ON THE ACTION OF GALOIS GROUPS ON $BU(n)^{\wedge}$

By ZDZISŁAW WOJTKOWIAK*

[Received 27 July 1979; in final form 10 February 1983]

Introduction

LET $\tilde{Q}$ be an algebraic closure of the rational numbers and let $X^{\wedge}$ (resp. $X_p^{\wedge}$) denote the finite completion of X (resp. the p-completion of X). In [12] Dennis Sullivan has defined a homotopy action of Gal $(\tilde{Q}; Q)$ on $BU^{\wedge}$ and $BU(n)^{\wedge}$ i.e. a homomorphism of Gal $(\tilde{Q}; Q)$ into a group of homotopy equivalences of $BU^{\wedge}$ and of $BU(n)^{\wedge}$. In this paper "action" will always mean this homotopy action of Galois groups. Sullivan showed that the action of Gal $(\hat{Q}; Q)$ on $BU^{\wedge}$ factors through an action of Gal $(\tilde{Q}; Q)/[\text{Gal}(\tilde{Q}; Q); \text{Gal}(\tilde{Q}; Q)] \approx \hat{Z}^*$. In the case of finite Grassmann manifolds and $BU(n)$ he asked whether the action of Gal $(\tilde{Q}; Q)$ is abelian as well. He also considered the actions of the groups Gal $(\bar{F}_p; F_p)$ (F_p being a finite field of p-elements) on the r-completion of $BU(n)$ for different p and asked which subgroup of the group of homotopy equivalences of $BU(n)_r^{\wedge}$ these actions generated. In this paper we show the following results which are partial answers to some of his questions.

THEOREM A. *The action of* Gal $(\tilde{Q}; Q)$ *on* $BU(n)^{\wedge}_{1/n!}$ *factors through an action of* $\hat{Z}^*$.

Let $S^{\infty}(X)$ denote a suspension spectrum and $(S^{\infty}(X))_r^{\wedge}$ the localization of the spectrum $S^{\infty}(X)$ with respect to the homology theory $H_*(\ ; Z/r)$.

THEOREM B. *For all primes r the action of* Gal $(\tilde{Q}; Q)$ *on* $(S^{\infty}BU(n))_r^{\wedge}$ *factors through an action of* $\hat{Z}^*$.

Our last result concerns the actions of Gal $(\bar{F}_p; F_p)$ on $BU(n)_r^{\wedge}$ for various p's. Let $\coprod_{p\neq r}$ Gal $(\bar{F}_p; F_p)$ be the direct sum. The group Gal $(\bar{F}_p; F_p)$ has a natural compact topology given by the inverse limit structure and we equip $\coprod_{p\neq r}$ Gal $(\bar{F}_p; F_p)$ with the topology of a direct limit of finite products. The set $[BU(n)_r^{\wedge}; BU(n)_r^{\wedge}]$ also has a compact topology and is a semigroup with composition as multiplication. We shall see further that the composition is a continuous operation in $[BU(n)_r^{\wedge}; BU(n)_r^{\wedge}]$.

* Whilst finishing this work the author was supported by the University of Oxford Mathematical Prizes Fund and The Royal Society.

THEOREM C. *Let r be a prime greater than n. The actions of* Gal $(\bar{F}_p; F_p)$ *on* $BU(n)^{\wedge}_r$ *yield a continuous homomorphism of semigroups*

$$\Phi\colon \coprod_{p\neq r} \mathrm{Gal}\,(\bar{F}_p; F_p) \to [BU(n)^{\wedge}_r; BU(n)^{\wedge}_r].$$

Φ *factors through a surjective map*

$$\varphi\colon \coprod_{p\neq r} \mathrm{Gal}\,(\bar{F}_p; \bar{F}_p) \to \hat{Z}^*_r,$$

where φ *maps the Frobenius automorphism of* $\bar{F}_p$ *onto p.*

Now we shall briefly explain the main ideas of the proofs. The algebraic action of Gal $(\bar{Q}; Q)$ on a simplicial scheme $\bar{W}GL_{n,\bar{Q}}$ induces an action of Gal $(\bar{Q}; Q)$ on the rigid etale homotopy type of $\bar{W}GL_{n,\bar{Q}}$. The comparison theorem allows one to define a homotopy action of Gal $(\bar{Q}; Q)$ on $BGL(n; C)^{\wedge}_r$. Let $\bar{W}(\Sigma_n \int Q/Z) \otimes \bar{Q}$ be a disjoint union of spec $\bar{Q}$ indexed by $\bar{W}(\Sigma_n \int Q/Z)$ with obvious faces and degeneracies. There is a map of simplicial schemes $\bar{W}(\Sigma_n \int Q/Z) \otimes \bar{Q} \to \bar{W}GL_{n,\bar{Q}}$. This map is Gal $(\bar{Q}, Q_{ab})$-equivariant where Q_{ab} is a maximal abelian extension of Q. The action of Gal $(\bar{Q}; Q_{ab})$ on $(\bar{W}(\Sigma_n \int Q/Z) \otimes \bar{Q})_{\mathrm{ret}}$ is trivial because $\bar{W}(\Sigma_n \int Q/Z) \otimes \bar{Q}$ is a disjoint union of copies of spec $\bar{Q}$. $(\bar{W}(\Sigma_n \int Q/Z) \otimes \bar{Q})_{\mathrm{ret}}$ is weakly homotopy equivalent to $\bar{W}\Sigma_n \int Q/Z$ and $(\bar{W}\Sigma_n \int Q/Z)^{\wedge}_r 1/n!$ is homotopy equivalent to $BU(n)^{\wedge}_r 1/n!$ for $r>n$. Hence it follows that the action of Gal $(\bar{Q}; Q_{ab}) =$ [Gal $(\bar{Q}, Q)$; Gal $(\bar{Q}, Q)$] on $BU(n)^{\wedge} 1/n!$ is trivial. These are the main ideas of the proof of Theorem A. The proof of Theorem C is quite similar though more complicated.

In the proof of Theorem B we use the stable splitting

$$\left(S^{\infty}B\Sigma_n \int Z/r^{\infty}\right)^{\wedge}_r \approx (S^{\infty}BU(n))^{\wedge}_r \vee \Gamma \quad (*),$$

which follows from the existence of the transfer.

We show that the induced action of Gal $(\bar{Q}; Q)$ on $(S^{\infty}B\Sigma_n \int Z/r^{\infty})^{\wedge}_r$ factors through $\hat{Z}^*$ and then using the splitting (*) we show that the action of Gal $(\hat{Q}; Q)$ on $(S^{\infty}BU(n))^{\wedge}_r$ also factors through $\hat{Z}^*$.

I would like to thank very much Professor E. Friedlander for his help in the proof of the key fact (Proposition 2.7), for his essential simplification of the arguments in Section 4 and for his help in several other points. I would also like to thank very much the referee for his very careful reading of the manuscript and for many valuable suggestions. Finally I would also like to thank Prof. A. Białynicki, R. Rubinstein and M. Hopkins for several useful discussions.

During the final stages of this work, the author was a visitor at Oxford

University and would like to thank very warmly Prof. Ioan James for his kind invitation and his support in finishing this work.

1. Simplicial spaces and notations

In [5] Eric Friedlander has defined the rigid etale homotopy type functor for a simplicial scheme. He denoted it by $(\ )_{ret}$. Using local homeomorphisms instead of etale maps we can define a functor $(\ )_{rlh}$ on the category of simplicial spaces in strict analogy with $(\ )_{ret}$. Let $X.$ be a simplicial space and let $S.X.$ be a bisimplicial set of singular simplexes of $S.$. Let Δ denote the diagonal functor. Then $\Delta(X.)_{rlh}$ and $\Delta S.X.$ are homotopy equivalent (see [6] p. 211).

If $X.$ is a simplicial scheme (resp. a simplicial space) then $(X.)_{ret}$ (resp. $(X.)_{rlh}$) is a pro-object in the category of bi-simplicial sets. To this pro-object we apply a diagonal functor, then the geometric realization functor and finally, to each space, the finite completion functor. Such an inverse system of spaces forms an inverse system of Brownian functors (see [13] § 3). We shall denote by $(X.)^\wedge_{ret}$ (resp. $(X.)^\wedge_{rlh}$) the space which represents the inverse limit of these functors. We shall also use the p-completion functor instead of the finite completion. In that case we shall denote the spaces which result by $((X.)_{ret})^\wedge_p$ and $((X.)_{rlh})^\wedge_p$.

We shall be concerned with a simplicial space $\bar{W}G$, G being a connected Lie group or a discrete group. We have the following proposition.

PROPOSITION 1.1. *$(\bar{W}G)_{rlh}$ is homotopy equivalent to BG and $(\bar{W}G)^\wedge_{rlh}$ (resp. $((\bar{W}G)_{rlh})^\wedge_p$) is homotopy equivalent to $BG^\wedge$ (resp. $BG^\wedge_p$).*

Proposition 1.1 follows from the discussion at the beginning of this section and the fact that the geometric realization of $\Delta S.\bar{W}G$ is homotopy equivalent to the geometric realization of $\bar{W}G$ (see [8] Theorem 11.13 and [10] Lemma p. 94 if G is connected). The geometric realization of $\bar{W}G$ is a classifying space of G (see [11] § 3). The proof for G discrete we leave to the reader.

2. The action of Gal $(\bar{Q}; Q)$ on $BU(n)^\wedge$

In this section we prove Theorem A. The following two propositions allow us to define the action of Gal $(\bar{Q}; Q)$ on $BU(n)^\wedge$.

PROPOSITION 2.1. *There exists a homotopy equivalence*

$$j\colon ((\bar{W}GL(n, C)_{rlh})^\wedge \to ((\bar{W}GL_{n:C})_{ret})^\wedge.$$

This map is given by interpreting an etale covering as a local homeomorphism. It is one of the homotopy equivalences from [5] Proposition 2.8.

PROPOSITION 2.2. *Let* $\alpha: \tilde{Q} \to C$ *be an inclusion. Then there exists a homotopy equivalence*

$$\alpha^*: ((\bar{W}GL_{n,C})_{ret})^\wedge \to ((\bar{W}GL_{n,\tilde{Q}})_{ret})^\wedge$$

induced by α.

Proof. It follows from **[14]** Expose XVI Corollaire 1.6 that $H^q(GL_{n,\tilde{Q}}, Z/m) \to H^q(GL_{n,C}, Z/m)$ is an isomorphism for each m. Hence it follows from [**5**] Lemma 3.4 and a spectral sequence argument of the type used in the proof of Proposition 2.8 of [**5**] that α^* is a homotopy equivalence.

The homotopy action of Gal $(\tilde{Q}; Q)$ on $BU(n)^\wedge$ is defined as follows. If $\sigma \in \text{Gal}\,(\tilde{Q}; Q)$ and $U. \to \bar{W}GL_{n,\tilde{Q}}$ is an etale covering then the pullback $\sigma!(U.)$ is also an etale covering of $\bar{W}GL_{n,\tilde{Q}}$. Hence σ induces a map of the category of rigid etale coverings of $\bar{W}GL_{n,\tilde{Q}}$ into itself. This map is an equivalence because σ has an inverse σ^{-1}. Hence σ induces a homotopy equivalence of $((\bar{W}GL_{n,\tilde{Q}})_{ret})^\wedge$ and therefore (by Proposition 2.1, 2.2 and the results of Section 1) a homotopy equivalence of $BU(n)^\wedge$ which we also denote by σ.

Now we begin the proof of Theorem A.

LEMMA 2.3. *Let* $X^\wedge$ *be the finite completion of a simply connected space* X. *If* $f: X^\wedge \to X^\wedge$ *then* $f = \prod_p f_p$ *where* $f_p: X_p^\wedge \to X_p^\wedge$.

Proof. The projection $X^\wedge = \prod_p X_p^\wedge \to X_p^\wedge$ is a p-completion. Let $f_p: X_p^\wedge \to X_p^\wedge$ be the p-completion of f. The lemma now follows from the homotopy commutative diagram

$$\begin{array}{ccc} X^\wedge & \xrightarrow{f} & X^\wedge \\ \downarrow & & \downarrow \\ \prod X_p^\wedge & \xrightarrow{\prod f_p} & \prod X_p^\wedge . \end{array}$$

Let P be the set of primes $\leq n$. Then

$$(BU(n)^\wedge)_{1/n!} = \Big(\prod_{p \notin P} BU(n)_p^\wedge\Big) \times \Big(\prod_{p \in P} \Big(\prod_{i=1}^{n} K(Q \otimes \hat{Z}_p; 2i)\Big)\Big)$$

because if $p > n$ then $1/n! \in \hat{Z}_p$ and if $p \leq n$ then $\hat{Z}_p \otimes Z_{[1/n!]} = Q \otimes \hat{Z}_p$.

If $\sigma \in \text{Gal}\,(\tilde{Q}; Q)$ then by Lemma 2.3 we have $\sigma = \prod_p \sigma_P: BU(n)^\wedge \to BU(n)^\wedge$. Let $[\text{Gal}\,(\tilde{Q}; Q); \text{Gal}\,(\tilde{Q}; Q)]$ be a commutator subgroup of Gal $(\tilde{Q}; Q)$.

PROPOSITION 2.4. *If* $p \leq n$ *and* $\sigma \in [\mathrm{Gal}(\bar{Q}; Q); \mathrm{Gal}(\bar{Q}; Q)]$ *then* $(\sigma_p)_{[1/n!]}$: $(BU(n)_p^\wedge)_{[1/n!]} \to (BU(n)_{p[1/n!]}^\wedge$ *is homotopic to the identity.*

Proof. First we shall show that the action of $\mathrm{Gal}(\hat{Q}; Q)$ on $H^*(((GL_{1,\bar{Q}})^n)_{\mathrm{ret}}; \hat{Z}_p)$ factors through $\hat{Z}^*$. Let

$$\mathcal{U}_{k_1,\ldots,k_n} = \{\mathrm{Spec}\,\bar{Q}[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}] \\ \to \mathrm{Spec}\,\bar{Q}[t_1, \ldots, t_n, t_1^{-1} \cdots t_n^{-1}]\} = (GL_{1,\bar{Q}})^n$$

be an etale map given by $t_i \to t_i^{-1}$. Then

$$\mathcal{U}_{k_1,\ldots,k_n} \underset{(GL_{1,Q})^n}{\times} \iota\text{-times} \cdots \underset{(GL_{1,Q})^n}{\times} \mathcal{U}_{k_1,\ldots,k_n}$$

has $(k_1 \cdots k_n)^\iota$ components. All components are equal $(GL_{1,\bar{Q}})^n$ and are indexed by elements of $(\mu_{k_1})^\iota \times \cdots \times (\mu_{k_n})^\iota$ where μ_{k_i} is the group of k_ith-roots of unity. Any elements $\sigma \in \mathrm{Gal}(\bar{Q}; Q)$ permutes the components according to its action on roots of unity. The Cech nerve of this covering is $\bar{W}^Z/k_1 \times \cdots \times \bar{W}^Z/k_n$. The homotopy inverse limit of the pro-simplicial set $\{\bar{W}^Z/k_1 \times \cdots \times \bar{W}^Z/k_n\}$ indexed by $\mathcal{U}_{k_1,\ldots,k_n}$ is weakly homotopy equivalent to $((GL_{1,\bar{Q}})^n)_{\mathrm{ret}}^\wedge$. The action of $\mathrm{Gal}(\bar{Q}; Q)$ on the pro-simplicial set $\{\bar{W}^Z/k_1 \times \cdots \times \bar{W}^Z/k_n\}_{\mathcal{U}_{k_1,\ldots,k_n}}$ factors through $\hat{Z}^*$. Hence the action of $\mathrm{Gal}(\bar{Q}; Q)$ on $H^*(((GL_{1,\bar{Q}})^n)_{\mathrm{ret}}^\wedge; \hat{Z}_p)$ also factors through $\hat{Z}^*$. There is a fibration $((GL_{1,\bar{Q}})^n)_{\mathrm{ret}} \to (W(GL_{1,\bar{Q}})^n)_{\mathrm{ret}} \to (\bar{W}(GL_{1,\bar{Q}})^n)_{\mathrm{ret}}$ and for any $\sigma \in \mathrm{Gal}(\bar{Q}; Q)$ we have a commutative diagram

$$\begin{array}{ccccc} ((GL_{1,\bar{Q}})^n)_{\mathrm{ret}} & \to & (W(GL_{1,\bar{Q}})^n)_{\mathrm{ret}} & \to & (\bar{W}(GL_{1,\bar{Q}})^n)_{\mathrm{ret}} \\ \sigma\uparrow & & \sigma\uparrow & & \sigma\uparrow \\ ((GL_{1,\bar{Q}})^n)_{\mathrm{ret}} & \to & (W(GL_{1,\bar{Q}})^n)_{\mathrm{ret}} & \to & (\bar{W}(GL_{1,\bar{Q}})^n)_{\mathrm{ret}} \end{array}$$

This implies that the action of $\mathrm{Gal}(\bar{Q}; Q)$ on $H^*(((\bar{W}(GL_{1,\bar{Q}})^n)_{\mathrm{ret}})_p^\wedge; \hat{Z}_p)$ also factors through $\hat{Z}^*$.

The natural map $\bar{W}(GL_{1,\bar{Q}})^n \to \bar{W}GL_{n,\bar{Q}}$ induces a monomorphism on cohomology. Therefore the action of $\mathrm{Gal}(\bar{Q}, Q)$ on $H^*(((\bar{W}GL_{n,\bar{Q}})_{\mathrm{ret}})_p^\wedge; \hat{Z}_p)$ also factors through $\hat{Z}^*$. It follows that σ_p induces the identity on the cohomology of $BU(n)_p^\wedge$ with $\hat{Z}_p$ coefficients. Hence after inverting $n!$ $\sigma_p[1/n!]$ is homotopic to the identity because then the space in question is a product of $K(\hat{Z}_p \otimes Q; n)$'s. (These are essentially arguments of Sullivan from [**12**].)

Now we examine the action on the first factor of $(BU(n)^\wedge)_{[1/n!]}$. We apply a standard trick of approximating $BU(n)$ by the classifying space of a discrete group (see [**1**]).

We set $S = \Sigma_n \int Q/Z$. The embedding i: $Q/Z \to \bar{Q}^*$ into roots of unity and the inclusion α: $\bar{Q} \to C$ induce ι_1: $S \to GL(n, \bar{Q})$ and ι: $S \to GL(n; C)$.

LEMMA 2.5. *The maps* ι_*: $(BS)^\wedge_r \to BGL(n; C)^\wedge_r$ *and* ι_*: $((\bar{W}S)_{rlh})^\wedge_r \to (\bar{W}GL(n; C)_{rlh})^\wedge_r$ *are homotopy equivalences for every prime* $r > n$.

Proof. The cohomology of BS with Z/r coefficients is $Z/r[t_1, \ldots, t_n]^{\Sigma_n}$. This can easily be seen from the spectral sequence of the fibration $B(Q/Z)^n \to BS \to B\Sigma_n$. The cohomology ring of $BGL(n; C)$ is the same. The map $B(Q/Z)^n \to B(C^*)^n$ is clearly an isomorphism on $H^*(\ ; Z/r)$ and therefore ι_*: $(BS)^\wedge_r \to BGL(n; C)$ is a homotopy equivalence. The fact just proved and the discussion in Section 1 imply that the second map is also a homotopy equivalence.

Let $S \otimes C$ (resp. $S \otimes \bar{Q}$) be a group scheme equal to the disjoint union of copies of spec C (resp. spec $\bar{Q}$) indexed by S. Then $\bar{W}(S \otimes C)$ and $\bar{W}(S \otimes \bar{Q})$ are simplicial schemes (see [7] Example 1.1). The maps of schemes $S \otimes K \to GL_{n,K}$ on the component indexed by s are given by $K[x_{ij}](\det(x_{ij})^{-1}) \ni f(x_{ij}) \to f(\zeta) \in K$ where $\zeta = \iota_1(s)$ if $K = \bar{Q}$ and $\zeta = \iota(s)$ if $K = C$. These maps induce maps of simplicial schemes ι_K: $\bar{W}(S \otimes K) \to \bar{W}GL_{n,K}$ for $K = C$ or $\bar{Q}$.

LEMMA 2.6. *The following diagram commutes up to homotopy and after applying the functor* $(\ )^\wedge_r$ $(r > n)$ *all arrows become homotopy equivalences.*

$$\begin{array}{ccc}
(\bar{W}S)_{rlh} & \xrightarrow{\iota_*} & (\bar{W}GL(n; C))_{rlh} \\
\downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j} \\
(\bar{W}S \otimes C)_{\text{ret}} & \xrightarrow{(\iota_C)_*} & (\bar{W}GL_{n,C})_{\text{ret}} \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha^*} \\
\bar{W}(S \otimes \bar{Q})_{\text{ret}} & \xrightarrow{(\iota_{\bar{Q}})_*} & (\bar{W}GL_{n,\bar{Q}})_{\text{ret}}
\end{array}$$

Proof. The commutativity of both small diagrams is clear. It follows from Propositions 2.1 and 2.2 that j and α^* are homotopy equivalences after applying $(\ )^\wedge_r$. The fact that $(\text{spec } K)_{\text{ret}}$ is contractible for algebraically closed fields $\bar{Q}$ and C implies that j_1 and α_1^* are weak homotopy equivalences. It follows from Lemma 2.5 that ι_* after r-completion is the homotopy equivalence. Hence all other arrows are homotopy equivalences after applying the functor $(\ )^\wedge_r$.

PROPOSITION 2.7. *If* $\sigma \in [\text{Gal}(\bar{Q}; Q); \text{Gal}(\bar{Q}; Q)]$ *then* $\sigma^\wedge_r$: $BU(n)^\wedge_r \to BU(n)^\wedge_r$ *is homotopic to the identity for every* $r > n$.

Proof. The action of $\text{Gal}(\bar{Q}; Q)$ on the set of closed points of the algebraic group $GL_{n,\bar{Q}}$ which is a subvariety of $(\bar{Q})^{n^2}$ is given in the following way. $\sigma \in \text{Gal}(\bar{Q}; Q)$ maps a point (v_{ij}) into a point $(\sigma(v_{ij}))$. In terms of the affine coordinates ring the action of σ is given in the following way. If $f=\sum_{i,j,k} a_{i,j,k}x_{i,j}^k$ then $f^\sigma=\sum_{i,j,k}\sigma(a_{i,j,k})x_{i,j}^k$. Similarly it acts on the affine coordinate ring of spec $\tilde{Q}$ i.e. on $\tilde{Q}$. Let Q_{ab} be a maximal abelian extension of Q. We shall show that the map of simplicial schemes $\iota_{\tilde{Q}}$: $\bar{W}(S\otimes\tilde{Q})\to\bar{W}GL_{n,\tilde{Q}}$ is Gal $(\tilde{Q}; Q_{ab})$-equivariant. The map of schemes $S\otimes\tilde{Q}\to GL_{n,\tilde{Q}}$ on the component of $S\otimes\tilde{Q}$ indexed by s is given by homomorphism

$$\iota_1^s\colon\ \tilde{Q}[x_{ij}](\det(x_{ij})^{-1})\to\tilde{Q},\qquad \iota_1^s(f)=f(\zeta)\text{ where }\zeta=\iota_1(s).$$

Hence for any $\sigma\in\text{Gal}\,(\tilde{Q}; Q_{ab})$ we have

$$\begin{aligned}\sigma(\iota_1^s(f))&=\sigma\Big(\sum_{i,j,k} a_{i,j,k}\zeta_{ij}^k\Big)=\sum_{i,j,k}\sigma(a_{i,j,k})\sigma(\zeta_{i,j}^k)\\&=\sum_{i,j,k}\sigma(a_{i,j,k})\zeta_{i,j}^k=f^\sigma(\zeta)=\iota_1^s(f^\sigma).\end{aligned}$$

This implies that the map of schemes $S\otimes\tilde{Q}\to GL_{n,\tilde{Q}}$ is Gal $(\tilde{Q}; Q_{ab})$-equivariant. Hence also a map of simplicial schemes $\bar{W}S\otimes\tilde{Q}\to\bar{W}GL_{n,\tilde{Q}}$ is Gal $(\tilde{Q}; Q_{ab})$-equivariant. Hence we have a commutative diagram

$$\begin{array}{ccc}(\bar{W}(S\otimes\tilde{Q}))_{\text{ret}} & \xrightarrow{(\iota_Q)_*} & (\bar{W}GL_{n,\tilde{Q}})_{\text{ret}}\\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma}\\ \bar{W}(S\otimes\tilde{Q})_{\text{ret}} & \xrightarrow{(\iota_Q)_*} & (\bar{W}GL_{n,\tilde{Q}})_{\text{ret}}\end{array}$$

for any $\sigma\in[\text{Gal}\,(\tilde{Q}; Q); \text{Gal}\,(\tilde{Q}; Q)]$ because

$$[\text{Gal}\,(\tilde{Q}, Q); \text{Gal}\,(\tilde{Q}, Q)]=\text{Gal}\,(\tilde{Q}; Q_{ab}).$$

The map σ: $(\bar{W}(S\otimes\tilde{Q}))_{\text{ret}}\to(\bar{W}(S\otimes\tilde{Q}))_{\text{ret}}$ is homotopic to the identity because σ preserves components spec $\tilde{Q}$ of $\bar{W}(S\otimes\tilde{Q})$ and (spec $\tilde{Q})_{\text{ret}}$ is a point. Now it follows from Lemma 2.6 that $\sigma_r^\wedge$: $(\bar{W}GL_{n,\tilde{Q}})_{\text{ret}})_r^\wedge\to((\bar{W}GL_{n,\tilde{Q}})_{\text{ret}})_r^\wedge$ is homotopic to the identity. This implies the proposition by Propositions 2.1 and 2.2. Theorem A now follows immediately from Propositions 2.4 and 2.7.

The action of Gal $(C; Q)$ on $BU(n)^\wedge$ is not interesting as we see from the following proposition.

Proposition 2.8. *The action of* Gal $(C; Q)$ *on* $BU(n)^\wedge$ *factors through* Gal $(\tilde{Q}; Q)$.

The proposition follows from Proposition 2.2 and the commutativity of the diagram

$$\begin{array}{ccc} (\bar{W}GL_{n,C})_{\text{ret}} & \xrightarrow{\alpha^*} & (\bar{W}GL_{n,\tilde{Q}})_{\text{ret}} \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\bar{\sigma}} \\ (\bar{W}GL_{n,C})_{\text{ret}} & \xrightarrow{\alpha^*} & (\bar{W}GL_{n,\tilde{Q}})_{\text{ret}} \end{array}$$

where $\bar{\sigma}$ is the restriction of $\sigma \in \text{Gal}\,(C; Q)$ to $\tilde{Q}$.

3. The stable action of Gal ($\tilde{Q}$; Q)

If X is a CW complex let $S^\infty X = \{S^n X\}_{n=1}^\infty$ be the suspension spectrum. Bousfield [4] has constructed localizations of the category of spectra with respect to all homology theories. The localization of a spectrum E with respect to the Eilenberg–MacLane spectrum $K(Z/r)$ we denote by $E_r^\wedge$ because of its behaviour on homotopy groups (see [4] Proposition 2.5 and Theorem 3.1). We shall be mainly concerned here with suspension spectra of $((\bar{W}G)_{\text{ret}})_r^\wedge$. Their $K(Z/r)$-localizations we shall denote by $(S^\infty(\bar{W}G)_{\text{ret}})_r^\wedge$.

Let $S = \Sigma_n \int Z/r^\infty$. The embedding i: $Z/r^\infty \to \tilde{Q}^*$ induces a map of simplicial schemes

$$\iota_{\tilde{Q}}\colon \bar{W}(S \otimes \tilde{Q}) \to \bar{W}GL_{n,\tilde{Q}} \quad \text{which is} \quad \text{Gal}\,(\tilde{Q}; Q_{(\mu_{r^\infty})})$$

equivariant where $Q_{(\mu_{r^\infty})}$ is an extension of Q obtained by adjoining all r^k-roots of unity. Hence for any $\sigma \in [\text{Gal}\,(\tilde{Q}; Q); \text{Gal}\,(\tilde{Q}; Q)]$ we have a commutative diagram

$$\begin{array}{ccc} \bar{W}(S \otimes \tilde{Q})_{\text{ret}} & \xrightarrow{(\iota_{\tilde{Q}})_*} & (\bar{W}GL_{n,\tilde{Q}})_{\text{ret}} \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ \bar{W}(S \otimes \tilde{Q})_{\text{ret}} & \xrightarrow{(\iota_{\tilde{Q}})_*} & (\bar{W}GL_{n,\tilde{Q}})_{\text{ret}}. \end{array}$$

Using the same argument as in the proof of Proposition 2.7 we obtain that σ: $\bar{W}(S \otimes \tilde{Q})_{\text{ret}} \to \bar{W}(S \otimes \tilde{Q})_{\text{ret}}$ is homotopic to the identity. If the group of coefficients of cohomology is Z/r, then $\bar{W}(S \otimes \tilde{Q})_{\text{ret}}$ can serve as a homological model for a classifying space of a normalizer of a maximal torus in $U(n)$. Hence it follows from [3] Theorem 5.5 that there is a transfer

$$\tau\colon\ (S^\infty(\bar{W}GL_{n,\tilde{Q}})_{\text{ret}})_r^\wedge \to (S^\infty \bar{W}(S \otimes \tilde{Q})_{\text{ret}})_r^\wedge$$

such that

$$j \circ \tau = \text{id}_{(S^\infty(\bar{W}GL_{n,\tilde{Q}})_{\text{ret}})_r^\wedge} \quad \text{where} \quad j = (S^\infty_{((\iota_{\tilde{Q}})_*)_r^\wedge})_r^\wedge.$$

Let $[X; Y]$ be the set of homotopy classes of maps between spectra. In order to prove Theorem B it is sufficient to show that for every spectrum X the map from $[X; (S^\infty(\bar{W}GL_{n,\bar{Q}})_{ret})_r^\wedge]$ into $[X; (S^\infty(\bar{W}GL_{n,\bar{Q}})_{ret})_r^\wedge]$ induced by $S^\infty(\sigma)_r^\wedge$ is the identity. Let $f \in [X; S^\infty(\bar{W}GL_{n,\bar{Q}})_{ret})_r^\wedge]$. Then $g = \tau \circ f \in [X; (S^\infty \bar{W}(S \otimes \bar{Q})_{ret})_r^\wedge]$ and $j \circ g = f$, so that $S^\infty(\sigma)_r^\wedge \circ f = S^\infty(\sigma)_r^\wedge \circ j \circ g = j \circ g = f$. This completes the proof.

4. The action of Gal $(\bar{F}_p; F_p)$ on $BU(n)_r^\wedge$

In this section we shall compare the homotopy self equivalences of $BU(n)_r^\wedge$ induced by Gal $(\bar{F}_p \cdot F_p)$ for different p. Let i: $Z/r^\infty \to C^*$ and k: $Z/r^\infty \to \bar{F}_p^*$ be embeddings into roots of unity. Set $S = \Sigma_n \int Z/r^\infty$. The map k induces a map of simplicial schemes K: $\bar{W}(S \otimes \bar{F}_p) \to \bar{W}GL_{n,\bar{F}_p}$. The map K on the component spec $\bar{F}_p$ of $S^n \otimes \bar{F}_p$ indexed by $s = (s_1, \ldots, s_n)$ is given by

$$\bigotimes_{\iota=1}^{n} F_p[x_{ij}^\iota]/(\det(x_{ij}^\iota) - 1) \ni f_1 \otimes \cdots \otimes f_n \to f_1(\zeta_1) \cdots f_n(\zeta_n)$$

where $\zeta_i = \bar{k}(s_i)$ and $\bar{k}$: $S \to GL(n, \bar{F}_p)$ is induced by k: $Z/r^\infty \to \bar{F}_p^*$. Let ι: $\bar{W}S \to \bar{W}GL(n, C)$ be a map induced by i: $Z/r^\infty \to C^*$. The following proposition will be fundamental in this section.

PROPOSITION 4.1. *Suppose that there are given two embeddings i: $Z/r^\infty \to C^*$ and k: $Z/r^\infty \to \bar{F}_p^*$ into roots of unity. Suppose also that $(r, p) = 1$ and $r > n$. Then these embeddings induce homotopy equivalences*

$$((\bar{W}GL_{n,\bar{F}_p})_{ret})_r^\wedge \xleftarrow{K_*} (\bar{W}(S \otimes \bar{F}_p)_{ret})_r^\wedge \xleftarrow{Jd} (\bar{W}S_{rth})_r^\wedge$$
$$\xrightarrow{\iota_*} (\bar{W}GL(n; C)_{rth})_r^\wedge \approx BU(n)_r^\wedge.$$

In particular, i and k naturally determine a homotopy equivalence

$$\theta_{(k,\iota)}: (\bar{W}GL_{n,\bar{F}_p})_{ret})_r^\wedge \to (\bar{W}GL(n, C)_{rth})_r^\wedge.$$

Proof. Let us notice that the pro-objects $\bar{W}(S \otimes \bar{F}_p)_{ret}$ and $(\bar{W}S)_{rth}$ are weakly homotopy equivalent. The map between $(\bar{W}(S \otimes \bar{F}_p)_{ret})_r^\wedge$ and $((\bar{W}S)_{rth})_r^\wedge$ we shall denote by Jd. The map Jd and ι_* are clearly homotopy equivalences. To show that K_* is a homotopy equivalence we must compare the situation over $\bar{F}_p$ via the Witt vectors of $\bar{F}_p$ with the situation over C. But in the characteristic zero the map $((\bar{W}GL_{n,C})_{ret})_r^\wedge \leftarrow (\bar{W}(S \otimes C)_{ret})_r^\wedge$ is a homotopy equivalence because $((\bar{W}GL(n, C))_{rth})_r^\wedge \leftarrow (\bar{W}S_{rth})_r^\wedge$ is a homotopy equivalence. This implies that K_* is a homotopy equivalence.

Remark. To compare $((\bar{W}G_{\bar{F}_p})_{ret})_r^\wedge$ and $((\bar{W}G_C)_{ret})_r^\wedge$ one uses the embedding f of the Witt vectors of $\bar{F}_p$ into C (see [5] Proposition 2.8).

However in our special case it is not necessary. The embedding f is replaced by embeddings $i\colon Z/r^{\infty}\to C^{*}$ and $k\colon Z/r^{\infty}\to \bar{F}_p$. One can show that the raising homotopy equivalences $\theta(k,i)$ are the same as one obtains using the Witt vectors. This implies that in our special case ($(r,p)=1$ and $r>n$) the homotopy equivalence θ from Proposition 2.8 of [5] depends only on the restriction of f to roots of unity.

If $\sigma\in \mathrm{Gal}(\bar{F}_p;F_p)$ we denote by σ the induced map of $((\bar{W}GL_{n,\bar{F}_p})_{\mathrm{ret}})^{\wedge}_r$.

Set $\psi_{(i,k)}(\sigma)=\theta(k,i)\circ\sigma\circ\theta(k,i)^{-1}$.

LEMMA 4.2. *Let $k\colon Z/r^{\infty}\to\bar{F}_p$ be an embedding into roots of unity and let $\varphi_p\colon \mathrm{Gal}(\bar{F}_p;F_p)\to \mathrm{Aut}(Z/r^{\infty})=\hat{Z}^{*}_r$ be the homomorphism given by restriction. If $\sigma\in\mathrm{Gal}(\bar{F}_p;F_p)$ then the following diagram commutes up to homotopy*

$$\begin{array}{ccc}
(\bar{W}GL_{n,\bar{F}_p})_{\mathrm{ret}} & \xleftarrow{K_*} & \bar{W}(S\otimes\bar{F}_p)_{\mathrm{ret}}\\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle \varphi_p(\sigma)_*=(\bar{W}\varphi_p(\sigma))_{\mathrm{ret}}}\\
(\bar{W}GL_{n,\bar{F}_p})_{\mathrm{ret}} & \xleftarrow{K_*} & \bar{W}(S\otimes\bar{F}_p)_{\mathrm{ret}}
\end{array}$$

where $\bar{W}\varphi_p(\sigma)\colon \bar{W}(S\otimes\bar{F}_p)\to\bar{W}(S\otimes\bar{F}_p)$ is induced by $\varphi_p(\sigma)\colon Z/r^{\infty}\to Z/r^{\infty}$. $\bar{W}\varphi_p(\sigma)$ maps a copy of $\mathrm{spec}\,\bar{F}_p$ *indexed by $s\in S^n$ into a copy of* $\mathrm{spec}\,\bar{F}_p$ *indexed by* $(\mathrm{id}\int\varphi_p(\sigma))(s)$ *by the identity map.*

Proof. For any $\sigma\in\mathrm{Gal}(\bar{F}_p;F_p)$ we have a commutative diagram

$$\begin{array}{ccc}
 & \swarrow^{K} & \bar{W}(S\otimes\bar{F}_p)\\
\bar{W}GL_{n,\bar{F}_p} & & \downarrow{\scriptstyle\sigma}\\
\downarrow{\scriptstyle\sigma} & & \bar{W}(S\otimes\bar{F}_p)\\
\bar{W}GL_{n,\bar{F}_p} & & \downarrow{\scriptstyle\bar{W}(\varphi_p(\sigma))}\\
 & \nwarrow_{K} & \bar{W}(S\otimes\bar{F}_p)
\end{array}$$

where $\sigma\colon\bar{W}(S\otimes\bar{F}_p)\to\bar{W}(S\otimes\bar{F}_p)$ is the Galois action. Let us notice that the map $\sigma\colon\bar{W}(S\otimes\bar{F}_p)_{\mathrm{ret}}\to\bar{W}(S\otimes\bar{F}_p)_{\mathrm{ret}}$ is the identity. This implies that the diagram in question is homotopy commutative.

Now we shall show that $\psi_{(i,k)}(\sigma)$ does not depend on the choice of i and k.

PROPOSITION 4.3. *Let $i\colon Z/r^{\infty}\to C^{*}$, $i_1\colon Z/r^{\infty}\to C^{*}$, $k\colon Z/r^{\infty}\to\bar{F}^{*}_p$ and $k_1\colon Z/r^{\infty}\to\bar{F}^{*}_p$ be embeddings into roots of unity. Then for any $\sigma\in$* $\mathrm{Gal}(\bar{F}_p;F_p)$ *we have*

$$\psi_{(i,k)}(\sigma)=\psi_{(i_1,k_1)}(\sigma).$$

Moreover $\psi_{(i,k)}(\sigma)=\iota_*\circ\varphi_p(\sigma)_{\#}\circ\iota_*^{-1}$ where $\varphi_p(\sigma)_{\#}\colon((\bar{W}S)_{r\mathrm{th}})^{\wedge}_r\to((\bar{W}S)_{r\mathrm{th}})^{\wedge}_r$ is induced by $\bar{W}(\mathrm{id}\,\Sigma_n\int\varphi_p(\sigma))$.

Proof. Let $a\colon Z/r^\infty \to Z/r^\infty$ be a homomorphism such that $i = i_1 \circ a$. From the definition of $\psi_{(i,k)}(\sigma)$ and $\theta(k, i)$ we have

$$\begin{aligned}\psi_{(i,k)}(\sigma) &= \theta(k, i)\circ\sigma\circ\theta(k, i)^{-1}\\ &= \iota_*\circ Jd^{-1}\circ K_*^{-1}\circ\sigma\circ K_*\circ Jd\circ\iota_*^{-1}.\end{aligned}$$

It follows from Lemma 4.2 that

$$\psi_{(i,k)}(\sigma) = \iota_*\circ Jd^{-1}\circ\varphi_p(\sigma)_*\circ Jd\circ\iota_*.$$

It is clear that the following diagram commutes

$$\begin{array}{ccc} (\bar{W}(S\otimes\bar{F}_p)_{\mathrm{ret}})_r^\wedge & \xleftarrow{Jd} & ((\bar{W}S)_{rlh})_r^\wedge \\ \Big\downarrow{\scriptstyle \varphi_p(\sigma)_*} & & \Big\downarrow{\scriptstyle \varphi_p(\sigma)} \\ (\bar{W}(S\otimes\bar{F}_p)_{\mathrm{ret}})_r^\wedge & \xleftarrow{Jd} & ((\bar{W}S)_{rlh})_r^\wedge \end{array}$$

where $\varphi_p(\sigma)_\# = \bar{W}(\mathrm{id}_{\Sigma_n}\int\varphi_p(\sigma))_r^\wedge$. This implies that

$$\psi_{(i,k)}(\sigma) = \iota_*\circ\varphi_p(\sigma)_\#\circ\iota_*^{-1}$$

and similarly

$$\psi_{(i_1,k_1)}(\sigma) = \iota_{1*}\circ\varphi_p(\sigma)_\#\circ\iota_{1*}^{-1}.$$

But the equality $\iota_{1*} = \iota_*\circ a_\#$ where $a_\# = (\bar{W}(\mathrm{id}_{\Sigma_n}\int a))_r^\wedge$ and the fact that Aut (Z/r^∞) is abelian implies that

$$\begin{aligned}\iota_{1*}\circ\varphi_p(\sigma)_\#\circ\iota_{1*}^{-1} &= \iota_*\circ a_\#\circ\varphi_p(\sigma)_\#\circ a_\#^{-1}\circ\iota_*^{-1}\\ &= \iota_*\circ\varphi_p(\sigma)_\#\circ a_\#\circ a_\#^{-1}\circ\iota_*^{-1} = \iota_*\circ\varphi_p(\sigma)_\#\circ\iota_*^{-1}\end{aligned}$$

i.e.

$$\psi_{(i_1,k_1)}(\sigma) = \psi_{(i,k)}(\sigma)$$

We see that $\psi_{(i,k)}(\sigma)$ does not depend on i and k. We set $\psi(\sigma) = \psi_{(i,k)}(\sigma)$.

Let us notice that the formula $\psi(\sigma) = \iota_*\circ\varphi_p(\sigma)_\#\circ\iota_*^{-1}$ implies the following corollary.

COROLLARY 4.4. *Suppose that $(r, p) = 1$ and $r > n$. Then the action of* Gal $(\bar{F}_p, F_p)$ *on* $BU(n)_r^\wedge$ *factors through* φ_p: Gal $(\bar{F}_p; F_p) \to \hat{Z}_r^*$.

Now we shall prove Theorem C. First we shall show that the actions of Gal $(\bar{F}_p; F_p)$ on $BU(n)_r^\wedge$ commute for all $p \neq r$ and that they generate the group $\hat{Z}_r^*$. Throughout the following discussion, r is a prime greater than n.

Fix an embedding $i\colon Z/r^\infty \to C^*$ and embeddings $k_p\colon Z/r^\infty \to \bar{F}_p$ for each prime $p \neq r$. Hence for each prime $p \neq r$ we have an action of $\mathrm{Gal}(\bar{F}_p; F_p)$ on $BU(n)_r^\wedge$.

Let us define a map

$$\Phi: \coprod_{p \neq r} \mathrm{Gal}(\bar{F}_p; F_p) \to [BU(n)_r^\wedge; BU(n)_r^\wedge]$$

by setting $\Phi(\sigma_1, \sigma_2, \ldots, \sigma_n, \mathrm{id}, \mathrm{id}, \ldots) = \psi(\sigma_1) \circ \cdots \circ \psi(\sigma_n)$.

Let us define also two other maps

$$\varphi: \coprod_{p \neq r} \mathrm{Gal}(\bar{F}_p, F_p) \to \mathrm{Aut}(Z/r^\infty) = \hat{Z}_r^*$$

and

$$\mu: \mathrm{Aut}(Z/r^\infty) \to [BU(n)_r^\wedge; BU(n)_r^\wedge].$$

If $\sigma_i \in \mathrm{Gal}(\bar{F}_{p_i}; F_{p_i})$ we set

$$\varphi(\sigma_1, \ldots, \sigma_n, \mathrm{id}, \ldots, \mathrm{id}, \ldots) = \varphi_{p_1}(\sigma_1) \circ \varphi_{p_2}(\sigma_2) \circ \cdots \circ \varphi_{p_n}(\sigma_n)$$

where $\varphi_{p_n}(\sigma_i)$ is the restriction of σ_i to Z/r^∞. It is clear that φ is a homomorphism and φ maps a Frobenius automorphism of $\bar{F}_p$ onto p. The map μ is defined in the following way:

$$\mu(u) = \iota_* \circ u_\# \circ \iota_*^{-1}$$

where $u_\#$: $((\bar{W}S)_{rlh})_r^\wedge \to ((\bar{W}S)_{rlh})_r^\wedge$ is induced by $\bar{W}(\mathrm{id}_{\Sigma_n} \int u)$. It is clear that μ is a homomorphism.

It follows immediately from the formula $\psi(\sigma) = \iota_* \circ \varphi_p(\sigma)_\# \circ \iota_*^{-1}$ from Proposition 4.3 that $\Phi = \mu \circ \varphi$. Both μ and φ are homomorphisms. Hence Φ is also homomorphism.

Now we shall show that φ is epimorphic. Suppose that $r > 2$. Let us notice that $\hat{Z}_r^* = \varprojlim (Z/r^n)^*$. The group $(Z/r^n)^*$ is a product of a cyclic group of order $r-1$ and of a cyclic group of order r^{n-1} generated by $r+1$. Suppose that a prime number q is a generator of $(Z/r)^*$ and $r+1 = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$. Let $\varphi_n: G = \mathrm{Gal}(\bar{F}_q : F_q) \times \mathrm{Gal}(\bar{F}_{p_1} : F_{p_1}) \times \cdots \times \mathrm{Gal}(\bar{F}_{p_m} : F_{p_m}) \to (Z/r^n)^*$ be a composition of φ restricted to G with a projection onto $(Z/r^n)^*$. Let $\mathscr{F}_p$ be the Frobenius automorphism of $\bar{F}_p$.

The image of $\mathscr{F}_q$ in $(Z/r)^*$ has order $r-1$. Hence some power of $\varphi_n(\mathscr{F}_q)$ has order $r-1$ in $(Z/r^n)^*$. Clearly $\varphi_n(\mathscr{F}_{p_1}^{\alpha_1}, \ldots, \mathscr{F}_{p_m}^{\alpha_m}) = r+1$ has order r^{n-1} in $(Z/r^n)^*$. Hence it follows that φ_n is onto for each n. This implies that $\varphi(G)$ is a dense subset of $\hat{Z}_r^*$. $\varphi(G)$ is also a closed subset of $\hat{Z}_r^*$ because G is compact and φ restricted to G is continuous. This implies that $\varphi(G) = \hat{Z}_r^*$.

To finish the proof we must show that the semigroup operations are continuous in $\coprod_{p \neq r} \mathrm{Gal}(\bar{F}_p; F)$ and $[BU(n)_r^\wedge; BU(n)_r^\wedge]$ and that Φ is a continuous map. First we show that Φ is continuous. We shall denote the geometric realization of $(\bar{W}G)_{rlh}$ by the abbreviated form $\bar{W}G$. Φ is equal to the following composition.

$$\coprod_{p\neq r} \mathrm{Gal}\,(\bar{F}_p; F_p) \xrightarrow{i} [\bar{W}S;\ \bar{W}S] \xrightarrow{j} [\bar{W}S^\wedge_r;\ \bar{W}S^\wedge_r]$$
$$\xrightarrow{\iota} [\bar{W}GL(n, C)^\wedge_r;\ \bar{W}GL(n, C)^\wedge_r]$$

where

$$i(\sigma_1, \sigma_2, \ldots, \sigma_n, \mathrm{id}, \ldots) = \bar{W}\Big(\mathrm{id}_{\Sigma_n} \int \varphi(\sigma_1)\Big)_{rlh} \circ \bar{W}\Big(\mathrm{id}_{\Sigma_n} \int \varphi(\sigma_2)\Big)_{rlh} \circ \cdots \circ j(f) = f^\wedge_r$$

and $\iota(h) = \iota_* \circ h \circ \iota_*^{-1}$. We topologized $[\bar{W}S;\ \bar{W}S] = \varprojlim_k [\bar{W}\Sigma_n \int Z/r^k;\ \bar{W}\Sigma_n \int Z/r^k]$ as an inverse limit of finite sets $[\bar{W}\Sigma_n \int Z/r^k;\ \bar{W}\Sigma_n \int Z/r^k]$. This definition of the topology on $[\bar{W}S;\ \bar{W}S]$ implies immediately that i is continuous. We recall [**12**] that $[Y; X^\wedge_r]$ is topologized as the inverse limit of finite sets $\varprojlim_{\{f\}} (\varprojlim_i [Y_i; F])$ where $\{Y_i\}$ runs through the finite subcomplexes of Y and $\{f\colon X \to F\}$ is the inverse system defining $X^\wedge_r$. $[Y; X^\wedge_r]$ is a compact Hausdorff space. Now we show that j is continuous. Let $c^*\colon [\bar{W}S^\wedge_r;\ \bar{W}S^\wedge_r] \to [\bar{W}S;\ \bar{W}S^\wedge_r]$ be a map induced by the r-completion map $c\colon \bar{W}S \to \bar{W}S^\wedge_r$. The map c^* is clearly continuous. It follows from the obstruction theory that c^* is a bijection. This implies that c^* is a homeomorphism because both spaces are compact and Hausdorff. Let $c_*\colon [\bar{W}S;\ \bar{W}S] \to [\bar{W}S;\ \bar{W}S^\wedge_r]$ be also induced by c. We have that $c_* = c^* \circ j$. Hence to show that j is continuous it is enough to show that c_* is continuous.

Let X_i be finite subcomplexes of $\bar{W}S = \varinjlim_i \bar{W}(\Sigma_n \int Z/r^i)$ such that

i) $X_i \subset \bar{W}(\Sigma_n \int Z/r^i)$ and i-skeleton of $\bar{W}(\Sigma_n \int Z/r^i) \subset X_i$,
ii) the homology groups of X_i are finite,
iii) $\bar{W}S = \varinjlim X_i$.

The topology we have defined on $[\bar{W}S;\ \bar{W}S]$ is equal to the inverse limit topology obtained from the isomorphism $[\bar{W}S;\ \bar{W}S] = \varprojlim [X_i;\ \bar{W}S]$. If $\{f\colon \bar{W}S \to F\}$ is an inverse system defining $(\bar{W}S)^\wedge_r$ then there is a map from the inverse system $[X_i;\ \bar{W}S]_i$ into the inverse system $[X_i; F]_{i,f}$ and the map of the inverse limits is continuous because maps between finite sets with discrete topologies are clearly continuous. But this map is c_*.

It remains to show that ι is continuous.

The map $\iota\colon \bar{W}S \to \bar{W}GL(n, C)$ induces continuous maps

$$J_*\colon [\bar{W}S^\wedge_r;\ \bar{W}S^\wedge_r;\ \bar{W}GL(n, C)^\wedge_r]$$

and

$$J^*: [\bar{W}GL(n, C)^\wedge_r; \bar{W}GL(n, C)^\wedge_r] \to [\bar{W}S^\wedge_r; \bar{W}GL(n, C)^\wedge_r.$$

These maps are homeomorphisms because all spaces are compact, Hausdorff. Let us notice that $J^* \circ \iota = J_*$. This implies that ι is continuous.

It is clear that the composition is continuous in all the sets involved except perhaps $[BU(n)^\wedge_r; BU(n)^\wedge_r]$. We shall give here a proof of a more general fact shown to us by Eric Friedlander. Our first proof of the continuity of the composition in $[BU(n)^\wedge_r; BU(n)^\wedge_r]$ used the properties of the quotient of the compact-open topology from [**9**].

PROPOSITION 4.4. (E. Friedlander) *Let ( )^ denote the Sullivan completion with respect to some subset of primes. Suppose that Y is a space of finite type with "good" homotopy groups (see* [**13**] *p.* 44–45). *Then the composition*

$$m: [X^\wedge; Y^\wedge] \times [Y^\wedge; Z^\wedge] \to [X^\wedge; Z^\wedge]$$

is a continuous map. (Each set of homotopy classes has the Sullivan topology.)

Proof. Let $\{Z_f\}_f$ be an inverse system defining $Z^\wedge$. We can suppose that each Z_f has only a finite number of non-zero homotopy groups. From the definition of the Sullivan completion we have immediately that

$$[Y^\wedge; Z^\wedge] = \varprojlim_f [Y^\wedge; Z_f] \quad \text{and} \quad [X^\wedge; Z^\wedge] = \varprojlim_f [X^\wedge; Z_f].$$

to show that m is continuous it is enough to show that

$$\rho_0 \circ m: [X^\wedge; Y^\wedge] \times \varprojlim_f [Y^\wedge; Z_f] \to [X^\wedge; Z_{f_0}]$$

is continuous for each f_0, where

$$\rho_0: \varprojlim_f [X^\wedge; Z_f] \to [X^\wedge; Z_{f_0}]$$

is a projection. The map $\rho_0 \circ m$ factors through a projection

$$[X^\wedge; Y^\wedge] \times \varprojlim_f [Y^\wedge; Z_f] \to [X^\wedge; Y^\wedge] \times [Y^\wedge; Z_{f_0}]$$

which is clearly continuous. Hence it is enough to show that the composition

$$[X^\wedge; Y^\wedge] \times [Y^\wedge; Z_{f_0}] \to [X^\wedge; Z_{f_0}]$$

is continuous. Our assumptions on Y and $\{Z_f\}_f$ imply that $[Y^\wedge; Z_{f_0}]$ is a finite set. Therefore it suffices to show that for any $y \in [Y^\wedge; Z_{f_0}]$ the map R_y: $[X^\wedge; Y^\wedge] \to [X^\wedge; Z_{f_0}]$ given by $R_y(x) = y \circ x$ is continuous. If Y has "good" homotopy groups we can find $z: Y \to Z_{f_0}$ such that $z^\wedge = y$. The maps of the form $R_{z^\wedge}$ are clearly continuous. This finishes the proof.

REFERENCES

1. J. F. Adams, 'Maps between classifying spaces II', *Inventiones Math.* 49, (1978), 1–65.
2. M. Artin and B. Mazur, *Etale Homotopy*, Lecture Notes in Mathematics no. 100, Springer-Verlag 1969.
3. J. C. Becker and D. H. Gottlieb, 'The transfer map and fiber bundles', *Topology* 14, pp. 1–12, 1975.
4. A. K. Bousfield, 'The localization of spectra with respect to homology', *Topology* 18, (1979), 257–281.
5. E. Friedlander, 'Computation of K-theories of finite fields', *Topology* 15, (1976), 87–109.
6. E. Friedlander, 'Maps between localized homogeneous spaces, *Topology* 16, (1977), 205–216.
7. E. Friedlander, *Etale Homotopy of Simplicial Schemes*, Annals of Math Studies, Princeton University Press, 1982.
8. J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics no. 271, Springer-Verlag 1972.
9. J. P. Pezennec, 'Propriétés topologiques de $[X; Y]$ et fantômes de finitude', *Bull. Soc. Math. France* 107, (1973), 113–126.
10. D. Quillen, *Higher algebraic K-theory I*, In: Lecture Notes in Mathematics no. 341, Springer-Verlag 1973.
11. G. Segal, *Classifying spaces and spectral sequences*, Publ. Math. Inst. des Hautes Etudes Scient. (Paris), 34, (1968).
12. D. Sullivan, *Geometric topology, localization, periodicity and Galois symmetry*, M.I.T. Press, Cambridge 1970.
13. D. Sullivan, 'Genetics homotopy theory and the Adams conjecture', *Ann. of Math.* 100, (1974), 1–74.
14. *Theorie des Topos et Cohomologie Etale des Schemas*, Tom 3, Lecture Notes in Mathematics no. 305, Springer-Verlag 1973

Forschungsinstitut für Mathematik
ETH-Zentrum
CH-8092 Zürich
Switzerland