FREQUENCY OF OSCILLATIONS OF AN ERROR TERM RELATED TO THE EULER FUNCTION

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Abstract. Let φ be the Euler function, and consider the error term H in the asymptotic formula

$$\sum_{n\leqslant x}\frac{\varphi(n)}{n}=\frac{6}{\pi^2}x+H(x).$$

It is proved that, for any fixed real number A, there are at least $C_AT+O(1)$ integers $n\in[1,T]$ such that $(H(n)-A)(H(n+1)-A)<0$, where $0<C_A<1$ is a constant depending on A.

Let φ be the Euler function (*i.e.*, $\varphi(n)$ denotes the number of integers not exceeding n which are relatively prime to n), and define

$$H(x)=\sum_{n\leqslant x}\frac{\varphi(n)}{n}-\frac{6}{\pi^2}x.$$

In **[2]**, it is shown that $H(x)$ has a large number (of order T) of sign changes on integers $n\leqslant T$. In this note, we prove that this phenomenon occurs as well for the changes in sign of $H_A(n)=H(n)-A$, where A is any fixed real number. The value $A=3/\pi^2$ plays a special role. It is indeed known that the distribution function Δ of the values taken by H_{3/π^2} at integers is symmetric **[3]**, whence in particular $\Delta(0)=1/2$, so one would expect the number of changes in sign of $H_A(n)$ to be particularly important when $A=3/\pi^2$. But the slightly surprising fact is that the only value of A for which a straightforward modification of the argument in **[2]** is inefficient is precisely $A=3/\pi^2$.

THEOREM. *Let A be a fixed number. For all sufficiently large T,*

$$|\{n\in[1,T]:(H(n)-A)(H(n+1)-A)<0\}|\geqslant C_AT,$$

where $|\{\cdots\}|$ denotes the cardinality of the set and $0<C_A<1$ is a constant (depending on A).

We separate the proof into three cases: (i) $A<3/\pi^2$, (ii) $A=3/\pi^2$ and (iii) $A>3/\pi^2$. Cases (i) and (iii) can be treated as in **[2]**, §3. For case (i) replace $D(0)$ in the argument there by $D(A)$, where $D(u)=\lim_{x\to\infty}x^{-1}|\{n\leqslant x: H(n)\leqslant u\}|$, and note that, if $H(n)<A$ and $H(m)<A$

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for all integral $m \in [n, n+2h]$, then, for any real $t \in [n, n+h)$, we have

$$\left| \int_t^{t+h} H(u)\, du \right| \geqslant \left(\frac{3}{\pi^2} - A \right) \frac{h}{2},$$

as soon as h is large enough. This comes from the fact that $H(x)$ is a straight line of slope $-6/\pi^2$ in every interval $[m, m+1)$ when m is an integer. For case (iii) consider instead the proportion $(1 - D(A))$ of integers n for which $H(n) > A$, and similarly note that, if $H(n) > A$ and $H(m) > A$ for all integral $m \in [n, n+2h)$, then, for any real $t \in [n, n+h)$, we have

$$\left| \int_t^{t+h} H(u) du \right| \geqslant \left(A - \frac{3}{\pi^2} \right) \frac{h}{2},$$

as soon as h is large enough. It is now clear why this method does not work when $A = 3/\pi^2$.

From **[1]** and **[3]**, we know that the distribution function $D(u)$ exists, $D(3/\pi^2) = 1/2$ and $D(u)$ is a continuous function of u. Hence, if C is a constant, then for all sufficiently large T we have

$$|\{T \leqslant n < 2T - C \colon H(n) \leqslant 3/\pi^2\}| \geqslant \frac{3T}{8}. \tag{1}$$

Let h be a large constant, which will be chosen later, and assume that T is large enough to satisfy (1) for $C = 10h$. We divide the interval $[T, 2T)$ into divisions $[T, T+h)$, $[T+h, T+2h), \ldots$ of length h, discarding the remaining interval of length $T - [T/h]h$, and group every 8 divisions to form an interval I, discarding the last $[T/h] - 8[T/(8h)]$ divisions. Then these newly formed intervals I cover the interval $[T, 2T - 8h)$, and their number is $[T/(8h)] \leqslant T/(8h)$. For convenience, we use the symbol $\mathcal{J}$ to designate a subinterval of I consisting of the initial 6 divisions. We define

$$\mathcal{C} = \{I \colon H(n) \leqslant 3/\pi^2 \quad \text{for some integer } n \in \mathcal{J}\},$$

and, for each I in $\mathcal{C}$, we choose one associated integer $n_0 = n_0(I)$ in $\mathcal{J}$ with $H(n_0) \leqslant 3/\pi^2$ (for instance the smallest such integer). By (1), if T is large enough, we have

$$|\mathcal{C}| \geqslant \frac{(3T/8 - (2h+1)(T/8h))}{6h+1} = \frac{T(h-1)}{8h(6h+1)},$$

so that $|\mathcal{C}| \geqslant T/(50h)$ if $h \geqslant 29$. From the continuity of $D(u)$, we can find $\varepsilon > 0$ such that the set $S = \{n \leqslant 2T \colon 3/\pi^2 - \varepsilon \leqslant H(n) \leqslant 3/\pi^2\}$ has cardinality $|S| \leqslant T/200$. Consider $J_1 = \{I \in \mathcal{C} \colon |I \cap S| \leqslant h/2\}$. Then

$$\frac{h}{2} |\mathcal{C} \setminus J_1| \leqslant \sum_{I \in \mathcal{C} \setminus J_1} |I \cap S| \leqslant |S| \leqslant \frac{T}{200}.$$

From this, we have $|J_1| \geq T/(100h)$. Then we can proceed with the argument in [**2**] on the collection J_1. Define

$$J_2 = \{I \in J_1 : H(m) \leq 3/\pi^2 \text{ for all integers } m \in [n_0, n_0 + 2h)\}.$$

We have, when h is a sufficiently large constant,

$$|J_2| \frac{\varepsilon^2 h^3}{16} \leq \sum_{I \in J_2} \int_{n_0}^{n_0+h} \left(\int_t^{t+h} H(u)du \right)^2 dt \leq \int_T^{2T} \left(\int_t^{t+h} H(u)du \right)^2 dt.$$

The first inequality comes from the facts that $I \in J_2$ has at most $h/2$ elements in S, that $H(m) < 3/\pi^2 - \varepsilon$ if $m \notin S$, and that $H(x)$ is a straight line of slope $-6/\pi^2$ in every interval $[m, m+1)$ when m is an integer. But the last integral at most KTh for some constant K by [**2**, Main Lemma]. Thus, when h is a sufficiently large constant,

$$|J_1 \setminus J_2| \geq \frac{T}{100h} - \frac{16KT}{\varepsilon^2 h^2} \geq \frac{T}{200h}.$$

In order to conclude the proof of the theorem, it is now of course sufficient to invoke the fact that $H(n) \neq 3/\pi^2$ when n is an integer. But, in view of the other error term $E(x)$ in the remark just below for which the equivalent property is not easy to establish (and may not be true), we may also argue that, by continuity of the distribution function D, if T is large enough then the set of integers $n \leq 2T$ for which $H(n) = 3/\pi^2$ is less than $T/(400h)$.

REMARK. This method can be applied to the error term

$$E(x) = \sum_{n \leq x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6} x + \frac{1}{2} \log x$$

associated with the sum-of-divisors function σ as well. In this case, the critical value for which the argument of case (ii) applies is $A = \pi^2/12$.

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