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Multiple decrements or competing risks

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SUMMARY

Given two states A and B such that individuals in state A have mutually exclusive probabilities, possibly dependent on the time spent in state A , of leaving that state because of (i) death, or (ii) passage to state B , what is the probability of an individual passing to state B and dying there within a given period? This problem has been of great interest and importance to actuaries for over 100 years and the solutions of their professional contemporaries have appeared in their textbooks. Twenty-five years ago statisticians felicitously named the technique the theory of competing risks (Neyman, 1950) and developed the formulae *ab initio* in the framework of Markov processes. Consideration of the problem started with Daniel Bernoulli's smallpox mathematics and continued with pension fund financing involving the payment of annuities to invalid lives. Most of the currently accepted techniques were developed during the nineteenth century. This article reviews the history both of actuarial and statistical contributions to the literature.

Some key words: Actuarial mathematics; Competing risk; Multiple decrement.

1. INOCULATION AGAINST SMALLPOX

In order to conform with the abundant actuarial literature of this century it is desirable to utilize a notation due to the Russian actuary Hamza (1900). Write l_x for the expected number of survivors at age x from l_0 , an initial group of entrants into state A at age zero which need not be the 'real' zero of birth. At age x this expected number has become subdivided into l_x^{aa} expected in state A and l_x^{ii} in state B , that is

$$l_x = l_x^{aa} + l_x^{ii}, \quad dl_x = dl_x^{aa} + dl_x^{ii}. \tag{1}$$

It is, of course, preferable to write $l_0 = 1$ and make l_x the probability of survival to at least age x . This is not, however, the way in which the earlier writers viewed l_x . While random variations were envisaged in the actual number of survivors to age x it was assumed that this number was so large that it converged to the 'true' value. Nowadays we can avoid ambiguity by referring to l_x as the expected number of survivors at age x . Now write μ_x^a for the force of decrement, or conditional instantaneous probability of exit, from state A , consisting of μ_x^{aa} , the force of mortality, and ν_x the force of passage to B . Let μ_x^i be the force of mortality affecting individuals in state B . No other type of movement from state B is envisaged. Then, considering the changes in the groups A and B in the infinitesimal interval of time following age x ,

$$-dl_x^{aa} = l_x^{aa} \mu_x^a dx = l_x^{aa} (\mu_x^{aa} + \nu_x) dx, \quad dl_x^{ii} = l_x^{aa} \nu_x dx - l_x^{ii} \mu_x^i dx. \tag{2}$$

Hence, from (1),

$$-dl_x = l_x^{aa} \mu_x^a dx + l_x^{ii} \mu_x^i dx \tag{3}$$

expressing the fact that deaths in the whole community are composed of deaths from A plus deaths from B . This can be rewritten as

$$-dl_x = l_x \mu_x^{aa} dx + l_x^i (\mu_x^i - \mu_x^{aa}) dx = l_x \mu_x^{aa} dx + l_x^i \phi_x dx, \quad (4)$$

and, on substituting this implicit value of μ_x^{aa} in the first equation (2), we have

$$d l_x^{aa} = \frac{l_x^{aa}}{l_x} (dl_x + l_x^i \phi_x dx) - l_x^{aa} \nu_x dx. \quad (5)$$

The solution, derived by Richard (1946, p. 79) without reference to any originator, is

$$l_x^{aa} = l_x \exp \left\{ - \int_0^x (\mu_u^a - \mu_u^i) du \right\} \left[1 + \int_0^x (\mu_u^i - \mu_u^{aa}) \exp \left\{ - \int_0^u (\mu_w^a - \mu_w^i) dw \right\} du \right]^{-1}. \quad (6)$$

The differential equation (5) is essentially that obtained by Daniel Bernoulli (1766) in his pathbreaking application of infinitesimal methods to a problem in probability (Westergaard, 1932, chapter IX). It is included in Laplace's textbook (1812, Livre II, Art. 36). Bernoulli's state A consisted of individuals who had never had smallpox while state B comprised those who had contracted smallpox and would either die from it very soon, or survive and no longer be subject to that disease. In Bernoulli's, and Laplace's, model there is thus an immediate heavy mortality affecting those who catch smallpox followed by normal mortality, μ_x^{aa} , thereafter. Both Bernoulli and Laplace took account of this slight modification in our general model by replacing the second term, $l_x^i \phi_x$, of (4) by $l_x^{aa} \nu_x \rho_x$, the deaths from smallpox, ρ_x being the proportion of smallpox sufferers at age x who die from that disease. In Bernoulli's time there were no age-specific statistics to estimate ν_x and ρ_x and the best he could do was to guess them both equal to 0.125, independent of age. Most of Bernoulli's 45-page memoir is concerned with the legitimacy of this assumption and the statistical conclusions to which it led. His only mathematical formulae consisted of the development of (6) which in his special case, with $\nu_x - \phi_x = 1/n$ and $\phi_x = -1/(nm)$, became

$$l_x^{aa} = l_x e^{-x/m} \left\{ 1 - \frac{1}{m} (1 - e^{-x/m}) \right\}^{-1}, \quad (7)$$

and the corresponding expression for the proportion of the l_x^{aa} individuals who have not had smallpox, namely

$$l_x^{aa} / \exp \left(- \int_0^x \nu_u du \right). \quad (8)$$

Bernoulli pointed out that with $\nu_u = 1/n$ and the foregoing expression for l_x^{aa} , (8) approaches $m/(m-1)$ as $x \rightarrow \infty$. With his choice of $m = n = 8$, he evaluated (7) and (8) for $x = 1(1) 24$ and $x = 1(1) 25$, respectively, and used Halley's Breslau table for l_x , overlooking that Halley had calculated $\int l_y dy$ with limits x and $x+1$, and not l_x (Westergaard, 1932). Laplace, also considering inoculation for smallpox, in effect put $\nu_x - \phi_x = \zeta_x$ and $\phi_x = -\xi_x$ and reached (6) without any attempt at numerical evaluation.

D'Alembert wrote at some length (1761) on the difficulty of comparing 'un danger présent', namely death from inoculation, then estimated at about 1/300, with 'un danger éloigné', eventual death from smallpox, and argued that a comparison of life expectancies was not the way to proceed. Todhunter (1865, Arts. 479-88, 506-7 and 516-25) provides a review of this author's work and in order to derive his principal relation it is convenient to change our problem and notation slightly.

Write $\mu_x^{(i)}$ for the force of mortality at age x on account of cause of death i and $\mu_x^{(-i)}$ for the force of mortality at that age due to all causes of death other than i . With l_x having the same

meaning as before

$$-dl_x = l_x(\mu_x^{(i)} + \mu_x^{(-i)}) dx \quad (9)$$

and, in an obvious notation,

$$-dl_x^{(-i)} = l_x^{(-i)} \mu_x^{(-i)} dx,$$

which leads to

$$dl_x^{(-i)} / l_x^{(-i)} = dl_x / l_x + \mu_x^{(i)} dx.$$

On integrating, we obtain

$$l_x^{(-i)} = l_x \exp\left(\int_0^x \mu_u^{(i)} du\right). \quad (10)$$

This, equivalent to (8) but free of ν_x , is the relation obtained by d'Alembert (1761) for the expected number of individuals in l_x who have not had smallpox, except that $\mu_x^{(i)} dx$ was there replaced by its equivalent, the differential du_x / l_x , where u_x is the aggregate expected number of deaths prior to age x on account of cause i since the l_0 children were born. Although d'Alembert proposed to cumulate the deaths from smallpox age by age Todhunter (1865) stated that (10) was 'not of practical use because the value of the integral ... is not known'. On the other hand Laplace (1812) had explained that approximate values of $\mu_x^{(i)}$ at successive integer ages could be obtained by dividing the number of deaths from cause i at age x by the number of survivors from l_0 infants between ages $x - \frac{1}{2}$ and $x + \frac{1}{2}$. The integral in (10) could then be obtained by trapezoidal quadrature. Karn (1931), who provides an excellent account of the history of inoculation for smallpox, used a similar estimation procedure except that she approximated the differential coefficient du_x / dx at age x by means of the operator relation $D = \log(1 + \Delta)$ through the term in Δ^3 . She then used the Euler-Maclaurin expansion for the integral through second differences.

We observe that if all the causes of death are denoted by i_j ($j = 1, \dots, k$) relation (10) can be written

$$l_x = \prod_{j=1}^k \exp\left(-\int_0^x \mu_u^{(i_j)} du\right) = \exp\left(-\sum_{j=1}^k \int_0^x \mu_u^{(i_j)} du\right). \quad (11)$$

This formula first appeared in Cournot (1843, p. 317). The English actuary Makeham apparently discovered it independently (1867) and used it to interpret (1874) the foregoing developments of Bernoulli, d'Alembert and Laplace. Kanner (1871) derived it again, without using single letters for the forces of exit, also without referring to any precursors.

Two other authors contributed to the inoculation controversy by producing statistics and questioning the age independence of Bernoulli's m and n . One was Lambert (1772, Chapter VII) and the other Duvillard (1806; Quiquet, 1934, Dossier XIII); neither added anything to the foregoing theory.

Finally, we mention a paper that foreshadowed the developments of the mid-nineteenth century. Trembley (1799) sought to derive Bernoulli's relation (7) without using infinitesimal calculus and to generalize it to apply to age-dependent n^{-1} and m^{-1} which we will here write as w_x and q_x^i , respectively. Although Trembley developed his formulæ by years of age and shifted all his smallpox cases to the year-end he emphasized, and demonstrated by numerical examples, that the unit of time could be much smaller, in particular 1/256th of a year. Trembley's result, obtained by simple arguments, was

$$l_{x+1}^{aa} = \frac{l_x^{aa} p_x (1 - w_x)}{l_x - l_x^{aa} w_x q_x^i},$$

and could be rewritten in the form

$$p_x^{aa} = \frac{p_x (1 - w_x)}{1 - l_x^{aa} w_x q_x^i / l_x}, \quad (12)$$

where $p_x^{aa} = l_{x+1}^{aa} / l_x^{aa}$. This relation would provide the series l_x^{aa} for $x = 1, 2, \dots$ in succession and

Trembley pointed out that when $w_x = 1/n$ and $q_x^i = 1/m$ an inductive argument produces

$$l_x^{aa} = \frac{l_x(1-n^{-1})^x}{1-m^{-1}+m^{-1}(1-n^{-1})^x}$$

and this tends to (7) when $n \rightarrow \infty$.

Trembley concludes his memoir by citing Lambert's (1772) and another series of values of w_x and q_x^i (Todhunter, 1865, Art. 787). The differences between the series are considerable.

2. INSURANCE AGAINST PERMANENT INVALIDITY

The conclusion to be drawn from the foregoing is that the basic mathematics of what we now call the continuous time version of a Markov chain model (Bartholomew, 1973) were developed during the eighteenth century. Moreover the methods were reported in the probability texts of the early nineteenth century. Although the concept of state B as that of permanent inability to work, or permanent and total disability, dates back to the earliest pension plans of the sixteenth century a serious demand for the estimation of, for example, the probability of a worker becoming permanently invalid at integer age x , only arose when the financial liabilities of the German railway pension plans were evaluated in the middle of the nineteenth century (Braun, 1925). No use was then made of the mathematics already in existence for a similar problem.

The earlier German railway pension fund actuaries like Heym and Wiegand (1859, 1865), who referred to Heym as the 'creator of invalidity insurance science', were faced with a body of employees who were depleted by deaths and permanent invalids. If q_x is written for the probability of death among those presently aged x in this group of active employees, and w_x for the corresponding probability of invalidity these authors wrote

$$p_x^{aa} = (1-q_x)(1-w_x) \quad (13)$$

for the transition probability of an active employee aged x surviving as an active employee through age $x+1$. The two probabilities on the right-hand side of (13) were to be estimated from relative frequencies using observations of deaths and invalidities in the employee body.

We may now extend our use of Hamza's notation to include three further transition probabilities: q_x^{aa} , the probability of an active individual aged x dying in an active state before age $x+1$; p_x^{ai} , the probability of an active individual aged x becoming invalid and surviving invalid to age $x+1$; and q_x^{ai} , the probability of an active individual aged x becoming invalid and dying in that state prior to age $x+1$. It follows that

$$p_x^{aa} + p_x^{ai} + q_x^{aa} + q_x^{ai} = 1, \quad p_x^{aa} + q_x^{aa} + w_x = 1, \quad p_x^{ai} + q_x^{ai} = w_x.$$

Wittstein (1862) pointed out that the two events whose probabilities are multiplied on the right-hand side of (13) are not independent. Instead he used an estimation formula derived for $q_x = 1 - p_x$, the probability of dying at age x , for a group of lives subject to accretion and depletion between ages x and $x+1$.

Wittstein (1862) supposes that A lives start the year under observation at age x , B lives enter during the year who were aged x at the beginning of the year, and C similar lives escape from observation. Among the changing group of lives there are d deaths during the course of the year. We are required to estimate q_x , the probability of death at age x .

It is assumed by Wittstein that the two components of the difference $B - C$ are each distributed uniformly over the year so that $(B - C)dt$ can be assumed to occur at epoch t ($0 < t < 1$) after the beginning of the year. If, then, $q(x+t, x+1)$ is written for the probability that an individual aged exactly $x+t$ will die in the interval of time $1-t$ before age $x+1$ the expected

number of lives under observation at age $x + 1$ is

$$A(1 - q_x) + (B - C) \int_0^1 \{1 - q(x + t, x + 1)\} dt = A + B - C - d. \quad (14)$$

The naive procedure adopted by Wittstein and by actuaries generally until the 1960's was to estimate q_x by the quantity derived for q_x from this or a similar equation. But before this could be done $q(x + t, x + 1)$ ($0 < t < 1$) had to be written in terms of q_x . Wittstein himself chose two alternatives:

$$q(x + t, x + 1) = \frac{q_x(1 - t)}{1 - tq_x}, \quad (15)$$

$$q(x + t, x + 1) = 1 - (1 - q_x)^{1-t}. \quad (16)$$

Both these forms assume the values of q_x and zero, respectively, when $t = 0$ and 1; the first is based on the assumption of a uniform distribution of expected deaths over the year of age, and the second assumes a constant force of mortality between ages x and $x + 1$. If either expression is inserted into (14) and the logarithmic series developed there results

$$q_x \simeq \frac{d}{A + \frac{1}{2}(B - C)} \quad (17)$$

to a first approximation. Wittstein pointed out that the first neglected term is

$$-\frac{1}{6}(B - C) \frac{d^2}{\{A + \frac{1}{2}(B - C)\}^2}, \quad (18)$$

when (15) is used and is half this for (16). In actuarial practice (18) would be very small.

The approximate relation (14) can be expressed exactly by noting that if N lives are observed for part of their year of life between ages x and $x + 1$ such that the putative period of observation for life j is from age $x + a_j$ to age $x + b_j$ ($0 \leq a_j \leq b_j \leq 1$; $j = 1, \dots, N$), then the expected number of deaths is

$$\sum_{j=1}^N q(x + a_j, x + b_j) \quad (19)$$

and can be equated to d , the number of deaths observed. Wittstein's assumption (15) is equivalent to

$$q(x + a, x + b) = \frac{(b - a)q_x}{1 - (1 - b)q_x} \simeq (b - a)q_x,$$

when the square and higher powers of q_x are ignored. A commonly used estimate of q_x is obtained by inserting this approximation into (19) to produce (Seal, 1962)

$$q_x \simeq d \bigg/ \sum_{j=1}^N (b_j - a_j). \quad (20)$$

The denominator of this expression is called the 'exact duration' form of the 'exposed to risk' and takes no account of the dates of death of those who die. When all values of $b_j - a_j$ that are not unity are set equal to one-half, (20) reduces to (17) and the denominator of (20) is called the 'mean duration' form of the 'exposed to risk'. Even in the latter case British and American actuaries have not used the denominator of (17) explicitly but it was named 'the actuarial method' by Berkson & Gage (1950) and this name unfortunately became incorporated into the statistical literature (Bartholomew, 1973). The concept of exposure to risk as the length of time a person is 'exposed' dates back at least to Woolhouse (1867) but not, I think, to Lambert (1772, pp. 578 ff.) even though that author deducted half the smallpox deaths from the original

group A in arriving at a probability of death from causes other than smallpox. Nevertheless it was Wittstein who gave the 'exposure' idea a probabilistic justification.

Wittstein (1862) argued that if B were zero and the lives escaping from observation consisted of individuals becoming disabled for work one could write $C = Aw_x$ in (14) and by using (15) obtain the approximate result

$$p_x^{aa} = (1 - q_x) \{1 - w_x(1 + \frac{1}{2}q_x)\}. \quad (21)$$

A similar result held for women subject both to mortality and earlier marriage, reasons for payment of contingent benefits in the early widows benefit plans.

Zeuner (1869) reported Wittstein's article in his book, the second with 'mathematical statistics' in its title, and proposed to evaluate the difference between entrants and exits at epoch t after age x as proportional to l_{x+t} , since both types of individual were in some sense 'survivors'. Once again the formula (17) results when squares and higher powers of q_x are ignored but it is interesting to note that before expansion (21) becomes

$$p_x^{aa} = (1 - q_x) \{1 - 2w_x / (2 - q_x)\}. \quad (22)$$

Both Wittstein and Zeuner were aware that (21) and (22) assumed that the mortality of active and invalid lives was the same. Kanner (1871) showed how (14) would need modification to recognize the different mortalities of these classes of individuals and, later, Küttner (1880) demonstrated that (22) would need correcting only by replacing q_x by q'_x given by (23) below.

Karup in 1875 began a series of articles to describe the estimation of 'independent' or 'pure' probabilities of mortality and disablement from (11), at this point ascribed to Kanner, if not to Karup himself. If we write

$$1 - q'_x = \exp\left(-\int_x^{x+1} \mu_x^{aa} dw\right), \quad 1 - w'_x = \exp\left(-\int_x^{x+1} \nu_w dw\right) \quad (23)$$

(11) gives

$$p_x^{aa} = (1 - q'_x) (1 - w'_x), \quad (24)$$

which, not surprisingly, results also from (6) when the mortality of active lives is the same as that of invalids, namely $\mu_x^{aa} = \mu_x^i$. Karup's major and most pleasing contribution (Karup, 1893) was to estimate q'_x by

$$q'_x \simeq \frac{d}{A - \frac{1}{2}j}, \quad (25)$$

where j is the number of cases of invalidity between ages x and $x + 1$, and w'_x by

$$w'_x \simeq \frac{j}{A - \frac{1}{2}d}. \quad (26)$$

Here B and C of Wittstein's case are supposed to be zero. They can be introduced in the denominators without difficulty. Incidentally, in this paper Karup expressed the forces of exit, for the first time, by single greek letters.

The ensuing furor among German actuaries was dramatic. It has been described at length by Spangenberg (1911) and summarized by Pasquier (1913, 1915) from that account. We limit ourselves to saying that some authors found it difficult to accept the idea of invalidity freed of mortality, one even writing that he would use (25) but not (26); another thought that eliminating one cause of exit would change the rest, a matter already raised in the smallpox controversy; and another argued that the force of a given type of exit would be different when acting alone or in the presence of other forces. The protagonists appeared to feel very deeply about their views and one even called Karup a 'conceited humbug'.

It was well into this century before the 'independent', 'pure', 'partial', 'simple', 'absolute',

or 'net' probabilities, as they have variously been called, were accepted as error-free concepts. As late as 1925 Berger in his mathematically oriented textbook found it necessary to prove that the use of such probabilities was equivalent to replacing all the actual decrements from other causes by a person of the same age and status as the decrement, a proposition advanced by Karup himself and widely disputed. Richard writing in 1946 said that until very recently the majority of actuaries thought that 'independent' probabilities were only of theoretical interest. One must admit that this is true so long as, for example, a probability of death 'independent' of disability has no intrinsic meaning. This was not the case in the original smallpox problem nor in 'causes of death' problems generally.

Nevertheless actuarial textbooks of many countries such as Blaschke (1906), Amtmann & Pfaffenberger (1907), Loewy (1924), Berger (1925), Hunter & Phillips (1932), Spurgeon (1932), Galbrun (1933), Steffensen (1934), Richard (1946), Böhm (1953), Saxer (1955), Hooker & Longley-Cook (1957), Dussart (1958), Zwinggi (1958), Risser (1965), Simonsen (1966), Jordan (1967), Coppini (1968) and Wolff (1970) have included substantial sections on multiple decrement mathematics. And there were over 50 articles published on multiple decrement, principally invalidity, mathematics in actuarial journals during the first half of this century. The only real advances derive from the introduction of 'revalidity' by Risser (1912) and his reference to a Volterra integral equation for the number of invalids, and from Böhmer's (1912) idea that since

$$1 - q(x, x + 1) = \prod_{j=0}^{n-1} \{1 - q(x + t_j, x + t_{j+1})\} \quad (t_0 = 0, t_n = 1),$$

for estimation purposes the year could be subdivided into n unequal intervals during each of which exactly one decrement occurred. The material included in the actuarial textbooks of those, and even more recent, days consists of the following.

(i) Differential equations are solved for the four fundamental transition probabilities, now extended to t years instead of one:

$$\begin{aligned} \frac{d_t P_x^{aa}}{dt} &= - {}_t P_x^{aa} (\mu_{x+t}^{aa} + \nu_{x+t}), & \frac{d_t P_x^{ai}}{dt} &= {}_t P_x^{aa} \nu_{x+t} - {}_t P_x^{ai} \mu_{x+t}^i, \\ \frac{d_t Q_x^{aa}}{dt} &= {}_t P_x^{aa} \mu_{x+t}^{aa}, & \frac{d_t Q_x^{ai}}{dt} &= {}_t P_x^{ai} \mu_{x+t}^i, \end{aligned}$$

with obvious boundary conditions at $t = 0$. A 'classic' exposition is that of Böhmer (1914), recently reprinted.

(ii) The approximate simple relations for these transition probabilities, such as (21), (22), (24), are given in terms of either 'independent' or 'dependent' basic probabilities.

(iii) Estimation formulae are given for the basic probabilities, such as (17), (25), (26).

Some of the textbooks (Berger, 1925; Zwinggi, 1958) first developed the mathematics in terms of n causes of decrement and then specialized to the invalidity case which is of special importance to actuaries.

Until the 1930's the mathematics of (i) and (ii) was restricted, with very few exceptions, to publications in German. The British, after Makeham's article assimilating decremental tables with the smallpox literature, limited themselves to the practical problem of estimating the basic decremental probabilities. Sprague (1879), writing just after the most heated publications in the 'independence' dispute and surely with knowledge of it, produced the standard article in English on the subject which showed how to estimate the decremental forces directly by deducting half all decrements in the denominator of expressions such as (25). Alternatively, this could be expressed by saying that the 'exposed to risk' of (20) was to be modified by replacing the $x + b_j$ of any decrement by the age at which the decrement occurred. Articles in French,

even in the Swiss actuarial *Mitteilungen*, which published the greatest number of contributions to this subject in the twentieth century, are conspicuously absent.

Pasquier's (1913) consideration of a force of 'revalidity' in the differential equations for l_x^{aa} and l_x^{ii} was shown by him to lead to a second order differential equation of Riccati type which he solved for the particular case of constant decremental forces. It is well known that the Riccati equation is a special case of the Volterra integral equation of the second kind. The latter arises quite naturally as follows.

Hamza (1900) had supposed that $p_x^{aa} = l_{x+1}^{aa}/l_x^{aa}$ had been obtained for all integer x from observations, e.g. by (21); this, in turn, led to l_x^{aa} and then $l_x^{ii} = l_x - l_x^{aa}$ or, independently,

$$l_x^{ii} = \int_0^x l_s^{aa} \nu_s \exp\left(-\int_s^x \mu_t^i dt\right) dz, \quad (27)$$

since invalid lives aged x are composed of the survivors of active lives who became invalid at age $z < x$. With l_x^{aa} , ν_x and μ_x^i all obtained from observation l_x^{ii} could be calculated from (27) by the Euler-Maclaurin expansion, for example. But (27) can be rewritten as

$$l_x = l_x^{aa} + \int_0^x l_s^{aa} \nu_s \exp\left(-\int_s^x \mu_t^i dt\right) dz, \quad (28)$$

and this is formally a Volterra integral equation of the second kind for l_x^{aa} (Risser, 1912). In this equation the mortality of invalids depends on their attained age only but if it is also a function of the time since invalidity, so that μ_t^i is replaced by $\mu^i(t, t-z)$, say, in (28), we obtain an equation

$$l_x = l_x^{aa} + \int_0^x l_s^{aa} \nu_s p^i(z, x) dz, \quad (29)$$

where

$$p^i(z, x) = \exp\left\{-\int_s^x \mu^i(t, t-z) dt\right\},$$

and must utilize the standard Volterra integral equation solution, namely

$$l_x^{aa} = l_x + \sum_{j=1}^{\infty} \phi_j(x),$$

where

$$\phi_j(x) = -\int_{x_0}^x \phi_{j-1}(t) \nu(t) p^i(t, x) dt \quad (j = 1, 2, \dots), \quad \phi_0(x) = l_x. \quad (30)$$

Schoenbaum (1931) used this procedure to derive the values of l_x^{aa} for Czechoslovakian social insurance. It is interesting that, in order to secure seven significant figures in l_x^{aa} , whereas the summation of the ϕ_j 's was as short as one term at age 21, by age 80 sixteen terms were needed. This was a considerable numerical achievement in precomputer days.

There have been about a dozen actuarial articles concerned with Volterra equations like (29) or the Riccati specialization. Both Richard (1946) and Risser (1965) report on these developments.

3. MAXIMUM LIKELIHOOD ESTIMATION

The first half of the twentieth century thus saw the mathematical theory of invalidity insurance, or of transitions between two states A and B , established on a sound base. Galbrun's (1933) monograph on the subject, in which a completely new system of notation was introduced, summarized previous developments in this essentially actuarial field. This work was an introduction to the author's later text (1934) in which attention was paid to the number of returns from state B to state A during a given period of time. The writer has dubbed this 'repeated sickness' and Seal (1970) gives a short history.

Reference to Galbrun (1933) or the more recent Zwinggi (1958) shows that actuaries thought that the 'expected value' technique of Wittstein, not necessarily given his name, was perfectly adequate for the estimation of q_x, w_x, q'_x, w'_x and other probabilities basic to numerical calculations. It was not until Sverdrup (1965) that maximum likelihood estimation of the transition probabilities was introduced to actuaries but by then statisticians had become interested in transitions between states in continuous time.

In 1950 Neyman devoted 27 pages of his new elementary probability text book to the 'Evaluation of competing risks'. Here we find, for example, the probability that an outgoing 'active' patient will survive a year, say, but in the meantime have suffered a relapse into his original disease, namely become 'invalid', written as

$$Q_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} (e^{-\lambda_2} - e^{-\lambda_1 - \lambda_2}),$$

where we can identify the age-independent forces, or intensities of risk, λ by comparison with Pasquier's (1913) formula

$$p_x^{\alpha t} = \frac{\nu}{\mu^{\alpha\alpha} + \nu - \mu^t} (e^{-\mu^t} - e^{-\mu^{\alpha\alpha} - \nu})$$

in which the three decremental forces are assumed to be constant throughout the year of age.

A year later Fix & Neyman (1951) produced maximum likelihood estimates of the transition probabilities and extended the mathematics to the 'repeated sickness' case. Referring to Sverdrup (1965) we see that the maximum likelihood estimate of a decremental force, assumed constant over the year of age, is that used by Sprague (1879) on intuitive grounds as applicable at exact age $x + \frac{1}{2}$. And by the invariance property of maximum likelihood estimators the estimates of $p_x^{\alpha\alpha}, p_x^{\alpha t}$, etc., obtained by inserting the maximum likelihood estimates of the decremental forces involved therein, are themselves maximum likelihood estimates. This is all rather disappointing for those who want to demonstrate the superiority of new statistical ideas.

We suppose that the neglect of a substantial actuarial literature by today's much larger population of mathematically trained statisticians is merely irritating. Consider, for example, Chiang's (1961) 'weaker assumption', in developing relations between 'net', i.e. 'independent', and 'crude', i.e. 'dependent', probabilities, namely that, in the notation of the present paper,

$$\mu_{x+t}^{(i)} = r_{x+t}^i \mu_{x+t}.$$

This assumption, now known in the statistical literature (David, 1970) as Chiang's proportionality assumption, was given explicitly in this form by Greville (1948, 1949) and was introduced as a reasonable hypothesis even earlier by Meier (1940). But how can actuaries complain when, except for Makeham's lip service in 1874, the original founders of the mathematical model of transitions between states have received no attention whatever in the actuarial literature.

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