

# DENSITIES WITH GAUSSIAN TAILS

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## ABSTRACT

Consider densities  $f_i(t)$ , for  $i = 1, \dots, d$ , on the real line which have thin tails in the sense that, for each  $i$ ,

$$f_i(t) \sim \gamma_i(t)e^{-\psi_i(t)},$$

where  $\gamma_i$  behaves roughly like a constant and  $\psi_i$  is convex,  $C^2$ , with  $\psi' \rightarrow \infty$  and  $\psi'' > 0$  and  $1/\sqrt{\psi''}$  is self-neglecting. (The latter is an asymptotic variation condition.) Then the convolution is of the same form

$$f_1 * \dots * f_d(t) \sim \gamma(t)e^{-\psi(t)}.$$

Formulae for  $\gamma$ ,  $\psi$  are given in terms of the factor densities and involve the conjugate transform and infimal convolution of convexity theory. The derivations require embedding densities in exponential families and showing that the assumed form of the densities implies asymptotic normality of the exponential families.

## 0. Introduction

The purpose of this paper is to explain the statement: ‘The class of densities with Gaussian tails is closed under convolution’. For the moment we think of a density with a Gaussian tail as a density with a thin tail in the sense that

$$f(t) \sim \gamma(t)e^{-\psi(t)} \quad \text{as } t \rightarrow \infty,$$

where  $\psi$  is a convex  $C^2$ -function with  $\psi''$  strictly positive and  $\psi' \rightarrow \infty$ , and where  $\gamma$  behaves more or less like a constant. The relation  $\sim$  denotes asymptotic equality: the quotient of the two sides of the equation tends to 1 as  $t \rightarrow \infty$ . We shall shortly impose certain regularity conditions on the functions  $\psi$  and  $\gamma$  which will ensure that any finite convolution of such functions again has this form. Moreover, we shall see that the functions  $\psi$  and  $\gamma$  for the convolution can be expressed in terms of the functions  $\psi_i$  and  $\gamma_i$  of the factors.

Let us first say a few words about the terminology ‘Gaussian tail’. Any distribution  $F$  with a moment-generating function generates an exponential family of distributions

$$dF_\lambda(x) = e^{\lambda x} dF(x)/C(\lambda)$$

where the norming factor  $C(\lambda) = \int e^{\lambda x} dF(x)$  is nothing but the moment-generating function of  $F$ . If  $F$  has a density  $f$  with a thin tail as above, then the moment-generating function  $C(\lambda)$  is finite for all  $\lambda \geq 0$  and hence the density  $f_\lambda(x) = e^{\lambda x} f(x)/C(\lambda)$  is well defined for all  $\lambda \geq 0$ . The conditions which we impose on  $\gamma$  and  $\psi$  imply that  $f_\lambda$  is asymptotically normal: there exist norming

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constants  $a_\lambda > 0$  and  $b_\lambda \in R$  such that the normalized densities  $g_\lambda$  satisfy

$$g_\lambda(x) = a_\lambda f_\lambda(b_\lambda + a_\lambda x) \rightarrow \frac{1}{\sqrt{(2\pi)}} e^{-x^2/2}, \quad \lambda \rightarrow \infty,$$

uniformly on  $R$ .

There is a second related reason for the name Gaussian tails. Originally we were interested in the distribution of extremes of linear combinations of independent identically distributed random variables. For distributions with thick tails there exists an elegant theory of subexponential distributions, which links sums and maxima. For thin tails there exist isolated results (see, for example, the excellent papers [4, 10]) but no general theory. By using Laplace's principle for estimating the density of the convolution  $f^{*2}(t) = \int f(t - x)f(x) dx$  for a density of the form above, we obtain

$$\begin{aligned} f^{*2}(t) &\sim \gamma^2(\frac{1}{2}t) \int e^{-(\psi(t/2+x)+\psi(t/2-x))} dx \\ &\sim \gamma^2(\frac{1}{2}t)e^{-2\psi(t/2)} \int e^{-x^2\psi''(t/2)} dx \\ &= \gamma^2(\frac{1}{2}t)(\sqrt{(\pi/\psi''(\frac{1}{2}t))})e^{-2\psi(t/2)}, \end{aligned}$$

at least if  $\sigma^2 := 1/\psi''$  (and  $\gamma$ ) behave like constants in an appropriate sense. Some reflection shows that a suitable condition on the functions  $\sigma$  and  $\gamma$  is

$$(0.1) \quad \begin{aligned} \sigma(t + x\sigma(t))/\sigma(t) &\rightarrow 1 \quad \text{as } t \rightarrow \infty, \\ \gamma(t + x\sigma(t))/\gamma(t) &\rightarrow 1 \quad \text{as } t \rightarrow \infty, \end{aligned}$$

both uniformly on bounded  $x$ -intervals. (A function  $\sigma$  which satisfies relation (0.1) above is said to be self-neglecting.) This argument remains valid if the random variables  $X$  and  $Y$  do not have the same distribution (and even if they are dependent) as long as the bivariate density  $h(x, y)$  has the form  $e^{-\psi(x,y)}$  for a 'smooth' convex function  $\psi$ . In this paper we consider sums of a finite number of independent random variables whose densities satisfy the asymptotic relation above where the functions  $\gamma_i$  and  $\psi_i$  may depend on the index  $i$ . The restriction to independent variables allows us to obtain precise results. An interesting application of our results to saddlepoint approximations is contained in [1].

Feigin and Yashchin [5] relate the behaviour of the tail of a density function to the behaviour of the Laplace transform in the case where the Laplace transform is asymptotically Gaussian. Our results are analogous. Since we work with densities, the proofs are straightforward and do not make use of transform theory. Indeed an exact formulation of our result in terms of Laplace transforms might be quite difficult.

### 1. Closure under convolution

We introduce a class of probability densities with thin tails which is closed under convolution. Theorem 1.1 is the main result of this paper. It provides a flexible method for obtaining the tail behaviour of a finite convolution based on knowledge of the tail behaviour of the convolution factors.

THEOREM 1.1. Let  $X_1, \dots, X_d$  be independent random variables with densities  $f_1, \dots, f_d$ . We assume that the densities  $f_i$  are strictly positive on a left neighbourhood of their upper endpoints  $t_{i\infty}$  and satisfy the asymptotic equality

$$(1.1) \quad f_i(t) \sim \gamma_i(t)e^{-\psi_i(t)}, \quad \text{as } t \uparrow t_{i\infty},$$

where the functions  $\psi_i$  satisfy

$$(1.2) \quad \psi_i \text{ is } C^2 \text{ and } \psi_i'' \text{ is positive}$$

(so that  $\psi_i$  is convex), and

$$(1.3) \quad \sigma_i := 1/\sqrt{\psi_i''} \text{ is self-neglecting.}$$

The functions  $\gamma_i$  satisfy

$$(1.4) \quad \frac{\gamma_i(t + x\sigma_i(t))}{\gamma_i(t)} \rightarrow 1 \quad \text{as } t \uparrow t_{i\infty}$$

uniformly on bounded  $x$ -intervals. Furthermore, we assume that

$$(1.5) \quad \sup_t \psi_i'(t) =: \tau_{i\infty} \leq \infty \text{ is independent of } i.$$

Then the density  $f_0 = f_1 * \dots * f_d$  of  $X_0 := X_1 + \dots + X_d$  has the same form

$$f_0(t) \sim \gamma_0(t)e^{-\psi_0(t)}, \quad \text{as } t \uparrow t_{0\infty} = t_{1\infty} + \dots + t_{d\infty}.$$

Explicit formulae for  $\gamma_0$  and  $\psi_0$  can be given in terms of the inverse functions  $q_i = (\psi_i')^{-1}$  to the strictly increasing derivatives  $\psi_i'$  as follows. Write  $t = q_1 + \dots + q_d$ . Then  $t = t(\tau)$  is a continuous strictly increasing function of  $\tau$  and  $t(\tau) \uparrow t_{0\infty}$  as  $\tau \uparrow \tau_{0\infty}$ . Now one may choose

$$(1.6) \quad \psi_0(t) = \psi_1(q_1) + \dots + \psi_d(q_d),$$

$$(1.7) \quad \sigma_0^2(t) = \sigma_1^2(q_1) + \dots + \sigma_d^2(q_d),$$

$$(1.8) \quad (\sqrt{2\pi})\sigma_0(t)\gamma_0(t) = \prod_{1 \leq i \leq d} (\sqrt{2\pi})\sigma_i(q_i)\gamma_i(q_i).$$

Then  $\sigma_0^2 = 1/\psi_0''$  and  $\sup \psi_0'(t) = \tau_{0\infty}$ .

Note that we have replaced the intuitive condition that the derivatives  $\psi_i'$  are unbounded by the formal condition that  $\psi_i'(t_{i\infty}) =: \tau_{i\infty} \leq \infty$  is independent of the index  $i$ . If  $\tau_{0\infty}$  is finite then all the upper endpoints  $t_{i\infty}$  are infinite (Proposition 5.5).

We give a sketch of the proof. Determine the minimum of the function

$$\psi(x_1, \dots, x_d) = \psi_1(x_1) + \dots + \psi_d(x_d)$$

on the hyperplane  $x_1 + \dots + x_d = t$  by Lagrange's method and evaluate the convolution integral by Laplace's method. The convexity of  $\psi$  supplies the necessary bounds on the tails of the integrand. The formal proof is presented in § 7. In the intervening §§ 3–5 we develop the theory of asymptotically parabolic functions. These are the functions  $\psi$  which satisfy the conditions (1.2) and (1.3) above. This class of functions is of sufficient interest to warrant some attention.

Section 4 uses the theory of conjugate convex functions to give two alternative descriptions of the function  $\psi_0$  in terms of the functions  $\psi_i$ , viz.

$$\psi_0^* = \psi_1^* + \dots + \psi_d^*,$$

$$\psi_0 = \psi_1 \square \dots \square \psi_d,$$

where  $\psi_i^*$  is the conjugate function to  $\psi_i$  for  $i = 0, \dots, d$  and where the infimal convolution  $\varphi_1 \square \varphi_2$  of two convex functions  $\varphi_1$  and  $\varphi_2$  is the (convex) function  $\varphi$  defined by  $\varphi(t) = \inf\{\varphi_1(x) + \varphi_2(y) : x + y = t\}$ .

Section 6 shows that for bounded densities  $f \sim \gamma e^{-\psi}$  with  $\psi$  and  $\gamma$  as above, the associated exponential family of densities  $f_\tau$  is asymptotically Gaussian for  $\tau \rightarrow \tau_\infty$ . The convexity of the exponent will ensure a very strong form of convergence of the normalized densities; see (6.1). This strong form of asymptotic normality not only gives Theorem 1.1, it also establishes asymptotic normality of the vector  $X = (X_1, \dots, X_d)$  conditioned on the sum  $X_0 = t$  for  $t \rightarrow t_\infty$ . These conditional distributions are of interest in large deviation theory.

Section 2 treats the following question: when is the tail of a convolution determined (up to asymptotic equality) by the tails of the factors?

We now consider some remarks and examples which are intended to clarify the construction of the functions  $\psi_0$  and  $\gamma_0$  in the asymptotic expression for the density of the sum. We begin by checking the two relations  $\sigma_0^2 = 1/\psi_0''$  and  $\sup \psi_0'(t) = \tau_\infty$  given after relation (1.8).

PROPOSITION 1.2. *We have that*

$$\sigma_0^2 = 1/\psi_0'' \quad \text{and} \quad \sup \psi_0'(t) = \tau_\infty.$$

*Proof.* It is convenient to take  $\tau$  as the independent variable. Note that  $\tau = \psi_i'(q_i)$ . Hence  $q_i'(\tau) = 1/\psi_i''(q_i) = \sigma_i^2(q_i)$ . Differentiation of the equality (1.6) with respect to  $\tau$  gives

$$\psi_0'(t)q_0'(\tau) = \psi_1'(q_1)q_1'(\tau) + \dots + \psi_d'(q_d)q_d'(\tau) = \tau q_0'(\tau),$$

where we write  $t = q_0(\tau)$ . Hence  $\psi_0'(t) = \tau$ . This proves the second relation. Differentiation of this last expression with respect to  $\tau$  gives  $\psi_0''(t)q_0'(\tau) = 1$  and hence

$$1/\psi_0''(t) = q_0'(\tau) = q_1'(\tau) + \dots + q_d'(\tau) = \sigma_1^2 + \dots + \sigma_d^2 = \sigma_0^2(t).$$

Let us now look briefly at the conditions of the theorem. The densities  $f_i$  need not be continuous or bounded. Indeed it suffices that the distributions  $F_i$  have a density of the form  $\gamma_i e^{-\psi_i}$  on a left neighbourhood of  $t_{i\infty}$ . In Proposition 5.7 we shall see that if a distribution has the form  $F_i = 1 - e^{-\psi_i}$  on a left neighbourhood of its upper endpoint and  $\psi_i$  satisfies (1.2) and (1.3) then (1.1) holds and the theorem applies.

The decomposition  $f_i(t) = c_i(t)\gamma_i(t)e^{-\psi_i(t)}$  with  $c_i(t) \rightarrow 1$  for  $t \rightarrow t_{i\infty}$  is far from unique. If desired, we may choose  $\gamma_i \equiv 1$ . Even in the independent identically distributed case, this does not lead to a substantial simplification of the expression for  $\gamma_0$ . See (1.11) below. Condition (1.5) is indispensable, as will be shown by an example in § 2.

The function class of self-neglecting functions used in condition (1.3) is discussed in greater detail in § 3. Recall that the function  $\sigma$  is self-neglecting with endpoint  $t_\infty$  if

$$(1.9) \quad \lim_{t \uparrow t_\infty} \sigma(t + x\sigma(t))/\sigma(t) = 1 \quad \text{locally uniformly in } x.$$

For now, think of self-neglecting functions as functions whose derivative goes to zero. In a similar spirit we observe that condition (1.4) is satisfied if  $\sigma_i$  is self-neglecting and  $\gamma_i$  has density  $\gamma'_i$  satisfying

$$(1.10) \quad \gamma'_i(t)\sigma_i(t)/\gamma_i(t) \rightarrow 0.$$

To see this, drop the subscript for typographical ease. The mean value theorem with  $s = t + \theta x\sigma(t)$  gives  $\log \gamma(t + x\sigma(t)) - \log \gamma(t) = x\sigma(t)(\log \gamma)'(s) \rightarrow 0$  by (1.9) and (1.10).

The following three examples will help to clarify the theorem's statement.

EXAMPLE 1.1. If the random variables  $X_i$  are independent and identically distributed with common density  $f \sim \gamma e^{-\psi}$  then the sum has density

$$(1.11) \quad f_0(t) \sim \frac{1}{\sqrt{d}} \left( \frac{2\pi}{\psi''(t/d)} \right)^{(d-1)/2} f\left(\frac{t}{d}\right) \quad \text{as } t \rightarrow t_\infty.$$

EXAMPLE 1.2. If the random variables  $X_i$  are independent with densities  $f_i \sim \gamma_i e^{-\psi}$  then the density of the sum satisfies

$$f_0(t) \sim \frac{1}{\sqrt{d}} \left( \frac{2\pi}{\psi''(t/d)} \right)^{(d-1)/2} e^{-\psi(t/d)} \prod_{i=1}^d \gamma_i(t/d).$$

EXAMPLE 1.3. Let  $f(t) \sim Ct^\beta e^{-ct^\alpha}$  with  $\alpha > 1$ ,  $c$  and  $C > 0$ ,  $\beta \in R$ . Take  $\psi(t) = ct^\alpha$ . Then  $\sigma(t) = 1/\sqrt{\psi''(t)}$  has a derivative which vanishes in  $\infty$  and  $\gamma(t) = Ct^\beta$  satisfies (1.10).

Now suppose  $f_i(t) \sim C_i t^{\beta_i} \exp(-c_i t^{\alpha_i})$  for  $i = 1, \dots, d$ . For simplicity assume  $c_i \alpha_i = 1$  and write  $\alpha_i = 1 + 1/\rho_i$ . Then  $\psi'_i(t) = t^{\alpha_i - 1} = t^{1/\rho_i}$  and therefore  $q_i = t^{\rho_i}$ . Expressions are simplest in terms of the variable  $\tau$ . Note that  $t = t(\tau) = \sum_{i=1}^d \tau^{\rho_i}$  with  $\rho_i > 0$ . This allows us to treat  $\tau$  as a function of the variable  $t$  in the expressions below.

We have  $\sigma_i^2(q_i) = \rho_i \tau^{\rho_i - 1}$  and  $\gamma_i(q_i) = C_i \tau^{\rho_i \beta_i}$ . Theorem 1.1 gives

$$\psi_0(t) = \sum_{i=1}^d c_i \tau^{\rho_i + 1}, \quad \sigma_0^2(t) = \sum_{i=1}^d \rho_i \tau^{\rho_i - 1}, \quad \gamma_0(t) = \frac{C \tau^{s-d/2}}{\sqrt{(\sum_{i=1}^d \rho_i \tau^{\rho_i - 1})}},$$

where  $s = \sum_{i=1}^d \rho_i (\beta_i + \frac{1}{2})$  and  $C = (2\pi)^{(d-1)/2} \prod_{i=1}^d C_i \sqrt{\rho_i}$ . In the independent identically distributed case these types of densities have been analysed by Rootzen [10].

### 2. The tail of a convolution

The conditions of Theorem 1.1 imply that the upper tail of the convolution  $f = f_1 * \dots * f_d$  is determined (up to asymptotic equality) by the upper tails of the densities  $f_i$  of the summands. This aspect of Theorem 1.1 will form the subject of the present section. There is considerable literature on this subject for regularly

varying and subexponential tails. See, for example, [2]. We shall consider distributions  $F_i$  with upper endpoint  $\sup\{F_i < 1\} = \infty$  and with the property that the density  $f_i \sim e^{-\psi_i}$  exists and is strictly positive on a neighbourhood of  $\infty$ . The next example shows what happens if condition (1.5) is violated.

EXAMPLE. Let  $f(x) \sim e^{(\sqrt{x})-x}$ . Write  $\psi(x) = x - \sqrt{x}$ . Then  $\psi''(x) = \frac{1}{4}x^{-\frac{3}{2}}$  and  $\sigma(x) = 2x^{\frac{3}{4}}$  is self-neglecting since the derivative vanishes in  $\infty$ . Let  $g(x) \sim x^{-m}e^{(\sqrt{x})-x}$ . The function  $\gamma(x) = x^{-m}$  satisfies (1.10). Example 1.2 implies that

$$(f * g)(t) \sim 2(\sqrt{\pi})(\frac{1}{2}t)^{\frac{3}{4}}(\frac{1}{2}t)^{-m}e^{(\sqrt{2t})-t} \text{ as } t \rightarrow \infty.$$

Now take a density  $g$  with a slightly thinner tail, so that

$$g(x) = O(e^{-x}) \text{ and } \int_0^\infty e^x g(x) dx < \infty.$$

Then

$$(2.1) \quad (f * g)(t) \sim Af(t) \text{ as } t \rightarrow \infty$$

with  $A = \int e^x dG(x)$ . To verify this, write the convolution as the sum of three terms (for  $t > c \geq 0$ ):

$$(f * g)(t) = \int_0^{t-c} f(t-x)g(x) dx + \int_{-\infty}^0 f(t-x) dG(x) + \int_{-\infty}^c g(t-x) dF(x).$$

For fixed  $x \in R$ , one has  $f(t-x)/f(t) \rightarrow e^x$  and one can choose  $c \geq 0$  so large that  $\frac{1}{2} < f(t)/e^{(\sqrt{t})-t} < 2$  for  $t \geq c$ . This implies that  $f(t-x)/f(t) \leq 4e^x$  for  $x \geq 0$ ,  $t-x \geq c$  and  $f(t-x)/f(t) \leq 4$  for  $t \geq c$  and  $x \leq 0$ . Lebesgue's theorem on dominated convergence with majorizing function  $4 \vee 4e^x$  gives

$$\frac{1}{f(t)} \int_{-\infty}^{t-c} f(t-x) dG(x) \rightarrow \int_{-\infty}^\infty e^x dG(x) = A \text{ as } t \rightarrow \infty.$$

The third integral above is bounded by  $F(c)O(e^{-t+c}) = O(e^{-t}) = o(f(t))$  for  $t \rightarrow \infty$ . This establishes (2.1).

In the situation sketched here we do obtain an asymptotic expression for the tail of the density of the convolution, but of a less symmetric form than the expression in Theorem 1.1. In particular, we see that the upper tail of the convolution depends on the whole distribution function  $G$  and not only on the asymptotic behaviour of the density  $g(x)$  for  $x \rightarrow \infty$ . The same argument establishes the following result:

PROPOSITION 2.1. *Let the distributions  $F$  and  $G$  have densities  $f(x) \sim e^{-\psi(x)}$  and  $g(x) = o(e^{-bx})$  on a neighbourhood of  $\infty$ . Assume that  $\psi'(x) \leq b$  for  $x > x_0$  and  $\psi'(x) \rightarrow \tau$  for  $x \rightarrow \infty$ . If  $\int_0^\infty e^{bx} dG(x)$  is finite then (2.1) holds with  $A = \int e^{bx} dG(x)$ .*

Analogues of this result for distribution functions are given in [4].

We shall now formulate conditions which ensure that the convolution has a density on a neighbourhood of  $\infty$  and that the asymptotic behaviour of this density is determined by the asymptotic behaviour of the densities of the summands in  $\infty$ .

**PROPOSITION 2.2.** *Let  $F_1$  and  $F_2$  be distributions on  $R$  with densities  $f_i \sim e^{-\psi_i}$  on a neighbourhood of  $\infty$  such that  $\psi'_i(x) \rightarrow \infty$  for  $x \rightarrow \infty$ . Then  $F_1 * F_2$  has a density on a neighbourhood of  $\infty$  whose asymptotic behaviour depends only on the asymptotic behaviour of the functions  $f_i$  in  $\infty$ .*

*Proof.* Choose  $c \geq 0$  so that the distributions  $F_i$  have a density on  $(c, \infty)$ . Then the convolution  $F_1 * F_2$  has a density  $f$  on  $(2c, \infty)$  given by

$$f(t) = \int_c^{t-c} f_1(t-x)f_2(x) dx + \int_{-\infty}^c f_1(t-x) dF_2(x) + \int_{-\infty}^c f_2(t-x) dF_1(x).$$

By symmetry it suffices to show that

$$A = \int_{-\infty}^c e^{-\psi_1(t-x)} dF_2(x) = o\left(\int_c^{t-c} f_1(t-x)f_2(x) dx\right)$$

for  $t \rightarrow \infty$ . Choose  $c$  so large that  $\alpha = \int_{c+1}^{c+2} f_2(x) dx > 0$ . Observe that  $A \leq e^{-\psi_1(t-c)}$  and

$$\int_{c+1}^{c+2} e^{-\psi_1(t-x)} f_2(x) dx \geq \alpha e^{-\psi_1(t-c-1)} = \alpha e^{-\psi_1(t-c)} e^{\psi_1(t-\xi)}$$

with  $|\xi| \leq c + 1$  and hence  $\psi'_1(t - \xi) \rightarrow \infty$ .

Note that this result depends only on the first derivative of the functions  $\psi_i$  in the exponent. We still need a result in the case where the derivatives  $\psi'_i$  have a finite limit independent of the index  $i$ .

**PROPOSITION 2.3.** *Let  $F_1$  and  $F_2$  be distributions on  $R$  with densities  $f_i \sim e^{-\psi_i}$  on a neighbourhood of  $\infty$  with  $\psi_i$  convex and  $\psi'_i(x) \rightarrow \tau < \infty$  for  $x \rightarrow \infty$ . Then  $F_1 * F_2$  has a density on a neighbourhood of  $\infty$  whose asymptotic behaviour depends only on the asymptotic behaviour of the functions  $f_i$  in  $\infty$ .*

*Proof.* First note that the derivatives  $\psi'_i$  are increasing, and hence  $\tau > 0$ . We may assume that the densities exist on  $[c, \infty)$ , where  $c > 0$ , and that the asymptotic equalities hold there within a factor 2. As above,  $f(t)$  is the sum of three integrals and the sum over the finite interval  $[c, t - c]$  dominates for  $t \rightarrow \infty$ . Let  $\alpha_i = e^{-\psi_i(c)} > 0$  for  $i = 1, 2$ . We have the bounds  $2e^{-\psi_i(t-c)}$  for the integral over the infinite intervals, whereas for  $t > 2c$ ,

$$\int_c^{t-c} e^{-(\psi_1(t-x) + \psi_2(x))} dx \geq \frac{1}{2}(t - 2c)(\alpha_2 e^{-\psi_1(t-c)} + \alpha_1 e^{-\psi_2(t-c)}),$$

by unimodality of the integrand on the interval  $[c, t - c]$ . Now let  $t \rightarrow \infty$ .

**EXAMPLE.** Let  $f_i(x) \sim a_i x^{r_i-1} e^{-x}$  for  $i = 1, \dots, d$  with  $a_i$  and  $r_i$  positive constants. Then

$$(f_1 * \dots * f_d)(t) \sim a_1 \dots a_d \frac{\Gamma(r_1) \dots \Gamma(r_d)}{\Gamma(r)} t^{r-1} e^{-t}$$

with  $r = r_1 + \dots + r_d$ . For  $r_i \geq 1$  one can apply the result above; for  $r_i < 1$  observe that the integral in the proof above divided by  $f_1(t - c) \vee f_2(t - c)$  diverges for  $t \rightarrow \infty$  even if we divide the integrand by the factor  $x(t - x)$ . Densities with gamma

tails fall outside the scope of the present paper. The function

$$\psi(x) = x - (r - 1)\log x$$

in the exponent is convex for  $r > 1$ , but  $\sigma(x) = 1/\sqrt{\psi''(x)}$  is not self-neglecting. In geometrical terms this is the hyperbolic case, and we are concerned with the parabolic case.

### 3. Self-neglecting functions

In this section we collect some needed preliminaries about self-neglecting functions and the relation (1.4). (Cf. [2, 7, 8].) A function  $\sigma$  defined on a left neighbourhood of a point  $t_\infty \leq \infty$  is *self-neglecting* (written  $\sigma \in SN$ ) if it is strictly positive and satisfies

$$(3.1) \quad \frac{\sigma(t + x\sigma(t))}{\sigma(t)} \rightarrow 1 \quad \text{as } t \uparrow t_\infty$$

uniformly on bounded  $x$ -intervals. (In particular, this requires that for fixed  $x > 0$  one has the inequality  $t + x\sigma(t) < t_\infty$  for  $t$  sufficiently close to  $t_\infty$ .) It helps to think of  $\sigma(t)$  as the scale in a neighbourhood of  $t$ . If  $t_\infty = \infty$  then sufficient for  $\sigma \in SN$  is that  $\sigma$  have a density  $\sigma'$  satisfying  $\sigma'(t) \rightarrow 0$  as  $t \rightarrow \infty$ . If  $t_\infty < \infty$  and both  $\sigma$  and  $\sigma'$  vanish at  $t_\infty$ , then  $\sigma \in SN$ . Membership in  $SN$  is preserved under asymptotic equivalence: if  $\sigma \in SN$  and  $\tau \sim \sigma$ , then  $\tau \in SN$ .

Less obvious are the following facts. First, if  $\sigma$  is continuous, then pointwise convergence in (3.1) implies that convergence is locally uniform and hence  $\sigma \in SN$ . (See [2, p. 120] and [6, p. 49].) Second, if  $\sigma \in SN$  then  $\sigma \sim \tau$  where  $\tau$  is  $C^1$  and  $\tau'$  vanishes at  $t_\infty$  (along with  $\tau$  itself if  $t_\infty$  is finite). The proof of the first fact carries over to the function  $\gamma$ :

**PROPOSITION 3.1.** *If  $\gamma$  is continuous and satisfies (1.4) with  $\sigma$  self-neglecting, then (1.4) holds uniformly on bounded  $x$ -intervals.*

We give a short proof of the second fact for the case that  $t_\infty = \infty$  since the basic argument is useful. The definition of a self-neglecting function supplies an  $s_0$  such that  $\sigma$  is strictly positive on  $[s_0, \infty)$  and

$$(3.2) \quad \frac{1}{2} < \sigma(t + x\sigma(t))/\sigma(t) < 2$$

for  $t \geq s_0$  and  $|x| \leq 1$ . Using  $s_0$  as the base of a recursion, define the increasing sequence  $(s_n)$  by  $s_{n+1} = s_n + \sigma(s_n)$ . Then  $s_n \rightarrow \infty$ . (Otherwise,  $s_\infty := \sup s_n < \infty$  and  $\sigma(s_n) = s_{n+1} - s_n \rightarrow 0$ . On the other hand,  $\sigma(s_\infty) = c > 0$  and (3.2) evaluated at  $t = s_\infty$  states that  $\sigma > \frac{1}{2}c$  on the interval  $(s_\infty - c, s_\infty + c)$ .) Relation (3.1) now implies that  $\sigma_n := \sigma(s_n) \sim \sigma_{n+1}$ . Let  $\tau$  be the piecewise linear continuous function which agrees with  $\sigma$  in the points  $s_n$ ; that is, define

$$\tau(s) = \sigma_n + (\sigma_{n+1} - \sigma_n)(s - s_n)/(s_{n+1} - s_n) \quad \text{for } s_n \leq s \leq s_{n+1}.$$

Since  $0 \leq (s - s_n)/(s_{n+1} - s_n) \leq 1$ , the right-hand side above divided by  $\sigma_n$  converges to 1 and (3.1) gives  $\sigma \sim \tau$ . Furthermore,  $\tau'(s) \rightarrow 0$ .

By pushing this construction further, one can make  $\tau \in C^\infty$  such that  $\tau \sim \sigma$  and such that the derivatives of  $\tau$  satisfy  $\sigma^k \tau^{(k+1)} \rightarrow 0$ . To do this one changes the definition of  $\tau$  by interpolating not with a linear function but rather with an



increasing  $C^\infty$ -function from  $[0, 1]$  onto  $[0, 1]$  all of whose derivatives vanish in the endpoints 0 and 1. If this  $C^\infty$  function is denoted by  $f$  then we define  $\tau$  on  $[s_n, s_{n+1}]$  by

$$\tau(s) = \sigma_n + f((s - s_n)/(s_{n+1} - s_n))(\sigma_{n+1} - \sigma_n).$$

If we start the sequence  $(s_n)$  with a point  $s_0$  determined by the function  $\gamma$  and define the function  $\beta$  by

$$\beta(s) = \gamma(s_n) + f((s - s_n)/(s_{n+1} - s_n))(\gamma(s_{n+1}) - \gamma(s_n))$$

for  $s \in [s_n, s_{n+1})$ , then the construction above yields the following result:

**PROPOSITION 3.2.** *Suppose  $\sigma$  is self-neglecting and  $\gamma$  satisfies (1.4) uniformly on bounded  $x$ -intervals. Then there exists a  $C^\infty$ -function  $\beta \sim \gamma$  such that  $\sigma^k \beta^{(k)} / \beta$  vanishes at  $t_\infty$  for  $k = 1, 2, \dots$ .*

The following closure result for  $SN$  will be needed.

**PROPOSITION 3.3.** *If  $\sigma$  and  $\tau$  are self-neglecting with the same upper endpoint  $t_\infty$ , then so is  $\rho$  where  $\rho$  is defined by*

$$\rho^{-2} = \sigma^{-2} + \tau^{-2}.$$

*Proof.* If  $\sigma$  and  $\tau$  are  $C^1$  with derivatives which vanish in  $t_\infty$ , then  $\rho$  is  $C^1$  and

$$\frac{\rho'}{\rho^3} = \frac{\sigma'}{\sigma^3} + \frac{\tau'}{\tau^3} \Rightarrow \rho' = \left(\frac{\rho}{\sigma}\right)^3 \sigma' + \left(\frac{\rho}{\tau}\right)^3 \tau' \rightarrow 0$$

since  $\rho \leq \sigma$  and  $\rho \leq \tau$ . In the general case, choose  $\bar{\sigma} \sim \sigma$  and  $\bar{\tau} \sim \tau$  so that  $\bar{\sigma}'$  and  $\bar{\tau}'$  vanish in  $t_\infty$ . Then  $\bar{\sigma}^{-2} \sim \sigma^{-2}$  and  $\bar{\tau}^{-2} \sim \tau^{-2}$ , whence  $\bar{\rho} \sim \rho$ .

#### 4. The conjugate transform

We need some simple facts about the conjugate (Legendre) transform of a convex function [2, 3, 9, 11].

A convex function  $\psi$  is defined on the whole real line  $R$  with values in  $(-\infty, \infty]$ . Its *domain* is the set on which it is finite. This is a connected set, which we shall assume to contain at least two points. The function  $\psi$  is continuous on the interior of the domain. For  $\xi \in R$  we define

$$(4.1) \quad \psi^*(\xi) = \sup_x (\xi x - \psi(x)).$$

Note that  $\psi^*$  is convex and lower semi-continuous since it is the supremum of affine functions. For all  $x$  and  $\xi$  one has the inequality

$$(4.2) \quad \xi x \leq \psi(x) + \psi^*(\xi).$$

The function  $\psi^*$  is called the *convex conjugate* of  $\psi$ . It is well known that  $\psi^{**} = (\psi^*)^*$  is the convex hull of  $\psi$ :

$$\psi^{**} = \sup\{\xi \mid \xi \text{ is affine and } \xi \leq \psi\},$$

where the inequality and supremum are defined pointwise in the previous display.

Now assume  $\psi$  is  $C^1$  and strictly convex on the open interval  $D$ . Then  $\psi'$  is continuous and strictly increasing on  $D$ . The image  $\Delta = \psi'(D)$  is an open interval and  $\psi^*$  is finite on  $\Delta$ . Each  $\xi \in \Delta$  is the slope of the tangent line at a unique point  $x = q(\xi) = \xi^T \in D$ . Also  $q = \psi'^{\leftarrow} = \psi^{*\prime}$ . The symmetry in this situation is reflected in the symmetric relation

$$(4.3) \quad \xi x = \psi(x) + \psi^*(\xi)$$

for  $\xi \in \Delta, x \in D$  related as above. Note that (4.3) is a local definition of  $\psi^*$  which does not depend on the behaviour of  $\psi$  outside the open set  $D$ , whereas (4.1) apparently is a global definition. These results also hold for convex functions of several variables:

**PROPOSITION 4.1.** *Let  $\psi$  be a convex function on  $R^d$  with convex conjugate  $\psi^*$  defined on the dual space  $R^{d*}$  of all linear functionals on  $R^d$  by (4.1). Let  $D \subset R^d$  be an open set. Assume that  $\psi$  is finite and  $C^2$  on  $D$ , and that  $\psi''(x)$  is positive definite in each point  $x \in D$ . Then there exist an open set  $\Delta \subset R^{d*}$  and a homeomorphism  $q: \Delta \rightarrow D$  with inverse  $\theta: D \rightarrow \Delta$  such that if  $x$  and  $\xi$  satisfy (4.3), then*

$$(4.4) \quad q(\xi) = x \iff \theta(x) = \xi.$$

*Proof.* The function  $\theta: x \rightarrow \psi'(x)$  is  $C^1$  with non-singular derivative  $\psi''(x)$  on  $D$ . The inverse function theorem implies that  $\Delta = \theta(D)$  is open and that  $\theta$  locally is a  $C^1$  diffeomorphism. Convexity ensures that the function is injective.

The relation between the variables  $x$  and  $\xi$  in (4.4) is symmetric. We shall express this symmetry in the notation

$$(4.5) \quad x = \xi^T, \quad \xi = x^T,$$

which states that  $x$  is the unique point in  $D$  where the tangent plane to  $\psi$  has slope  $\xi$ .

The formula for  $\psi_0$  in the statement of Theorem 1.1 has an elegant interpretation in terms of conjugate transforms. The *infimal convolution* of two convex functions  $\psi_1$  and  $\psi_2$  is defined in [9] as

$$\psi_1 \square \psi_2(x) = \inf_y \{ \psi_1(x - y) + \psi_2(y) \}$$

and it is shown there (Theorem 16.4) that

$$(4.6) \quad \psi_1 \square \psi_2 = (\psi_1^* + \psi_2^*)^*.$$

**THEOREM 4.2.** *Let the conditions of Theorem 1.1 hold. Then*

$$(4.7) \quad \psi_0 = \psi_1 \square \dots \square \psi_d$$

*on a left neighbourhood of  $t_\infty = t_{1\infty} + \dots + t_{d\infty}$ .*

*Proof.* Set  $\psi(x) = \psi_1(x_1) + \dots + \psi_d(x_d)$ , define  $\psi_0$  as in Theorem 1.1, and set

$$\chi(t) = \inf \{ \psi(x) \mid x_1 + \dots + x_d = t \}.$$

We shall construct  $t_0 < t_\infty$  such that  $\chi \equiv \psi_0$  on the interval  $(t_0, t_\infty)$ .

The functions  $\psi'_i$  are continuous and strictly increasing on a left neighbourhood of the upper endpoints  $t_{i\infty}$  with limit  $\tau_\infty$  in  $t_{i\infty}$ . Hence the  $d$  inverse functions  $q_i = (\psi'_i)^{-1}$  are defined, continuous and strictly increasing on an interval  $[\tau_0, \tau_\infty)$ . Set  $t_{i0} = q_i(\tau_0)$ . Then  $t = q_1 + \dots + q_d$  is a continuous strictly increasing function from the interval  $(\tau_0, \tau_\infty)$  onto  $(t_0, t_\infty)$  where  $t_0 = t_{10} + \dots + t_{d0}$ .

Let  $\tau \in (\tau_0, \tau_\infty)$ . Then  $\psi_0(t) = \psi(q)$  where  $q = (q_1(\tau), \dots, q_d(\tau))$  and  $t = q_1 + \dots + q_d$ . The function  $\varphi(x) = \psi(x) - (x_1 + \dots + x_d)\tau$  is convex and the derivative  $\varphi'$  vanishes in  $q = (\tau, \dots, \tau)^T$ . Since the matrix  $\varphi''(q) = \psi''(q)$  is positive definite, the function  $\varphi$  has a unique minimum in  $q$ . *A fortiori* the function  $\varphi$  restricted to the set  $S_t = \{x_1 + \dots + x_d = t\}$  achieves its minimum in the point  $q \in S_t$ , and so does  $\psi$  since the linear function  $x_1 + \dots + x_d$  is constant on  $S_t$ . This proves that  $\chi(t) = \psi(q)$ .

### 5. Asymptotically parabolic functions

In §4 we have studied the convexity of  $\psi$ . We shall now investigate the implications of the extra condition (1.3) that  $\sigma = 1/\sqrt{\psi''}$  is self-neglecting. The class of asymptotically parabolic (AP) functions defined below is natural for our problem. It possesses a number of pleasing and useful closure properties which we explore before describing the asymptotic behaviour of the functions  $\psi$  in this class. These closure properties further illuminate the statement of Theorem 1.1.

**DEFINITION.** A function  $\psi$  is *asymptotically parabolic* (AP) if it is a convex  $C^2$ -function which is defined on an open interval  $D$  such that  $\psi''$  is strictly positive on  $D$  and such that  $\sigma := 1/\sqrt{\psi''}$  is self-neglecting in the upper endpoint  $t_\infty \leq \infty$  of the open interval  $D$ . We call  $\sigma$  the *auxiliary* or *scale function*. Sometimes we shall write  $AP(t_\infty)$  for the set of asymptotically parabolic functions with common endpoint  $t_\infty$ .

**EXAMPLES.** Asymptotically parabolic functions:

| $\psi(t)$        | parameter        | $t_\infty$ | $\sigma(t)$         |                                   |
|------------------|------------------|------------|---------------------|-----------------------------------|
| $t^\alpha$       | $\alpha > 1$     | $\infty$   | $ct^{1-\alpha/2}$   | $c = 1/\sqrt{\alpha(\alpha - 1)}$ |
| $e^t$            |                  | $\infty$   | $e^{-t/2}$          |                                   |
| $t - t^\alpha$   | $0 < \alpha < 1$ | $\infty$   | $ct^{1-\alpha/2}$   | $c = 1/\sqrt{\alpha(1 - \alpha)}$ |
| $t \log t$       |                  | $\infty$   | $\sqrt{t}$          |                                   |
| $(-t)^{-\alpha}$ | $\alpha > 0$     | 0          | $c t ^{1+\alpha/2}$ | $c = 1/\sqrt{\alpha(1 + \alpha)}$ |

The functions  $\psi(t) = t - \log t$  and  $\psi(t) = 1/t$  are not asymptotically parabolic.

We begin with some closure properties of the class  $AP$ . The first proposition has an obvious proof and the second is a simple consequence of Lemma 3.3.

**PROPOSITION 5.1.** *The set of AP functions is closed for addition of constants and linear functions: if  $\psi$  is AP then so is  $\psi(t) + \rho + \lambda t$  for any  $\rho$  and  $\lambda$  in  $R$ .*

**PROPOSITION 5.2.** *The set of AP functions is a convex cone:*

- (a) *if  $\psi$  is AP with scale function  $\sigma$  then for any  $c > 0$  the function  $c\psi$  is AP with scale function  $\sigma/\sqrt{c}$ ;*
- (b) *suppose  $\psi_1, \psi_2 \in AP(t_\infty)$  with scale functions  $\sigma_1, \sigma_2$ ; then  $\psi_1 + \psi_2 \in AP(t_\infty)$  with scale function  $\sigma = (\sigma_1^{-2} + \sigma_2^{-2})^{-\frac{1}{2}}$ .*

**THEOREM 5.3.** *The class of asymptotically parabolic functions is closed under conjugation in the following sense: if  $\psi$  is AP with domain  $D$  and endpoint  $t_\infty$  and scale function  $\sigma$ , then the restriction of  $\psi^*$  to  $\Delta = \psi'(D)$  is AP with endpoint  $\tau_\infty = \psi'(t_\infty)$  and scale function  $s(\tau) = 1/\sigma(q)$  with  $q = \psi^{*\prime}(\tau)$ .*

*Proof.* The relation  $q = (\psi')^\leftarrow(\tau) = (\psi^*)'(\tau)$  determines a diffeomorphism  $q \leftrightarrow \tau$  between points  $q \in D$  and points  $\tau \in \Delta = \psi'(D)$  (see § 4). On the one hand,

$$q'(\tau) = (\psi^*)''(\tau) = 1/(s(\tau))^2,$$

and on the other,

$$q'(\tau) = ((\psi')^\leftarrow(\tau))' = 1/\psi''((\psi')^\leftarrow(\tau)) = \sigma^2(q).$$

This shows that the asserted relationship between  $\sigma$  and  $s$  holds. We still have to prove that the function  $s$  is self-neglecting. Observe that for  $|\theta| \leq 1$  we have

$$\begin{aligned} \psi'(q + u\sigma(q)) &= \psi'(q) + u\sigma(q)\psi''(q + \theta u\sigma(q)) \\ &= \psi'(q) + (u/\sigma(q))(\sigma^2(q)/\sigma^2(q + \theta u\sigma(q))) \\ &= \psi'(q) + (1 + o(1))u/\sigma(q) \end{aligned}$$

uniformly on bounded  $u$ -sets, by the self-neglecting property of  $\sigma$ . So given  $u = u(q) \rightarrow u_0$ , we may find  $v(q) \sim u(q)$  such that

$$(5.1) \quad \psi'(q) + u(q)/\sigma(q) = \psi'(q + v(q)\sigma(q)).$$

This relation implies that  $s \in SN$  as follows:

$$\begin{aligned} s(\tau + us(\tau)) &= 1/\sigma((\psi')^\leftarrow(\tau + us(\tau))) \\ &= 1/\sigma((\psi')^\leftarrow(\psi'(q) + u/\sigma(q))) \\ &= 1/\sigma((\psi')^\leftarrow(\psi'(q + v\sigma(q)))) \\ &= 1/\sigma(q + v\sigma(q)) \\ &\sim 1/\sigma(q) = s(\tau). \end{aligned}$$

We shall now derive some useful asymptotic properties of asymptotically parabolic functions.

**THEOREM 5.4.** *Suppose  $\psi$  is AP with endpoint  $t_\infty$ . Then  $\psi(t_\infty)$  is infinite.*

*Proof.* This is obvious if  $t_\infty = \infty$  and  $\tau_\infty = \sup \psi'(t) \neq 0$ . We are left with two cases:  $t_\infty$  is finite, and  $\tau_\infty = 0$ .

First assume that  $t_\infty$  is finite. Since  $\sigma = 1/\sqrt{\psi''}$  is self-neglecting with finite endpoint  $t_\infty$ , it follows from the parenthetical remark after (3.1) that  $\sigma(t) = o(t_\infty - t)$ . Hence for any  $M > 1$ , eventually we have  $\psi''(t) > M/(t_\infty - t)^2$  and  $\psi'(t) - \psi'(t_0) > M_1/(t_\infty - t)$ . This implies that  $\psi' \rightarrow \infty$  and in fact

$$(5.2) \quad (t_\infty - t)\psi'(t) \rightarrow \infty \quad \text{as } t \uparrow t_\infty.$$

Next assume that  $t_\infty = \infty$  and  $\tau_\infty = 0$ . Then  $\sigma(t) = o(t)$  implies that for any  $M > 0$  eventually  $\psi''(t) > M/t^2$ ,  $\psi'(t) < -M/t$  and hence we obtain the limit relation

$$(5.3) \quad (\tau_\infty - \psi'(t))t \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

PROPOSITION 5.5. *Suppose  $\psi$  is asymptotically parabolic with endpoint  $t_\infty$ . If  $t_\infty$  is finite then  $\tau_\infty = \sup \psi'(t) = \infty$  and (5.2) holds. If  $\tau_\infty$  is finite then (5.3) holds.*

*Proof.* See the proof of Theorem 5.4 above.

Theorem 5.4 can be regarded as a regularity condition of the convex function  $\psi$  in its right endpoint, and has an important consequence for the domain of the conjugate function  $\psi^*$ . By definition, the domain  $D$  of an AP function is an open interval. The domain of the conjugate function is a connected subset of  $R$  and contains the open interval  $\Delta = \psi'(D)$ . Now assume  $\tau_\infty = \sup \Delta$  is finite. Then  $\psi^*(\tau_\infty) = -\lim(\psi(t) - \tau_\infty t)$  for  $t \rightarrow t_\infty$ . This limit is infinite by Theorem 5.4 and hence  $\tau_\infty$  is not an element of the domain of  $\psi^*$ . (If the limit were finite then  $\tau_\infty$  would lie in the domain of  $\psi^*$ , and it would even be an interior point if  $t_\infty$  were finite too.) The domain of  $\psi^*$  need not be equal to  $\Delta = \psi'(D)$ , but for any  $\tau \in \Delta$  the two sets agree on the half line  $[\tau, \infty)$ , and since we are only concerned with the asymptotics in the upper endpoint, this is exactly what we need.

Suppose  $\psi$  is AP with scale function  $\sigma$ . A positive function  $\gamma$  defined on the domain of  $\psi$  is *flat* (for  $\psi$ ) if (1.4) holds uniformly on bounded  $x$ -intervals.

EXAMPLE. If  $\psi$  is asymptotically parabolic then  $\psi''$  is flat and so is  $\sigma = 1/\sqrt{\psi''}$ . Any continuous positive function  $\gamma$  which satisfies (1.4) is flat. A product of flat functions is flat. Since flatness is an asymptotic property, we shall often relax the positivity condition: the function should be strictly positive on a left neighbourhood of the endpoint  $t_\infty$  of  $\psi$ .

PROPOSITION 5.6. *Suppose  $\psi$  is AP. Let  $\gamma$  be a positive function defined on the domain  $D$  of  $\psi$ . Define  $\gamma^T$  on  $\Delta = \psi'(D)$  by*

$$(5.4) \quad \gamma^T(\xi) = \gamma(\xi^T).$$

*Then  $\gamma^T$  is flat for  $\psi^*$  if  $\gamma$  is flat for  $\psi$ .*

*Proof.* Let  $\tau = q^T$ . Then  $\tau + u\sigma(\tau) = (q + v\sigma(q))^T$  by (5.1) with  $v \sim u$ . This gives

$$\gamma^T(\tau + u\sigma(\tau)) = \gamma(q + v\sigma(q)) \sim \gamma(q) = \gamma^T(\tau).$$

The next result explains the terminology ‘asymptotically parabolic’ and ‘flat’.

THEOREM 5.7. *Suppose  $\psi$  is asymptotically parabolic with endpoint  $t_\infty$  and scale function  $\sigma$ , and  $\gamma$  is flat. Given  $t_0 < t_\infty$ ,  $M > 1$  and  $\varepsilon > 0$  there exists  $t_1 \in (t_0, t_\infty)$  such that for any  $t \in (t_1, t_\infty)$ ,*

$$\begin{aligned} J_t &= [t - M\sigma(t), t + M\sigma(t)] \subset (t_0, t_\infty), \\ 1 - \varepsilon &< \gamma(t + u\sigma(t))/\gamma(t) < 1 + \varepsilon, \quad |u| \leq M, \\ |\varphi(u) - \frac{1}{2}u^2| &\leq \frac{1}{2}\varepsilon u^2, \quad |u| \leq M, \end{aligned}$$

where  $\varphi$  is the normalized function

$$\varphi(u) = \psi(t + u\sigma(t)) - \psi(t) - u\sigma(t)\psi'(t).$$

*Proof.* The first relation is a consequence of the assumption that  $\sigma \in SN$ ; the second follows from the definition of ‘flat’. It holds in particular for the flat function  $\psi''$ . This implies that  $|\varphi'' - 1| < \varepsilon$  on  $[-M, M]$ . Repeated integration gives the third relation.

The next two results quantify the behaviour of  $\psi$  and its first two derivatives at the upper endpoint.

**PROPOSITION 5.8.** *Suppose  $\psi$  is AP with scale function  $\sigma$  and right endpoint  $t_\infty$ . Then  $\psi'$  is flat and*

$$(5.5) \quad \lim_{t \uparrow t_\infty} \psi'(t)\sigma(t) = \infty.$$

*Proof.* Note that  $\psi'$  is increasing. By Theorem 5.4,  $\psi(t_\infty) = \infty$  and hence the derivative  $\psi'$  is eventually strictly positive. Let  $x = x(t) \rightarrow x_0 \in R$ . Then  $\sigma \in SN$  gives

$$\begin{aligned} \psi'(t + x\sigma(t)) - \psi'(t) &= \int_0^x \psi''(t + y\sigma(t))\sigma(t) dy \\ &= \int_0^x (\sigma(t)/\sigma^2(t + y\sigma(t))) dy \\ &\sim \int_0^x (\sigma(t)/\sigma^2(t)) dy = x/\sigma(t) \quad \text{as } t \rightarrow t_\infty. \end{aligned}$$

Thus we have

$$\psi'(t + x\sigma(t))/\psi'(t) - 1 \sim x/(\sigma(t)\psi'(t)),$$

and it only remains to show that  $\sigma(t)\psi'(t) \rightarrow \infty$ . For any  $M > 1$ ,

$$\begin{aligned} \psi'(t)\sigma(t) &\geq (\psi'(t) - \psi'(t - M\sigma(t)))\sigma(t) \\ &= \sigma(t) \int_{t-M\sigma(t)}^t \psi''(u) du \\ &= \sigma^2(t) \int_{-M}^0 (1/\sigma^2(t + u\sigma(t))) du \rightarrow M. \end{aligned}$$

So  $\liminf_{t \uparrow t_\infty} \psi'(t)\sigma(t) \geq M$  for any  $M$  and this suffices to establish (5.5).

The examples at the beginning of this section show that the long term behaviour of asymptotically parabolic functions  $\psi$  may be far from parabolic.

Consider the derivative. The derivative  $\psi'$  is continuous and strictly increasing from  $D$  onto  $\Delta$ . One can construct an increasing sequence of points  $p_n = (t_n, \lambda_n)$  on the graph of  $\psi'$  with  $t_n \uparrow t_\infty$  such that the area of the rectangle  $[p_n, p_{n+1}]$  tends to 1. Then  $t_{n+1} - t_n =: \sigma_n \sim \sigma(t_n)$ , and hence  $\lambda_{n+1} - \lambda_n \sim 1/\sigma_n$ . (Indeed if we choose  $t_{n+1} - t_n \sim \sigma(t_n)$  then  $\psi'(t_{n+1}) - \psi'(t_n) \sim \sigma(t_n)\psi''(t_n) = 1/\sigma(t_n)$  since  $\sigma(t) = 1/\sqrt{\psi''(t)}$  is self-neglecting. Compare the construction of the sequence  $(s_n)$  in § 3.) This gives a graphic illustration of the symmetry between the function  $\psi$  (with derivative  $\psi'$ ) and the conjugate function  $\psi^*$  (with derivative  $\psi^{*\prime} = (\psi')^\leftarrow$ ).

Now let  $m \geq 1$  be a fixed integer. The area of the rectangle  $[p_n, p_{n+m}]$  will tend to  $m^2$ . Since  $\psi''$  is flat, it is asymptotically constant over the interval  $[t_n, t_{n+m}]$  and  $\psi'$  is asymptotically linear on  $[t_n, t_{n+m}]$ . Hence the area of the rectangle below the curve  $y = \psi'(x)$  will tend to  $\frac{1}{2}m^2$ . This yields the following useful inequality.

**PROPOSITION 5.9.** *Let  $\psi$  be AP and let  $t_n$  be as above. For any  $M > 1$  there exists an index  $n_0$  such that*

$$\psi(t_{n+k}) - \psi(t_n) \geq (t_{n+k} - t_n)\psi'(t_n) + kM$$

for  $n \geq n_0, k > 2M$ .

*Proof.* Set  $R(n, k) = \psi(t_{n+k}) - \psi(t_n) - (t_{n+k} - t_n)\psi'(t_n)$ . For fixed  $k \geq 1$  we have seen that  $R(n, k) \rightarrow \frac{1}{2}k^2$ . Let  $m = [2M + 1]$  denote the integer part of  $2M + 1$ . Then  $R(n, m) > mM$  eventually. The area of the oblong rectangle  $[t_n, t_{n+1}] \times [\lambda_{n-m}, \lambda_n]$  tends to  $m$  and hence exceeds  $M$  eventually.

We now return to Theorem 1.1.

Let  $\psi$  be AP with endpoint  $t_\infty$  and scale function  $\sigma$  and let  $f(t) \sim \gamma(t)e^{-\psi(t)}$ , for  $t \rightarrow t_\infty$ , be a probability density. We assume that  $\gamma$  is continuous and flat. In § 3 we have shown that we may then assume that  $\gamma$  is  $C^2$  and that  $\sigma^k(t)\gamma^{(k)}(t)/\gamma(t)$  vanishes in  $t_\infty$  for  $k = 1, 2$ . This implies that  $\varphi = \psi - \log \gamma$  is AP with scale function  $1/\sqrt{(\varphi''(t))} \sim \sigma(t)$ . Integrability of  $f$  implies that  $\varphi(t_\infty) = \infty$  (by Theorem 5.4 above) and hence that  $\sup \varphi'(t)$  is strictly positive (by convexity). The limit relation (5.5),  $\varphi'(t)\sigma(t) \rightarrow \infty$ , together with the relation  $\sigma(t)\gamma'(t)/\gamma(t) \rightarrow 0$ , implies that  $(\log \gamma(t))' = o(\varphi'(t))$  and hence  $\tau_\infty = \psi'(t_\infty) = \varphi'(t_\infty) > 0$ . The relations (5.2) and (5.3) on the derivatives  $\psi'_i$  give superlogarithmic increase for the functions  $\psi_i$  and hence determine the tail behaviour of the densities  $f_i$ . We combine these results in the following proposition.

**PROPOSITION 5.10.** *Let the conditions of Theorem 1.1 hold. Then  $\tau_\infty > 0$ , and  $\psi_i(t_\infty) = \infty$  for  $i = 1, \dots, d$ . If  $t_{i\infty}$  is finite then  $f_i(t) = o(t_{i\infty} - t)^{-n}$  for all  $n$ ; if  $\tau_\infty$  is finite then  $t^{-n}e^{\tau_\infty t}f_i(t) \rightarrow \infty$  for  $t \rightarrow \infty$ , for all  $n$ .*

*There exist flat functions  $\beta_i \sim \gamma_i$  such that  $\varphi_i = \psi_i - \log \beta_i$  is AP with upper endpoint  $t_{i\infty}$  and such that  $1/\varphi_i''(t) \sim \sigma_i^2(t)$  for  $t \rightarrow t_{i\infty}$  and  $\sup_t \varphi_i'(t) = \tau_\infty$ , and such that  $e^{-\varphi_i} = \beta_i e^{-\psi_i}$  is a probability density.*

In view of the results of § 2, it suffices to prove Theorem 1.1 in the case where  $f_i = \gamma_i e^{-\psi_i} = e^{-\varphi_i}$  where the functions  $\gamma_i$  are flat and the functions  $\psi_i$  and  $\varphi_i$  are asymptotically parabolic.

Our last result shows that the domain of the conjugate transform  $\psi^*$  and of the cumulant generating function of  $f \sim \gamma e^{-\psi}$  coincide, at least on  $[0, \infty)$ .

**PROPOSITION 5.11.** *Let  $X$  have density  $f \sim \gamma e^{-\psi}$  where  $\psi$  is AP and the function  $\gamma$  is flat. Let  $\tau_\infty = \sup \psi'(t)$ . Then  $C(\tau) = Ee^{\tau X}$  is finite for  $0 \leq \tau < \tau_\infty$ . If  $\tau_\infty$  is finite then  $Ee^{\tau X} = \infty$  for  $\tau = \tau_\infty$ .*

*Proof.* First assume that  $\tau < \tau_\infty$ . Set  $\varphi = \psi - \log \gamma$ . If  $\psi'(t) \rightarrow \tau_\infty < \infty$ , then by Proposition 5.8,  $\sigma(t) \rightarrow \infty$  and thus by Proposition 3.2 we have  $(\log \gamma)'(t) \rightarrow 0$ .

Hence  $\varphi(t) - \tau t > \varepsilon t$  eventually for some  $\varepsilon > 0$  and  $\int_0^\infty e^{\tau t - \varphi(t)} dt$  converges. If  $\psi'(t) \rightarrow \infty$ , then  $\varphi'(t) = \psi'(t) - (\log \gamma)'(t) \rightarrow \infty$  and the result follows similarly. If  $\tau = \tau_\infty < \infty$  then  $\tau t - \varphi(t) \rightarrow \infty$  by Theorem 5.4 and  $Ee^{\tau X} = \infty$ . This completes the proof.

6. Asymptotic normality of the exponential family

Before proceeding with the proof of Theorem 1.1 we make a brief digression and discuss the asymptotic normality of the exponential family  $f_\lambda$ , for  $\lambda \in \Lambda$ , generated by a  $d$ -dimensional density  $f$ . We begin by defining precisely what we mean by asymptotic normality with exponential tails.

DEFINITION. A sequence of random vectors  $X_n$  in  $R^d$  is asymptotically normal (AN) if there exist affine transformations  $\alpha_n$  on  $R^d$ ,  $\alpha_n(x) = A_n^{-1}(x - b_n)$  with  $A_n$  an invertible linear transformation on  $R^d$ , such that  $\alpha_n(X_n) \xrightarrow{d} U$  where the random vector  $U$  is standard normal. The sequence  $(X_n)$  is asymptotically normal with exponential tails (ANET) if the vectors  $\alpha_n(X_n)$  have densities  $g_n$  which satisfy the condition: for any  $\varepsilon > 0$  there exists an index  $n_0$  such that for  $n \geq n_0$ ,

$$(6.1) \quad |g_n(x) - \nu_d(x)| < \varepsilon e^{-\|x\|/\varepsilon} \quad \text{for } x \in R^d,$$

where  $\nu_d$  is the standard normal density on  $R^d$  and  $\|x\|$  denotes the Euclidean norm. We shall sometimes say that the sequence of densities  $(g_n)$  is ANET.

The reason for the name ‘with exponential tails’ is contained in the following result, which once stated hardly needs proof.

PROPOSITION 6.1. *The sequence  $(g_n)$  is ANET if and only if*

$$(6.2) \quad g_n(x) \rightarrow \nu_d(x)$$

locally uniformly in  $x$ , and given  $\varepsilon > 0$  there exist an index  $n_0$  and constants  $M > 1$  and  $C > 1$  such that

$$(6.3) \quad g_n(x) < C e^{-\|x\|/\varepsilon}, \quad \|x\| \geq M, n \geq n_0.$$

One may take  $C = 1$  in (6.3) without loss of generality since  $C e^{-t/\varepsilon} \leq e^{-t/2\varepsilon}$  for  $t \geq 2\varepsilon \log C$ .

REMARK 1. If  $(X_n)$  is asymptotically normal under two different affine transformations  $\alpha_n$  and  $\beta_n$ , and if the sequence is ANET with respect to one set of scaling transformations, then it is also ANET with respect to the other set.

REMARK 2. If a norm  $\|x\|_1$  on  $R^d$  other than the Euclidean norm is chosen then the right-hand side of (6.1) may be replaced by  $\varepsilon e^{-\|x\|_1/\delta}$  with  $\delta = c\varepsilon$  where the constant  $c$  depends on the new norm  $\|\cdot\|_1$ . In particular, if we take the norm  $\|x\|_1 = |x_1| + \dots + |x_d|$  then there is a nice relation between the bounds in  $d$  dimensions and in  $d - 1$  dimensions: if  $g_n$  satisfies (6.1), then the marginal density  $h_n$  on  $R^{d-1}$  defined by  $h_n(y) = \int_R g_n(y, u) du$ ,  $y \in R^{d-1}$ , satisfies

$$|h_n(y) - \nu_{d-1}(y)| < \varepsilon \int e^{-\|y\|_1/\delta - |u|/\delta} du = 2\varepsilon \delta e^{-\|y\|_1/\delta}.$$



This proves that the property of ANET is perserved under affine surjections. If the random vectors  $(X_n)$  are ANET and if  $(\gamma_n)$  is a sequence of affine surjections from  $R^d$  to  $R^m$  with  $m \leq d$  then the sequence of random vectors  $\gamma_n(X_n)$  in  $R^m$  is ANET.

**THEOREM 6.2.** *Suppose that the random vectors  $X_n$  in  $R^d$  are ANET. Let  $\gamma_n: R^d \rightarrow R^m$  with  $m \leq d$  be affine surjections such that  $W_n := \gamma_n(X_n) \xrightarrow{d} W$  where  $W$  is an  $m$ -dimensional normal vector with non-singular covariance matrix. Then the sequence  $(W_n)$  is ANET. Moreover, the conditional densities  $f_{n|w}$  of  $X_n$  given  $W_n = w$  are ANET and the inequality (6.1) holds uniformly on bounded  $w$ -sets.*

*Proof.* We may assume that  $W \in R^m$  has a standard normal density, that  $X_n \xrightarrow{d} U \in R^d$  where  $U$  has a standard normal density on  $R^d$ , and that  $\gamma_n$  is the projection on the first  $m$  coordinates. Let  $X_n$  have density  $f_n$  so that

$$(6.4) \quad f_{n|w}(y) = f_n(w, y) / \int_{R^{d-m}} f_n(w, y) dy$$

for  $w \in R^m$  and  $y \in R^{d-m}$ . Fix  $r > 0$  and restrict  $w$  to the closed ball with radius  $r$  around the origin in  $R^m$ . The marginal in the denominator of (6.4) is ANET by Remark 2 above, and with  $w$  restricted to the ball, the denominator is bounded away from zero. Also  $f_n(w, y)$  converges to  $v_d(w, y)$ , the standard normal density on  $R^d$ , locally uniformly and thus  $f_{n|w}$  satisfies (6.2). To prove the property of exponential tails it is necessary to verify (6.3). Since the denominator of (6.4) is bounded away from zero, it suffices to show for given  $\epsilon > 0$  that there exists a constant  $r' > 0$  such that, for large  $n$ ,

$$f_n(w, y) \leq e^c e^{-\|y\|/\epsilon}$$

for  $\|w\| \leq r, \|y\| \geq r'$ . This follows from the fact that  $f_n$  is ANET and

$$\{\|w\| \leq r, \|y\| \geq r'\} \subset \{\|(w, y)\| \geq r''\}.$$

The next result discusses the behaviour of moment generating functions and moments when there is convergence to normality with exponential tails.

**PROPOSITION 6.3.** *Suppose  $X_n$  in  $R^d$  is ANET and  $Y_n := A_n(X_n - b_n) \xrightarrow{d} U$  where  $U$  has a standard normal density on  $R^d$ . Then the moment generating function of  $Y_n$  converges to that of the standard normal density. For each  $t \in R^d$ ,*

$$Ee^{(t, Y_n)} \rightarrow e^{\|t\|^2/2}.$$

*Proof.* Take  $1/\epsilon \geq 1 + \|t\|$ . Eventually the density  $g_n$  of  $Y_n$  satisfies

$$e^{(t, y)} g_n(y) \leq e^{\|t\| \|y\|} K e^{-\|y\|/\epsilon} = K e^{-\|y\|}.$$

Now use dominated convergence.

**COROLLARY.** *The moments of  $Y_n$  converge to those of  $U$ . If  $\mu_n$  denotes the expectation of  $X_n$  and  $\Sigma_n$  the covariance, then  $A_n(\mu_n - b_n) \rightarrow 0$  and  $A'_n \Sigma_n A_n \rightarrow I$  where  $I$  is the identity matrix.*

The next result shows that ANET is precisely the convergence concept we need.

**THEOREM 6.4.** *Let  $f_n = e^{-\psi_n}$  be probability densities on  $R^d$  with  $\psi_n$  convex and  $C^2$ . If  $\psi'_n(0) \rightarrow 0$  and if  $\psi''_n(x) \rightarrow I$  uniformly on bounded  $x$ -sets, where  $I$  is the identity matrix, then the sequence  $(f_n)$  is ANET.*

*Proof.* This follows from the next more technical result by a Taylor expansion.

**PROPOSITION 6.5.** *Let  $f_n$  be probability densities on  $R^d$  such that  $f_n(x) \sim c_n e^{-\|x\|^2/2}$  uniformly on bounded  $x$ -sets in  $R^d$  for  $n \rightarrow \infty$ . If  $f_n \leq e^{-\psi_n}$  on  $R^d$  with  $\psi_n$  convex, and if there exists a constant  $M > 1$  such that for each  $x \in R^d$  eventually  $f_n(x) > e^{-\psi_n(x)}/M$ , then  $c_n \rightarrow (2\pi)^{-d/2}$  and the sequence  $(f_n)$  is ANET.*

*Proof.* Integration over a ball with radius 1 around the origin shows that the sequence  $(c_n)$  is bounded by a constant  $c$ . For fixed  $x \in R^d$  eventually

$$\frac{1}{2}\|x\|^2 - 1 - \log M < \psi_n(x) + \log c_n < \frac{1}{2}\|x\|^2 + 1.$$

The second inequality eventually holds uniformly on any closed ball

$$B = \{\|x\| \leq r\}$$

by assumption, and so does the first by convexity of the functions  $\psi_n$ ; see [9, Corollary 10.8.1]. The convex function  $\varphi_n = \psi_n + \log c_n + 1 + \log M$  lies between the two paraboloids  $\frac{1}{2}\|x\|^2$  and  $\frac{1}{2}\|x\|^2 + A$  on the ball  $\{\|x\| \leq r\}$ , with  $A = 4 + \log M$ . We may assume that  $r \geq 2\sqrt{A}$ . Let  $\|y\| > r$ . Since  $\varphi_n \geq \frac{1}{2}\|y\|^2$ , we get

$$\begin{aligned} \varphi_n(y) &\geq (\|y\|/r)(\frac{1}{2}r^2 - A) \geq \frac{1}{4}\|y\|, r, \\ f_n(y) &\leq ce^{-\|y\|^2/2} \leq ce^A e^{-\varphi_n(y)} \leq cMe^A e^{-r\|y\|/4}, \quad \|y\| > r. \end{aligned}$$

The integral of  $f_n$  over the complement of a large ball is small, and hence the integral over the ball is close to 1. This implies that  $c_n$  is close to  $(2\pi)^{-d/2}$ . Thus  $f_n$  converges to the normal density uniformly on bounded sets. Since the constant  $cM$  in the inequality above is fixed, and  $r$  is arbitrary, we have asymptotic normality with exponential tails.

Now consider the exponential family generated by a vector  $X$  in  $R^d$  with density  $f$ . It is well known that the set

$$\Lambda = \{\lambda \in R^d \mid C(\lambda) = Ee^{(\lambda, X)} < \infty\}$$

is convex and that the cumulant generating function  $\lambda \mapsto \log C(\lambda)$  is a convex function on  $\Lambda$ . With the density  $f$  we associate the exponential family of densities  $f_\lambda$ , for  $\lambda \in \Lambda$ , by defining

$$(6.5) \quad f_\lambda(x) = e^{(\lambda, x)}f(x)/C(\lambda), \quad \text{for } x \in R^d.$$

For the proof of Theorem 1.1 we need consider only the one-dimensional case.

**THEOREM 6.6.** *Let the random variable  $X$  have bounded density  $f \sim \gamma e^{-\psi}$  where  $\psi$  is AP with endpoint  $t_\infty$  and scale function  $\sigma$ , and  $\gamma$  is flat. Let  $\tau_\infty = \sup \psi'$ . Define  $f_\tau$ , with  $\tau \in \Lambda$ , by (6.5). Then  $\Lambda \cap [0, \infty) = [0, \tau_\infty)$  and  $q = \tau^T \rightarrow t_\infty$  is equivalent with  $\tau = q^T \rightarrow \tau_\infty$ . The normalized densities  $g_\tau(u) = \sigma f_\tau(q + \sigma u)$  are ANET. Note that  $q = \tau^T = (\psi')^-(\tau)$  depends on  $\tau$  and that  $\sigma = \sigma(q)$  depends on  $q$ . The moment generating function  $C(\tau)$  of the density  $f$  and the conjugate transform  $\psi^*$  of the function  $\psi$  in the exponent of the density  $f$  are related by the asymptotic equality*

$$(6.6) \quad C(\tau) \sim \gamma(q)\sigma(q)(\sqrt{2\pi})e^{\psi^*(\tau)}.$$

*Proof.* Let  $0 < \tau < \tau_\infty$  and assume  $\tau \rightarrow \tau_\infty$ . Proposition 5.11 gives  $\Lambda \cap [0, \infty) = [0, \tau_\infty)$ . Define  $q = \tau^T$ , so that  $\psi'(q) = \tau$ . Then  $q \rightarrow t_\infty$  if and only if  $\tau \rightarrow \tau_\infty$  by strict monotonicity of  $\psi'$  on  $D$  and by definition of  $\tau_\infty = \sup \psi'(D)$ . Define normalized densities  $g_\tau(u) = \sigma f_\tau(q + \sigma u)$  with  $\sigma = \sigma(q)$ ,  $q = \tau^T$ . For  $x = q + \sigma u$ ,

$$\begin{aligned} \tau x - \psi(x) &= \tau q + \tau \sigma u - \psi(q) - \psi'(q) \sigma u - \frac{1}{2} \psi''(q) \sigma^2 u^2 + r_\tau(u) \\ &= \psi^*(\tau) - \frac{1}{2} u^2 + r_\tau(u). \end{aligned}$$

The remainder term vanishes uniformly on bounded intervals for  $\tau \rightarrow \tau_\infty$  by Theorem 5.7. By the same theorem

$$g_\tau(u) \sim c(\tau) e^{-u^2/2} \quad \text{as } \tau \rightarrow \tau_\infty$$

uniformly on bounded  $u$ -sets where  $c(\tau) = \gamma(q) \sigma(q) e^{\psi^*(\tau)} / C(\tau)$ .

If  $-\log f$  is convex then the functions  $-\log g_\tau$  are convex and  $g_\tau$  is ANET by Proposition 6.5 for  $\tau \rightarrow \tau_\infty$ . In particular,  $c(\tau) \rightarrow 1/\sqrt{(2\pi)}$  which gives (6.6).

In the general case we only have asymptotic equality. We then enclose  $-\log g_\tau$  between two convex functions. This is done as follows:  $f \sim e^{-\varphi}$  where  $\varphi$  is asymptotically parabolic, by Proposition 5.10. We may assume that  $t_\infty$  is positive and that  $\frac{1}{2} \leq f e^\varphi \leq 2$  on a left neighbourhood of  $t_\infty$ , say on  $[0, t_\infty)$  for convenience. Choose  $M > 1 - \varphi(0)$  so that  $f \leq e^M$  on  $R$ , and let  $\varphi_0$  be the convex hull of the function  $\varphi_1$  which is identically  $-M$  on  $(-\infty, 0]$  and  $\varphi - 1$  on  $(0, t_\infty)$ . Then  $\varphi_0 \equiv \varphi - 1$  on an interval  $(c, t_\infty)$ . (The function  $\varphi^*$  is AP. This implies by Theorem 5.4 that the tangent line to  $\varphi - 1$  in  $t$  intersects the  $y$ -axis in a point  $y_t \rightarrow -\infty$  for  $t \rightarrow t_\infty$ .) Now apply Proposition 5 above:  $g_\tau \leq e^{-\varphi_\tau}$  with  $\varphi_\tau(u) = \varphi(q + u\sigma(q)) - \log \sigma(q) - 1$  convex. For any  $r > 1$  we have the inequality  $g_\tau \geq e^{-\varphi_\tau - r}$  on  $[-r, r]$  if we choose  $\tau$  so large that  $[q - r\sigma(q), q + r\sigma(q)] \subset (c, t_\infty)$ . This is possible by Theorem 5.7.

### 7. The proof

Let  $f$  denote the density of the random vector  $X = (X_1, \dots, X_d)$  with independent components  $X_i$  having density  $f_i \sim \gamma_i e^{-\psi_i}$  as in the statement of Theorem 1.1. The exponential family of densities  $f_\lambda$  is defined in (6.5). These densities are products. Choose  $\lambda = (\tau, \dots, \tau)$  with  $\tau \rightarrow \tau_\infty$ , and apply Theorem 6.6 to the components  $X_i$  to find that the normalized densities  $g_{i\tau}(u_i) = \sigma_i f_{i\tau}(q_i + \sigma_i u_i)$  are ANET for  $\tau \rightarrow \tau_\infty$  (equivalently for  $q_i \rightarrow t_{i\infty}$  where  $q = \lambda^T$ , that is,  $\psi'_i(q_i) = \tau$  for  $i = 1, \dots, d$ ). Hence the multivariate density  $\prod_{i=1}^d g_{i\tau}$  is ANET and

$$(7.1) \quad \prod_{i=1}^d g_{i\tau}(u_i) = \prod_{i=1}^d \sigma_i f_{i\tau}(q_i + \sigma_i u_i) \rightarrow \prod_{i=1}^d e^{-u_i^2/2} / \sqrt{(2\pi)}.$$

If the  $X_{i\tau}$  are independent variables with densities  $f_{i\tau}$ , then the corollary to Proposition 6.3 gives

$$(7.2) \quad \text{Var}(X_{i\tau}) \sim \sigma_i^2, \quad (EX_{i\tau} - q_i) / \sigma_i \rightarrow 0$$

for  $\tau \rightarrow \tau_\infty$ . The sum  $X_{0\tau} = X_{1\tau} + \dots + X_{d\tau}$  with density  $f_{0\tau}$ , say, is ANET for  $\tau \rightarrow \tau_\infty$  by Theorem 6.2; hence

$$(7.3) \quad \sigma_0 f_{0\tau}(t + \sigma_0 u) \rightarrow e^{-u^2/2} / \sqrt{(2\pi)} \quad \text{as } \tau \rightarrow \tau_\infty$$

where, by (7.2) and the convergence-of-types theorem, we may choose  $\sigma_0^2 = \sigma_1^2 + \dots + \sigma_d^2$  and  $t = q_1 + \dots + q_d$  (see the corollary to Proposition 6.3).

Setting  $u_i = 0, i = 1, \dots, d$  in (7.1) and (7.3) we find that

$$(7.4) \quad f_\lambda(q) = \prod_{i=1}^d f_{i\tau}(q_i) = e^{(\lambda,q)} f(q) / C(\lambda) \sim \prod_{i=1}^d 1/\sigma_i \sqrt{(2\pi)},$$

and

$$(7.5) \quad f_{0\tau}(t) = e^{t} f_0(t) / C_0(\tau) \sim 1/\sigma_0 \sqrt{(2\pi)}.$$

Observe that  $(\lambda, q) = \tau t$  and  $C(\lambda) = E e^{(\lambda, X)} = E e^{\tau X_0} = C_0(\tau)$ . The relations (7.4) and (7.5) now give

$$f_0(t) \sim \frac{\prod_{i=1}^d (\sigma_i \sqrt{(2\pi)})}{\sigma_0 \sqrt{(2\pi)}} f(q).$$

Substitute  $f(q) = \prod_{i=1}^d \gamma_i(q_i) e^{-\psi_i(q_i)}$  to obtain

$$f_0(t) \sim \frac{\prod_{i=1}^d ((\sqrt{2\pi}) \sigma_i(q_i) \gamma_i(q_i))}{\sigma_0 \sqrt{(2\pi)}} \exp\left\{-\sum_{i=1}^d \psi_i(q_i)\right\}.$$

If we define  $\psi_0$  and  $\gamma_0$  by (1.6) and (1.8), that is,

$$\psi_0(t) = \sum_{i=1}^d \psi_i(q_i), \quad \gamma_0(t) = \frac{\prod_{i=1}^d (\sigma_i(q_i) \gamma_i(q_i) \sqrt{(2\pi)})}{\sigma_0(t) \sqrt{(2\pi)}},$$

then we obtain the desired asymptotic identity

$$f_0(t) \sim \gamma_0(t) e^{-\psi_0(t)}.$$

We still have to show that  $\psi_0$  is asymptotically parabolic and that  $\gamma_0$  is flat.

The first statement holds since the functions  $\psi_i^*$  are asymptotically parabolic with common endpoint  $\tau_\infty$  by Theorem 5.3. This implies that  $\psi_0^* = \psi_1^* + \dots + \psi_d^*$  (see (4.6) and Theorem 4.2) is AP by Proposition 5.2, and hence also  $\psi_0 = \psi_0^{**}$  by Theorem 5.3.

Define

$$\delta_i(q_i) = (\sqrt{2\pi}) \sigma_i(q_i) \gamma_i(q_i) \quad \text{for } i = 1, \dots, d,$$

$$\delta_0(t) = \prod \delta_i(q_i) = (\sqrt{2\pi}) \sigma_0(t) \gamma_0(t).$$

The function  $\delta_i$  is flat for  $\psi_i$  since the factors  $\sigma_i$  and  $\gamma_i$  are flat by assumption (see Proposition 3.1). The function  $\delta_i^T$  is flat for  $\psi_i^*$  (see Proposition 5.6) and hence for  $\psi_0^* = \sum \psi_i^*$ . (Note that  $1/s_0^2 = \psi_0^{**} = \sum \psi_i^{**} = \sum 1/s_i^2$  implies  $s_0 < \min s_i$ ; compare Proposition 3.3.) The product  $d_0 = \delta_1^T \dots \delta_d^T$  is flat for  $\psi_0^*$  and hence  $d_0^T$  is flat for  $\psi_0$ . Now observe that

$$d_0^T(t) = d_0(\tau) = \delta_1^T(\tau) \dots \delta_d^T(\tau) = \delta_1(q_1) \dots \delta_d(q_d) = \delta_0(t),$$

which shows that  $\delta_0$  is flat for  $\psi_0$ , and hence also  $\gamma_0 = \delta_0/\sigma_0 \sqrt{(2\pi)}$ .

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