$\lim_{\lambda\to 0+}\limsup_{t\to\infty} \max_{t \leqslant u \leqslant t+\lambda\phi^{*\prime\prime}(t)^{-1/2}} |h(u)-h(t)|/h(t) = 0$$

suffices. This can also be used here; consider (4.4) and let $(w-v)$ be small. For related one-sided Tauberian conditions, compare [19].

COROLLARY 4.6. *Let the assumptions of Theorem* 4.4 *hold. If f does not have a Gaussian tail then there exist sequences* $t_n \to \infty$ *and* $a_n > 0$ *such that* $a_n/s(t_n) \to 0$ *and* $\liminf_{n\to\infty} h(t_n+a_n)/h(t_n) > 1$ *with* $h = fe^{\phi^*}$ *as above. There exist similar sequences such that* $\limsup_{n\to\infty} h(t_n+a_n)/h(t_n) < 1$.

We return now to the criterion of asymptotic convexity in Section 2. We claim that condition (2.1) in Theorem 2.2 may be replaced by the weaker condition of asymptotic midpoint convexity:

$$\inf_{\varepsilon>0} \limsup_{t\to\infty} \sup_{0<a<\varepsilon t} \frac{f(t)f(t+2a)}{f^2(t+a)} \leqslant 1. \tag{4.5}$$

THEOREM 4.7. *Suppose that the distribution function F has a very thin tail and that the cumulant generating function χ is asymptotically parabolic. Assume that F has a density f which does not have a Gaussian tail. Then there exist sequences $w_n \to \infty$ and $a_n > 0$ with $a_n = o(w_n)$ such that*

$$\limsup_{n\to\infty} \frac{f(w_n - a_n)f(w_n + a_n)}{f^2(w_n)} > 1.$$

Furthermore, the corresponding weak concavity condition is also violated.

Proof. The relation (4.1) with $f_n = f^*_{\tau_n}$ and $\tau_n \to \infty$ will hold for any interval (v, w). By Theorem 3.3 there exists a sequence $\tau_n \to \infty$ such that $f_n(0)$ does not converge to $n_{0,1}(0) = 1/\sqrt{(2\pi)}$. For a suitable subsequence there exists a constant $\eta > 0$ such that $|\psi_n(0)| > 11\eta$, where $\psi_n(u) = \log(n_{0,1}(u)/f_n(u))$. The functions ψ_n have spikes or oscillate. For large n the spikes are sharp and the frequency of the oscillation is high since they have to average out over any interval (v, w) by Lemma 4.1.

Let $\delta_n \to 0$ so that (4.1) holds on the intervals $((k-1)\delta_n, (k+1)\delta_n)$ for $k = -4, -3, \ldots, 3, 4$. By Lemma 4.9 below there exist sequences $x_n < y_n$ which converge to 0 so that

$$\psi_n(x_n) + \psi_n(y_n) + \eta < 2\psi_n(\tfrac{1}{2}(x_n + y_n)). \tag{4.6}$$

A change of variables gives $\psi(w_n - a_n) + \psi(w_n + a_n) + \eta < 2\psi(w_n)$, where $\psi = -\log f$, and $a_n = o(s_n)$ with $s_n = \sigma(\tau_n) \sim s(w_n)$, yielding the desired result.

REMARK 4.8. From the proof it is seen that we may choose $a_n = o(s(w_n))$, where s is the scale function of the Legendre transform of the cumulant generating function χ. Note that s is self-neglecting and hence $s(w) = o(w)$ for $w \to \infty$.

LEMMA 4.9. *Let $f\colon(-5, 5) \to \mathbb{R}$ be measurable and suppose that $|f(u)| > 10$ for some $u \in (-1, 1)$ and that the function f achieves a value less than 1 and a value greater than -1 in each of the nine intervals $(k-1, k+1)$ for $k = -4, \ldots, 4$. (This will be the case if the average of the function over each of these intervals is less than 1 in absolute value.) Then there exist $x < y$ such that $f(x) + f(y) + 1 < 2f(z)$ for $z = \frac{1}{2}(x+y)$.*

Proof. First assume that $f(x) < -10$ in some point $x \in (-1, 1)$ and $f \geqslant -4$ on $(1, 3)$. Choose $y \in (3, 5)$ so that $f(y) < 1$. Then the midpoint z lies in $(1, 3)$ and $f(x) + f(y) + 1 \leqslant -8 < 2f(z)$.

Hence we may assume the existence of three points $u_1 < u_0 < u_2$ in $(-3, 3)$ such that

$$f(u_1) \vee f(u_2) = \alpha < f(u_0) - 3.$$

Let σ denote the supremum of f over the interval $I = (u_1, u_2)$. If σ is finite we choose $z \in I$ so that $f(z) > \sigma - 1$ and x and y are equidistant, with $x = u_1$ if u_1 is closer to z than

u_2, and otherwise $y = u_2$. Then $f(x)+f(y)+1 \leqslant \alpha+\sigma+1 < \sigma-3+\sigma+1 < 2f(z)$. If σ is infinite we choose γ so large that $\lambda\{f>\gamma\} < 1$. Choose $z \in I$ with $f(z) > \gamma+1$. The set $A = \{t \mid 0 < |t-z| < 1,\ f(t) \leqslant \gamma\}$ has measure $\lambda(A) > 1$, and so has the set A' obtained by reflection in z. Hence $\lambda(A \cap A') > 0$. Choose $a \in A \cap A'$. Then z lies midway between a and its reflection point a' and $f(a)+f(a')+1 \leqslant 2\gamma+1 < 2f(z)$.

5. *Tauberian conditions on the characteristic function*

In this section we formulate Tauberian conditions on the characteristic function (cf) $\hat{f}(\mathrm{i}y)$ for $y \in \mathbb{R}$, which will ensure the existence of a density with a Gaussian tail.

In a recent paper [3] it was shown that the class of bounded densities with Gaussian tails is closed under convolution. A natural question is: does there exist a Tauberian theorem which allows a simple proof of the convolution closure. Here we shall obtain a partial result in this direction. However, establishing the existence of a Gaussian tail by inspection of the moment generating function is difficult: we are able to construct a C^∞-density f which satisfies (0.2) and (0.3) and which does not have a Gaussian tail. Yet the difference between the moment generating function $\hat{f}_\tau^*$ of the random variable U_τ^* in (1.8) and the standard Gaussian moment generating function $\hat{n}_{0,1}(z) = e^{z^2/2}$ is very small for $\tau \to \infty$. The difference $\hat{f}_\tau^*(z)-\hat{n}_{0,1}(z)$, and the derivatives of this difference of any order, vanish faster than $e^{-\tau^2/3}$ for $\tau \to \infty$. Moreover this order of convergence holds uniformly for z in each vertical strip $|\Re z| < M$ in the complex plane. Even in the simple case $\hat{f}(\tau) \sim \exp(\tau^2/2)$ for $\tau \to \infty$, and under the excessive magnification (by $\exp(\frac{1}{3}\tau^2)$) the transform $\hat{f}_\tau$ in (0.1) does not reveal whether the source function has the desired form $f \sim e^{-\psi}$ with ψ asymptotically parabolic.

THEOREM 5.1. *Suppose that the distribution function F has a very thin tail. Let $\hat{f}_\tau^*$ denote the moment generating function of the normalized random variable U_τ^* in* (1.8). *If $\hat{f}_\tau^*(\mathrm{i}y) \to e^{-y^2/2}$ in $L^1(\mathbb{R}, dy)$ then F has a continuous density f with Gaussian tail.*

Proof. Convergence of the characteristic functions in L^1 implies uniform convergence of the densities by the inversion theorem. In particular f_τ is a bounded continuous density for $\tau \geqslant \tau_0$. Hence the density f of F exists and is continuous. By Proposition 1.5, f has a Gaussian tail.

The Tauberian condition above is very similar to [**8**, Theorem 1(iii)]. It is of interest since it yields the convolution property below as a simple corollary.

COROLLARY 5.2. *If two distribution functions F_1 and F_2 satisfy the conditions of Theorem* 5.1 *then so does their convolution $F_0 = F_1 * F_2$.*

The proof follows from the next lemma, observing that the associated random variables are $U_{0,\tau}^* = (U_{1,\tau}+U_{2,\tau}-\mu_1(\tau)-\mu_2(\tau))/\sigma_0(\tau)$, where $\sigma_0^2 = \sigma_1^2+\sigma_2^2$.

LEMMA 5.3. *Let $\alpha_n > 0$ and $\beta_n > 0$ satisfy $\alpha_n^2+\beta_n^2 = 1$ and let ϕ_n and ψ_n be characteristic functions which converge to $\theta(y) := \exp(-\frac{1}{2}y^2)$ in the L_1-norm. Then the product $\chi_n(y) = \phi_n(\alpha_n y)\,\psi_n(\beta_n y)$ converges to $\theta(y)$ in the L_1-norm as well.*

Proof. We assume that $\beta_n \geqslant \frac{1}{2}$. Then we have

$$|\chi_n(y)-\theta(y)| = |\phi_n(\alpha_n y)|\,|\psi_n(\beta_n y)-\theta(\beta_n y)|+|\phi_n(\alpha_n y)-\theta_n(\alpha_n y)|\,|\theta_n(\beta_n y)|.$$

The L_1-norm of the first term on the right is bounded by $\|\psi_n-\theta_n\|_1/\beta_n$, which tends to zero; the L_1-norm of the second term on the right vanishes asymptotically by the dominated convergence theorem, since the expression is bounded by $2\theta(\beta_n y) \leqslant \exp(-\frac{1}{4}y^2)$ and since $|\phi_n(y)-\theta(y)|$ converges to zero uniformly on bounded intervals.

The next two results contain sufficient conditions for the L_1-convergence of the characteristic functions.

THEOREM 5.4. *Suppose that the distribution function F has a very thin tail and its cumulant generating function χ is asymptotically parabolic. If there exist $M > 0$ and $\lambda > \frac{1}{2}$ such that for τ large enough*

$$\Re(\chi(\tau+\mathrm{i}y)-\chi(\tau))+\lambda\log(y^2\chi''(\tau)) \leqslant M \quad \textit{for } y\in\mathbb{R}, \tag{5.1}$$

then F has a continuous density f with Gaussian tail.

Proof. The difference $\phi_\tau(y) = \chi(\tau+\mathrm{i}y)-\chi(\tau)$ is the logarithm of the characteristic function of F_τ. Hence (using $\sigma^2(\tau) = \chi''(\tau)$) (5.1) gives

$$|\hat{f}_\tau(\mathrm{i}y/\sigma(\tau)+\mu(\tau))| = e^{\Re\phi_\tau(y/\sigma(\tau))}\, e^{\lambda\log(y/(\sigma(\tau))^2\chi''(\tau))}y^{-2\lambda} \leqslant e^M y^{-2\lambda}\in L_1(\mathbb{R}).$$

By assumption $\hat{f}^*_\tau(\mathrm{i}y) \to e^{-y^2/2}$ converges pointwise and Lebesgue's Theorem gives convergence in L^1. Hence Theorem 5.1 applies.

REMARK 5.5. If two distribution functions F_1 and F_2 both satisfy (5.1) with constants M_1 and M_2 and some $\lambda > \frac{1}{2}$, then the convolution $F_0 = F_1 * F_2$ satisfies (5.1) with constant $M_0 = M_1+M_2$. Thus within the class of densities with a Gaussian tail we have a decreasing family of subclasses, indexed by λ, of densities with a Gaussian tail, each of which is closed under convolution.

The following result is related to a theorem of Berg [**4**].

THEOREM 5.6. *Suppose that the distribution function F has a very thin tail and that the cumulant generating function is asymptotically parabolic. If there exist constants $c > 0$ and τ_0 such that for $\tau \geqslant \tau_0$ we have*

$$\Re\left(\frac{\chi''(\tau+\mathrm{i}\xi)}{\chi''(\tau)}\right) \geqslant c > 0 \quad \textit{for } \xi\in\mathbb{R}, \tag{5.2}$$

then F has a continuous density with Gaussian tail.

Proof. We know by Theorem 1.2 that $U^*_\tau = (U_\tau-\chi'(\tau))\,\delta(\tau) \xrightarrow{d} N_{0,1}$, and hence with $\delta(t) = (\chi''(t))^{-1/2}$ we have

$$\hat{f}^*_\tau(\mathrm{i}y) = e^{-\mathrm{i}y\chi'(\tau)\delta(\tau)}\hat{f}(\tau+\mathrm{i}y\delta(\tau))/\hat{f}(\tau) \longrightarrow e^{-y^2/2} \quad \text{as } \tau \longrightarrow \infty \quad \text{for any } y\in\mathbb{R}.$$

Furthermore by our assumption we find that

$$\log \hat{f}_\tau^*(\mathrm{i}y) = -\mathrm{i}y\chi'(\tau)\,\delta(\tau)+\chi(\tau+\mathrm{i}y\delta(\tau))-\chi(\tau) = -\chi''(\tau+\mathrm{i}\theta y\delta(\tau))\,\delta^2(\tau)\frac{y^2}{2},$$

so $\Re(\log \hat{f}_\tau^*(\mathrm{i}y)) \leqslant -\frac{1}{2}cy^2$ for $y\in\mathbb{R}$, $\tau \geqslant \tau_0$. Hence for τ large enough $|\hat{f}_\tau^*(\mathrm{i}y)| \leqslant e^{-c_1 y^2}$ with some $c_1 > 0$ and by the Dominated Convergence Theorem our Theorem 5.1 applies.

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