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The existence of infinitely many bifurcating branches

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Synopsis

We consider the non-linear problem $-\Delta u(x)-f(x, u(x))=\lambda u(x)$ for $x\in\mathbb{R}^N$ and $u\in W^{1,2}(\mathbb{R}^N)$. We show that, under suitable conditions on f, there exist infinitely many branches all bifurcating from the lowest point of the continuous spectrum $\lambda=0$. The method used in the proof is based on a theorem of Ljusternik-Schnirelman type for the free case.

1. Introduction

We consider the following non-linear problem:

$$-\Delta u(x)-f(x, u(x))=\lambda u(x) \quad \text{for } x\in\mathbb{R}^N.$$

This problem has been treated by many authors including Berger, Strauss, Berestycki and Lions. In this paper we follow Stuart [**4**, **5**].

We prove the following theorem.

THEOREM 1.1. *Suppose* $f(x, u(x))=q(x)\,|u(x)|^\sigma u(x)$ *where* $q\in L^p(\mathbb{R}^N)$ *with* $\max\{N/2, 2\}\leqq p\leqq\infty$, $0<\sigma<2(2-N/p)/(N-2)$, *and* $q(x)>0$ *for almost all* $x\in\mathbb{R}^N$.

Suppose further that there exist constants A, $t>0$ *such that* $q(x)\geqq A/(1+|x|)^t$ *for almost all* $x\in\mathbb{R}^N$ *and that* $0<t<2-N\sigma/2$.

Then,

(i) *for* $\lambda<0$, *the equation*

$$-\Delta u(x)-q(x)\,|u(x)|^\sigma u(x)=\lambda u(x) \quad \text{for } x\in\mathbb{R}^N$$

has infinitely many distinct pairs of (generalised) solutions $\{(\lambda, \pm u_k^\lambda)\}_{k\in\mathbb{N}}$;

(ii) *the lowest point of the continuous spectrum is a bifurcation point; in fact all solutions* $(\lambda, \pm u_k^\lambda)$ *bifurcate from* $\lambda=0$:

$$\|u_k^\lambda\|_T\to 0 \quad \text{as} \quad \lambda\to 0^-.$$

We prove this theorem even for more generalised non-linearities $f(x, u(x))$ such as used by Stuart [**4**]. (See Conditions (A1*), (A2*), (A3*) and (A4*) below.)

The existence of an infinite number of solutions for each negative value of λ has been established by Berestycki and Lions [**2**], at least when $f(x, u(x))=g(u(x))$. Stuart has shown that $\lambda=0$ is a bifurcation point and that there exists a branch of solutions bifurcating from $\lambda=0$ [**4**, **5**]. What we show is that in fact there exist infinitely many branches *all* bifurcating from the lowest point of continuous spectrum $\lambda=0$.

The main tool is a generalised result of Ambrosetti and Rabinowitz [**1, 3**] concerning the existence of an infinite number of critical points of a functional. It involves the investigation of the functional on sets of arbitrary genus and we construct such sets using functions of the following type:

$$u(x)=p(|x|^2)e^{-|x|^2} \quad \text{for } x\in\mathbb{R}^N \tag{1.1}$$

where p is a polynomial. This construction seems simpler than that used previously for problems of this kind [**2**].

An alternative approach is discussed in Section 6.

2. The equation $T'Tu-F(u)=\lambda u$

We consider the equation

$$-\Delta u(x)-f(x,u(x))=\lambda u(x), \quad x\in\mathbb{R}^N, \quad N\geqq 2 \tag{2.1}$$

and the corresponding bifurcation problem, but first we give a precise meaning to this equation. (This section follows Stuart [**4**].)

Let us begin with the operator $-\Delta$.

We put

$$H:=L^2(\mathbb{R}^N)=L^2, \quad \|u\|:=\left\{\int u(x)^2\,dx\right\}^{\frac{1}{2}},$$

$$\mathscr{D}(S):=\left\{u\in H\colon \sum_{i=1}^N D_i^2u\in H\right\}, \qquad Su:=-\sum_{i=1}^N D_i^2u,$$

i.e. S is the self-adjoint extension of the negative Laplacian in H. When no domain of integration is indicated, it is understood that the integration is over all of $\mathbb{R}^N$. Let $(H_2,\|.\|_2)$ be the Hilbert space obtained by equipping $\mathscr{D}(S)$ with the graph norm

$$\|u\|_2:=\{\|u\|^2+\|Su\|^2\}^{\frac{1}{2}}, \quad \forall\, u\in\mathscr{D}(S).$$

Then, up to equivalence of norms, $H_2=W^{2,2}(\mathbb{R}^N)$.

We now take $T=S^{\frac{1}{2}}$, the positive self-adjoint square root of S. Let $(H_T,\|.\|_T)$ be the Hilbert space obtained by equipping $\mathscr{D}(T)$ with the graph norm

$$\|u\|_T:=\{\|u\|^2+\|Tu\|^2\}^{\frac{1}{2}}, \quad \forall\, u\in\mathscr{D}(T).$$

Then $H_T=W^{2,1}(\mathbb{R}^N)$ and $\|Tu\|=\||\nabla u|\|$, $\forall\, u\in H_T$ where $\nabla u=(D_1u,\dots,D_Nu)$.

By identifying H with H^*, we can write $H_T\subset H=H^*\subset(H_T)^*$ and use $\langle\,.\,,.\rangle$ for the duality between $(H_T)^*$ and H_T. Since $T\colon H_T\to H$ is bounded, it has a conjugate $T'\colon H^*=H\to(H_T)^*$ which is also bounded. Then $T'T\colon H_T\to(H_T)^*$ is a bounded linear operator such that $T'Tu=Su$, $\forall\, u\in\mathscr{D}(S)$ and $\mathscr{D}(S)=\{u\in H_T\colon T'Tu\in H\}$.

These results are discussed in more detail in [**4**].

We now turn to f in (2.1) and make the following basic assumption.

(A1) *The function f can be written as a sum, $f=\sum_{i=1}^m f_i$, of a finite number of functions f_i where, for $1\leqq i\leqq m$, $f_i\colon\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is of Carathéodory type such* *that*

$$|f_i(x, s)| \leqq A_i(x)\,|s|^{1+\sigma_i}$$

for all $s \in \mathbb{R}$ *and almost all* $x \in \mathbb{R}^N$, *where* $A_i \in L^{p_i}(\mathbb{R}^N)$ *for some* p_i *such that* $\max\{N/2, 2\} \leqq p_i \leqq \infty$ *and* $0 < \sigma_i < 2(2 - N/p_i)/(N-2)$.

(When $p = \infty$, $1/p$ is understood to be 0 and when $N = 2$, $1/(N-2)$ is understood to be $+\infty$.)

This assumption guarantees a well-posed problem (2.1) in the sense that

$$f(x, u(x)) \in (H_T)^* \quad \text{whenever } u \in H_T.$$

In fact, if (A1) is satisfied, we set

$$\mathcal{F}_i(x, s) := \int_0^s f_i(x, r)\,dr \quad \text{and} \quad \mathcal{F} := \sum_{i=1}^m \mathcal{F}_i.$$

For $u : \mathbb{R}^N \to \mathbb{R}$, let

$$F_i(u)(x) := f_i(x, u(x)) \quad \text{and} \quad F := \sum_{i=1}^m F_i,$$

$$\varphi_i(u) := \int \mathcal{F}_i(x, u(x))\,dx \quad \text{and} \quad \varphi := \sum_{i=1}^m \varphi_i.$$

The problem (2.1) is then equivalent to

$$Su - F(u) = \lambda u, \quad u \in H_2. \tag{2.2}$$

The following result is also given by Stuart [**4**].

PROPOSITION 2.1. *Let Condition* (A1) *hold.*

(i) *For* $1 \leqq i \leqq m$, F_i *maps* L^{τ_i} *boundedly and continuously to* L^{q_i} *where*

$$\tau_i := (2 + \sigma_i)/(1 - 1/p_i) \quad \text{and} \quad q_i := \tau_i/(\tau_i - 1),$$

and F *maps* H_T *boundedly and continuously to* $(H_T)^*$. *Further,*

$$|\langle F_i(u), u\rangle| \leqq K_i\,\|Tu\|^{\alpha_i}\,\|u\|^{\beta_i}, \quad \forall\, u \in H_T,$$

where $\alpha_i := N(\sigma_i/2 + 1/p_i)$, $\beta_i := 2 + \sigma_i - \alpha_i$ *and* $K_i > 0$ *and so*

$$|\langle F(u), u\rangle| \leqq \max_{1 \leqq i \leqq m}\{mK_i\,\|Tu\|^{\alpha_i}\,\|u\|^{\beta_i}\}, \quad \forall\, u \in H_T.$$

(ii) *If in addition to Assumption* (A1) *we have*

$$0 < \sigma_i < 2(1 - N/p_i)/(N-2) \quad \textit{for} \quad 1 \leqq i \leqq m,$$

then F_i *maps* L^{τ_i} *boundedly and continuously into* H.

(iii) *For* $1 \leqq i \leqq m$, $\varphi : H_T \to \mathbb{R}$ *is continuously Fréchet differentiable and* $\varphi'(u)v = \langle F(u), v\rangle$ *for all* $u, v \in H_T$.

Remark. We note that $2 < \tau_i < 2N/(N-2)$ and so H_T is continuously embedded in L^{τ_i} for $1 \leqq i \leqq m$ by the Sobolev embedding. It follows that L^{q_i} is continuously embedded in $(H_T)^*$.

A pair (λ, u) is now called a (*generalised*) *solution* of (2.2) if
(i) $(\lambda, u) \in \mathbb{R} \times H_T$
and
(ii) $T'Tu - F(u) = \lambda u$ holds in $(H_T)^*$, i.e.

$$\langle T'Tu, v\rangle - \langle F(u), v\rangle = \lambda\langle u, v\rangle$$

for all $v \in H_T$. λ is called a (L^2-) *bifurcation point* for (2.2) if there exists a sequence $\{(\lambda_n, u_n)\}$ of (generalised) solutions to (2.2) such that

(i) $$u_n \neq 0, \quad \forall\, n \in \mathbb{N},$$

(ii) $$\lambda_n \to \lambda \quad \text{and} \quad \|u_n\|_T \to 0 \quad \text{for } n \to \infty.$$

We want to show that $\lambda = 0$ is a bifurcation point for (2.2) and that there exist infinitely many bifurcating branches; to do this we need φ to be weakly sequentially continuous. This is guaranteed by the following assumption.

(A1*) *f satisfies Condition* (A1) *and $A_i(x) \to 0$ for $|x| \to \infty$ whenever $p_i = \infty$ and $f(x, s)$ is odd with respect to s.*

Then the following result holds.

PROPOSITION 2.2. *Let Condition* (A1*) *hold. Then F is completely continuous and compact; more precisely*

$$u_n \rightharpoonup u \text{ in } H_T \Rightarrow F(u_n) \to F(u) \text{ in } (H_T)^* \quad \textit{for } n \to \infty.$$

COROLLARY 2.3. *Let Condition* (A1*) *hold. Then $\varphi: H_T \to \mathbb{R}$ is weakly sequentially continuous.*

Proof. This follows from the compactness of $\varphi' = F$. See [**6**, Satz 39.22].

3. The functional J_λ

For $\lambda < 0$, we put

$$|||u|||_\lambda = \{\|Tu\|^2 - \lambda\, \|u\|^2\}^{\frac{1}{2}}, \quad \forall\, u \in H_T.$$

Note that $|||\cdot|||_\lambda$ and $\|\cdot\|_T$ are equivalent norms in H_T. We now define a functional J_λ whose critical points are (generalised) solutions to (2.1); critical points in turn will be found via a theorem of Ljusternik–Schnirelman type for the free case.

We put

$$J_\lambda: H_T \to \mathbb{R}, \quad u \mapsto J_\lambda(u) := \tfrac{1}{2}\, |||u|||_\lambda^2 - \varphi(u).$$

In order to control the radial behaviour of J_λ, we make the following assumption on f.

(A2*) *There exist constants $\bar{\sigma} \geqq \underline{\sigma} > 0$ such that for every $s \in \mathbb{R}$*

$$\mathcal{F}(x, ts) \geqq t^{2+\underline{\sigma}}\mathcal{F}(x, s) \geqq 0 \quad \textit{whenever } t \geqq 1$$

and

$$\mathcal{F}(x, ts) \geqq t^{2+\bar{\sigma}}\mathcal{F}(x, s) \geqq 0 \quad \textit{whenever } 0 \leqq t \leqq 1.$$

Further,

$$\varphi(u) > 0 \quad \textit{for all} \quad u \in H_T \backslash \{0\}.$$

Note that Condition (A2*) will be satisfied if the function f is of the form

$$f(x, s) = \sum_{i=1}^{m} q_i(x)\,|s|^{\sigma_i} s \tag{3.1}$$

with $q_i(x) > 0$ for almost all $x \in \mathbb{R}^N$ and $\sigma_i > 0$. (Simply take $\underline{\sigma} := \min\{\sigma_i : 1 \leqq i \leqq m\}$, $\bar{\sigma} := \max\{\sigma_i : 1 \leqq i \leqq m\}$.)

In order to have a quantitative control on the radial behaviour of J_λ, we introduce a second functional on H_T:

$$I_\lambda : H_T \to \mathbb{R},\ u \mapsto I_\lambda(u) = \begin{cases} 0 & (u = 0), \\ \frac{1}{2}|||u|||_\lambda^2 - |||u|||_\lambda^{2+\underline{\sigma}}\psi(u) & (u \in H_T,\ |||u|||_\lambda \geqq 1), \\ \frac{1}{2}|||u|||_\lambda^2 - |||u|||_\lambda^{2+\bar{\sigma}}\psi(u) & (u \in H_T,\ 0 < |||u|||_\lambda \leqq 1), \end{cases}$$

where

$$\psi(u) := \varphi(u/|||u|||_\lambda) \quad \text{for} \quad u \in H_T \backslash \{0\}.$$

Then I_λ is a majorant functional for J_λ whose radial behaviour can be completely controlled. In fact, the following lemma follows immediately from the definition of I_λ and Condition (A2*).

LEMMA 3.1. *Let Condition* (A2*) *be satisfied. Then*

(i) $$J_\lambda(u) = I_\lambda(u) \quad \textit{whenever} \quad |||u|||_\lambda = 1.$$

(ii) $$J_\lambda(u) \leqq I_\lambda(u) \quad \textit{for all} \quad u \in H_T.$$

Let us now have a look at the radial behaviour of I_λ. For any fixed $u \in H_T \backslash \{0\}$, we put

$$a : [0, \infty) \to \mathbb{R}, \qquad t \mapsto a(t) := I_\lambda(tu),$$

i.e.

$$a(t) = \begin{cases} 0 & (t = 0), \\ \frac{1}{2}t^2\,|||u|||_\lambda^2 - t^{2+\bar{\sigma}}\,|||u|||_\lambda^{2+\bar{\sigma}}\,\psi(u) & (0 < t \leqq 1/|||u|||_\lambda), \\ \frac{1}{2}t^2\,|||u|||_\lambda^2 - t^{2+\underline{\sigma}}\,|||u|||_\lambda^{2+\underline{\sigma}}\,\psi(u) & (t \geqq 1/|||u|||_\lambda). \end{cases}$$

Then

$$a'(t) = \begin{cases} 0 & (t = 0), \\ t\,|||u|||_\lambda^2\{1 - (2+\bar{\sigma})t^{\bar{\sigma}}\,|||u|||_\lambda^{\bar{\sigma}}\,\psi(u)\} & (0 < t < 1/|||u|||_\lambda), \\ t\,|||u|||_\lambda^2\{1 - (2+\underline{\sigma})t^{\underline{\sigma}}\,|||u|||_\lambda^{\underline{\sigma}}\,\psi(u)\} & (t > 1/|||u|||_\lambda), \end{cases}$$

and

$$a'_-(1/|||u|||_\lambda) = |||u|||_\lambda\,\{1 - (2+\bar{\sigma})\psi(u)\},$$
$$a'_+(1/|||u|||_\lambda) = |||u|||_\lambda\,\{1 - (2+\underline{\sigma})\psi(u)\},$$

where a'_- and a'_+ are the left and right derivatives. Hence a can be extremal for $t = 1$ only if

$$|||u|||_\lambda = \{(2+\underline{\sigma})\psi(u)\}^{-1/\underline{\sigma}} > 1$$

or

$$|||u|||_\lambda = \{(2+\bar{\sigma})\psi(u)\}^{-1/\bar{\sigma}} < 1.$$

In the case where $\underline{\sigma}=\bar{\sigma}=:\sigma$, both conditions reduce to

$$1=(2+\sigma)\psi(u)\,|||u|||_\lambda^\sigma.$$

We put

$$M_\lambda:=M_\lambda^{(i)}\cup M_\lambda^{(ii)}$$

where

$$M_\lambda^{(i)}:=\{u\in H_T\backslash\{0\}:|||u|||_\lambda=\{(2+\underline{\sigma})\psi(u)\}^{-1/\underline{\sigma}}>1\},$$
$$M^{(ii)}:=\{u\in H_T\backslash\{0\}:|||u|||_\lambda=\{(2+\bar{\sigma})\psi(u)\}^{-1/\bar{\sigma}}<1\}.$$

(In the case where $\underline{\sigma}=\bar{\sigma}=\sigma$, simply set

$$M_\lambda:=\{u\in H_T\backslash\{0\}:1=(2+\sigma)\psi(u)\,|||u|||_\lambda^\sigma.)$$

The two following lemmas give the central properties of these sets.

LEMMA 3.2. *Let $u\in H_T\backslash\{0\}$ be fixed. If $I_\lambda(tu)$ is maximal for $t=1$ then $u\in M_\lambda$.*

LEMMA 3.3.

(i) *If $u\in M_\lambda^{(i)}$, then $I_\lambda(u)=\underline{\sigma}/(4+2\underline{\sigma})\,|||u|||_\lambda^2>\underline{\sigma}/(4+2\underline{\sigma})$.*
(ii) *If $u\in M_\lambda^{(ii)}$, then $I_\lambda(u)=\bar{\sigma}/(4+2\bar{\sigma})\,|||u|||_\lambda^2<\bar{\sigma}/(4+2\bar{\sigma})$.*
(iii) *If $\bar{\sigma}=\underline{\sigma}=\sigma$ and $u\in M_\lambda$, then $I_\lambda(u)=\sigma/(4+2\sigma)\,|||u|||_\lambda^2$.*

As mentioned above, we investigate the critical points of J_λ via a theorem of Ljusternik–Schirelman type given by Ambrosetti and Rabinowitz; (I_1)–(I_5) will therefore refer to conditions on J_λ given by these authors in [**1**]. We now show that J_λ in fact satisfies these conditions if the following assumption is made on f.

(A3*) *There exists $q>2$ such that for all $s\in\mathbb{R}$ and almost all $x\in\mathbb{R}^N$*

$$f(x,s)s\geqq q\mathcal{F}(x,s)\geqq 0.$$

Note that f satisfies this condition if f is of the form given in (3.1); simply set $q:=2+\underline{\sigma}$. Assumption (A3*) means that

$$\langle F(u),u\rangle\geqq q\varphi(u)\geqq 0\quad\text{for all}\quad u\in H_T;$$

Assumptions (A2*) and (A3*) together give that

$$\langle F(u),u\rangle\geqq 2\varphi(u)>0\quad\text{for all}\quad u\in H_T\backslash\{0\}.\tag{3.2}$$

We now suppose that f satisfies Conditions (A1*), (A2*) and (A3*) and show that J_λ satisfies Conditions (I_1)–(I_5) in [**1**].

(I_1) *There exists $\rho,\alpha>0$ such that $J_\lambda>0$ on $B_\rho\backslash\{0\}$ and $J_\lambda\geqq\alpha>0$ on S_ρ where $B_\rho:=\{u\in H_T:|||u|||_\lambda<\rho\}$ and $S_\rho:=\partial B_\rho$.*

Proof. The proof can be found in [**5**]. For completeness, we just recall that, for $u\in H_T$,

$$J_\lambda(u)\geqq\tfrac{1}{2}|||u|||_\lambda^2-\frac{1}{2}\sum_{i=1}^m K_i\,\|Tu\|^{\alpha_i}\,\|u\|^{\beta_i}$$

by (3.2) and Proposition 2.1. Hence,

$$J_\lambda(u) \geqq \tfrac{1}{2}|||u|||_\lambda^2 [1 - \max_{1\leqq i\leqq m} \{mK_i\, |||u|||_\lambda^{\alpha_i+\beta_i-2}\, |\lambda|^{-\beta_i/2}\}]$$

and, since $\alpha_i + \beta_i > 2$, the proof is complete. □

(I_2) *There exists* $e \in H_T\backslash\{0\}$ *such that* $J_\lambda(e) = 0$.

Proof. This follows immediately from (I_1) and (I_5) (see below). In fact there exist infinitely many such elements. □

(I_3) *If* $\{u_n\}$ *is a sequence in* H_T *such that*

$$0 < J_\lambda(u_n) \leqq \sup J_\lambda(u_n) < \infty$$

and

$$\|J'(u_n)\|_{(H_T)^*} \to 0 \quad \textit{for} \quad n \to \infty,$$

then there exists a subsequence $\{u_{n'}\}$ *such that* $u_{n'}$ *converges in* H_T *to some* $\bar{u}$.

Proof. For a proof see [**5**].

Remark. Condition (I_3) is the Palais–Smale condition $(PS)^+$.

(I_4) J_λ *is even*: $J_\lambda(u) = J_\lambda(-u)$ *for all* $u \in H_T$.

Proof. By (A1*), φ is an even functional. Therefore J_λ is also even. □

(I_5) *For any finite dimensional subspace* Z *of* H_T, *the set* $Z \cap \{u \in H_T\colon J_\lambda(u) \geqq 0\}$ *is bounded.*

Proof. For a proof see [**5**].

4. The existence of infinitely many solutions

According to [**1**], we set

$$\Gamma := \{g \in C([0,1], H_T)\colon g(0) = 0 \quad \text{and} \quad g(e) = 1\}$$

where e is the element whose existence is given by (I_2);

$$\Gamma_* := \{h \in C(H_T, H_T)\colon h(0) = 0,$$
$$h \text{ is a homeomorphism from } H_T \text{ to } H_T \text{ and } h(B) \subset \hat{A}_0\}$$

where

$$B := \{u \in H_T\colon |||u|||_\lambda < 1\} \quad \text{and} \quad \hat{A}_0 := \{u \in H_T\colon J_\lambda(u) \geqq 0\};$$
$$\Gamma^* := \{h \in \Gamma_*\colon h \text{ is odd}\};$$
$$\Gamma_k := \{K \subset H_T\colon K \text{ is compact in } H_T,\ K \text{ is a symmetric set i.e. } -K = K,$$
$$\gamma(K \cap h(\partial B)) \geqq k, \quad \forall\, h \in \Gamma^* \quad (k \in \mathbb{N})\}$$

where γ is the genus of a set

If Conditions (A1*), (A2*) and (A3*) are satisfied, equation (2.1) has for each $\lambda < 0$ infinitely many (generalised) solutions corresponding to the critical values b^λ

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and $b_k^\lambda (k \in \mathbb{N})$, where

(i) $$b^\lambda := \inf_{g\in\Gamma} \max_{u\in g([0,1])} J_\lambda(u),$$

(ii) $$b_k^\lambda := \inf_{K\in\Gamma_k} \max_{u\in K} J_\lambda(u).$$

5. Behaviour of the solutions as $\lambda \to 0^-$

We discuss the behaviour of the solutions u_k^λ to equation (2.1) which correspond to the critical values b_k^λ when $\lambda \to 0$. Accordingly, we make the following assumption on f:

(A4*) *There exist constants A, δ, $t > 0$ such that*

$$\mathscr{F}(x, s) \geqq A(1+|x|)^{-t}\,|s|^{2+\bar{\sigma}}$$

for all $|s| < \delta$ and for almost all $x \in \mathbb{R}^N$ where δ and t satisfy the inequalities

$$0 < \delta < 1 \quad \text{and} \quad 0 < t < 2 + \underline{\sigma} - \bar{\sigma} - N\bar{\sigma}/2,$$

where $\bar{\sigma}$ and $\underline{\sigma}$ are the constants of Condition (A2*).

Note that Condition (A4*) is satisfied by a function f of the form (3.1) if

$$q_i(x) \geqq A(1+|x|)^{-t} \quad \text{for almost all } x \in \mathbb{R}^N \quad (i = 1 \ldots m).$$

What we want to show is that under Assumptions (A1*)–(A4*) bifurcation will occur at the point $\lambda = 0$.

Let $p(t) := \sum_{i=0}^{k-1} a_i t^i$ be a polynomial of degree $\leqq k-1$. We often identify $p(t)$ and $p := (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{R}^k$. The space $\mathbb{R}^k$ will be considered to be equipped with the norm

$$\|p\|_k := \max\{|a_i| : i = 0, \ldots, k-1\}, \quad \forall\, p = (a_0, \ldots, a_{k-1}) \in \mathbb{R}^k;$$

this norm is equivalent to the usual one

$$|||p||| := \left\{\sum_{i=0}^{k-1} a_i^2\right\}^{\frac{1}{2}}, \qquad \forall\, p = (a_0, \ldots, a_{k-1}) \in \mathbb{R}^k.$$

For $p \in \mathbb{R}^k$, we set

$$d_p := \left\{\int p^2(|y|^2) e^{-2|y|^2}\, dy\right\}^{\frac{1}{2}},$$

$$L_p := 4 \int |y|^2 [p'(|y|^2) - p(|y|^2)]^2 e^{-2|y|^2}\, dy,$$

$$K_p := 2^{-t} A \int_{|y|\geqq 1} |y|^{-t}\, |p(|y|^2)|^{2+\bar{\sigma}} e^{-(2+\bar{\sigma})|y|^2}\, dy.$$

We first give some properties of d_p, L_p and K_p.

Lemma 5.1. *d_p depends continuously on p, for all $p \in \mathbb{R}^k$.*

Proof. Let

$$p := (a_0, \dots, a_{k-1}) \in \mathbb{R}^k,$$
$$\varepsilon := (\varepsilon_0, \dots, \varepsilon_{k-1}) \in \mathbb{R}^k,$$
$$p+\varepsilon := (a_0+\varepsilon_0, \dots, a_{k-1}+\varepsilon_{k-1}) \in \mathbb{R}^k.$$

Then

$$\begin{aligned} d^2_{p+\varepsilon} - d^2_p &= 2\int \varepsilon(|y|^2)p(|y|^2)e^{-2|y|^2}\,dy + \int \varepsilon^2(|y|^2)e^{-2|y|^2}\,dy \\ &= \sum_{i=0}^{2k-2}\sum_{j=0}^{i}\left\{2\int \varepsilon_j a_{i-j}\,|y|^{2i}e^{-2|y|^2}\,dy \right. \\ &\quad \left. + \int \varepsilon_j\varepsilon_{i-j}\,|y|^{2i}e^{-2|y|^2}\,dy\right\} \\ &= \sum_{i=0}^{2k-2}\sum_{j=0}^{i}\varepsilon_j(2a_{i-j}+\varepsilon_{i-j})\int |y|^{2i}e^{-2|y|^2}\,dy, \end{aligned}$$

where $a_s = \varepsilon_s = 0$ for $s \geqq k$. Therefore,

$$d^2_{p+\varepsilon} - d^2_p \to 0 \quad \text{as} \quad \|\varepsilon\|_k \to 0.$$

This proves the continuity of d_p. □

LEMMA 5.2. *There exists a constant $\mathscr{L} > 0$ such that*

$$0 \leqq L_p \leqq \mathscr{L}\,\|p\|_k^2, \quad \forall\, p \in \mathbb{R}^k.$$

Proof. Let $p = (a_0, \dots, a_{k-1}) \in \mathbb{R}^k$. Then

$$\begin{aligned} 0 \leqq L_p &= 4\int |y|^2[p'(|y|^2) - p(|y|^2)]^2 e^{-2|y|^2}\,dy \\ &= 4\int |y|^2\left\{\sum_{i=0}^{k-1}[(i+1)a_{i+1} - a_i]\,|y|^{2i}\right\}^2 e^{-2|y|^2}\,dy, \end{aligned}$$

where a_k is taken to be zero. Since

$$\begin{aligned} |(i+1)a_{i+1} - a_i| &\leqq (k+1)\max\{|a_i| : i = 0, \dots, k-1\} \\ &= (k+1)\,\|p\|_k, \end{aligned}$$

we have

$$0 \leqq L_p \leqq 4(k+1)^2 \int |y|^2\left\{\sum_{i=0}^{k-1}|y|^{2i}\right\}^2 e^{-2|y|^2}\,dy \,.\, \|p\|_k^2.$$

If we put

$$\mathscr{L} := 4(k+1)^2\int |y|^2\left\{\sum_{i=0}^{k-1}|y|^{2i}\right\}^2 e^{-2|y|^2}\,dy,$$

the lemma is proved. □

LEMMA 5.3. (*i*) *K_p depends continuously on p, for all $p \in \mathbb{R}^k$.* (*ii*) *$K_p = 0$ if and only if $p = 0$.*

Proof. (i) Let $p := (a_0, \dots, a_{k-1}) \in \mathbb{R}^k$,

$$\varepsilon := (\varepsilon_0, \dots, \varepsilon_{k-1}) \in \mathbb{R}^k \quad \text{with} \quad \|\varepsilon\|_k < 1.$$

Then

$$K_{p+\varepsilon} = 2^{-t} A \int_{|y| \geq 1} |y|^{-t} \, |(p+\varepsilon)(|y|^2)|^{2+\bar{\sigma}} e^{-(2+\bar{\sigma})|y|^2} \, dy.$$

If one takes as dominating function

$$y \mapsto |y|^{-t} \, |\tilde{p}(|y|^2)|^{2+\bar{\sigma}} e^{-(2+\bar{\sigma})|y|^2}, \qquad y \in \mathbb{R}^N,$$

where

$$\tilde{p}(t) := \sum_{i=0}^{k-1} (|a_i| + 1) t^i,$$

the Lebesgue dominated convergence theorem gives

$$K_{p+\varepsilon} \to K_p \quad \text{as} \quad \|\varepsilon\|_k \to 0.$$

(ii) This follows from the definition of K_p. □

For $p \in \mathbb{R}^k$ and $\lambda < 0$, we put

$$u_{p,\lambda}(x) := p(-\lambda \, |x|^2) e^{\lambda |x|^2}, \qquad x \in \mathbb{R}^N,$$

and we consider the following subspace

$$Z(k, \lambda) := \{u_{p,\lambda}(x) : p \in \mathbb{R}^k\} \subset H_T.$$

By Condition (I_5), the set

$$Z(k, \lambda)_+ := Z(k, \lambda) \cap \{u \in H_T : J_\lambda(u) \geqq 0\}$$

is bounded in H_T and hence in $Z(k, \lambda)$. The following proposition shows that this boundedness is uniform if $|\lambda|$ is small enough.

PROPOSITION 5.4. *Let Conditions* (A1*)–(A4*) *hold and put*

$$\tilde{Z}(k, \lambda) := \{u_{p,\lambda}(x) \in Z(k, \lambda) : \|p\|_k \leqq 1\}.$$

Then there exists a constant $\lambda_0 \in [-1, 0)$ *such that*

$$Z(k, \lambda)_+ \subset \tilde{Z}(k, \lambda) \quad \textit{whenever} \quad \lambda \in (\lambda_0, 0).$$

Proof. Let $Z := \{u_{p,\lambda}(x) \in Z(k, \lambda) : \|p\|_k = 1\}$. For the majorant functional I_λ we show that

$$I_\lambda|_Z(u) < 0 \quad \text{whenever} \quad \lambda \in (\lambda_0, 0).$$

The conclusion of the proof follows from the radial behaviour of I_λ and the connection between J_λ and I_λ.

We first remark that

$$\|u_{p,\lambda}\|^2 = |\lambda|^{-N/2} d_p^2,$$

$$\|Tu_{p,\lambda}\|^2 = |\lambda|^{1-N/2} L_p,$$

and

$$|||u_{p,\lambda}|||_\lambda^2 = |\lambda|^{1-N/2}(L_p + d_p^2) \quad \text{for all} \quad u_{p,\lambda}(x) \in Z(k, \lambda).$$

These equalities can easily be verified by a direct computation. We now put

$$y(p):=2\max_{x\in\mathbb{R}^N}|u_{p,\lambda}(x)|,\quad \forall\, p\in\mathbb{R}^k$$

and

$$y_0:=\min\{y(p)\colon p\in\mathbb{R}^k \quad\text{with}\quad \|p\|_k=1\}.$$

Note that $y(p)$ is always finite and depends continuously on p. Since $y(p)>0$ whenever $p\neq 0$, we thus have $y_0>0$. If δ is the constant in Condition (A4*), we have, by (A2*),

$$\begin{aligned}\varphi(u_{p,\lambda})&=\varphi\left(\frac{y(p)}{\delta}\cdot\frac{\delta}{y(p)}u_{p,\lambda}\right)\\&\geqq[y(p)/\delta]^{2+\sigma}\varphi\left(\frac{\delta}{y(p)}u_{p,\lambda}\right)\end{aligned}$$

where

$$\sigma=\begin{cases}\bar\sigma & \text{if}\quad y(p)/\delta<1\\ \underline{\sigma} & \text{otherwise.}\end{cases}$$

Therefore, by (A4*),

$$\begin{aligned}\varphi(u_{p,\lambda})&\geqq[y(p)/\delta]^{(2+\sigma)/2+\bar\sigma)}2^{-t}A\int_{|x|\geqq 1}|x|^{-t}\,|p(\lambda|x|^2)|^{2+\bar\sigma}e^{\lambda(2+\bar\sigma)|x|^2}\,dx\\&\geqq[y(p)/\delta]^{(2+\sigma)/(2+\bar\sigma)}2^{-t}A\,|\lambda|^{(t-N)/2}\int_{|y|\geqq 1}|y|^{-t}\,|p(-|y|^2)^{2+\bar\sigma}e^{-(2+\bar\sigma)}\,|y|^2\,dy\end{aligned}$$

if $|\lambda|\leqq 1$. Hence, for $|\lambda|\leqq 1$,

$$\varphi(u_{p,\lambda})\geqq C\cdot K_p\cdot|\lambda|^{(t-N)/2}$$

where

$$C:=\min\{[y_0/\delta]^{(2+\sigma)/(2+\bar\sigma)},\,y_0/\delta\}>0.$$

If we set

$$\begin{aligned}D&:=\max\{L_p+d_p^2\colon p\in\mathbb{R}^k \quad\text{with}\quad \|p\|_k=1\},\\K&:=\min\{K_p\colon p\in\mathbb{R}^k\quad\text{with}\quad\|p\|_k=1\},\end{aligned}$$

we have, by Lemmas 5.1, 5.2 and 5.3, that D and K are finite and positive. Hence, for $\lambda\in[-1,0)$,

$$\begin{aligned}I_\lambda(u_{p,\lambda})&\leqq\tfrac12|\lambda|^{1-N/2}(L_p+d_p^2)-CK_p\,|\lambda|^{(t-N)/2}\\&\leqq|\lambda|^{(t-N)/2}(\tfrac12|\lambda|^{1-t/2}D-C\cdot K).\end{aligned}$$

Therefore, there exists some $\lambda_0\in[-1,0)$ such that

$$I_\lambda(u_{p,\lambda})<0\quad\text{for all}\quad u_{p,\lambda}(x)\in Z$$

where $\lambda\in(\lambda_0,0)$. □

Remark. By the proof of Lemma 2.7 in [**1**],

$$\tilde Z(k,\lambda)\in\Gamma_k\ (k\in\mathbb{N})\quad\text{for}\quad \lambda\in(\lambda_0,0).$$

PROPOSITION 5.5. *Let Conditions* (A1*)–(A4*) *hold and let* λ_0 *be given by Proposition* 5.4. *Then there exists a constant* $\lambda_1 \in [\lambda_0, 0)$ *such that*

$$\psi(u_{p,\lambda}) \geqq 2 \quad \text{for all} \quad u_{p,\lambda} \in \tilde{Z}(k,\lambda)\backslash\{0\}$$

where $\lambda \in (\lambda_1, 0)$.

Proof. For $u_{p,\lambda} \in \tilde{Z}(k,\lambda)\backslash\{0\}$, we have

$$\psi(u_{p,\lambda}) = \varphi(u_{p,\lambda}/|||u_{p,\lambda}|||_\lambda) \geqq |||u_{p,\lambda}|||_\lambda^{-(2+\sigma)}\varphi(u_{p,\lambda})$$

where

$$\sigma = \begin{cases} \bar{\sigma} & \text{if} \quad |||u_{p,\lambda}|||_\lambda \geqq 1, \\ \underline{\sigma} & \text{otherwise} \end{cases}$$

Hence,

$$\psi(u_{p,\lambda}) \geqq \{|\lambda|^{1-N/2}(L_p + d_p^2)\}^{-1-\sigma/2} \cdot C_p \cdot K_p \cdot |\lambda|^{(t-N)/2}$$

where

$$C_p := \min\{[y(p)/\delta]^{(2+\sigma)/(2+\bar{\sigma})}, \qquad y(p)/\delta\}.$$

But since

$$\psi(tu_{p,\lambda}) = \psi(u_{p,\lambda}) \quad \text{for all} \quad t > 0,$$

we get

$$\psi(u_{p,\lambda}) \geqq D^{-1-\sigma/2}CK\,|\lambda|^\kappa$$

where C, D and K are the same as in the proof of Proposition 5.4 and where κ is given by

$$\kappa = (N/2-1)(1+\sigma/2) + (t-N)/2 < -1 + N\bar{\sigma}/4 - \underline{\sigma}/2 + t/2 < -\bar{\sigma}/2 < 0.$$

So

$$\psi(u_{p,\lambda}) \to \infty \quad \text{for} \quad \lambda \to 0^- \quad \text{uniformly on} \quad \tilde{Z}(k,\lambda)\backslash\{0\}. \quad \square$$

PROPOSITION 5.6. *Let Conditions* (A1*)–(A4*) *hold. Let* u_k^λ *be the* (*generalised*) *solution to* (2.1) *corresponding to the critical value* $b_k^\lambda(\lambda < 0,\ k \in \mathbb{N})$. *Then*

$$b_k^\lambda \to 0 \quad \text{as} \quad \lambda \to 0^-.$$

More precisely,

$$b_k^\lambda = o(\lambda) \quad \text{for} \quad \lambda \to 0^-.$$

Proof. Suppose that $\underline{\sigma} < \bar{\sigma}$. For $\underline{\sigma} = \bar{\sigma}$, the proof remains the same except that $M_\lambda^{(ii)}$ is replaced by M_λ. Suppose $\lambda \in (\lambda_1, 0)$. Then

$$\begin{aligned} 0 < b_k^\lambda &= \inf_{K \in \Gamma_k} \max_{u \in K} J_\lambda(u) \\ &\leqq \max_{u \in \tilde{Z}(k,\lambda)} I_\lambda(u) \\ &\leqq \max_{u \in \tilde{Z}(k,\lambda) \cap M_\lambda^{(ii)}} I_\lambda(u) \end{aligned}$$

since $\tilde{Z}(k,\lambda) \cap M_\lambda^{(i)} = \varnothing$ by Proposition 5.5. If we put

$$t(u) := [(2+\bar{\sigma})\psi(u)]^{-1/\bar{\sigma}}(L_p + d_p^2)^{-\frac{1}{2}}|\lambda|^{-\frac{1}{2}+N/4}$$

and

$$Z_\lambda := \{u_{p,\lambda} \in Z(k,\lambda) : \|p\|_k = 1\},$$

then

$$t(u_{p,\lambda})u_{p,\lambda} \in M_\lambda^{(ii)} \quad \text{whenever} \quad u_{p,\lambda} \in Z_\lambda$$

and so

$$\begin{aligned} 0 < b_k^\lambda &\leqq \max_{u \in Z_\lambda} I_\lambda(t(u)u) \\ &= \bar{\sigma}/(4+2\bar{\sigma}) \max_{u \in Z_\lambda} |||t(u)u|||_\lambda^2 \\ &= \bar{\sigma}/(4+2\bar{\sigma}) \max_{u \in Z_\lambda} [(2+\bar{\sigma})\psi(u)]^{-2/\bar{\sigma}} \\ &\leqq \text{const}\, |\lambda|^{-2\kappa/\bar{\sigma}} \end{aligned}$$

by the proof of Proposition 5.5. Since $-2\kappa/\bar{\sigma} > 1$, we have

$$b_k^\lambda = o(\lambda) \quad \text{for} \quad \lambda \to 0^-. \quad \square$$

We are now ready to prove the main theorem.

THEOREM 5.7. *Let Conditions* (A1*)–(A4*) *hold. Then* (i) *for* $\lambda < 0$, *equation* (2.1) *has infinitely many distinct pairs of (generalised) solutions* $\{(\lambda, \pm u_k^\lambda)\}_{k \in \mathbb{N}}$; (ii) $\lambda = 0$ *is a bifurcation point for equation* (2.1), *i.e.*

$$\|u_k^\lambda\|_T \to 0 \quad \textit{as} \quad \lambda \to 0^-.$$

Proof. (i) This is a result of Section 4.

(ii) Let u_k^λ be a critical point of J_λ:

$$J_\lambda(u_k^\lambda) = b_k^\lambda, \qquad J'_\lambda(u_k^\lambda) = 0.$$

Therefore,

$$\frac{d}{dt} J_\lambda(tu_k^\lambda)|_{t=1} = 0, \quad \text{i.e.} \quad |||u_k^\lambda|||_\lambda^2 = \langle F(u_k^\lambda), u_k^\lambda \rangle.$$

But

$$\begin{aligned} b_k^\lambda &= \tfrac{1}{2} |||u_k^\lambda|||_\lambda^2 - \varphi(u_k^\lambda) \\ &= \tfrac{1}{2}\langle F(u_k^\lambda), u_k^\lambda\rangle - \varphi(u_k^\lambda) = o(\lambda) \quad \text{for} \quad \lambda \to 0^- \end{aligned}$$

and thus by (A3*)

$$b_k^\lambda \geqq (\tfrac{1}{2}q - 1)\varphi(u_k^\lambda) \geqq 0 \to \varphi(u_k^\lambda) = o(\lambda) \quad \text{for} \quad \lambda \to 0^-.$$

Therefore,

$$\langle F(u_k^\lambda), u_k^\lambda \rangle = o(\lambda) \quad \text{for} \quad \lambda \to 0^-,$$

i.e.

$$|||u_k^\lambda|||_\lambda^2 = o(\lambda) \quad \text{for} \quad \lambda \to 0^-.$$

Since $|||u|||_\lambda \geqq \|Tu\|$ and $|||u|||_\lambda \geqq \sqrt{|\lambda|}\, \|u\|$ for all $u \in H_T$, we have

$$\begin{aligned} \|Tu_k\|^2 &= o(\lambda), \\ \|u_k\|^2 &= o(1), \\ \|u_k\|_T^2 &= o(1) \quad \text{for} \quad \lambda \to 0^-. \quad \square \end{aligned}$$

6. An alternative approach

There is another approach to the problem (2.1) given by Stuart [**4**]. We put

$$J(u):=\tfrac{1}{2}\|Tu\|^2-\varphi(u) \quad \text{for} \quad u\in H_T$$

and

$$M_r:=\{u\in H_T:\|u\|=r\} \quad \text{for} \quad r>0.$$

Then critical points of $J|_{M_r}$ are (generalised) solutions to (2.1) and it is sufficient to verify the hypotheses of Theorem 4 of [**5**]. The main point is to show that Assumption (S4) is satisfied for all $j\in\mathbb{N}$ and this can be done (under the assumptions on f given below) using functions of type (1.1) and calculations similar to those of Section 5. The details will appear in [**7**].

Let us make the following (weaker) assumptions on f:

(A2) *$\mathcal{F}(x,ts)\geqq t^2\mathcal{F}(x,s)\geqq 0$ for all $s\in\mathbb{R}$ all $t\geqq 1$ and almost all $x\in\mathbb{R}^N$.*

(A3) *$f(x,s)s\geqq 2\mathcal{F}(x,s)\geqq 0$ for all $s\in\mathbb{R}$ and almost all $x\in\mathbb{R}^N$.*

(A4) *There exist positive constants σ, δ, A and t such that $0<\sigma<2(2-t)/N$ and $\mathcal{F}(x,s)\geqq A(1+|x|)^{-t}\,|s|^{2+\sigma}$ for all $0\leqq s\leqq\delta$ and almost all $x\in\mathbb{R}^N$.*

THEOREM 6.1. *Let Conditions* (A1*), (A2)–(A4) *hold. Then, for $r>0$ small enough, there exist infinitely many distinct pairs of (generalised) solutions $(\lambda_n^r, \pm u_n^r)\in\mathbb{R}\times H_T$ for equation* (2.1) *such that*

$$\|u_n^r\|=r, \qquad \lambda_n^r<0 \qquad \text{for all } n\in\mathbb{N}.$$
$$\|u_n^r\|_T\to 0, \qquad \lambda_n^r\to 0^- \qquad \text{as } r\to 0^+.$$

Let us remark that the question as to whether Condition (A1*) can be weakened remains open.

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