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An accuracy barrier for stable three-time-level difference schemes for hyperbolic equations

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We consider three-time-level difference schemes for the linear constant coefficient advection equation $u_t = cu_x$. In 1985 it was conjectured that the barrrier to local order p of schemes which are stable is given by

$$p \leqslant 2\min\{R, S\}.$$

Here R and S denote the number of downwind and upwind points, respectively, in the difference stencil with respect to the characteristic of the differential equation through the update point. Here we prove the conjecture for a class of explicit and implicit schemes of maximal accuracy. In order to prove this result, the existing theory on order stars has to be generalized to the extent where it is applicable to an order star on the Riemann surface of the algebraic function associated with a difference scheme. Proof of the conjecture for all schemes relies on an additional conjecture about the geometry of the order star.

We dedicate this paper to the memory of Professor Peter Henrici. With his excellent books on numerics and complex analysis he has helped us all to understand the subjects better.

1. Introduction

Suppose we have a difference scheme for an initial-boundary value problem for a system of hyperbolic partial differential equations. A *global difference scheme* for the solution of this problem generally consists of an *interior scheme* and a *boundary scheme*. By the Lax–Richtmyer Equivalence Theorem these difference schemes result in solutions which are convergent to the exact solution only when they are consistent with the initial-boundary value problem and are stable. Consistency is the minimal requirement that the order of accuracy p is one for interior and boundary schemes. Their stability in the global framework was investigated by Kreiss (1966, 1968) and in the influential paper by Gustafsson, Kreiss and Sundström (1972). In the latter paper the following necessary condition was given for the global scheme to be stable, namely that the corresponding interior scheme has to be stable in the Von Neumann sense when applied to the pure Cauchy problem for the scalar advection equation. Goldberg and Tadmor (Goldberg & Tadmor (1987), Goldberg (1991)) gave more practical sufficient conditions for stability of global schemes. These conditions entail, among others, that the boundary scheme also has to be stable in the same sense as mentioned above for the interior scheme.

From these results it can be concluded that accurate and stable difference schemes for

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the scalar advection equation are of fundamental importance in the construction of useful global schemes. For this reason we consider a Cauchy problem for the scalar advection equation

$$\frac{\partial}{\partial t}\, u(t,x) = c\, \frac{\partial}{\partial x}\, u(t,x), \quad x \in \mathbb{R},\ t \geqslant 0, \tag{1}$$

$$u(0,x) = u_0(x) \text{ given,}$$

and a class of multistep ($(k+1)$-time-level) difference schemes of the form

$$\sum_{i=0}^{k} \sum_{j=-r_i}^{s_i} a_{ij}\, u_{n+i,m+j} = 0 \tag{2}$$

which are used to determine an approximate solution of (1). The coefficients a_{ij} are responsible for the two above-mentioned features, namely the *accuracy* and *stability* of the scheme. In general the requirement of stability imposes a bound on the order of a scheme. This paper focuses on this barrier to the order imposed by the requirement of stability for schemes of type (2).

One-step schemes ($k = 1$) have been extensively studied in Iserles (1985, 1982), Iserles & Strang (1983), Jeltsch (1985), Jeltsch & Smit (1987), Smit (1985) and results for multistep schemes were given in Jeltsch (1988), Jeltsch *et al* (1988), Jeltsch & Smit (1992), Strang & Iserles (1983). In Jeltsch (1988) and Jeltsch & Smit (1987) it was conjectured that the order barrier for stable multi-time-level schemes (2) should be

$$p \leqslant 2 \min\{R, S\}. \tag{3}$$

Here, for counting purposes, we let the 'zero line' be the characteristic through the point on the new time level for which one solves. Then R denotes the number of downwind points and S the number of upwind points of a given scheme with respect to this zero line. This means that a stable scheme of order p needs to have on each side of the characteristic at least $\lceil p/2 \rceil$ points in the stencil. (Here $\lceil \alpha \rceil$ denotes the smallest integer which is not smaller than α.) If $p = 1$, this conjecture reduces to the Courant–Friedrichs–Lewy condition. Hence (3) has the quality of being an extension of the Courant–Friedrichs–Lewy condition (Courant *et al* (1928)).

The bound (3) was proved in Jeltsch & Smit (1987) for two-time-level schemes. In Jeltsch & Smit (1992) it was partially proved for a small subclass of explicit three-time-level schemes. In Jeltsch (1988) and Jeltsch *et al* (1988) many examples in support of (3) were given for multi-time-level schemes. In Jeltsch & Kiani (1991) the first lower bound in (3) for the $(k+1)$-level case was given by actually showing stability of schemes with long and slender stencils (only one step in space). Such schemes may be useful as high-order boundary schemes. In Sections 3–8 of this paper we generalize the results in Jeltsch & Smit (1992) for convex maximal order explicit and implicit three-time-level schemes (see Section 2). The results for all other schemes follow from a conjecture presented in Section 6.

The analysis in this paper is based on the order star technique, which was introduced in Wanner *et al* (1978) and treated extensively in Hairer & Wanner (1991) and Iserles &

Nørsett (1991). These ideas have to be generalized for order stars on a Riemann surface defined by an algebraic function. This algebraic function is treated in Section 4. An additional complication is that our order stars are defined with respect to the comparison function z^μ. This comparison function was first used in Smit (1985) and Iserles (1985). The analysis in Sections 3–8 is a continuation of the work in Jeltsch & Smit (1991, 1992), namely a study of the order stars on a two-sheeted Riemann surface. Since z^μ is multiple-valued with a logarithmic singularity at $z = 0$, extreme care has to be taken with the integration path used for the application of the argument principle. Notwithstanding these complications the order stars basically retain the elegant features which make them so useful in the sense that they allow a simple geometrical interpretation of the relationship between accuracy and stability.

For explicit schemes the order of the logarithmic singularity at $z = 0$ determines the maximum multiplicity of components of the order star. The various possible geometric configurations and the corresponding multiplicities of these geometries are investigated in Section 6. Except for a small subclass of schemes the derived bounds on order of the schemes do not lead to a proof of (3). This leads to the introduction of a conjecture that certain geometric configurations are not possible. A proof of the conjecture is provided for a subset of schemes of maximal order, see Section 7.

For implicit schemes the poles of the algebraic function also play an important role. The geometry of components containing poles is investigated in Section 8. Section 9 combines the results of the previous sections to provide the proof of (3).

We believe that this paper indicates the direction which the generalization of (3) to the $(k+1)$-time-level case will take. Since we restrict ourselves to schemes which can be considered convex, i.e. with an increasing stencil, we only work with poles of the algebraic function. In order to allow convexity for negative time as well, i.e. convexity in the reverse time direction, equivalently, concave schemes, the zeros of the algebraic function also have to be taken into account. Note that by excluding concave schemes we exclude the possibility of a branch point of the algebraic function added to the logarithmic singularity at $z = 0$.

In a parallel investigation in Jeltsch *et al* (1993) concerning three-time-level schemes for the wave equation we build on work started in Renaut (1989) and Renaut & Smit (1992). In that case the symmetry properties of the schemes lead to a considerable simplification of the order star theory as treated in Sections 5–9. By also taking into account the role of the zeros of all of the polynomials defining the algebraic function the class of schemes can be treated there without imposing a restriction such as convexity. Furthermore, because of symmetry, there is no possibility of a branch point at $z = 0$.

2. Order, stability and normalization of schemes

We consider three-time-level difference schemes of the form

$$\sum_{j=-r_2}^{s_2} a_{2j}\, u_{n+2,m+j} + \sum_{j=-r_1}^{s_1} a_{1j}\, u_{n+1,m+j} + \sum_{j=-r_0}^{s_0} a_{0j}\, u_{n,m+j} = 0 \tag{4}$$

$$n = 0, 1, 2, ..., \quad m = 0, \pm 1, \pm 2, ...$$

The step sizes in the time and space variables are denoted by Δt and Δx, resp., while

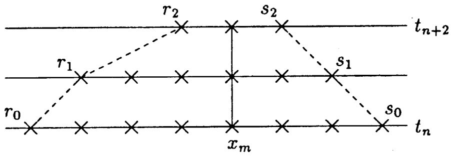

FIG. 1. Convex stencil.

$\mu = c\Delta t/\Delta x$ denotes the Courant number which is assumed to be fixed. The coefficients a_{ij} are real and depend in general on μ, $a_{ij} = a_{ij}(\mu)$. Further r_i, $s_i \in \mathbb{Z}$ with $r_2 \geqslant 0$, $s_2 \geqslant 0$ and $-r_i \leqslant s_i$, $i = 0, 1$ and $a_{i,-r_i} \neq 0$, $a_{i,s_i} \neq 0$ for $i = 0, 1, 2$. The value u_{nm} approximates $u(n\Delta t, m\Delta x)$. If $r_2 = s_2 = 0$ a scheme (4) is said to be *explicit*. Otherwise it is called *implicit*.

A scheme with a stencil satisfying

$$\begin{cases} 0 \leqslant r_0 - r_1 \leqslant r_1 - r_2 \\ 0 \leqslant s_0 - s_1 \leqslant s_1 - s_2 \end{cases} \tag{5}$$

is called a *convex scheme* with an *increasing stencil* (Fig. 1). From a computational point of view these schemes seem to yield the most interesting stencils.

A Fourier transform enables us to associate with (4) on time level $n + i$ a function

$$a_i(z) = \sum_{j=-r_i}^{s_i} a_{ij}\, z^j, \qquad i = 0, 1, 2 \tag{6}$$

and to introduce the *characteristic function*

$$\Phi(z, w) = a_2(z)\, w^2 + a_1(z)w + a_0(z),$$

which is assumed to be irreducible (see Jeltsch *et al* (1988)).

In order to be able to solve (4) for the values on the new time level in the implicit case, we impose the necessary and sufficient condition

$$a_2(z) \neq 0 \qquad \text{for } |z| = 1.$$

We also require our schemes to satisfy the following *normalization condition* (see Iserles & Strang (1983), Jeltsch *et al* (1988)):

$$\begin{cases} r_2 = \text{number of zeros of } a_2(z) \text{ with } |z| < 1 \\ s_2 = \text{number of zeros of } a_2(z) \text{ with } |z| > 1. \end{cases} \tag{7}$$

A scheme (4) is said to be *stable* if

$$\left.\begin{matrix} \Phi(z, w) = 0 \\ |z| = 1 \end{matrix}\right\} \Longrightarrow \begin{cases} |w| \leqslant 1 \text{ and if } |w| = 1, \\ \text{then } w \text{ is a simple root.} \end{cases} \tag{8}$$

A scheme (4) has *error order p* if for any smooth solution $u(t, x)$ of (1) we have

$$\sum_{i=0}^{2} \sum_{j=-r_i}^{s_i} a_{ij}\, u(t + i\Delta t,\ x + j\Delta x) = C\, \frac{\partial^{p+1}}{\partial x^{p+1}}\, u(t, x)(\Delta x)^{p+1} + O((\Delta x)^{p+2})$$
$$\text{if } \Delta x \to 0 \text{ and } \mu = \text{constant}.$$

Since we are interested only in schemes with positive order, we assume that

$$\Phi(1,1) = \sum_{i=0}^{2} \sum_{j=-r_i}^{s_i} a_{ij} = 0 .$$

The next result expresses the order of a scheme as a property of the solution w of $\Phi(z, w) = 0$.

PROPOSITION 2.1 (*Equivalent order conditions*, Jeltsch *et al* (1988), Strang & Iserles (1983)) Let a scheme (4) with characteristic function $\Phi(z, w)$ and Courant number μ be stable and satisfy $\Phi(1, 1) = 0$. Then the following three conditions are equivalent.

a) The scheme has order p.
b) $\Phi(z, z^\mu) = O((z-1)^{p+1})$ as $z \to 1$.
c) The algebraic function w given by $\Phi(z, w(z)) \equiv 0$ has exactly one branch w_1 which is analytic in a neighbourhood of $z = 1$ and satisfies

$$z^\mu - w_1(z) = O((z-1)^{p+1}) \quad \text{as } z \to 1.$$

The next theorem gives the highest possible order that a scheme can have if stability is ignored. We introduce the index set of the difference stencil

$$I = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : \; 0 \leqslant i \leqslant 2, \; -r_i \leqslant j \leqslant s_i\}.$$

A scheme (4) is said to be *regular* if a characteristic line through any given stencil point does not pass through any other point of the difference stencil (see Jeltsch *et al* (1988)).

PROPOSITION 2.2 (*Regular stencil*, Jeltsch *et al* (1988)) Let a scheme (4) have a regular difference stencil with index set I. Then the highest possible order that the scheme can have is

$$p = |I| - 2,$$

where $|I|$ denotes the number of indices in I.

3. Main result: bound on order of stable schemes

Suppose we have a convex scheme (4) which is also regular. The characteristic through the point (t_{n+2}, x_m) (of the normalized scheme (4)) will be taken as the zero line. Then it is possible to interpret the order bound of stable schemes in a simple geometrical way such that for the highest order the number of points on each side of the zero line is balanced. If R denotes the total number of downwind points and S the total number of upwind points with respect to the zero line, then the order p of a stable scheme satisfies

$$p \leqslant 2 \min\{R, S\}.$$

This result can be related to the indices r_i, s_i of (4) in the following way: Define

$$R_1 = \begin{cases} 0 & \text{if } \mu < -r_1 \\ \lfloor r_1 + \mu \rfloor + 1 & \text{if } -r_1 < \mu < s_1, \\ r_1 + s_1 + 1 & \text{if } \mu > s_1 \end{cases} \qquad S_1 = r_1 + s_1 + 1 - R_1$$

$$R_0 = \begin{cases} 0 & \text{if } 2\mu < -r_0 \\ \lfloor r_0 + 2\mu \rfloor + 1 & \text{if } -r_0 < 2\mu < s_0, \\ r_0 + s_0 + 1 & \text{if } 2\mu > s_0 \end{cases} \qquad S_0 = r_0 + s_0 + 1 - R_0,$$

where $\lfloor \alpha \rfloor$ denotes the largest integer not exceeding α, and

$$R = R_0 + R_1 + r_2, \quad S = S_0 + S_1 + s_2. \tag{9}$$

Then the main result is as follows.

THEOREM 3.1 (*Maximal order of stable convex schemes*) Let a convex scheme (4) with an increasing stencil be normalized and have a fixed Courant number μ satisfying $0 < |\mu| < \frac{1}{2}$. If the scheme is stable, then the order p of the scheme is bounded by

$$p \leqslant 2 \min\{R, S\}.$$

REMARK 3.2

a) In Jeltsch & Smit (1992) the bound (3) was proved for a small subclass of explicit schemes of type (4). In this paper we generalize it, making use of a conjecture introduced in Section 6, for the class of (explicit and implicit) schemes of type (4) which are convex and have an increasing stencil.
b) In Section 7 we provide a partial proof of (3). In particular, for the maximal order schemes, $p = |I| - 2$,

$$p \leqslant \begin{cases} 2R & -\frac{1}{2} < \mu < 0 \\ 2S & 0 < \mu < \frac{1}{2}. \end{cases}$$

c) The result (3) can be extended to $|\mu| > \frac{1}{2}$ by making use of the following transformation. Assume that a stable scheme is represented by $\Phi(z, w)$, where $w(z)$ approximates z^μ in a neighbourhood of the point $z = 1, w = 1$. Then we consider the scheme represented by the characteristic function

$$\widetilde{\Phi}(z, u) = z^2 \, \Phi(z, u/z).$$

Since $u = zw$, the new scheme is stable and approximates $z^{\widetilde{\mu}} = z^{\mu+1}$ with the same order as the original scheme. The stencil undergoes the following transformations:

$$\widetilde{r}_1 = r_1 - 1, \quad \widetilde{s}_1 = s_1 + 1, \quad \widetilde{r}_0 = r_0 - 2, \quad \widetilde{s}_0 = s_0 + 2.$$

4. Properties of the algebraic function w

The algebraic function w, satisfying $\Phi(z, w(z)) \equiv 0$, is multiple-valued, consisting in general, for a given z, of two values $w_1(z)$ and $w_2(z)$. Associated with this algebraic function is the Riemann surface M,

$$M = \{(z, w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} : \ \Phi(z, w) = 0\},$$

consisting of two sheets, one above the other, interacting at a finite number of branch points z_i (where $w_1(z_i) = w_2(z_i)$). The surface M is a closed connected set on which w is single-valued and, except for a finite number of singular points, also analytic.

REMARK 4.1 (*Branch points of w*)

a) The branch points of w (except those that can occur at 0 and ∞) occur at points z_i where $a_1^2(z_i) - 4a_2(z_i)\, a_0(z_i) = 0$. Since the coefficients of this polynomial equation are real, the branch points are either real or they occur in complex conjugate pairs. Branch cuts along which the two sheets of M are connected can therefore always be taken to be straight lines which either fall on the real axis, or are orthogonal and symmetric to the real axes or occur in conjugate pairs.
b) If a scheme is stable, the corresponding algebraic function cannot have a branch point at $z = 1$ (see (8)). The sheet of M on which the point $z = 1$, $w = 1$ occurs, is called the *principal sheet*. Since the Riemann surface is connected, this notion is basically a local property in a neighbourhood of $z = 1$. We make the convention that the principal sheet refers to that part of M which can be connected to $z = 1$, $w = 1$ without crossing a branch cut. The remaining part of M will be called the *secondary sheet.*

REMARK 4.2 (*Poles of w*) The function w has a pole at every point where $a_2(z) = 0$. By the normalization condition (7) there are in total r_2 poles of w with $|z| < 1$ away from $z = 0$ on the two sheets of M. Moreover, if $\max\{r_0, r_1, r_2\} > 0$, then w can have a pole at $z = 0$ on one or both sheets of M (see Proposition 4.4).

REMARK 4.3 (*Zeros of w*) The finite points where w has zeros coincide with the points where the function $v(z) = 1/w(z)$ has poles. These points occur where $a_0(z)$ has zeros and occasionally also at $z = 0$.

The expansion of $w(z)$ around $z = 0$, determined by use of Newton's polygons, is important in the subsequent discussion.

PROPOSITION 4.4 (*Expansion at $z = 0$*) Let

$$\Phi(z, w) = (a_{2,-r_2} z^{-r_2} + \cdots + a_{2,s_2} z^{s_2})\, w^2 \ + (a_{1,-r_1} z^{-r_1} + \cdots + a_{1,s_1} z^{s_1})\, w + (a_{0,-r_0} z^{-r_0} + \cdots + a_{0,s_0} z^{s_0})$$

be the characteristic function of a convex scheme (4) with an increasing stencil. Then the algebraic function w satisfying $\Phi(z, w) = 0$ does not have a branch point at $z = 0$ and has the following expansions at $z = 0$:

a) If $r_1 - r_2 > r_0 - r_1$, then

$$w_1(z) = z^{-(r_1-r_2)}(c_0 + c_1 z + c_2 z^2 + \cdots)$$
$$w_2(z) = z^{-(r_0-r_1)}(d_0 + d_1 z + d_2 z^2 + \cdots),$$

where

$$c_0 = -\frac{a_{1,-r_1}}{a_{2,-r_2}} \quad \text{and} \quad d_0 = -\frac{a_{0,-r_0}}{a_{1,-r_1}}.$$

b) If $r_1 - r_2 = r_0 - r_1$, then

$$w_{1,2}(z) = z^{-(r_1-r_2)}(-a_1(z)z^{r_1} \pm \sqrt{D})/(2a_2(z)z^{r_2})$$
$$D(z) = (a_1^2(z) - 4a_2(z)\,a_0(z))z^{2r_1}$$
$$= d + O(z)$$

where $d = a_{1,-r_1}^2 - 4a_{2,-r_2}\,a_{0,-r_0}$.

REMARK 4.5 From Proposition 4.4 we observe that $z = 0$ is not a branch point of w if $r_1 - r_2 > r_0 - r_1$. If $2r_1 = r_0 + r_2$ and $d \neq 0$ then again $z = 0$ is not a branch point. If $d = 0$ then $z = 0$ can be a branch point. In the following we shall restrict ourselves to schemes where $z = 0$ is not a branch point in which case one has the two expansions

$$w_1(z) = z^{-(r_1-r_2)}(c_0 + c_1 z + c_2 z^2 + \cdots)$$
$$w_2(z) = z^{-(r_0-r_1)}(d_0 + d_1 z + d_2 z^2 + \cdots).$$

5. Order stars

An order star is defined on the Riemann surface M of the algebraic function w in the following way. Define the function φ by

$$\varphi(z, w) = z^{-\mu}\, w, \qquad (z, w) \in M$$

and the *order star* Ω by

$$\Omega = \{(z, w) \in M : \quad |\varphi(z, w)| > 1\}.$$

Because of the factor $z^{-\mu}$ the function φ is multiple-valued on M. However, the order star Ω, being defined by means of the modulus of φ, is again well defined on M. Ω^c denotes the complement of Ω, i.e. $\Omega^c = M \backslash \Omega$. Because the coefficients a_{ij} are real, Ω is symmetric with respect to the real axis.

The order and stability of a scheme, which were interpreted in Section 2 as properties of the function w, can be reinterpreted as properties of the order star. We give without proof those properties which are standard results in investigations involving order stars (see e.g. Wanner *et al* (1978), Iserles & Nørsett (1991)).

LEMMA 5.1 (*Stability*) If a scheme is stable, then

$$\Omega \cap \{(z, w) \in M : \quad |z| = 1\} = \emptyset.$$

LEMMA 5.2 (*Order*) A scheme (4) has order p if and only if at the point $z = 1$ on the principal sheet of M the order star consists of $p + 1$ sectors of angle $\pi/(p + 1)$, separated by $p + 1$ sectors of Ω^c, each with the same angle.

A subset A (with boundary ∂A) of Ω is said to be an *Ω-component* if $\partial A \subset \partial\Omega$ and A is connected. Ω^c-components are defined similarly. An Ω-component is said to be of *multiplicity* m if it contains m Ω-sectors at $z = 1$ on the principal sheet. Similarly for Ω^c-components.

Note that the curve on M which has the projection $|z| = 1$ in the z-plane separates M into two well-defined subsets. The set in M with $|z| < 1$ is called the unit disk Δ and the set with $|z| > 1$ is called the outside of the unit disk. By Lemma 5.1 there is a clear distinction between the portion of the order star inside and the portion outside the unit disk. The components inside Δ are bounded, where a component Ω_1 is said to be *bounded* if $\sup_{(z,w)\in\Omega_1} |z| < \infty$.

In order to emphasize important features, our pictures of Ω-components will not always be the exact geometrical embeddings of M into $\mathbb{R}^3$. They will, however, display the basic connectivity relations and cuts, and elucidate the important properties of both macroscopic and microscopic scale.

According to Remark 4.5 we restrict ourselves to schemes where there is no branch point of w at $z = 0$. Thus there are two values w_1^0 and w_2^0 with $\Phi(0, w_i^0) = 0$, i.e. there are two zero points $(0, w_1^0)$ and $(0, w_2^0)$ on M. Depending on the values of the indices r_i we know from Proposition 4.4 that there can be a pole of w at one or both of the zero points. Further, because of the factor $z^{-\mu}$ occurring in φ, this function in general no longer has an integer-valued leading exponent of z at the zero points.

We know from Remark 4.2 that there are r_2 poles of w away from $z = 0$ inside Δ (if our scheme is normalized). Since the expansion of $z^{-\mu}$ from any point $z_0 \neq 0$ has the form

$$z^{-\mu} = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots, \qquad |z - z_0| < |z_0|,$$

there will be poles of φ of exactly the same orders at the points with these z-values on one of the sheets of M.

The influence of poles and the behaviour of φ at $z = 0$ on the multiplicity of the components in which they occur is studied by means of the argument principle with respect to the function φ (see Wanner *et al* (1978)). In this regard the factor $z^{-\mu}$ of φ introduces onto M a new structure in the sense that it defines on M another Riemann surface which in general has infinitely many sheets. To define $z^{-\mu}$ uniquely on M, branch cuts, L_i, from $(0, w_i^0)$ to (∞, w_i^∞), $i = 1, 2$ are made. These cuts are made according to the following rules.

Rule 1 for cuts L_i: The branch cuts L_i have to be such that their projections onto the z-plane are either identical, or 'enclose' a 'sector' of $\mathbb{C}$ which does not contain a branch cut of M.

If we adhere to Rule 1, then $z^{-\mu}$ is defined uniquely on M (see Section 5.1 of Jeltsch & Smit (1992)), even if the cuts L_i are allowed to cross a branch cut of M. In the present context, however, we can always avoid this. To this extent we introduce the following rule.

Rule 2 for cuts L_i: The cuts L_i have to be such that each cut occurs only on one sheet of M, i.e. L_1 between $(0, w_1^0)$ and (∞, w_1^∞) on the principal sheet and L_2 between $(0, w_2^0)$ and (∞, w_2^∞) on the secondary sheet.

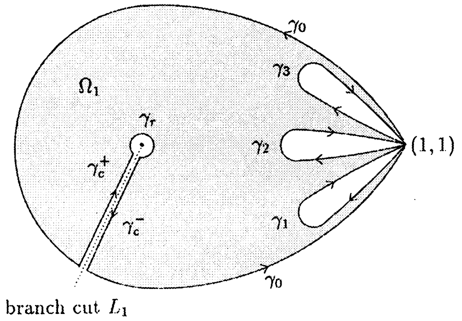

FIG. 2. Component Ω_1 illustrating the integration path.

We can always adhere to Rules 1 and 2 by choosing the branch cuts L_i to go along two radial lines which have the same projection onto the z-plane and for which the projection onto the z-plane does not pass through the point $z = 1$. This convention for making cuts L_i is adhered to unless explicitly indicated otherwise.

6. The role of the zero points on multiplicity

We start by restricting ourselves to the order stars of explicit schemes. Thus $r_2 = 0$ and there are no poles away from $z = 0$. Hence, inside Δ, every bounded Ω-component must contain at least one of the points $(0, \omega_i^0)$, $i = 1, 2$.

Our investigation of the relationship between the multiplicity of a component and the total order of poles/singularities of φ that it contains begins with a very simple type of Ω-component, Ω_1 (say), which occurs only on the principal sheet of M and which contains no branch points of w. This type of component was treated in Jeltsch & Smit (1992), but the proof is repeated because it illustrates the appropriate application of the argument principle.

PROPOSITION 6.1 (*Multiplicity*) Let Ω_1 be such that the principal branch can be defined as a single-valued function on the projection of Ω_1 onto the z-plane. Assume φ has a leading exponent of $-\alpha$ at $z = 0$. Then the multiplicity m of Ω_1 satisfies

$$m \leqslant \lfloor \alpha \rfloor + 1.$$

Proof. If Ω_1 is of multiplicity m, there are $m - 1$ Ω^c-components emerging from $(1, 1)$ to the 'inside' of Ω_1. We evaluate

$$\frac{1}{2\pi i} \int_\gamma \frac{\varphi'(z, w)}{\varphi(z, w)} \, dz$$

where γ is the closed curve which consists of the positively oriented boundary of Ω_1 and a portion going around the zero point (see Fig. 2):

γ_0: positively oriented (w.r.t. $z = 0$) 'outward' boundary of Ω_1. According to Wanner *et al* (1978) (proof of Proposition 4) the argument of φ decreases along γ_0.

γ_c^+, γ_c^-: two sides of a Jordan curve which connects the 'outward' boundary of Ω_1 with a circle around $z = 0$. According to Jeltsch & Smit (1992) (Lemma 4.4) the contributions to the integral along γ_c^+ and γ_c^- cancel out.

γ_r: circular curve with small radius r, traversed clockwise. According to Jeltsch & Smit (1992) (Lemma 4.3) the contribution of this curve to the integral is α.

$\gamma_1, \dots, \gamma_{m-1}$: boundary of Ω_1 along $m-1$ Ω^c-components emerging from $(1, 1)$ to the 'inside' of Ω_1. Again the argument of φ decreases along each path γ_i and, because φ is single-valued in $\Omega_1 \backslash L_1$, every time the boundary γ returns to $(1, 1)$ the argument has decreased by at least 2π.

Then

$$\gamma = \gamma_0 + \gamma_c^+ + \gamma_r + \gamma_c^- + (\gamma_1 + \gamma_2 + \cdots + \gamma_{m-1}).$$

By application of the argument principle, and because there are no zeros or poles of φ inside γ, we have

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_\gamma \frac{\varphi'(z, w)}{\varphi(z, w)} \, dz \\ &= \underbrace{\frac{1}{2\pi i} \int_{\gamma_0}}_{<0} + \underbrace{\frac{1}{2\pi i} \left(\int_{\gamma_c^+} + \int_{\gamma_c^-} \right)}_{=0} + \underbrace{\frac{1}{2\pi i} \int_{\gamma_r}}_{=\alpha} + \underbrace{\frac{1}{2\pi i} \int_{\gamma_1 + \cdots + \gamma_{m-1}}}_{\leqslant -(m-1)} . \end{aligned}$$

By combining the first three terms and introducing the notation γ_0^E to indicate the positively oriented curve

$$\gamma_0^E = \gamma_0 + \gamma_c^+ + \gamma_r + \gamma_c^-,$$

it follows that

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \frac{\varphi'(z, w)}{\varphi(z, w)} \, dz \leqslant \lfloor \alpha \rfloor .$$

Hence

$$m \leqslant \lfloor \alpha \rfloor + 1.$$

□

Clearly the component being treated in Proposition 6.1 involves only one sheet of M, although nothing prevents it in general from crossing a branch cut of M from one sheet to the other. We shall refer to a component of this kind as a *non-binary component*. With two sheets of M available there also exist components which involve both sheets of M in a very specific manner and which will be called binary. We shall make these statements more precise in the following definition.

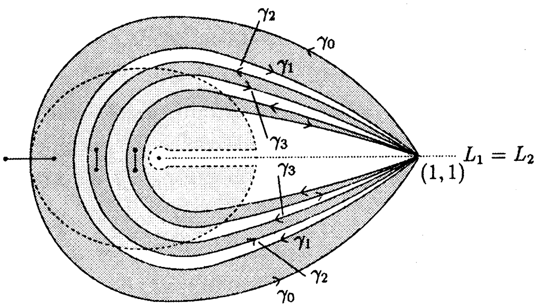

FIG. 3. Symmetric binary component and cuts L_i going through $z = 1$.

DEFINITION 6.2 (*Binary/non-binary components*) Let Ω_1 be an Ω-component containing exactly one zero point. Assume the branch cuts L_i are radial lines with the same projection L onto the z-plane and this projection does not pass through the point $z = 1$. We modify Ω_1 into $\widetilde{\Omega}_1$ by making cuts along L_i and encircling the zero points with infinitesimal small circles, see e.g. Fig. 2 and Fig. 4, such that $\widetilde{\Omega}_1$ satisfies the following properties

i) $\widetilde{\Omega}_1$ is connected.
ii) No closed curve in $\widetilde{\Omega}_1$ whose interior is contained completely in $\widetilde{\Omega}_1$ contains the zero point.
iii) No projection of $\partial\widetilde{\Omega}_1$ onto the z-plane intersects with L.

Ω_1 is called *non-binary* if the zero point is encircled once by such an infinitesimal circle of $\partial\widetilde{\Omega}_1$. In all other cases the component is called *binary*.

If a component Ω_1 has multiplicity m its boundary $\partial\Omega_1$ can be decomposed naturally into m curves γ_i which connect the point (1, 1). $\partial\widetilde{\Omega}_1$ consists also of m curves $\widetilde{\gamma}_i$ which connect (1, 1). These $\widetilde{\gamma}_i$ are either identical to γ_i or are extended, γ_i^E, by a cut along L_i and a circle around a zero point as was done with γ_0 in the previous proof.

LEMMA 6.3 (*Symmetric binary*) Let Ω_1 be a symmetric binary component containing one zero point $(0, w_2^0)$ (say), with leading exponent $-\alpha_2$ of φ at $(0, w_2^0)$, while the leading exponent of φ at $(0, w_1^0) \notin \Omega_1$ is $-\alpha_1$. Let δ_i denote the non-integer part of α_i, i.e.

$$\delta_i = \alpha_i - \lfloor \alpha_i \rfloor, \qquad i = 1, 2.$$

Then the multiplicity m of Ω_1 satisfies

$$m \leqslant 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor. \tag{10}$$

Proof. The proof is conducted in two different ways depending on the way in which the branch cuts L_i are chosen. The first version highlights the binary character of the component, while the second leads to the bound (10).

Version 1: We first deviate from our convention of making the cuts L_i by choosing them such that their projection onto the z-plane is the positive semi axis (see Fig. 3). With the integration along γ_0^E we have the situation that, on the principal sheet, we have gone once

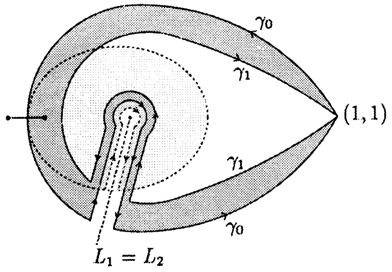

FIG. 4. Symmetric binary component and cuts L_i made according to convention.

around the zero point $(0, w_1^0)$ without crossing L_1. Hence we end up with γ_0^E at a point where $z^{-\mu}$, and therefore also φ, has a value which differs from the value with which it started at the point $(1, 1)$. We then have to return along γ_1 to our starting point at $(1, 1)$. Only then has the complete boundary of a component with respect to the function φ been traversed. Hence, in terms of counting the multiplicity of Ω_1 at $(1, 1)$, we can regard the point where γ_0^E went over into γ_1 as a point away from $(1, 1)$. When we apply the argument principle as we did in Proposition 6.1, this only accounts for the sectors of Ω_1 'on one side' of the cut L_1. Disregarding the sectors in the lower halfplane $Im\ z < 0$ we have a component with $m/2$ sectors in the upper halfplane which is, with respect to these $m/2$ sectors, a non-binary component. Hence

$$\frac{1}{2\pi i} \int_{\gamma_0^E+\gamma_1} \frac{\varphi'(z, w)}{\varphi(z, w)}\, dz \leqslant \lfloor \alpha_2 \rfloor$$

and, as in the proof of Proposition 6.1,

$$0 \leqslant \lfloor \alpha_2 \rfloor - (m/2 - 1) \tag{11}$$

from which we obtain

$$m \leqslant 2\lfloor \alpha_2 \rfloor + 2.$$

In this case the integration along $\gamma_2 + \gamma_3$, which has led to a drop in the argument by 2π, has contributed two sectors as compared to just one for a non-binary component.

Version 2: In Version 1 of the proof the cuts L_i were not made according to convention. By making the cuts according to convention (see Fig. 4) we find

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \frac{\varphi'(z, w)}{\varphi(z, w)}\, dz \leqslant \lfloor \alpha_1 + \alpha_2 \rfloor, \qquad \frac{1}{2\pi i} \int_{\gamma_1^E} \frac{\varphi'(z, w)}{\varphi(z, w)}\, dz \leqslant \lfloor -\alpha_1 \rfloor,$$

and

$$\frac{1}{2\pi i} \int_{\gamma_2^E} \frac{\varphi'(z, w)}{\varphi(z, w)}\, dz + \frac{1}{2\pi i} \int_{\gamma_3^E} \frac{\varphi'(z, w)}{\varphi(z, w)}\, dz \leqslant \lfloor \alpha_1 \rfloor + \lfloor -\alpha_1 \rfloor = -1.$$

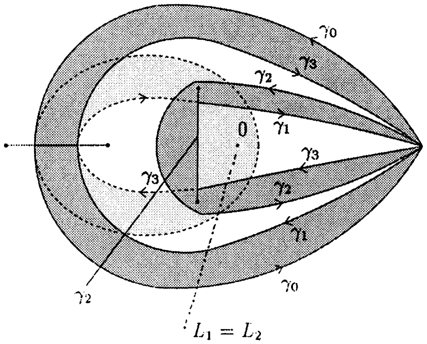

FIG. 5. Symmetric binary component with three 'inner' boundary curves.

Assuming $m/2 - 1$ pairs of curves like $\gamma_2 + \gamma_3$, application of the argument principle yields

$$0 \leqslant \lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor - (m/2 - 1). \tag{12}$$

Hence

$$m \leqslant 2\{\lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor) + \lfloor -\alpha_1 \rfloor\} + 2$$

and

$$m \leqslant 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor,$$

where we have made use of the fact that

$$\lfloor \alpha \rfloor + \lfloor -\alpha \rfloor = -1 \qquad \text{if } \alpha \notin \mathbb{Z}.$$

□

REMARK 6.4

a) The zero point not inside Ω_1 can be excluded by traversal of more than one 'inner' boundary curve which crosses the negative real axis; see Fig. 5.
If the integration process is carried out as in the proof of Lemma 6.3, we obtain

$$\frac{1}{2\pi i} \int_{\gamma_0^E} \leqslant \lfloor \alpha_1 + \alpha_2 \rfloor, \quad \frac{1}{2\pi i} \int_{\gamma_1^E} \leqslant \lfloor -\alpha_1 \rfloor, \quad \frac{1}{2\pi i} \int_{\gamma_2^E} \leqslant \lfloor \alpha_1 \rfloor,$$
$$\frac{1}{2\pi i} \int_{\gamma_3^E} \leqslant \lfloor -\alpha_1 \rfloor,$$

leading to the inequality

$$0 \leqslant \{\lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor + \lfloor \alpha_1 \rfloor + \lfloor -\alpha_1 \rfloor\} - \frac{(m-4)}{2},$$

with m again bounded by

$$m \leqslant 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor.$$

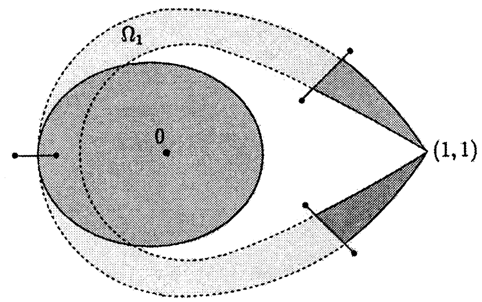

FIG. 6. Binary component containing the zero point on the principal sheet.

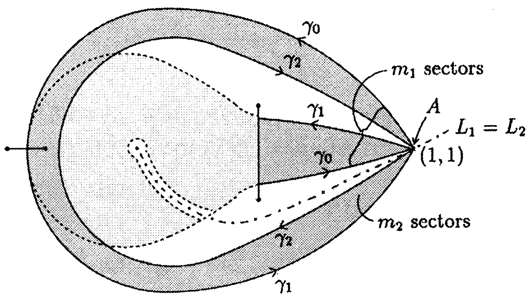

FIG. 7. Non-symmetric binary component and cuts L_i going through $z = 1$.

b) If $(0, w_1^0)$ is contained in Ω_1 and $(0, w_2^0)$ is excluded the multiplicity of Ω_1 is determined in the same way but with $\lfloor \alpha_2 \rfloor$ replaced by $\lfloor \alpha_1 \rfloor$, see Fig. 6.

LEMMA 6.5 (*Non-symmetric binary*) Let Ω_1 be a non-symmetric binary component containing one zero point $(0, w_2^0)$ (say), with leading exponent $-\alpha_2$ of φ at $(0, w_2^0)$. Then the multiplicity m of Ω_1 satisfies

$$m \leqslant 2\lfloor \alpha_2 \rfloor + 1.$$

Proof. The proof is again conducted in two ways as a result of two different choices of the branch cuts L_i.

Version 1: If cuts L_1 and L_2 were chosen as in Version 1 of the proof of Lemma 6.3 they would intersect a branch cut of M. Therefore the cuts L_1 and L_2 are made such that the projection winds from $(0, 0)$ to $(1, 1)$ without intersecting either the projection of $\partial\Omega$ onto $\mathbb{C}$ or the projection of any cut of M onto $\mathbb{C}$, see Fig. 7. Then Ω_1 is non-symmetric with respect to L_1. The argument principle is applied along the positively oriented boundary $\gamma = \gamma_0^E + \gamma_1 + \gamma_2$ which starts out on one side of the cut L_1 and is assumed to consist of

m_1 sectors at (1, 1). The contributions of the integrals are

$$\frac{1}{2\pi i}\int_{\gamma_0^E}\frac{\varphi'(z,w)}{\varphi(z,w)}\,dz \leqslant \lfloor\alpha_2\rfloor, \quad \frac{1}{2\pi i}\int_{\gamma_1+\gamma_2}\frac{\varphi'(z,w)}{\varphi(z,w)}\,dz \leqslant -1.$$

Application of the argument principle leads to

$$0 \leqslant \lfloor\alpha_2\rfloor - 1 - (m_1 - 2). \tag{13}$$

Observing that in the optimal case $m_1 = (m-1)/2 + 1$ we obtain the result.

The process is now repeated for the portion of Ω_1, 'on the other side' of L_1, where we assume a total of m_2 sectors. Integrating along γ_2 we end up at a point A (say) where φ is different to the initial value. Continuing along γ_0^E to A again, and then along γ_1 to return to our starting point we obtain

$$\frac{1}{2\pi i}\int_{\gamma_2+\gamma_0^E+\gamma_1}\frac{\varphi'(z,w)}{\varphi(z,w)}\,dz \leqslant \lfloor\alpha_2 - 1\rfloor,$$

where the -1 accounts for the fact that we returned to A, on which occasion the argument must have decreased by at least 2π. Application of the argument principle leads to

$$0 \leqslant \lfloor\alpha_2\rfloor - 1 - (m_2 - 1). \tag{14}$$

By combining (13) and (14) we obtain the following bound on the total number $m = m_1 + m_2$ of sectors of Ω_1:

$$m \leqslant 2\lfloor\alpha_2\rfloor + 1.$$

Version 2: By choosing the branch cuts L_i according to convention (see Fig. 8), we obtain the following for the integrals:

$$\frac{1}{2\pi i}\int_{\gamma_0^E}\frac{\varphi'}{\varphi}\,dz \leqslant \lfloor\alpha_2\rfloor, \quad \frac{1}{2\pi i}\int_{\gamma_1^E}\frac{\varphi'}{\varphi}\,dz \leqslant \lfloor\alpha_1\rfloor, \quad \frac{1}{2\pi i}\int_{\gamma_2^E}\frac{\varphi'}{\varphi}\,dz \leqslant \lfloor-\alpha_1\rfloor.$$

Application of the argument principle leads to

$$0 \leqslant \{\lfloor\alpha_2\rfloor + \lfloor\alpha_1\rfloor + \lfloor-\alpha_1\rfloor\} - \frac{(m-3)}{2},$$

where the factor 2 accounts for the binary nature of the component and 3 is subtracted from m because γ_0, γ_1 and γ_2 contribute 3 sectors. Hence, we again obtain

$$m \leqslant 2\lfloor\alpha_2\rfloor + 1.$$

□

In view of Lemmas 6.3 and 6.5 we have to conclude that the bound (10) is in general too sharp if the non-integer parts of α_1 and α_2 satisfy $0 < \delta_1 + \delta_2 < 1$. A combination of Lemmas 6.3 and 6.5 leads to the following general result for binary components.

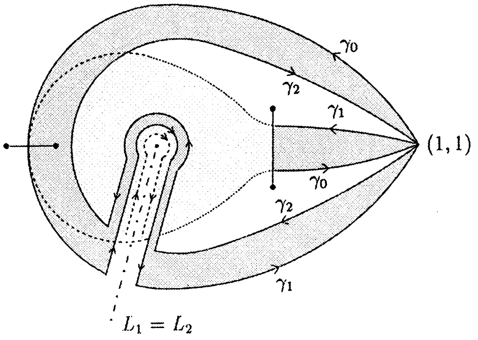

FIG. 8. Non-symmetric binary component and cuts L_i made according to convention.

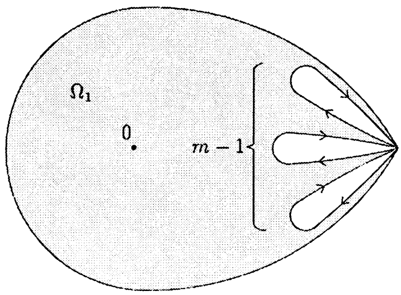

FIG. 9. Non-binary component with $m - 1$ Ω^c-components.

PROPOSITION 6.6 (*Binary*) Let Ω_1 be a binary component containing one zero point. Assume that φ has a leading exponent of $-\alpha_1$ at the zero point inside Ω_1 and $-\alpha_2$ at the other zero point. Let δ_1 and δ_2 be the non-integer parts of α_1 and α_2, respectively, i.e.

$$\delta_1 = \alpha_1 - \lfloor \alpha_1 \rfloor, \qquad \delta_2 = \alpha_2 - \lfloor \alpha_2 \rfloor.$$

Then the multiplicity m of Ω_1 satisfies

$$m \leqslant 2\lfloor \alpha_1 \rfloor + \max\{1,\ 2\lfloor \delta_1 + \delta_2 \rfloor\}.$$

REMARK 6.7 (*Efficiency*)

a) In view of the factor 2 accompanying $\lfloor \alpha \rfloor$ in the bound for binary components, we say that the zero point inside a binary component has a *higher efficiency* than the zero point inside a non-binary component. The zero point is in this case regarded as contributing twice to the multiplicity of the component.
b) The multiplicity of a non-binary component Ω_1 is achieved by $(m - 1)$ Ω^c-components which are bounded by Ω_1 and do not loop around either zero point, see Fig. 9. On the contrary, the multiplicity of a binary component is achieved because

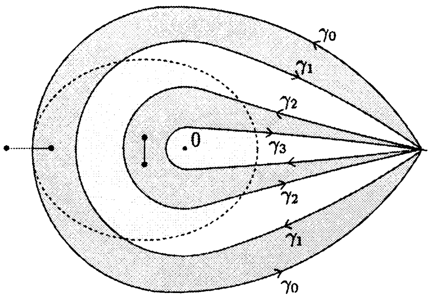

FIG. 10. Binary component with one 'binary loop'.

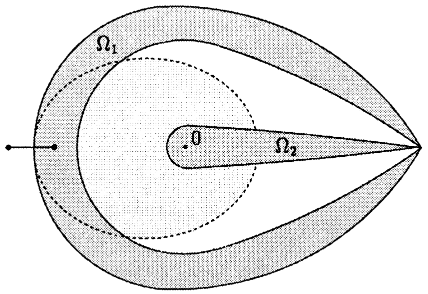

FIG. 11. Binary component Ω_1 combined with non-binary component Ω_2.

of Ω^c-components which do loop around one of the zero points (Fig. 10). In Lemma 5.10 of Jeltsch & Smit (1992) these were referred to as binary loops. Moreover, the connectedness of Ω_1 requires branch cuts.

PROPOSITION 6.8 There can be at most one binary component containing one zero point inside the unit disk Δ.

Proof. Let Ω_1 be a binary component inside Δ and say $(0, w_2^0)$ is contained in Ω_1, while $(0, w_1^0)$, which does not belong to Ω_1, is enclosed by (an) 'inner' boundary curve(s) of Ω_1 (see Definition 6.2). Suppose $(0, w_1^0)$ belongs to a second Ω-component Ω_2 (say). For Ω_2 to be binary, it has to have (an) 'outward' boundary curve(s) going through a branch cut on the negative real axis and then enclosing $(0, w_1^0)$. However, this is impossible since $(0, w_1^0)$ is already enclosed by (an) 'inner' boundary curve(s) of Ω_1. Hence, Ω_2 cannot be binary. □

A binary component Ω_1 can be combined with a non-binary component Ω_2 (say), as illustrated in Fig. 11. For such a combination the following theorem follows as a consequence of Propositions 6.1 and 6.6.

THEOREM 6.9 (*Binary plus non-binary*) Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$

at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M and suppose $(0, w_1^0)$ belongs to a non-binary component and $(0, w_2^0)$ to a binary component. Then the highest total multiplicity m that these two components can contribute at (1, 1) is given by

$$m \leqslant (\lfloor\alpha_1\rfloor + 1) + (2\lfloor\alpha_2\rfloor + \max\{1, 2\lfloor\delta_1 + \delta_2\rfloor\}), \tag{15}$$

where $\delta_i = \alpha_i - \lfloor\alpha_i\rfloor,\ i = 1, 2.$

REMARK 6.10 The bound (15) is a sharper bound than (5.20) in Jeltsch & Smit (1992, p 29, Proposition 5.15) because the contribution due to $\lfloor\alpha_1\rfloor$ is not doubled.

6.1 *Components containing two zero points*

An obvious way of obtaining a component with two zero points is to connect two components with one zero point each by means of a branch cut of M. We shall show that our technique to prove bounds for the multiplicity in such a situation will give a bound which is larger than what one would obtain by ignoring the connecting cut and applying the results of the previous section to each component separately. Since we have not found any example which shows that this higher bound is sharp we conjecture that the smaller bound (15) is correct in all cases.

CONJECTURE 6.11 Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M. Then the multiplicity m of the Ω-component Ω_1 containing both zero points satisfies

$$m \leqslant \lfloor\alpha_1\rfloor + 1 + 2\lfloor\alpha_2\rfloor + \max\{1, 2\lfloor\delta_1 + \delta_2\rfloor\}.$$

Clearly, if Ω_1 in the conjecture can be separated into two components then the conjecture is proved.

We can prove this conjecture for a class of schemes of maximal order, $p = |I| - 2$. The proof is obtained by contradiction. Hence we assume the converse and examine its implications.

DEFINITION 6.12 (*Double-binary component*) An Ω-component Ω_1 which contains both zero points is called double binary if the contribution to its multiplicity by both zero points occurs via a doubling of $\lfloor\alpha_1\rfloor$ and $\lfloor\alpha_2\rfloor$.

A double-binary component can be either symmetric or non-symmetric. We first consider Fig. 12 which depicts a modification of the symmetric binary component illustrated in Figs. 3, 4, in the sense that the 'inner' boundary curve is moved inward so as to include the second zero point, and hence give a component containing both zero points.

LEMMA 6.13 (*Symmetric double-binary component*) Let Ω_1 be a symmetric double-binary component containing both zero points, $(0, w_1^0)$ and $(0, w_2^0)$ with leading exponents $-\alpha_1$ and $-\alpha_2$ of φ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively. Then the multiplicity m of Ω_1 satisfies

$$m \leqslant 2\lfloor\alpha_1\rfloor + 2\lfloor\alpha_2\rfloor + 1 + 2\lfloor\delta_1 + \delta_2\rfloor.$$

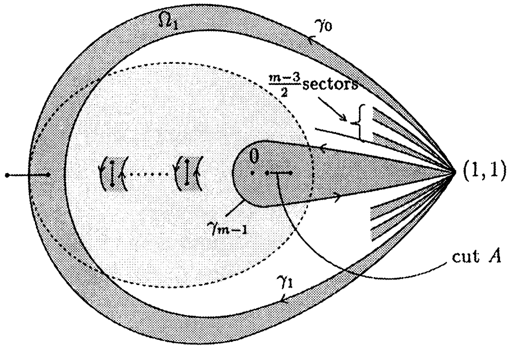

FIG. 12. Symmetric double-binary component.

Proof. Clearly if cut A were not present one could apply Theorem 6.9 and we would have Conjecture 6.11. But, since the cut A can't be removed we apply the argument principle to the whole component Ω_1. Hence

$$\begin{aligned}
0 &= \frac{1}{2\pi i}\int \frac{\gamma'}{\gamma}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\gamma_0^E} + \frac{1}{2\pi i}\int_{\gamma_1^E} + \frac{1}{2\pi i}\sum_{j=1}^{\frac{m-3}{2}}\left(\int_{\gamma_{2j}^E} + \int_{\gamma_{2j+1}^E}\right) + \frac{1}{2\pi i}\int_{\gamma_{m-1}^E} \\
&\leqslant \lfloor\alpha_1+\alpha_2\rfloor + \lfloor-\alpha_1\rfloor + \frac{m-3}{2}(-1) + \lfloor\alpha_1\rfloor
\end{aligned}$$

and this gives the bound

$$m \leqslant 2\lfloor\alpha_1\rfloor + 1 + 2\lfloor\alpha_2\rfloor + 2\lfloor\delta_1+\delta_2\rfloor. \tag{16}$$

□

Clearly, if $\lfloor\alpha_1\rfloor > 0$ then the bound (16) is not as sharp as the conjecture bound. While in this example the component could have been separated by removing cut A, such a separation is not as evident in the example depicted in Fig. 13. A comparison with Figs. 7 and 8 reveals that Fig. 13 is obtained from a non-symmetric binary component by replacing the cut locally orthogonal to the real axis by a cut, cut A, lying on the projection of the real axis onto $\mathbb{C}$. Therefore a component of this kind will be called non-symmetric double-binary.

LEMMA 6.14 (*Non-symmetric double-binary*) Let Ω_1 be a non-symmetric double-binary component containing both zero points, $(0, w_1^0)$ and $(0, w_2^0)$, with leading exponents $-\alpha_1$ and $-\alpha_2$ of φ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively. Then the multiplicity m of Ω_1 satisfies

$$m \leqslant 2(\lfloor\alpha_1\rfloor + \lfloor\alpha_2\rfloor + 1).$$

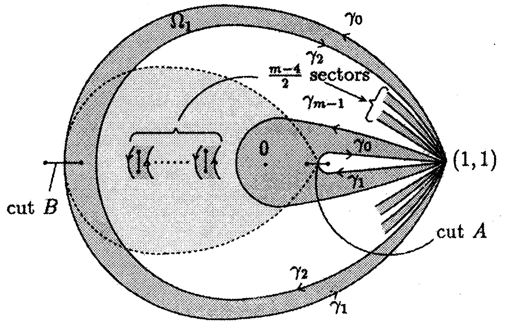

FIG. 13. Non-symmetric double-binary.

Proof. As usual we apply the argument principle to the whole component Ω_1. Hence

$$0 = \frac{1}{2\pi i}\int \frac{\gamma'}{\gamma}\,dz$$
$$= \frac{1}{2\pi i}\int_{\gamma_0^E} + \frac{1}{2\pi i}\int_{\gamma_1^E} + \frac{1}{2\pi i}\int_{\gamma_2^E} + \frac{1}{2\pi i}\sum_{j=1}^{\frac{m-4}{2}}\left(\int_{\gamma_{2j+1}^E} + \int_{\gamma_{2j+2}^E}\right) + \frac{1}{2\pi i}\int_{\gamma_{m-1}^E}$$
$$\leqslant \lfloor\alpha_2\rfloor + \lfloor\alpha_1\rfloor + \lfloor-\alpha_1\rfloor + \frac{m-4}{2}\left(\lfloor\alpha_1\rfloor + \lfloor-\alpha_1\rfloor\right) + \lfloor\alpha_1\rfloor$$

and this gives the bound

$$m \leqslant 2\lfloor\alpha_1\rfloor + 1 + 2\lfloor\alpha_2\rfloor + 1. \qquad (17)$$

□

Again, if $\lfloor\alpha_1\rfloor > 0$ then the bound (17) is not as sharp as the conjectured bound. As in the previous example the bound is wrong by the factor 2 in the term containing $\lfloor\alpha_1\rfloor$.

We observe that the results suggested by Lemmas 6.13 and 6.14 lead to the following general result for a double-binary component.

THEOREM 6.15 (*Double-binary*) Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M, and suppose that both $(0, w_1^0)$ and $(0, w_2^0)$ belong to one double-binary component, Ω_1. Then the highest multiplicity, M, that this component can contribute at $(1, 1)$ is given by

$$m \leqslant 2(\lfloor\alpha_1\rfloor + \lfloor\alpha_2\rfloor) + 1 + \max\{1, 2\lfloor\delta_1 + \delta_2\rfloor\}.$$

7. The role of the branch cuts on multiplicity

Close inspection of the results derived until now will reveal that these results actually rely on the occurrence of branch cuts to connect binary loops to the portion of the component containing one or both zero points. If there are insufficient branch cuts, or equivalently

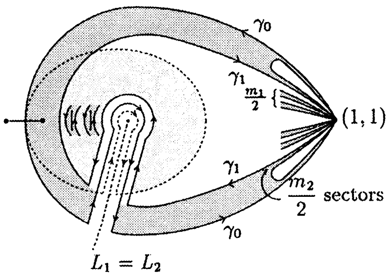

FIG. 14. Suboptimal symmetric binary.

insufficient branch points, these multiplicities will not be achievable. It therefore becomes appropriate to reformulate the earlier results in terms of the minimum number of branch points used by a component in order to achieve a certain multiplicity. The implication is that we now consider components which we will call *suboptimal* (Fig. 14); for example, for a binary component, Ω_1, we allow for the possibility that there are insufficient branch points for the zero point to be completely binary, and hence that there are also sectors of Ω_1 at $z = 1$ which contribute only a factor 1 rather than a factor 2 to the multiplicity.

To standardize our approach we will adopt the following notation:

$$K := \text{number of branch points utilized by the component}$$
$$m_1 := \text{number of sectors at } z = 1 \text{ due to binary loops}$$
$$m_2 := \text{number of sectors at } z = 1 \text{ due to non-binary loops.}$$

We also make the assumption, without loss of generality, $\lfloor\alpha_2\rfloor \geqslant \lfloor\alpha_1\rfloor$. Furthermore, to ease comparison with the previous results, we will refer to the components as classified as in the earlier sections. In each case we will employ the 'Version 2'-type proofs since these give the tighter bounds.

LEMMA 7.1 (*Suboptimal symmetric binary component, cf Lemma 6.3*) Let Ω_1 be a symmetric binary component containing one zero point $(0, w_2^0)$, say, with leading exponent $-\alpha_2$ of φ at $(0, w_2^0)$, while the leading exponent of φ at $(0, w_1^0) \notin \Omega_1$ is $-\alpha_1$. Further, suppose that Ω_1 contains at most K branch points of $w(z, \mu)$. Then the multiplicity m of Ω_1 satisfies

$$m \leqslant \lfloor\alpha_2\rfloor + \lfloor\delta_1 + \delta_2\rfloor + \min\left\{\lfloor\alpha_2\rfloor + \lfloor\delta_1 + \delta_2\rfloor, \left\lfloor\frac{K+1}{2}\right\rfloor\right\}.$$

Proof. The proof follows as for Lemma 6.3 but note now that the number of binary loops is limited by the number of branch points K. Each of the $m_1/2$ binary loops contributes -1 to the argument but two sectors at $z = 1$. The m_2 non-binary loops also contribute -1 to the argument decrease but one sector at $z = 1$. Hence, the total number of sectors at $z = 1$ is

$$m = m_1 + m_2 + 2.$$

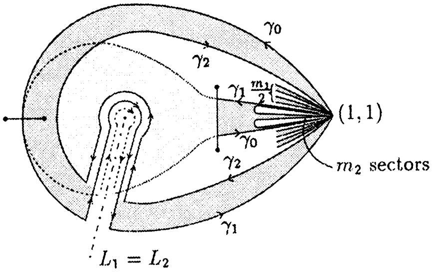

FIG. 15. Suboptimal non-symmetric binary.

Further, by the argument principle

$$0 \leqslant \lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor - \frac{m_1}{2} - m_2.$$

Therefore

$$\begin{aligned} m = m_1 + m_2 + 2 &\leqslant \lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor + \frac{m_1}{2} + 2 \\ &= \lfloor \alpha_2 \rfloor - 1 + \lfloor \delta_1 + \delta_2 \rfloor + 2 + \frac{m_1}{2}. \end{aligned}$$

Now each binary loop uses at least one branch cut to connect that loop to Ω_1. Also one branch point is required to make the component binary. Therefore

$$m_1 \leqslant K - 1,$$

and

$$\frac{m_1}{2} \leqslant \left\lfloor \frac{K-1}{2} \right\rfloor.$$

Thus

$$m \leqslant \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor + 1 + \left\lfloor \frac{K-1}{2} \right\rfloor$$

and by Lemma 6.3 the result follows. □

COROLLARY 7.2 The minimum number of branch points, $K_{\min}$, contained in a component Ω_1, described as in Lemma 7.1, for which the maximum multiplicity, as indicated by Lemma 6.3, is obtained, is given by

$$\left\lfloor \frac{K_{\min}+1}{2} \right\rfloor = \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor.$$

Proof. This follows immediately by observing that when $m_2 = 0$, $K \geqslant m - 1$, and hence $K_{\min} = m_{\max} - 1$, where $m_{\max} = 2\lfloor \alpha_2 \rfloor + 2\lfloor \delta_1 + \delta_2 \rfloor$. □

A non-symmetric binary component can also be suboptimal (Fig. 15).

LEMMA 7.3 (*Suboptimal non-symmetric binary component, cf Lemma* 6.5) Let Ω_1 be a non-symmetric binary component containing one zero point $(0, w_2^0)$, say, with leading exponent $-\alpha_2$ of φ at $(0, w_2^0)$, while the leading exponent of φ at $(0, w_1^0) \notin \Omega_1$ is $-\alpha_1$. Further, suppose that Ω_1 contains at most K branch points of $w(z, \mu)$. Then the multiplicity m of Ω_1 satisfies

$$m \leqslant \lfloor\alpha_2\rfloor + \min\left\{\lfloor\alpha_2\rfloor + 1, \left\lfloor\frac{K+1}{2}\right\rfloor\right\}.$$

COROLLARY 7.4 The minimum number of branch points, $K_{\min}$, contained in a component Ω_1, described as in Lemma 7.3, for which the maximum multiplicity, as indicated by Lemma 6.5, is obtained, is given by

$$\left\lfloor\frac{K_{\min}+1}{2}\right\rfloor = \lfloor\alpha_2\rfloor + 1.$$

These suboptimal binary components can be combined with a non-binary component in exactly the same way as a binary component is combined with a non-binary component in Theorem 6.9:

THEOREM 7.5 (*Suboptimal binary plus non-binary*) Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M, and suppose $(0, w_1^0)$ belongs to a non-binary component while $(0, w_2^0)$ belongs to a binary component containing K branch points. Then the highest total multiplicity m that these two components can contribute at $(1, 1)$ is given by

$$m \leqslant (\lfloor\alpha_1\rfloor + 1) + \left(\lfloor\alpha_2\rfloor + \min\left\{\left\lfloor\frac{K+1}{2}\right\rfloor + \lfloor\delta_1+\delta_2\rfloor, \lfloor\alpha_2\rfloor + \max\{1, 2\lfloor\delta_1+\delta_2\rfloor\}\right\}\right)$$

Theorem 7.5 implies that there is actually a combination of two components which yields a multiplicity between that which would be indicated by (i) both components of non-binary type, and (ii) one component optimal binary and the other non-binary. In the same way double-binary components can also be suboptimal with a multiplicity greater than that indicated by Theorem 6.9 but less than that indicated by Theorem 6.15. Such components are again limited in multiplicity by the number of branch points they contain.

LEMMA 7.6 (*Suboptimal symmetric double-binary, cf Lemma* 6.13) Let Ω_1 be a symmetric double-binary component (Fig. 16) containing both zero points, $(0, w_1^0)$ and $(0, w_2^0)$, with leading exponents $-\alpha_1$ and $-\alpha_2$ of φ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively. Further, suppose that Ω_1 contains at most K branch points of $w(z, \mu)$. Then the multiplicity m of Ω_1 satisfies

$$m \leqslant \lfloor\alpha_1\rfloor + \lfloor\alpha_2\rfloor + \lfloor\delta_1+\delta_2\rfloor + \min\left\{\lfloor\alpha_1\rfloor + \lfloor\alpha_2\rfloor + 1 + \lfloor\delta_1+\delta_2\rfloor, \left\lfloor\frac{K+1}{2}\right\rfloor\right\}.$$

Proof. The argument principle is applied assuming m_2 non-binary loops of Ω_1 at $z = 1$, and $m_1/2$ binary loops of Ω_1 at $z = 1$. In this case

$$m = m_2 + m_1 + 3,$$

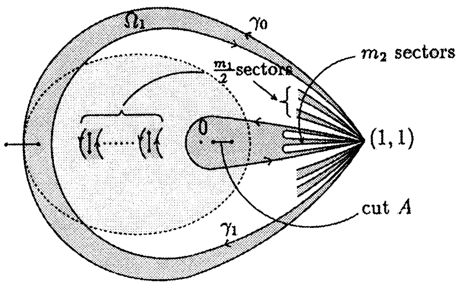

FIG. 16. Suboptimal symmetric double-binary component.

and by the argument principle,

$$0 \leqslant \lfloor \alpha_1 + \alpha_2 \rfloor + \lfloor -\alpha_1 \rfloor + \lfloor \alpha_1 \rfloor - m_2 - \frac{m_1}{2}.$$

Therefore

$$m \leqslant \lfloor \alpha_1 + \alpha_2 \rfloor - 1 + 3 + \frac{m_1}{2},$$

where m_1 is limited by the total number of branch points K,

$$m_1 \leqslant K - 3.$$

Therefore,

$$m \leqslant \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor + \left\lfloor \frac{K+1}{2} \right\rfloor,$$

and the result follows in combination with Lemma 6.13. □

Again there is a minimum number of branch points for which an optimal double-binary component can be obtained.

COROLLARY 7.7 The minimum number of branch points, $K_{\min}$, contained in a component Ω_1, described by Lemma 7.6, for which the maximum multiplicity, as indicated by Lemma 6.13, is obtained, is given by

$$\left\lfloor \frac{K_{\min}+1}{2} \right\rfloor = \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + \lfloor \delta_1 + \delta_2 \rfloor + 1.$$

The non-symmetric double-binary component, as illustrated by Fig. 13, may also be suboptimal, see Fig. 17.

LEMMA 7.8 (*Suboptimal non-symmetric double-binary, cf Lemma* 6.14) Let Ω_1 be a non-symmetric double-binary component containing both zero points, $(0, w_1^0)$ and $(0, w_2^0)$, with leading exponents $-\alpha_1$ and $-\alpha_2$ of φ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively. Further, suppose that Ω_1 contains at most K branch points of $w(z, \mu)$. Then the multiplicity m of Ω_1 satisfies

$$m \leqslant \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1 + \min\left\{ \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1, \left\lfloor \frac{K+1}{2} \right\rfloor \right\}.$$

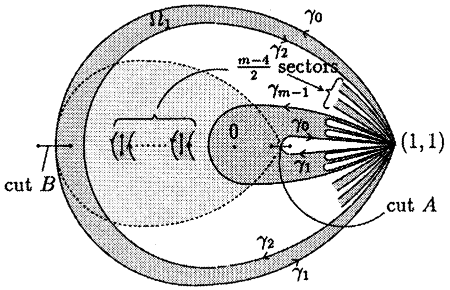

FIG. 17. Suboptimal non-symmetric double-binary component.

COROLLARY 7.9 The minimum number of branch points, $K_{\min}$, contained in a component Ω_1, described as in Lemma 7.8, for which the maximum multiplicity, as indicated by Lemma 6.14, is obtained, is given by

$$\left\lfloor \frac{K_{\min}+1}{2} \right\rfloor = \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1.$$

Combining the conclusions of Lemmas 7.6 and 7.8 we deduce the following theorem.

THEOREM 7.10 (*Suboptimal double-binary*) Let φ have leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M, and suppose that both zero points belong to a double-binary component Ω_1 which also contains K branch points. Then the highest total multiplicity, m, that this component can contribute at $(1, 1)$ is given by

$$m \leqslant \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + \max \left\{ \min \left\{ \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 2, \left\lfloor \frac{K+1}{2} \right\rfloor + 1 \right\}, \right.$$

$$\left. \min \left\{ \lfloor \alpha_1 \rfloor + \lfloor \alpha_2 \rfloor + 1 + 2\lfloor \delta_1 + \delta_2 \rfloor, \left\lfloor \frac{K+1}{2} \right\rfloor + \lfloor \delta_1 + \delta_2 \rfloor \right\} \right\}.$$

7.1 *Proof of Conjecture* 6.11 *for maximum order schemes*

Here we derive bounds on the order, p, of explicit schemes, under the assumption that the double-binary components described in Sections 6.1 and 7, exist. But we note that their multiplicities depend very intimately on the number of branch points contained inside the unit disk. These components cannot use branch cuts completely outside the unit disk since then stability would be violated. We also know, by Lemma 5.2, that the total number, m, of sectors at $z = 1$, both from inside and from outside the unit disk, satisfies $p + 1 = m$. Let m_I and m_O be the number of sectors of Ω at $z = 1$ from inside, and outside the unit disk, respectively. Then $m = m_I + m_O$, and by Lemma 5.2,

$$m_O - 1 \leqslant m_I \leqslant m_O + 1.$$

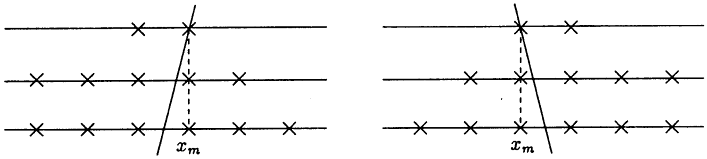

FIG. 18. Stencil of reversed scheme.

Therefore, if considered independently

$$p \leqslant \max\{2m_I, 2m_O\}. \tag{18}$$

But, since m_O can be seen to be limited by the number of branch points of $w(z, \mu)$ outside the unit disk, the bound on m_O actually depends on the bound on m_I, via the number of branch points, K_I, inside the unit disk, limiting the number of branch points, K_O, outside the unit disk, because for convex schemes

$$2(r_1 + s_1) = K_I + K_O.$$

We will show that for schemes of maximal order, $p = |I| - 2$, an argument in which both m_I and m_O are determined is required to derive a tight bound on p. But first we have to consider how to obtain m_O.

Instead of repeating the analyses of the earlier sections we apply a symmetry argument for the Ω-sectors outside Δ. For generality, we consider here implicit schemes. The functions a_i defined in (6) will be written here in the form $a_i(z, \mu)$ to emphasize the μ-dependence of the coefficients a_{ij}. Suppose we have a stable scheme (4) and that its stencil is regular for a certain value of μ, with R downwind and S upwind stencil points according to (9). Let ℓ denote the number of Ω-sectors of the corresponding order star Ω emerging from the point $(1, 1)$ outside the unit disk Δ. Then consider the reversed scheme

$$\begin{aligned} &\sum_{j=-s_2}^{r_2} a_{2,-j}(-\mu)\, u_{n+2,m+j} + \sum_{j=-s_1}^{r_1} a_{1,-j}(-\mu) u_{n+1,m+j} \\ &\quad + \sum_{j=-s_0}^{r_0} a_{0,-j}(-\mu) u_{n,m+j} = 0. \end{aligned} \tag{19}$$

This scheme can be thought of as being obtained by a transformation of the space variable x into $-x$. Hence the stencil is reflected about the line $x = x_m$ (see Fig. 18).

The new scheme has $R^* = S$ downwind and $S^* = R$ upwind stencil points with respect to the characteristic $\mu^* = -\mu$. The characteristic function Φ^* of the reversed scheme is obtained from the characteristic function Φ by using the transformation

$$z \to \frac{1}{z} \qquad \text{and} \qquad \mu \to -\mu.$$

Hence

$$\begin{aligned}\Phi^*(z, w^*, \mu^*) &= \Phi(\tfrac{1}{z}, w^*, -\mu)\\ &= a_2(\tfrac{1}{z}, -\mu)(w^*)^2 + a_1(\tfrac{1}{z}, -\mu)w^* + a_0(\tfrac{1}{z}, -\mu)\\ &= a_2^*(z, \mu^*)w^{*2} + a_1^*(z, \mu^*)w^* + a_0^*(z, \mu^*).\end{aligned}$$

Therefore the algebraic function w^* satisfies

$$w^*(z, \mu) = w(\tfrac{1}{z}, -\mu). \tag{20}$$

From this relationship it follows that the reversed scheme is stable and of order p if and only if the original scheme is stable and of order p. The order star Ω^* of the reversed scheme (19) is related to Ω of (4) in the sense that the portion of Ω outside the unit disk Δ is mapped to the inside of Δ and vice versa by the mapping $z \to \frac{1}{z}$.

Therefore, in order to determine m_O for a given value of μ, we should map $z \to \frac{1}{z}$ and investigate the portion of Ω inside Δ for $\mu^* = -\mu$. For explicit schemes the corresponding leading exponents of φ will be $-\beta_1$ and $-\beta_2$ at $(0, w_1^0)$ and $(0, w_2^0)$, respectively, where $\beta_1 = s_0 - s_1 + \mu^*$, $\beta_2 = s_1 - s_2 + \mu^*$. Effectively, this maps the pole at infinity to zero, so that the effects of the infinity points can be examined via the effects of the zero points for μ^*. Note here that the form of β_1 and β_2 explicitly assumes $s_1 - s_2 \geqslant s_0 - s_1$, see Proposition 4.4 and Remark 4.5. Hence the argument we adopt enforces convexity on both sides of the characteristic line.

To complete the methodology we also need to be able to count the number of branch points of $w^*(z, \mu^*)$ inside the unit disk for μ^*. But, by (20) this is just the number of branch points of $\omega(\frac{1}{z}, -\mu^*)$, which we have already denoted by K_O.

LEMMA 7.11 (*Conjecture* 6.11 *for explicit maximal order schemes*) Suppose we have an explicit stable scheme of type (4) of maximal order, $p = |I| - 2$, with a convex increasing stencil, with a fixed Courant number μ, $-\frac{1}{2} < \mu < 0$. Assume that the algebraic function w of $\Phi(z, w) = 0$ has no branch point at $z = 0$ and that φ has leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M. Then the multiplicity m of the Ω-component Ω_1 containing both zero points satisfies

$$m \leqslant \lfloor\alpha_1\rfloor + 1 + 2\lfloor\alpha_2\rfloor + \max\{1, 2\lfloor\delta_1 + \delta_2\rfloor\}.$$

Proof. Note that this is a statement of Conjecture 6.11 for the schemes of maximal order, $p_{\text{opt}} = |I| - 2$. Hence what we seek to prove is that for these schemes there cannot exist components of double-binary or suboptimal-binary-type. In particular we show that if these components exist the order is necessarily less that p_{opt}.

From Proposition 4.4 and Remark 4.5 we have the following expansions of $\varphi(z, w_i(z))$ at $z = 0$:

$$\begin{aligned}\varphi(z, w_1(z)) &= z^{-r_1-\mu}(b_0 + b_1 z + b_2 z^2 + \cdots),\\ \varphi(z, w_2(z)) &= z^{-(r_0-r_1)-\mu}(c_0 + c_1 z + c_2 z^2 + \cdots).\end{aligned}$$

It should be noted here that we have no means of associating a certain expansion with the zero point on a specific sheet. Hence the expansions will be associated with the zero points in the way which leads to the highest possible multiplicity.

Equivalently, for $\mu^* = -\mu$ we have the expansions of $\varphi(z, w_i^*(z))$ at $z = 0$

$$\varphi(z, w_1^*(z)) = z^{-s_1-\mu^*}(b_0^* + b_1^* z + b_2^* z^2 + \cdots),$$
$$\varphi(z, w_2^*(z)) = z^{-(s_0-s_1)-\mu^*}(c_0^* + c_1^* z + c_2^* z^* + \cdots),$$

obtained via the transformation $z \to \frac{1}{z}$. To avoid confusion we denote the exponents associated with μ by

$$\alpha_1 = r_0 - r_1 + \mu$$
$$\alpha_2 = r_1 + \mu, \tag{21}$$

and those associated with μ^* by

$$\beta_1 = s_0 - s_1 + \mu^*,$$
$$\beta_2 = s_1 + \mu^*. \tag{22}$$

By convexity, $\lfloor\alpha_2\rfloor \geqslant \lfloor\alpha_1\rfloor$ and $\lfloor\beta_2\rfloor \geqslant \lfloor\beta_1\rfloor$. Furthermore, we explicitly assume $\lfloor\alpha_1\rfloor > 0$. Otherwise the zero point at $(0, w_1^0)$ does not lie inside Ω and the argument is considerably simplified. Similarly, assume $\lfloor\alpha_2\rfloor > \lfloor\alpha_1\rfloor$.

Clearly $-\frac{1}{2} < \mu < 0$ implies $\lfloor\delta_1 + \delta_2\rfloor = \lfloor 2\delta_1\rfloor = \lfloor 2(1+\mu)\rfloor = 1$.

(i) By Theorems 7.5 and 7.10 we see that the optimal configuration is dependent on the number of branch points utilized by the components. In particular define K_L and K_U to be the minimum number of branch points for which an optimal binary–non-binary configuration, and a double-binary configuration is possible, respectively. Then

$$m_I \leqslant \begin{cases} \lfloor\alpha_1\rfloor + \lfloor\alpha_2\rfloor + 2 + \lfloor\frac{K_I+1}{2}\rfloor, & K_I < K_L \\ \lfloor\alpha_1\rfloor + 1 + 2\lfloor\alpha_2\rfloor + 2, & K_I = K_L \\ \lfloor\alpha_1\rfloor + \lfloor\alpha_2\rfloor + 1 + \lfloor\frac{K_I+1}{2}\rfloor, & K_L < K_I < K_U \\ 2(\lfloor\alpha_1\rfloor + \lfloor\alpha_2\rfloor + 1) + 1, & K_I \geqslant K_U. \end{cases} \tag{23}$$

Note that the bounds in (23) imply that the binary components are symmetric. Hence by Corollaries 7.2 and 7.7

$$\left\lfloor\frac{K_L+1}{2}\right\rfloor = \lfloor\alpha_2\rfloor + 1 \quad \text{and} \quad \left\lfloor\frac{K_U+1}{2}\right\rfloor = \lfloor\alpha_1\rfloor + \lfloor\alpha_2\rfloor + 2. \tag{24}$$

Substituting for $\lfloor\alpha_1\rfloor$ and $\lfloor\alpha_2\rfloor$ in (23) and (24) we obtain

$$m_I \leqslant \begin{cases} r_0 + \lfloor\frac{K_I+1}{2}\rfloor, & \lfloor\frac{K_I+1}{2}\rfloor < r_1 \\ r_0 + r_1, & \lfloor\frac{K_I+1}{2}\rfloor = r_1 \\ r_0 - 1 + \lfloor\frac{K_I+1}{2}\rfloor, & r_1 < \lfloor\frac{K_I+1}{2}\rfloor < r_0 \\ 2r_0 - 1, & \lfloor\frac{K_I+1}{2}\rfloor \geqslant r_0. \end{cases} \tag{25}$$

(ii) Now we consider the outside of the unit disk and apply an equivalent argument with respect to β_1, β_2 and $\mu^* = -\mu$. In this case in the bound for m_O we use $\lfloor \delta_1 + \delta_2 \rfloor = \lfloor 2\delta_1 \rfloor = \lfloor 2\mu \rfloor = 0$. Thus by Theorems 7.5 and 7.10

$$m_O \leqslant \begin{cases} \lfloor \beta_1 \rfloor + \lfloor \beta_2 \rfloor + 1 + \lfloor \frac{K_O+1}{2} \rfloor, & K_O < K_L \\ \lfloor \beta_1 \rfloor + 1 + 2\lfloor \beta_2 \rfloor + 1, & K_O = K_L \\ \lfloor \beta_1 \rfloor + \lfloor \beta_2 \rfloor + \lfloor \frac{K_O+1}{2} \rfloor + 1, & K_L < K_O < K_U \\ 2(\lfloor \beta_1 \rfloor + \lfloor \beta_2 \rfloor + 1), & K_O \geqslant K_U. \end{cases} \tag{26}$$

These components are non-symmetric and therefore, by Corollaries 7.4 and 7.9,

$$\left\lfloor \frac{K_L+1}{2} \right\rfloor = \lfloor \beta_2 \rfloor + 1 \quad \text{and} \quad \left\lfloor \frac{K_U+1}{2} \right\rfloor = \lfloor \beta_1 \rfloor + \lfloor \beta_2 \rfloor + 1. \tag{27}$$

As in (i), substitution of values for $\lfloor \beta_1 \rfloor$ and $\lfloor \beta_2 \rfloor$ in (26) and (27) leads to

$$m_O \leqslant \begin{cases} s_0 + 1 + \lfloor \frac{K_O+1}{2} \rfloor, & \lfloor \frac{K_O+1}{2} \rfloor < s_1 + 1 \\ s_0 + s_1 + 2, & \lfloor \frac{K_O+1}{2} \rfloor = s_1 + 1 \\ s_0 + \lfloor \frac{K_O+1}{2} \rfloor + 1, & s_1 + 1 < \lfloor \frac{K_O+1}{2} \rfloor < s_0 + 1 \\ 2(s_0 + 1), & \lfloor \frac{K_O+1}{2} \rfloor \geqslant s_0 + 1. \end{cases} \tag{28}$$

(iii) We now combine the results from inside and outside Δ.
First, observe that if K_I is even, so is K_O, and because the total number of branch points is $2(r_1 + s_1)$,

$$\left\lfloor \frac{K_I+1}{2} \right\rfloor + \left\lfloor \frac{K_O+1}{2} \right\rfloor = r_1 + s_1,$$

whereas, if K_I and K_O are odd,

$$\left\lfloor \frac{K_I+1}{2} \right\rfloor + \left\lfloor \frac{K_O+1}{2} \right\rfloor = r_1 + s_1 + 1.$$

Therefore bounds on K_I imply bounds on K_O, and vice versa. Hence, only certain combinations of components inside and outside Δ are possible.
In particular, suppose that inside Δ there is a double-binary or suboptimal double-binary configuration. Then by (25) $\lfloor \frac{K_I+1}{2} \rfloor > r_1$ and

$$\left\lfloor \frac{K_O+1}{2} \right\rfloor < s_1 + \begin{cases} 0 & K_I \text{ even} \\ 1 & K_I \text{ odd.} \end{cases}$$

Therefore outside Δ there can be at most a suboptimal binary configuration and

$$\begin{aligned} m_I + m_O &\leqslant r_0 - 1 + s_0 + 1 + \left\lfloor \frac{K_O+1}{2} \right\rfloor + \left\lfloor \frac{K_I+1}{2} \right\rfloor \\ &= r_0 + s_0 + r_1 + s_1 + \begin{cases} 0 & K_I \text{ even} \\ 1 & K_I \text{ odd.} \end{cases} \end{aligned}$$

Hence

$$m_I + m_O \leqslant |I| - 3 + \begin{cases} 0 & K_I \text{ even} \\ 1 & K_I \text{ odd.} \end{cases}$$

By Lemma 5.2, therefore

$$p \leqslant |I| - 3,$$

and (15) follows for $-\frac{1}{2} < \mu < 0$.

□

We could now repeat the arguments for $0 < \mu < \frac{1}{2}$ but it is sufficient to examine the order star outside Δ for $-\frac{1}{2} < \mu < 0$. Hence again we would like to show that the bounds on m_O, (28), for double-binary components, lead to a contradiction. For these components $\lfloor \frac{K_O+1}{2} \rfloor \geqslant s_1 + 1$, and hence $\lfloor \frac{K_I+1}{2} \rfloor \leqslant r_1$. Thus by (25)

$$m_I \leqslant r_0 + \left\lfloor \frac{K_I + 1}{2} \right\rfloor,$$

and

$$\begin{aligned} m_O + m_I &\leqslant r_0 + s_0 + \left\lfloor \frac{K_I + 1}{2} \right\rfloor + \left\lfloor \frac{K_O + 1}{2} \right\rfloor + 1 \\ &\leqslant r_0 + s_0 + r_1 + s_1 + 1 + \begin{cases} 0 & K_O \text{ even} \\ 1 & K_O \text{ odd.} \end{cases} \end{aligned}$$

This time

$$m_O + m_I \leqslant |I| - 2 + \begin{cases} 0 & K_O \text{ even} \\ 1 & K_O \text{ odd,} \end{cases}$$

and we do not obtain the required contradiction, unless it can be demonstrated that K_O is even.

Note that at no point did we explicitly impose stability in the above proof, because K_I and K_O are simply the number of branch points used by Ω from inside and outside Δ, respectively. But if a component inside Δ utilized a cut outside Δ then this would in fact require that the stability condition, Lemma 5.1, be violated. Hence K_I and K_O do actually refer to the number of branch points inside and outside Δ, respectively, and stability is required.

8. Implicit schemes: the role of poles away from $z = 0$ on multiplicity

8.1 *Components containing zero points and poles*

Assume we have inside a binary component poles of total multiplicity p. When applying the argument principle $-p$ is added on the left side of the equation, e.g. (11) or (12). This leads to a bound for m with an additional term $2p$. If the component is non-binary then this additional term is clearly only p. Hence, having poles in a non-binary component is less efficient than having one in a binary component. However, we shall show that we can then get a contribution $3p$ if the poles have multiplicity 1. To be able to do this the component is not allowed to contain a zero point. Such components are treated in the next section.

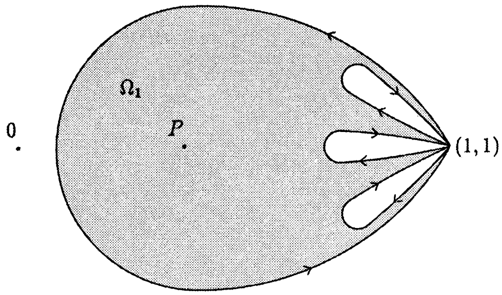

FIG. 19. Component with pole P on positive real axis on principal sheet of M.

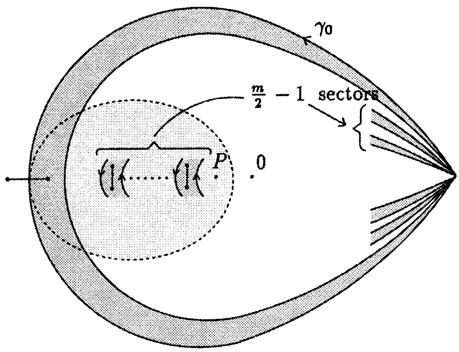

FIG. 20. Binary component with pole P on negative real axis and one zero point encircled.

8.2 *Components containing only poles away from $z = 0$*

We consider the relationship between the number of poles inside a component Ω_1 and its multiplicity under the assumption that Ω_1 contains neither zero point. Suppose a single pole P of multiplicity p lies inside Ω_1.

EXAMPLE 8.1 The pole P occurs on the positive real axis (for simplicity we locate it on the principal sheet in Fig. 19), or at any other location away from the real axis inside Δ. In this case the zero points do not have any effect on the component Ω_1, since the cuts L_i can be chosen such that they do not interact with Ω_1 in any way. Then Ω_1 corresponds to the type of component treated in Wanner *et al* (1978), where it was shown that the multiplicity m of Ω_1 is bounded by

$$m \leqslant p.$$

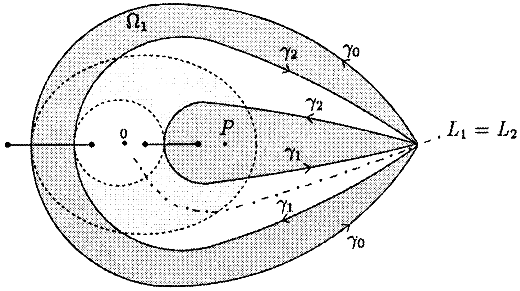

FIG. 21. Component with pole P and with both zero points encircled.

EXAMPLE 8.2 For the components in Fig. 20 the argument principle yields

$$-p = \frac{1}{2\pi i}\int_{\gamma_0^E} + \frac{1}{2\pi i}\int_{\gamma_1^E} + \frac{1}{2\pi i}\sum_{j=1}^{m/2-1}\int_{\gamma_{2j}^E+\gamma_{2j+1}^E}$$
$$\leqslant \lfloor\alpha_1\rfloor + \lfloor-\alpha_1\rfloor - (m/2-1).$$

Hence, $m \leqslant 2p$ as expected.

EXAMPLE 8.3 Let Ω_1 be such as in Fig. 21, with the pole P on the positive real axis on the secondary sheet of M and both zero points excluded from Ω_1. The branch cuts L_i are chosen such that their projection onto the z-plane passes through $z = 1$. From one side of L_1 we obtain

$$\frac{1}{2\pi i}\int_{\gamma_0^E+\gamma_1}\frac{\varphi'}{\varphi}\,\mathrm{d}z \leqslant \lfloor\alpha_2\rfloor, \quad \frac{1}{2\pi i}\int_{\gamma_2^E}\frac{\varphi'}{\varphi}\,\mathrm{d}z \leqslant \lfloor-\alpha_2\rfloor$$

and

$$-p \leqslant \lfloor\alpha_2\rfloor + \lfloor-\alpha_2\rfloor - (m_1 - 2).$$

From the other side of L_1 we obtain

$$\frac{1}{2\pi i}\int\limits_{\gamma_2+\gamma_1^E+\gamma_2^E}\frac{\varphi'}{\varphi}\,\mathrm{d}z \leqslant \lfloor-\alpha_2+\alpha_2-1\rfloor.$$

By applying the argument principle with the pole P inside the component and m_2 sectors at $z = 1$, we obtain

$$-p \leqslant -1 - (m_2 - 1).$$

A combination of these results leads to the following bound on the total multiplicity $m = m_1 + m_2$ of the component:

$$m \leqslant 2p + 1. \tag{29}$$

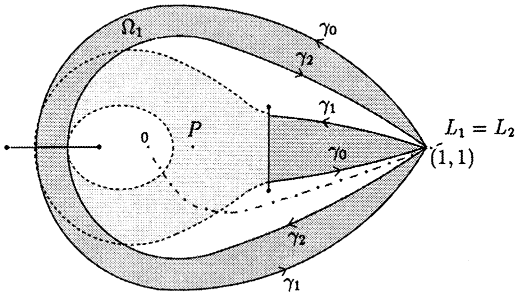

FIG. 22. Component with pole P and with both zero points encircled.

EXAMPLE 8.4 Let Ω_1 be such as in Fig. 22, with the pole P again occurring on the positive real axis on the secondary sheet of M and both zero points excluded from Ω_1. Then by choosing the branch cuts L_i to go through $z = 1$ and by applying the argument principle in exactly the same way as in Example 8.3, we again obtain the bound $m \leqslant 2p+1$ occurring in (29).

The question arises whether a pole P away from $z = 0$ can yield multiplicity higher than in Examples 8.3 and 8.4.

LEMMA 8.5 (*Multiplicity of a single pole*) Let P be a pole of order p away from $z = 0$ inside an Ω-component Ω_1 from which the two zero points are excluded by positively oriented portions of $\partial\Omega$ which encircle both zero points. Then the multiplicity m of Ω_1 is bounded by

$$m \leqslant 2\,p + 1.$$

We can deduce from Lemma 8.5 that the highest possible multiplicity of a component, Ω_1, relative to the order p of a pole P away from $z = 0$ inside it, is obtained if $p = 1$. Or, directly formulated: the most efficient poles away from $z = 0$ are simple poles. For such a simple pole we obtain the bound $m \leqslant 3$ on the multiplicity of the corresponding component. This entails that, if we have a normalized scheme which introduces into the corresponding order star poles of total order $p > 1$ inside Δ, then the highest possible contribution of these poles to the number of Ω-sectors inside Δ is obtained if these poles are simple and occur on the real axis. But then the complication occurs that the symmetry of components with respect to the real axis does not allow the simultaneous occurrence of two separate components, each with a multiplicity of 3. As before this problem is overcome by means of two components of multiplicity 3 which are joined via a branch cut to yield one component of multiplicity 6. In this new component each simple pole still contributes 3 to the multiplicity of the component. This is illustrated in Fig. 23, where there are three simple poles, each leading to a contribution of 3 sectors to the total multiplicity inside Δ. The rightmost pole belongs to a separate component, while the other two have joined to form a component of multiplicity 6. This situation is generalized.

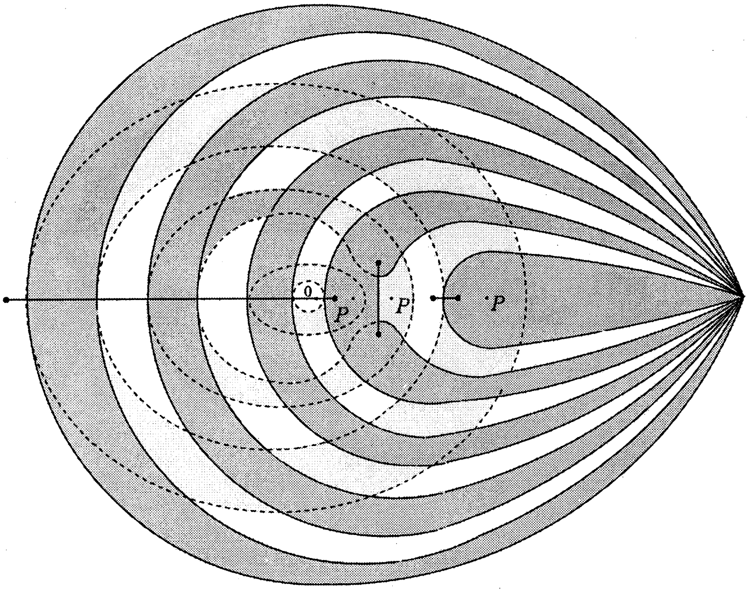

FIG. 23. Three simple poles, each leading to multiplicity 3, inside Ω-components.

PROPOSITION 8.6 (*Maximum multiplicity of poles*) Let the order star of a stable, normalized scheme have poles of total multiplicity p away from $z = 0$ inside Δ. Then the highest possible contribution of these poles to the multiplicity m of components inside Δ is obtained if the poles are simple and real, leading to a multiplicity bounded by

$$m \leqslant 3p. \tag{30}$$

REMARK 8.7 Instead of two components with simple poles being joined to form one component such as in Fig. 23, a component containing a simple pole can also be joined with a component containing a zero point, such as in Fig. 24. There the combined multiplicity is 4. (The zero point $(0, w_2^0)$ belongs to a separate binary component of multiplicity 2.) It can be seen that the efficiency of the pole and of the zero point $(0, w_1^0)$ remain unchanged as if they occur in separate components.

REMARK 8.8 Observe from Figs. 21–24 that in order for the poles in these components to each contribute a maximum multiplicity, of 3, branch cuts are required. In particular, for a single pole a minimum of 3 branch points inside Δ are utilized. But for components containing more than one pole the branch points are used more efficiently. For two poles again just 3 branch points in Δ are sufficient. As further poles are added to the component each requires an additional cut inside Δ. We therefore deduce the following corollary:

COROLLARY 8.9 Let the order star of a stable, normalized scheme have n_p poles away from $z = 0$ inside a component Ω_1, inside Δ. Then the minimum number of branch points,

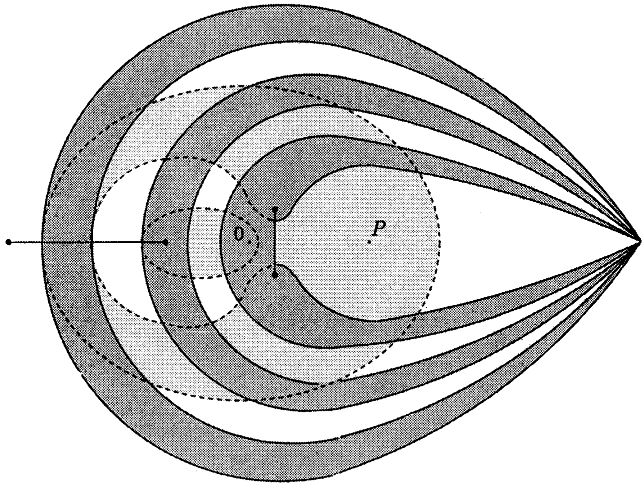

FIG. 24. Component with a pole and one with a zero point which have been joined to form one component.

$K_{\min}$, contained in Ω_1 such that the multiplicity of Ω_1 is given by $m = 3n_p$ satisfies

$$K_{\min} = 2n_p - 1.$$

The question then arises as to whether Conjecture 6.11 can still be proved for the schemes of maximal order. This, however, turns out not to be so difficult. First let us consider both the order of the zero points and the number of branch points for the implicit schemes. When $r_2 > 0, s_2 > 0$ the exponents of φ in the expansion around $z = 0$ are given by

$$\alpha_1 = r_0 - r_1 + \mu$$
$$\alpha_2 = r_1 - r_2 + \mu$$

and

$$\beta_1 = s_0 - s_1 + \mu^*$$
$$\beta_2 = s_1 - s_2 + \mu^*,$$

as compared with (21) and (22), respectively, when $r_2 = s_2 = 0$. But, by convexity, we still have $\lfloor \alpha_2 \rfloor \geqslant \lfloor \alpha_1 \rfloor$ and $\lfloor \beta_2 \rfloor \geqslant \lfloor \beta_1 \rfloor$, even though $\lfloor \alpha_2 \rfloor$ and $\lfloor \beta_2 \rfloor$ are reduced by r_2 and s_2, respectively. Hence the number of branch points K utilized by the zero points is reduced by $2r_2 - 1$ if K is odd and $2r_2$ if K is even. But, by Corollary 8.9, the r_2 poles inside Δ need at least $2r_2 - 1$ branch points in order to contribute maximum multiplicity. Note further that for K even, it was demonstrated in Lemma 7.11 that $p = |I| - 2$ could not be achieved. Hence the branch points left unutilized by the reduction of $\lfloor \alpha_2 \rfloor$ and $\lfloor \beta_2 \rfloor$, when $r_2, s_2 > 0$, are immediately required to contribute maximum multiplicity from the poles. Furthermore, since the contribution due to the poles gives a factor 3, rather than 2, in front of n_p and $\lfloor \alpha_1 \rfloor$, respectively, we deduce that the optimal configuration uses branch points to maximize multiplicity due to the poles rather than due to the zero points. This leads us to conclude that Conjecture 6.11 is also valid for convex implicit schemes:

LEMMA 8.10 (*Conjecture* 6.11 *for implicit schemes of maximal order*) Suppose we have an implicit stable scheme of type (4) of maximal order, $p = |I| - 2$, with a convex increasing stencil, with a fixed Courant number μ, $-\frac{1}{2} < \mu < 0$. Assume that the algebraic function w of $\Phi(z, w) = 0$ has no branch point at $z = 0$ and that φ has leading exponents of $-\alpha_1$ and $-\alpha_2$ at the zero points $(0, w_1^0)$ and $(0, w_2^0)$, respectively, on the two sheets of M. Then the multiplicity m of the Ω-component Ω_1 containing both zero points and no poles satisfies

$$m \leqslant \lfloor \alpha_1 \rfloor + 1 + 2 \lfloor \alpha_2 \rfloor + \max\{1, 2 \lfloor \delta_1 + \delta_2 \rfloor\}.$$

9. Proof of the main theorem

The proof of Theorem 3.1 is divided into stages. One part of it is proved in Lemma 9.1 and the other part in Lemma 9.2. The numbers R and S denote the number of downwind and upwind stencil points, respectively, with respect to the characteristic through (t_{n+2}, x_m) as defined in (9).

LEMMA 9.1 (*Maximal order with stability*) Suppose we have a convex, normalized scheme of type (4) with a fixed Courant number μ, $0 < |\mu| < \frac{1}{2}$. If the scheme is stable, and Conjecture 6.11 is satisfied, then the order p of the scheme is bounded by

$$p \leqslant 2R. \tag{31}$$

Proof. Since the scheme is stable, there will be a clear distinction between the portion of the corresponding order star Ω inside Δ and the portion outside Δ. In this proof we restrict ourselves to the portion inside Δ.

From Proposition 4.4 we have the following expansions of $\varphi(z, w_i(z))$ at $z = 0$:

$$\varphi(z, w_1(z)) = z^{-(r_1 - r_2) - \mu}(b_0 + b_1 z + b_2 z^2 + \cdots),$$
$$\varphi(z, w_2(z)) = z^{-(r_0 - r_1) - \mu}(c_0 + c_1 z + c_2 z^2 + \cdots).$$

Again we have no means of associating a certain expansion with the zero point on a specific sheet. Hence the expansions will be associated with the zero points in the way which leads to the highest possible multiplicity.

In the remainder of the proof we have to work separately with the cases where $\mu < 0$ and where $\mu > 0$. Note also that where we assume Conjecture 6.11, Lemmas 7.11 and 8.10 give the result for $p = p_{\text{opt}}$, and $-\frac{1}{2} < \mu < 0$.

a) We first assume $-\frac{1}{2} < \mu < 0$. Then the following choices of the indices r_0, r_1, r_2 lead to different combinations of Ω-components inside Δ.

   (i) $r_0 = r_1 = r_2 = 0$. Then also $R = 0$. According to the Courant–Friedrichs–Lewy condition the scheme cannot be convergent, i.e. it is impossible to have order $p \geqslant 1$ and stability simultaneously.

(ii) $r_0 = r_1 = r_2 > 0$. There are r_2 poles away from $z = 0$ inside Δ, while both $\varphi(z, w_1(z))$ and $\varphi(z, w_2(z))$ have positive leading exponents of $-\alpha_i = -\mu$ at $z = 0$, implying that both zero points belong to Ω^c. We apply Proposition 8.6 to obtain

$$m \leqslant 3r_2 = r_0 + r_1 + r_2 = R.$$

(iii) $0 = r_0 - r_1 < r_1 - r_2$. Then $-\alpha_1 = -\mu > 0$, implying that $(0, w_1^0) \in \Omega^c$, and $-\alpha_2 = -(r_1 - r_2) - \mu < 0$, implying that $(0, w_2^0) \in \Omega$. The highest possible multiplicity is obtained if $(0, w_2^0)$ belongs to a binary component, in which case we apply Proposition 6.6. If $r_2 > 0$, the poles away from $z = 0$ are again treated according to Proposition 8.6. Then we have

$$\begin{aligned} m &\leqslant 3r_2 + 2\lfloor r_1 - r_2 + \mu\rfloor + 2\lfloor 2 + 2\mu\rfloor \\ &= 3r_2 + 2(r_1 - r_2 - 1) + 2 \\ &= r_2 + 2r_1 = r_2 + r_1 + r_0 = R. \end{aligned}$$

(iv) $0 < r_0 - r_1 < r_1 - r_2$. Then $-\alpha_1 = -(r_0 - r_1) - \mu < 0$ and $-\alpha_2 = -(r_1 - r_2) - \mu < 0$, implying that both $(0, w_1^0)$ and $(0, w_2^0)$ belong to Ω. Since $\alpha_1 < \alpha_2$, the highest possible multiplicity is obtained by applying Conjecture 6.11, with $(0, w_1^0)$ inside a non-binary and $(0, w_2^0)$ inside a binary component. If $r_2 > 0$, the poles away from $z = 0$ are again treated according to Proposition 8.6. Then the total multiplicity m inside Δ is bounded by

$$\begin{aligned} m &\leqslant 3r_2 + \{2\lfloor r_1 - r_2 + \mu\rfloor + 2\lfloor 2 + 2\mu\rfloor\} + \{\lfloor r_0 - r_1 + \mu\rfloor + 1\} \\ &= 3r_2 + \{2(r_1 - r_2 - 1) + 2\} + \{(r_0 - r_1 - 1) + 1\} \\ &= r_0 + r_1 + r_2 = R. \end{aligned}$$

b) Assume $0 < \mu < \frac{1}{2}$. Then we have $-\alpha_1 = -(r_0 - r_1) - \mu < 0$ and $-\alpha_2 = -(r_1 - r_2) - \mu < 0$, implying that both $(0, w_1^0)$ and $(0, w_2^0)$ belong to Ω. Since $\alpha_1 \leqslant \alpha_2$, the highest possible multiplicity is obtained, assuming Conjecture 6.11, if α_1 is inside a non-binary and α_2 is inside a binary component. If $r_2 > 0$, the poles away from $z = 0$ are treated according to Proposition 8.6. This leads to the bound

$$\begin{aligned} m &\leqslant 3r_2 + \{2\lfloor r_1 - r_2 + \mu\rfloor + 1\} + \{\lfloor r_0 - r_1 + \mu\rfloor + 1\} \\ &= 3r_2 + \{2(r_1 - r_2) + 1\} + \{(r_0 - r_1) + 1\} \\ &= r_0 + r_1 + r_2 + 2 = R. \end{aligned}$$

In all the foregoing cases we obtained

$$m \leqslant R.$$

The remainder of the proof makes use of Lemma 5.2 and hence equation (18) to give for the order p of the scheme

$$p + 1 \leqslant m + (m + 1) \leqslant 2R + 1,$$

which leads to $p \leqslant 2R$. □

Concerning the upwind points of a difference stencil we now prove the following lemma.

LEMMA 9.2 (*Maximum order with stability*) Suppose we have a convex and normalized scheme of type (4) with a fixed Courant number μ satisfying $0 < |\mu| < \frac{1}{2}$. If the scheme is stable, then the order p of the scheme is bounded by

$$p \leqslant 2S.$$

Proof. Instead of repeating the argument of Lemma 9.1 for the Ω-sectors outside Δ, we apply the symmetry argument introduced in Section 7 to prove this result.

Hence, if (31) is proved for a value of μ for which the stencil is regular, we obtain by the mapping $z \to \frac{1}{z}$ and $\mu \to -\mu$ for $-\mu$ that

$$\ell \leqslant R^* = S.$$

By making use of Lemma 5.2 this result leads to

$$p \leqslant 2S.$$

□

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