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Conditional probabilities in the excursion set theory: generic barriers and non-Gaussian initial conditions

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ABSTRACT

The excursion set theory, where density perturbations evolve stochastically with the smoothing scale, provides a method for computing the dark matter halo mass function. The computation of the mass function is mapped into the so-called first-passage time problem in the presence of a moving barrier. The excursion set theory is also a powerful formalism to study other properties of dark matter haloes such as halo bias, accretion rate, formation time, merging rate and the formation history of haloes. This is achieved by computing conditional probabilities with non-trivial initial conditions, and the conditional two-barrier first-crossing rate. In this paper we use the path integral formulation of the excursion set theory to calculate analytically these conditional probabilities in the presence of a generic moving barrier, including the one describing the ellipsoidal collapse, and for both Gaussian and non-Gaussian initial conditions. While most of our analysis associated with Gaussian initial conditions assumes Markovianity (top-hat in momentum space smoothing, rather than generic filters), the non-Markovianity of the random walks induced by non-Gaussianity is consistently accounted for. We compute, for a generic barrier, the first two scale-independent halo bias parameters, the conditional mass function and the halo formation time probability, including the effects of non-Gaussianities. We also provide the expression for the two-constant-barrier first-crossing rate when non-Markovian effects are induced by a top-hat filter function in real space.

Key words: cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

The distribution in mass of dark matter haloes, as well as their clustering properties, formation history, and merging rate, play an important role in many problems of modern cosmology, because of their relevance to the formation and evolution of galaxies and clusters, and of their sensitivity to the statistical properties of the primordial density field. In particular, the most massive haloes evolved from rare fluctuations in the primordial density field, so their abundance and clustering properties are sensitive probes of primordial non-Gaussianities (Grinstein & Wise 1986; Matarrese, Lucchin & Bonometto 1986; Lucchin, Matarrese & Vittorio 1988; Moscardini et al. 1991; Koyama, Soda & Taruya 1999; Matarrese, Verde & Jimenez 2000; Robinson & Baker 2000; Robinson, Gawiser & Silk 2000; LoVerde et al. 2008; Lam & Sheth 2009; Giannantonio & Porciani 2010; Maggiore & Riotto 2010c), which could be detected or significantly constrained by various planned large-scale galaxy surveys, see e.g. Dalal et al. (2008) and Carbone, Verde & Matarrese (2008). Furthermore, the primordial non-Gaussianities (NG) alters the clustering of dark matter haloes inducing a scale-dependent bias on large scales (Afshordi & Tolley 2008; Dalal et al. 2008; Matarrese & Verde 2008; Slosar et al. 2008) while even for small primordial NG the evolution of perturbations on super-Hubble scales yields extra contributions on smaller scales (Bartolo, Matarrese & Riotto 2005; Matarrese & Verde 2009; Bartolo, Matarrese & Riotto 2010).

The halo mass function can be written as

$$\frac{\mathrm{d}n(M)}{\mathrm{d}M} = f(\sigma)\frac{\bar{\rho}}{M^2}\frac{\mathrm{d}\ln\sigma^{-1}(M)}{\mathrm{d}\ln M}, \qquad (1)$$

where $n(M)$ is the number density of dark matter haloes of mass M, $\sigma(M)$ is the variance of the linear density field smoothed on a scale R corresponding to a mass M, and $\bar{\rho}$ is the average density of the universe. The basic problem is therefore the computation of the function $f(\sigma)$. Analytical computations of the halo mass function are typically based on Press–Schechter (PS) theory (Press & Schechter 1974) and its extension (Peacock & Heavens 1990; Bond et al. 1991) known as excursion set theory (see Zentner 2007 for a review). In

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excursion set theory the density perturbation depends stochastically with the smoothing scale, and the problem of computing the probability of halo formation is mapped into the so-called first-passage time problem in the presence of a barrier. With standard manipulations (see e.g. Zentner 2007), the function $f(\sigma)$ which appears in (1) is related to the first-crossing rate $\mathcal{F}$ by $f(\sigma) = 2\sigma^2\mathcal{F}(\sigma^2)$.

In a recent series of papers (Maggiore & Riotto 2010a,b,c) (hereafter MR1, MR2 and MR3, respectively), the original formulation of excursion set theory has been extended to deal analytically with the non-Markovian effects which are induced either by the use of a realistic filter function, or by non-Gaussianities in the primordial density field. In the original formulation of Bond et al. (1991) the problem with the density field smoothed using a top-hat filter in wavenumber space was solved analytically and numerical techniques were adopted for the case of non-Markovian noises. The use of a top-hat window function in momentum space has the technical advantage that the evolution of the smoothed density field with the smoothing scale becomes Markovian, but its important drawback is that is it is not possible to associate a well-defined mass to a region smoothed with such a filter (see Bond et al. 1991; Zentner 2007; Maggiore & Riotto 2010a). For any other choice of filter function such as a top-hat function in real space [for which the relation between the mass M and the smoothing scale R is well defined and is simply $M = (4/3)\pi R^3\bar{\rho}$] the actual evolution of the smoothed density field with R is non-Markovian.

The same happens if the initial conditions for the gravitational potential and/or the density contrast are non-Gaussian and the problem was solved first in MR3. The basic idea is to reformulate the first-passage time problem in the presence of a barrier in terms of the computation of a path integral with a boundary (i.e. over a sum over all 'trajectories' $\delta(S)$ that always stay below the barrier), and then to use standard results from quantum field theory and statistical mechanics to express this path integral in terms of the connected correlators of the theory. This allows us to include the effect of non-Markovianities arising, e.g., from the non-Gaussianities. In particular, in MR3 we have shown how to include the effect of a non-vanishing bispectrum, while the case of a non-vanishing trispectrum was considered in Maggiore & Riotto (2010d) (see also D'Amico et al. 2010 for an approach to non-Gaussianities which combines our technique with the saddle point method developed in Matarrese et al. 2000).

An essential ingredient of excursion set theory is a model for the collapse of a dark matter halo. In its simplest implementation, one uses the spherical collapse model. This model, however, is certainly a significant oversimplification of the complicated dynamics leading to halo formation and can be improved in different, complementary, ways. A crucial step was taken by Sheth, Mo & Tormen (2001) who took into account the fact that actual haloes are triaxial (Bardeen et al. 1986; Bond & Myers 1996) and showed that an ellipsoidal collapse model can be implemented, within the excursion set theory framework, by computing the first-crossing rate in the presence of a barrier $B_{\rm ST}(S)$,

$$B_{\rm ST}(S) \simeq \sqrt{a}\delta_{\rm c}(z)\left[1 + 0.4\left(\frac{S}{a\delta_{\rm c}^2(z)}\right)^{0.6}\right], \qquad (2)$$

which depends on $S \equiv \sigma^2$ ('moving barrier'), rather than taking the value $\delta_{\rm c}(z)$ of the spherical collapse, which is redshift dependent, but independent of S. Physically this reflects the fact that low-mass haloes (which corresponds to large S) have larger deviations from sphericity and significant shear, that opposes collapse.

Note that, to improve the agreement between the prediction from the excursion set theory with an ellipsoidal collapse and the N-body simulations, Sheth et al. (2001) also found that it was necessary to multiply $\delta_{\rm c}(z)$ by $\sqrt{a}$, where $\sqrt{a} \simeq 0.84$ was obtained by requiring that their mass function fits the GIF simulation. In MR2 we proposed a physical justification for the introduction of this parameter in the halo mass function, suggesting that some of the physical complications inherent to a realistic description of halo formation could be included in the excursion set theory framework, at least at an effective level, by treating the critical threshold for collapse as a stochastic variable, whose scatter reflects a number of complicated aspects of the underlying dynamics (see also Audit, Teyssier & Alimi 1997; Lee & Shandarin 1998; Sheth et al. 2001, for earlier related ideas). Solving the first-passage time problem in the presence of a barrier which is diffusing around its mean value, it was found in MR2 that the coefficient a can be related to the diffusion coefficient D_B of the stochastic barrier as $a = 1/(1 + D_B)$. The numerical value of D_B, and therefore the corresponding value of a, depends among other things on the algorithm used for identifying haloes. From recent N-body simulations that studied the properties of the collapse barrier, a value $D_B \simeq 0.25$ was deduced in MR2, predicting $a \simeq 0.80$, in agreement within the accuracy of the MR1 prediction ($\sim$20 per cent) with the value of a extracted directly from a fit to the mass function (see also Corasaniti & Achitouv 2011 for recent related work).

The path-integral formulation developed in MR1 and MR3 was restricted to the case of a constant barrier $\delta_c(z)$ and it was subsequently generalized to the case of the ellipsoidal moving barrier in De Simone, Maggiore & Riotto (2010). In the present paper we further develop the path-integral formulation of the excursion set theory to calculate, for a generic moving barrier and for Gaussian and non-Gaussian initial conditions, other basic quantities necessary to characterize the physics of dark matter haloes like halo bias, accretion rates, formation times, merging, halo assembly bias and so on.

We know that dark matter haloes typically form at sites of high-density peaks. The spatial distribution of dark matter haloes is therefore a biased tracer of the underlying mass distribution. A standard way to quantify this difference between haloes and mass is to use a bias parameter b_h, which can be defined as the ratio of the overdensity of haloes to mass, or as the square root of the ratio of the two-point correlation function (or power spectrum) of haloes to mass. Like the halo mass function, analytic expressions for the halo bias can be obtained from the excursion set theory based on the spherical gravitational collapse model (Cole & Kaiser 1989; Bond et al. 1991; Mo & White 1996) and for the ellipsoidal one (Sheth et al. 2001). The approach to the clustering evolution is based on a generalization of the so-called peak-background split (Bardeen et al. 1986) which basically consists in splitting the mass perturbations in a fine-grained (peak) component filtered on a scale R and a coarse-grained (background) component filtered on a scale $R_0 \gg R$. The underlying idea is to ascribe the collapse of objects on small scales to the high-frequency modes of the density fields, while the action of large-scale structures of these non-linear condensations is due to a shift of the local background density. In the excursion set theory the problem of computing the probability of halo formation is mapped into the first-passage time problem of a random walk which starts from a given value of the density contrast δ_0 at a given radius R_0 corresponding to a given value of the variance $\sigma(M_0)$. When the random walk performed by the smoothed density contrast is Markovian, the first-crossing rate is easily computed by a simple shift of the initial conditions. This is due to the fact that, being the noise white, the memory about the way the system arrived at the point δ_0 at a given time is lost. On the contrary, when

the random walk is non-Markovian, the system has memory effects and it remembers how it arrived at δ_0. This influences the subsequent first-crossing rate. The computation of the halo bias mass function in the case in which the non-Markovianity is induced by the choice of a top-hat window function in real space, and within a spherical collapse model, has been recently performed in Ma et al. (2010). In this paper we perform the calculation of the halo bias parameters for the ellipsoidal barrier and when non-Gaussian initial conditions introduce non-Markovianity, see also Giannantonio & Porciani (2010) for a treatment of NG halo bias.

The excursion set theory is also a powerful formalism for studying the formation history of haloes. The most immediate quantity of interest is the conditional mass function. Given a halo of mass M_0 at redshift z_a, one can compute the average manner in which this mass was partitioned among smaller haloes at some higher redshift $z_b > z_a$. The conditional mass function is simply the average number of haloes of mass M_n at redshift z_b that are incorporated into an object of mass M_0 at redshift z_a. In the language of excursion set theory this can be formulated as a two-barrier problem, i.e. in terms of the conditional first crossing rate, $\mathcal{F}(B_b(S_n), S_n|B_a(S_0), S_0)$, describing the rate at which trajectories make their first crossing of the barrier $B_b(S) \equiv B(S, z = z_b)$ at a value $S = S_n$, corresponding to the mass M_n, under the condition that, at an earlier 'time' $S = S_0$ [corresponding to the mass M_0; recall that decreasing the variance S the corresponding mass $M(S)$ increases, so $S_0 < S_n$ means $M_0 > M_n$], they crossed the threshold $B_a(S) \equiv B(S, z = z_a)$. Then, a halo of mass M_0 has its mass partitioned on average among a spectrum of haloes at redshift z_b as (Lacey & Cole 1993; Zentner 2007)

$$\frac{\mathrm{d}n(M_n|M_0)}{\mathrm{d}M_n} = \frac{M_0}{M_n}\,\mathcal{F}(B_b(S_n), S_n|B_a(S_0), S_0)\left|\frac{\mathrm{d}S_n}{\mathrm{d}M_n}\right|. \tag{3}$$

The function $\mathcal{F}(S_n, B_b(S_n)|S_0, B_a(S_0))$ gives the probability of the second barrier first crossing at a particular value of S_n, while the factor (M_0/M_n) converts it from a probability per unit mass of halo M_0 into the number of haloes of mass M_n. The two-barrier result can also be manipulated to yield the average mass accretion rate, halo formation time and so on.

The relationship between the unconditional mass function and the first-crossing distribution associated with barrier-crossing random walks has been extended to obtain the conditional mass function of haloes by Bond et al. (1991) and Lacey & Cole (1993) within the spherical collapse (the so-called extended Press–Schechter model). The two-barrier first-rate probability has a simple analytic form in the constant barrier spherical collapse model. Again, this is because the random walk performed by the smoothed density contrast is Markovian. For a moving barrier (such as the ellipsoidal collapse model), however, exact analytic forms have been found only for the special case of a linear barrier (Sheth 1998) while Lam & Sheth (2009) have proposed a better motivated approach to the problem by offering a Taylor series-like approximation for a general moving barrier, see also Giocoli et al. (2007). Zhang, Ma & Fakhouri (2008) have provided analytical expressions for the two-barrier first-crossing rate for the ellipsoidal collapse and Gaussian initial conditions. In this paper we compute this conditional probability using the path-integral formulation for a generic moving barrier for Gaussian and non-Gaussian initial conditions. A by-product of such a calculation is the determination of the halo formation time probability.

The paper is organized as follows. In Section 2 we summarize the basic ingredients for the calculation of the first-crossing rate from the excursion set theory and a generic moving barrier. In Section 3 we compute the conditional probability necessary to deduce the halo bias parameters, both in the Gaussian and non-Gaussian case. Section 4 contains the computation of the two-barrier first-crossing rate for a generic moving barrier and again for both Gaussian and non-Gaussian initial conditions. In Section 5 we present our results for the halo formation time probability. Finally, Section 6 contains our conclusions and a summary of the main results, while some technical material is collected in the appendices. In particular, Appendix A contains some useful numerical fits, while Appendix B contains the computation of the two-barrier first crossing rate including the non-Markovian effects coming from the choice of a top-hat filter in real space.

2 PATH INTEGRAL FORMULATION OF EXCURSION SET THEORY FOR A MOVING BARRIER

Let us discuss the basic points of the original formulation of excursion set theory for a moving barrier. We will closely follow MR1 and De Simone et al. (2010); at the expense of being repetitive, we will report here various details that the reader can find in these references. This will hopefully help to follow and speed up the calculations of the subsequent sections.

In the excursion set theory, one considers the density field δ smoothed over a radius R, and studies its stochastic evolution as a function of the smoothing scale R. As it was found in the classical paper by Bond et al. (1991), when the density $\delta(R)$ is smoothed with a sharp filter in wavenumber space, and the density fluctuations have Gaussian statistics, the smoothed density field satisfies the equation

$$\frac{\partial\delta(S)}{\partial S} = \eta(S)\,, \tag{4}$$

where $S = \sigma^2(R)$ is the variance of the linear density field smoothed on the scale R and computed with a sharp filter in wavenumber space, while $\eta(S)$ is a stochastic variable that satisfies

$$\langle\eta(S_1)\eta(S_2)\rangle = \delta_D(S_1 - S_2)\,, \tag{5}$$

where δ_D denotes the Dirac-delta function. Equations (4) and (5) are the same as a Langevin equation with a Dirac-delta noise $\eta(S)$, with the variance S formally playing the role of time. Let us denote by $\Pi(\delta, S)\,\mathrm{d}\delta$ the probability density that the variable $\delta(S)$ reaches a value between δ and $\delta + \mathrm{d}\delta$ by 'time' S. In the general non-Markovian case it is not possible to derive a simple, local, differential equation for $\Pi(\delta, S)$ [indeed, it can be shown that $\Pi(\delta, S)$ rather satisfies a complicated integro-differential equation which is non-local with respect to 'time' S, see equation (83) of MR1], so one cannot proceed as in the Markovian case where, as we will review below, $\Pi(\delta, S)$ is determined by the solution of the Fokker–Planck equation with appropriate boundary conditions. Rather, we construct the probability distribution $\Pi(\delta, S)$ directly by summing over all paths that never exceeded the corresponding threshold, i.e. by writing $\Pi(\delta, S)$ as a path integral with a boundary. To obtain such a representation, we consider an ensemble of trajectories all starting at $S_0 = 0$ from an initial position $\delta(0) = \delta_0$ and we follow them for a 'time' S. We discretize the interval $[0, S]$ in steps $\Delta S = \epsilon$, so $S_k = k\epsilon$ with $k = 1, \ldots, n$ and $S_n \equiv S$. A trajectory is then defined by the collection of values $\{\delta_1, \ldots, \delta_n\}$, such that $\delta(S_k) = \delta_k$ and $B(S_i) = B_i$. The probability density in the space of trajectories is

$$W(\delta_0; \delta_1, \ldots, \delta_n; S_n) \equiv \langle\delta_D(\delta(S_1) - \delta_1)\cdots\delta_D(\delta(S_n) - \delta_n)\rangle\,, \tag{6}$$

where δ_D denotes the Dirac delta. Then the probability of arriving in δ_n in a 'time' S_n, starting from an initial value δ_0, without ever

going above the threshold, is (Bond et al. 1991)

$$\Pi_{\rm mb}(\delta_n; S_n) \equiv \int_{-\infty}^{B_1} {\rm d}\delta_1 \cdots \int_{-\infty}^{B_{n-1}} {\rm d}\delta_{n-1} \times W(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n). \quad (7)$$

The label 'mb' in $\Pi_{\rm mb}$ stands for moving barrier. The function $W(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n)$ can be expressed in terms of the connected correlators of the theory,

$$W(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \int \mathcal{D}\lambda \, {\rm e}^Z \,, \quad (8)$$

where

$$\int \mathcal{D}\lambda \equiv \int_{-\infty}^{\infty} \frac{{\rm d}\lambda_1}{2\pi} \cdots \frac{{\rm d}\lambda_n}{2\pi} \,, \quad (9)$$

and

$$Z = i \sum_{i=1}^{n} \lambda_i \delta_i + \sum_{p=2}^{\infty} \frac{(-i)^p}{p!} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \lambda_{i_1} \cdots \lambda_{i_p} \langle \delta_{i_1} \cdots \delta_{i_p} \rangle_c \,. \quad (10)$$

Here $\langle \delta_1 \cdots \delta_n \rangle_{\rm c}$ denotes the connected n-point correlator. So

$$\Pi_{\rm mb}(\delta_0; \delta_n; S_n) = \int_{-\infty}^{B_1} {\rm d}\delta_1 \cdots \int_{-\infty}^{B_{n-1}} {\rm d}\delta_{n-1} \int \mathcal{D}\lambda \, {\rm e}^Z \,. \quad (11)$$

When $\delta(S)$ satisfies equations (4) and (5) (which is the case for sharp filter in wavenumber space) the two-point function can be easily computed, and is given by

$$\langle \delta(S_i) \delta(S_j) \rangle = \min(S_i, S_j) \,. \quad (12)$$

In the rest of this section we will restrict ourselves to the Gaussian and Markovian case.

2.1 The case of Markovian noise

Taking the derivative with respect to the time $S_n \equiv S$ of equation (11) and using the fact that, when multiplying $\exp\{i\sum_i \lambda_i \delta_i\}$, $i\lambda_j$ ($j = 1, \ldots, n$) can be replaced $\partial_j \equiv \partial/\partial\delta^j$, we discover that $\Pi_{\rm mb}(\delta_n; S_n)$ satisfies the Fokker–Planck (FP) equation

$$\frac{\partial \Pi_{\rm mb}(\delta_n; S_n)}{\partial S_n} = \frac{1}{2} \frac{\partial^2 \Pi_{\rm mb}(\delta_n; S_n)}{\partial \delta^2} \,. \quad (13)$$

In the continuum limit, the boundary condition to be imposed on the solution of equation (13) is

$$\Pi_{\rm mb}(\delta_n; S_n) = 0 \qquad \text{for} \quad \delta_n \geq B_n. \quad (14)$$

In the continuum limit the first-crossing rate is then given by

$$\mathcal{F}_{\rm mb}(S_n) = -\frac{\partial}{\partial S} \int_{-\infty}^{B_n} {\rm d}\delta_n \, \Pi_{\rm mb}(\delta_n; S_n) = -\frac{{\rm d}B_n}{{\rm d}S_n} \Pi_{\rm mb}(B_n, S_n) - \int_{-\infty}^{B_n} {\rm d}\delta_n \, \frac{\partial \Pi_{\rm mb}(\delta_n; S_n)}{\partial S_n}. \quad (15)$$

The first term on the right-hand side vanishes because of the boundary condition, while the second term can be written in a more convenient form using the FP equation (13), so

$$\mathcal{F}_{\rm mb}(S_n) = -\frac{1}{2} \int_{-\infty}^{B_n} {\rm d}\delta \, \frac{\partial^2 \Pi_{\rm mb}(\delta_n; S_n)}{\partial \delta^2} = -\frac{1}{2} \left. \frac{\partial \Pi_{\rm mb}(\delta_n; S_n)}{\partial \delta_n} \right|_{\delta = B_n} . \quad (16)$$

To compute the probability $\Pi_{\rm mb}(\delta_n, S_n)$ we proceed in the following way. At every ith step of the path integral we Taylor expand the barrier around its final value, as first suggested by Lam & Sheth (2009),

$$B_i = B_n + \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} \, (S_i - S_n)^p \,, \quad (17)$$

where

$$B_n^{(p)} \equiv \frac{{\rm d}^p B(S_n)}{{\rm d}S_n^p}, \quad (18)$$

[so in particular $B_n^{(0)} = B(S_n)$]. We now perform a shift in the integration variables δ_i ($i = 1, \ldots, n-1$) in the path integral

$$\delta_i \to \delta_i - \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} \, (S_i - S_n)^p \,. \quad (19)$$

Then $\Pi_{\rm mb}(\delta_n; S_n)$ can be written as

$$\Pi_{\rm mb}(\delta_n; S_n) = \int_{-\infty}^{B_n} {\rm d}\delta_1 \cdots \int_{-\infty}^{B_n} {\rm d}\delta_{n-1} \int \mathcal{D}\lambda \, {\rm e}^Z, \quad (20)$$

where

$$Z = i \sum_{i=1}^{n} \lambda_i \delta_i - \frac{1}{2} \sum_{i,j=1}^{n} \lambda_i \lambda_j \min(S_i, S_j) + i \sum_{i=1}^{n-1} \lambda_i \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} \, (S_i - S_n)^p \,. \quad (21)$$

We next expand

$$\exp\left\{ i \sum_{i=1}^{n-1} \lambda_i \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} \, (S_i - S_n)^p \right\} \simeq 1 + i \sum_{i=1}^{n-1} \lambda_i \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} \, (S_i - S_n)^p - \frac{1}{2} \sum_{i,j=1}^{n-1} \lambda_i \lambda_j \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{p! q!} \, (S_i - S_n)^p \left(S_j - S_n\right)^q + \cdots, \quad (22)$$

and we write $\Pi_{\rm mb}(\delta_n; S_n)$ as

$$\Pi_{\rm mb}(\delta_n; S_n) = \Pi_{\rm mb}^{(0)}(\delta_n; S_n) + \Pi_{\rm mb}^{(1)}(\delta_n; S_n) + \Pi_{\rm mb}^{(2)}(\delta_n; S_n) + \cdots \,. \quad (23)$$

For the zeroth order term $\Pi_{\rm mb}^{(0)}$ we can immediately take the continuum limit, using the results of MR1, and we get the standard probability density of excursion set theory in the Markovian and Gaussian case,

$$\Pi_{\rm mb}^{(0)}(\delta_n; S_n) = \frac{1}{\sqrt{2\pi S_n}} \left[{\rm e}^{-\delta_n^2/(2S_n)} - {\rm e}^{-(2B_n - \delta_n)^2/(2S_n)} \right] . \quad (24)$$

The terms $\Pi_{\rm mb}^{(1)}$ and $\Pi_{\rm mb}^{(2)}$ are given by

$$\Pi_{\rm mb}^{(1)}(\delta_n; S_n) = \sum_{i=1}^{n-1} \int_{-\infty}^{B_n} {\rm d}\delta_1 \cdots {\rm d}\delta_{n-1} \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} \times (S_i - S_n)^p \, \partial_i W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n) \,, \quad (25)$$

and

$$\Pi_{\rm mb}^{(2)}(\delta_n; S_n) = \frac{1}{2} \sum_{i,j=1}^{n-1} \int_{-\infty}^{B_n} {\rm d}\delta_1 \cdots {\rm d}\delta_{n-1} \sum_{p,q=1}^{\infty} \frac{B_n^{(p)} B_n^{(q)}}{p! q!} \times (S_i - S_n)^p \left(S_j - S_n\right)^q \partial_i \partial_j W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n) \,, \quad (26)$$

where

$$W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_n;S_n)=\frac{1}{(2\pi\epsilon)^{n/2}}\,{\rm e}^{-\frac{1}{2\epsilon}\sum_{i=0}^{n-1}(\delta_{i+1}-\delta_i)^2}, \tag{27}$$

and superscript 'gm' (Gaussian–Markovian) reminds us that this value of W is computed for Gaussian fluctuations, and when the evolution with respect to the smoothing scale is Markovian. Their continuum limit is more subtle, and can be computed using the technique developed in MR1, as we review below.

We have therefore formally expanded $\Pi_{\rm mb}(\delta_n, S_n)$ in a series of terms $\Pi^{(1)}_{\rm mb}$, $\Pi^{(2)}_{\rm mb}$, etc. in which each term is itself given by an infinite sum over indices $p, q, \ldots$ We have to evaluate the continuum limit of objects such as

$$\sum_{i=1}^{n-1}F(S_i)\int_{-\infty}^{B_n}{\rm d}\delta_1\cdots{\rm d}\delta_{n-1}\,\partial_i W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_n;S_n)\,, \tag{28}$$

where F denotes a generic function. To compute this expression we integrate ∂_i by parts,

$$\begin{aligned}&\int_{-\infty}^{B_n}{\rm d}\delta_1\cdots{\rm d}\delta_{n-1}\,\partial_i W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_n;S_n)\\&\quad=\int_{-\infty}^{B_n}{\rm d}\delta_1\cdots\widehat{{\rm d}\delta_i}\cdots{\rm d}\delta_{n-1}\\&\qquad\times W(\delta_0;\delta_1,\ldots,\delta_i=B_n,\ldots,\delta_{n-1},\delta_n;S_n)\,,\end{aligned} \tag{29}$$

where the notation $\widehat{{\rm d}\delta_i}$ means that we must omit ${\rm d}\delta_i$ from the list of integration variables. We next observe that $W^{\rm gm}$ satisfies

$$\begin{aligned}&W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_i=B_n,\ldots,\delta_n;S_n)\\&\quad=W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_{i-1},B_n;S_i)\\&\qquad\times W^{\rm gm}(B_n;\delta_{i+1},\ldots,\delta_n;S_n-S_i)\,,\end{aligned} \tag{30}$$

as can be verified directly from its explicit expression (27). Then

$$\begin{aligned}&\int_{-\infty}^{B_n}{\rm d}\delta_1\cdots{\rm d}\delta_{i-1}\int_{-\infty}^{B_n}{\rm d}\delta_{i+1}\cdots{\rm d}\delta_{n-1}\\&\qquad\times W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_{i-1},B_n;S_i)\\&\qquad\times W^{\rm gm}(B_n;\delta_{i+1},\ldots,\delta_n;S_n-S_i)\\&\quad=\Pi^{\rm gm}(\delta_0;B_n;S_i)\Pi^{\rm gm}(B_n;\delta_n;S_n-S_i),\end{aligned} \tag{31}$$

and to compute the expression given in equation (28) we must compute objects such as

$$\sum_{i=1}^{n-1}F(S_i)\Pi^{\rm gm}(\delta_0;B_n;S_i)\Pi^{\rm gm}(B_n;\delta_n;S_n-S_i). \tag{32}$$

We then need to know $\Pi^{\rm gm}(\delta_0;B_n;S_i)$. By definition, in the continuum limit this quantity vanishes, since its second argument is equal to the the threshold value B_n. However, in the continuum limit the sum over i becomes $1/\epsilon$ times an integral over an intermediate time variable S_i,

$$\sum_{i=1}^{n-1}\to\frac{1}{\epsilon}\int_o^{S_n}{\rm d}S_i, \tag{33}$$

so we need to know how $\Pi^{\rm gm}(\delta_0;B_n;S_i)$ approaches zero when $\epsilon\to 0$. In MR1 it was proven that it vanishes as $\sqrt{\epsilon}$, and that

$$\Pi^{\rm gm}(\delta_0;B_n;S_n)=\sqrt{\epsilon}\,\frac{B_n-\delta_0}{\sqrt{\pi}\,S_n^{3/2}}{\rm e}^{-(B_n-\delta_0)^2/(2S_n)}+\mathcal{O}(\epsilon). \tag{34}$$

Similarly, for $\delta_n<B_n$,

$$\Pi^{\rm gm}(B_n;\delta_n;S_n)=\sqrt{\epsilon}\,\frac{B_n-\delta_n}{\sqrt{\pi}\,S_n^{3/2}}{\rm e}^{-(B_n-\delta_n)^2/(2S_n)}+\mathcal{O}(\epsilon). \tag{35}$$

In the following, we will also need the expression for $\Pi^{\rm gm}$ with the first and second argument both equal to B_n, which is given by (see again MR1)

$$\Pi^{\rm gm}(B_n;B_n;S)=\frac{\epsilon}{\sqrt{2\pi}S_n^{3/2}}\,. \tag{36}$$

In order to finalize the computation, we must either perform some approximation, or identify a suitable small parameter, and organize the terms in a systematic expansion in such a small parameter. In De Simone et al. (2010) we have discussed in detail two different expansion techniques (one based on a systematic expansion in derivatives for a slowly varying barrier, and the other in which a large number of terms are resummed), which were shown to provide very close numerical results. Furthermore, it was found that the results obtained with these systematic expansions are in the end numerically very close to that obtained with a simpler albeit more empirical procedure, which amounts to approximating $(S_n-S_i)^{p-1}\simeq S_n^{p-1}$ inside the integrals in equations (25) and (26), and at the same time truncating the sum over p in equation (25) to $p=5$ [while, in this approximation, $\Pi^{(2)}_{\rm mb}$ does not contribute]. This is in fact equivalent to the approximation made by Sheth & Tormen (2002) (ST in the following) and gives the same results. However, as discussed in Section 3.1 of De Simone et al. (2010), when one makes the approximation $(S_n-S_i)^{p-1}\simeq S_n^{p-1}$, one must also necessarily truncate the sum to a maximum value, otherwise the first-crossing rate resums to a trivial result, where all corrections due to the ellipsoidal barrier disappear. Therefore, the procedure of replacing $(S_n-S_i)^{p-1}\to S_n^{p-1}$ inside the integrals and, *at the same time*, truncating the sum, must be viewed as a simple heuristic procedure to get a result which is numerically close to the result of more systematic expansions. Since this procedure is technically much simpler than the systematic expansions discussed in De Simone et al. (2010), and works well numerically, we will adopt it in the following.

We first compute $\Pi^{(1)}_{\rm mb}$. Before performing the above approximation, the expression of $\Pi^{(1)}_{\rm mb}(\delta_n;S_n)$ in equation (25) can be rewritten as

$$\begin{aligned}\Pi^{(1)}_{\rm mb}(\delta_n;S_n)=\frac{B_n(B_n-\delta_n)}{\pi}&\sum_{p=1}^{\infty}\frac{(-1)^p}{p!}B_n^{(p)}\\&\times\int_0^{S_n}{\rm d}S_i\,\frac{(S_n-S_i)^{p-(3/2)}}{S_i^{3/2}}\\&\times{\rm e}^{-B_n^2/(2S_i)}{\rm e}^{-(B_n-\delta_n)^2/[2(S_n-S_i)]}.\end{aligned} \tag{37}$$

Since this integral is finite in the limit $\delta_n\to B_n$, taking the approximation $(S_n-S_i)^{p-1}\simeq(S_n)^{p-1}$ does not alter the convergence properties of the integral, but simplifies significantly its computation, since

$$\begin{aligned}&\int_0^{S_n}{\rm d}S_i\,\frac{1}{S_i^{3/2}(S_n-S_i)^{1/2}}\\&\qquad\times{\rm e}^{-B_n^2/(2S_i)}{\rm e}^{-(B_n-\delta_n)^2/(2(S_n-S_i))}\\&\quad=\frac{\sqrt{2\pi}}{B_n}\frac{1}{S_n^{1/2}}\exp\left\{-\frac{(2B_n-\delta_n)^2}{2S_n}\right\},\end{aligned} \tag{38}$$

so in this approximation $\Pi^{(1)}_{\rm mb}(\delta_n;S_n)$ is given by

$$\begin{aligned}\Pi^{(1,{\rm ST})}_{\rm mb}(\delta_n;S_n)=\frac{2(B_n-\delta_n)}{\sqrt{2\pi}S_n^{3/2}}&{\rm e}^{-(2B_n-\delta_n)^2/(2S_n)}\\&\times\sum_{p=1}^{5}\frac{(-S_n)^p}{p!}B_n^{(p)},\end{aligned} \tag{39}$$

where the superscript 'ST' reminds us that we have performed the approximations that are equivalent to those which give the ST mass function. A reason why this approximation works well is that [at least for what concerns $\Pi^{(1)}_{\rm mb}$] the terms which are neglected give contributions proportional to higher powers of $(B_n - \delta_n)$. Since in the end the mass function is obtained from the first-crossing rate (16), we actually only need the first derivative of $\Pi_{\rm mb}(\delta_n; S_n)$ evaluated at $\delta_n = B_n$, and terms proportional to $(B_n - \delta_n)^N$ with $N \geq 2$ give a vanishing contribution.

Higher order contributions to the first-crossing rate vanish. In fact, in the same approximation one finds (De Simone et al. 2010) that $\Pi^{(n,{\rm ST})}_{\rm mb}$ vanishes as $(B_n - \delta_n)^n$ for $\delta_n \to B_n$, so its first derivative $\partial \Pi^{(n),{\rm ST}}_{\rm mb}/\partial\delta_n$ evaluated in $\delta_n = B_n$, which according to equation (16) gives its contribution to the first-crossing rate, vanishes for all $n \geq 2$.

The total first-crossing rate for a moving barrier, in the approximation discussed above, is therefore given by

$$\mathcal{F}_{\rm mb}({\rm S_n}) = \frac{{\rm e}^{-B_n^2/(2S_n)}}{\sqrt{2\pi} S_n^{3/2}} \sum_{p=0}^{5} \frac{(-S_n)^p}{p!} \frac{\partial^p B_n}{\partial S_n^p} = \frac{{\rm e}^{-B_n^2/(2S_n)}}{\sqrt{2\pi} S_n^{3/2}} \left(B_n + \mathcal{P}(S_n)\right), \tag{40}$$

where

$$\mathcal{P}(S_n) \equiv \mathcal{P}_n = \sum_{p=1}^{5} \frac{(-S_n)^p}{p!} \frac{\partial^p B_n}{\partial S_n^p}. \tag{41}$$

When applied to the ellipsoidal barrier given in equation (2), one recovers the ellipsoidal collapse result of Sheth & Tormen (2002).

$$\mathcal{F}_{\rm ST}(S_n) \simeq \frac{\sqrt{a}\,\delta_c(z)}{\sqrt{2\pi} S_n^{3/2}} {\rm e}^{-B_n^2/(2S_n)} \left[1 + 0.4 \sum_{p=0}^{5} (-1)^p \binom{0.6}{p} \left(\frac{S_n}{a\delta_{\rm c}^2(z)}\right)^{0.6}\right] = \frac{\sqrt{a}\,\delta_{\rm c}(z)}{\sqrt{2\pi} S_n^{3/2}} {\rm e}^{-B_n^2/(2S)} \left[1 + 0.067 \left(\frac{S_n}{a\delta_{\rm c}^2(z)}\right)^{0.6}\right]. \tag{42}$$

As it is well known (Sheth & Tormen 2002), this first-crossing rate is not normalized to unity. This is a basic difference between the moving barrier and the constant (spherical) barrier model. When the barrier height is constant, all random walks are guaranteed to cross the barrier because the rms height of random walks at S_n is proportional to $\sqrt{S_n}$. At sufficiently large S_n, all walks will have crossed the constant barrier. In the moving barrier case, in which the barrier diverges when $S_n \to \infty$, not all trajectories intersect it. This is because the rms height of the random walk grows more slowly than the rate at which the barrier height increases and there is no guarantee that all random walks will intercept the barrier. It seems reasonable to associate the fraction of random walks that do not cross the barrier with the particles that in N-body simulations are not associated to bound states (Sheth & Tormen 2002).

After this rather long and technical summary of how to compute the first-crossing rate for a generic moving barrier, we are ready to compute conditional probabilities.

3 HALO BIAS

We now apply the technique of the previous section to the computation of the halo bias, including the non-Markovian corrections coming from the NG. We will use a top-hat window function in wavenumber space. The calculation of the non-Markovian effects on the bias from a top-hat window function in real space can be found in Ma et al. (2010).

3.1 Conditional probability: the moving barrier case and Gaussian initial conditions

We begin our analysis with the simpler case in which the density field is Gaussian. Since we are also taking a top-hat filter in wavenumber space, the evolution is the Markovian. To compute the bias, we need the probability of forming a halo of mass M, corresponding to a smoothing radius R, under the condition that the smoothed density contrast on a much larger scale R_m has a specified value $\delta_m = \delta(R_m)$. We use $\mathcal{F}_{\rm mb}(S_n|\delta_m, S_m)$ to denote the corresponding conditional first-crossing rate, i.e. the rate at which trajectories first cross the barrier $\delta = B(S)$ at time S_n, under the condition that they passed through the point $\delta = \delta_m$ at an earlier time S_m. We also use the notation $\mathcal{F}_{\rm mb}(S_n|0) \equiv \mathcal{F}_{\rm mb}(S_n|\delta_m = 0, S_m = 0)$, so $\mathcal{F}_{\rm mb}(S_n|0)$ is the first-crossing rate when the density approaches the cosmic mean value on very large scales.

The halo overdensity in Lagrangian space is given by (Mo & White 1996; see also Zentner 2007 for a review)

$$1 + \delta^L_{\rm halo} = \frac{\mathcal{F}_{\rm mb}(S_n|\delta_m, S_m)}{\mathcal{F}_{\rm mb}(S_n|0)}. \tag{43}$$

The relevant quantity for our purposes is the halo conditional probability

$$\Pi_{\rm halo}(\delta_n, S_n|\delta_m, S_m) \equiv \frac{\int_{-\infty}^{B_1} {\rm d}\delta_1 \cdots \widehat{{\rm d}\delta_m} \cdots \int_{-\infty}^{B_{n-1}} {\rm d}\delta_{n-1} W(\delta_0 = 0; \delta_1, \ldots, \delta_n; S_n)}{\int_{-\infty}^{B_1} {\rm d}\delta_1 \cdots \int_{-\infty}^{B_{m-1}} {\rm d}\delta_{m-1} W(\delta_0 = 0; \delta_1, \ldots, \delta_m; S_m)}, \tag{44}$$

where the hat over ${\rm d}\delta_m$ means that ${\rm d}\delta_m$ must be omitted from the list of integration variables. The numerator is a sum over all trajectories that start from $\delta_0 = 0$ at $S = 0$, have a given fixed value δ_m at S_m, and a value δ_n at S_n, while all other points of the trajectory, $\delta_1, \ldots, \delta_{m-1}, \delta_{m+1}, \ldots \delta_{n-1}$ are integrated up to the corresponding value of barrier, and we use the notation $B_i \equiv B(S_i)$. The denominator gives the appropriate normalization to the conditional probability.

The conditional first-crossing rate $\mathcal{F}_{\rm mb}(S_n|\delta_m, S_m)$ is obtained from the conditional probability $\Pi_{\rm halo}(\delta_n, S_n|\delta_m, S_m)$ using

$$\mathcal{F}_{\rm mb}(S_n|\delta_m, S_m) = -\frac{\partial}{\partial S_n} \int_{-\infty}^{B_n} {\rm d}\delta_n\, \Pi_{\rm halo}(\delta_n, S_n|\delta_m, S_m). \tag{45}$$

Since we are considering the Gaussian case, with a top-hat filter in wavenumber space, the probability density W factorizes:

$$W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_m, \ldots, \delta_n; S_n) = W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_{m-1}, \delta_m; S_m) \times W^{\rm gm}(\delta_m; \delta_{m+1}, \ldots, \delta_n; S_n - S_m), \tag{46}$$

and the halo probability $\Pi_{\rm halo}(\delta_n, S_n|\delta_m, S_m)$ in equation (44) becomes identical to the probability of arriving in δ_n at time S_n, starting from δ_m at time S_m for the moving barrier, reflecting the fact that the evolution of $\delta(S)$ is in this case Markovian. We can then compute $\Pi_{\rm halo}$ as in the previous section, performing a shift of the remaining integration variables δ_i with $i = (m+1, \ldots, n-1)$,

$$\delta_i \to \delta_i - \sum_{p=1}^{\infty} \frac{B_n^{(p)}}{p!} (S_i - S_n)^p, \tag{47}$$

and we get

$$\Pi_{\rm halo}(\delta_n, S_n|\delta_m, S_m) = \frac{1}{\sqrt{2\pi(S_n-S_m)}}\left(\mathrm{e}^{-(\delta_n-\delta_m)^2/(2(S_n-S_m))} - \mathrm{e}^{-(2B_n-\delta_n-\delta_m)^2/(2(S_n-S_m))}\right) + \frac{2(B_n-\delta_n)}{\sqrt{2\pi}(S_n-S_m)^{3/2}}\mathrm{e}^{-(2B_n-\delta_n-\delta_m)^2/(2(S_n-S_m))}\mathcal{P}_{mn} - \frac{2(B_n-\delta_n)^2}{\sqrt{2\pi}(S_n-S_m)^{5/2}}\mathrm{e}^{-(2B_n-\delta_n-\delta_m)^2/(2(S_n-S_m))}\mathcal{P}_{mn}^2\,, \quad (48)$$

where

$$\mathcal{P}_{mn} \equiv \mathcal{P}(S_m, S_n) = \sum_{p=1}^{\infty}\frac{(S_m-S_n)^p}{p!}B_n^{(p)}\,. \quad (49)$$

In the following, we will use this quantity with the sum truncated to $p = 5$, as discussed in Section 2 and as advocated by Sheth & Tormen (2002) for the conditional mass function.

The calculation of the conditional first-crossing rate $\mathcal{F}_{\rm mb}(S_n|\delta_m, S_m)$ proceeds by taking the derivative with respect to S_n

$$\mathcal{F}_{\rm mb}(S_n|\delta_m, S_m) = \frac{(B_n-\delta_m)+\mathcal{P}(S_m,S_n)}{\sqrt{2\pi}(S_n-S_m)^{3/2}}\mathrm{e}^{-\frac{(B_n-\delta_m)^2}{2(S_n-S_m)}}\,. \quad (50)$$

In a sufficiently large region $S_m \ll S_n$ and $\delta_m \ll \delta_n$. Then, expanding to quadratic order in δ_m and after mapping to Eulerian space, we find the first two Eulerian bias coefficients

$$b_1 \simeq 1 + \frac{B_n}{S_n} - \frac{1}{B_n+\mathcal{P}(0,S_n)}\,. \quad (51)$$

and

$$b_2 \simeq \frac{B_n^2}{S_n^2} - \frac{1}{S_n} - 2\frac{B_n}{(B_n+\mathcal{P}(0,S_n))S_n}\,. \quad (52)$$

The above results hold for a generic barrier. We now examine it for different collapse models.

3.1.1 Constant barrier

We first apply these results to the spherical collapse model where the barrier is constant $\delta_c(z)$. Using the standard notation $\nu \equiv \delta_c(z)/\sigma$ we get, for the bias coefficients,

$$b_1 \simeq 1 + \frac{\nu^2}{\delta_c} - \frac{1}{\delta_c}\,, \quad (53)$$

and

$$b_2 \simeq \frac{\nu^4}{\delta_c^2} - 3\frac{\nu^2}{\delta_c^2}\,. \quad (54)$$

3.1.2 Ellipsoidal barrier

For the the ellipsoidal barrier (2), we get, for the bias coefficients,

$$b_1 \simeq 1 + \sqrt{a}\frac{\nu^2}{\delta_c}\left[1+0.4\left(\frac{1}{a\nu^2}\right)^{0.6}\right] - \frac{1}{\sqrt{a}\delta_c\left[1+0.067\left(\frac{1}{a\nu^2}\right)^{0.6}\right]}\,. \quad (55)$$

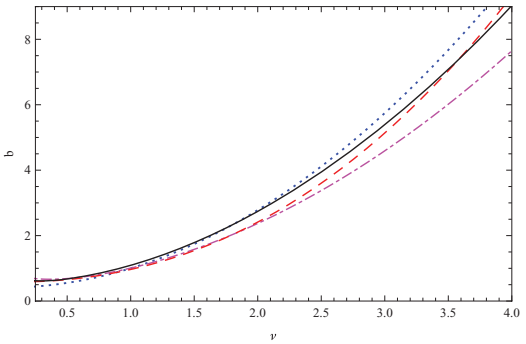

Figure 1. The halo bias at linear order as a function of $\nu = \delta_c/\sigma$. Our formula in equation (55) (solid black line) is compared to other results: the fit to N-body simulations as in Tinker et al. (2010) (dashed red), and two standard approximations: spherical (dotted blue) and the SMT expression for the bias (dot–dashed magenta), which is also predicted by the diffusing barrier model of Section 3.1.3.

and

$$b_2 \simeq a\frac{\nu^4}{\delta_c^2}\left[1+0.4\left(\frac{1}{a\nu^2}\right)^{0.6}\right]^2 - \frac{\nu^2}{\delta_c^2} - 2\frac{\nu^2}{\delta_c^2}\frac{1+0.4\left(\frac{1}{a\nu}\right)^{0.6}}{1+0.067\left(\frac{1}{a\nu^2}\right)^{0.6}}\,. \quad (56)$$

For large masses, $\nu^2 \gg 1$, the bias coefficient scales like

$$b_1 \simeq \sqrt{a}\,\frac{\nu^2}{\delta_c}\,. \quad (57)$$

Observe that this result differs from that proposed by Sheth et al. (2001), which in the large mass limit rather scales like $b_1 \sim a(\nu^2/\delta_c)$, i.e. it is smaller than (57) by a factor $\sqrt{a}$. The proportionality to $\sqrt{a}$ in equation (57) can be traced to the fact that, in the large mass limit, b_1 is dominated by the term B_n/S_n in equation (51) and, for the barrier given in equation (2), B_n is proportional to $\sqrt{a}$. In other words, this dependence is a consequence of the fact that, in the large mass limit, the barrier of Sheth et al. (2001) does not reduce to the spherical collapse model barrier δ_c, but rather to $\sqrt{a}\delta_c$, so the large mass limit (57) can be formally obtained from the spherical collapse model by performing the replacement $\delta_c \to \sqrt{a}\delta_c$. For Sheth et al. (2001), the physics of spherical collapse always gives δ_c, not $\sqrt{a}\delta_c$, so it is really only the variances which were rescaled, $\sigma^2 \to \sigma^2/a$. Since the unconditional mass functions are the same in our case and in the one discussed by Sheth et al. (2001), but the bias factors are not, this means that one could, in principle, discriminate between whether a should be thought of as rescaling δ_c or σ^2.

In Fig. 1 we show our result (55) for the bias and compare it to the fit to N-body simulations of Tinker et al. (2010) and to two standard approximations: spherical collapse and the SMT result. Our result for the bias is manifestly in better agreement with N-body simulations than the commonly-used approximations.

3.1.3 The diffusing barrier

In MR2 it has been proposed a model in which the barrier performs a diffusing motion, with diffusion coefficient D_B, around an average value which for the spherical collapse model is simply $\delta_c(z)$ (without any factor of $\sqrt{a}$) while for the ellipsoidal model is given

$$B(S_n) \simeq \delta_c(z)\left[1+0.4\left(\frac{S_n}{a\delta_c^2(z)}\right)^{0.6}\right]\,, \quad (58)$$

i.e. by equation (2) without the factor $\sqrt{a}$. Equation (58) is in fact the barrier which actually follows for an ellipsoidal model for collapse, while the factor $\sqrt{a}$ in equation (2) was simply inserted by hand, in order to fit the data. Including the stochastic motion of the barrier, which is meant to mimic a number of random effects in the process of halo formation, it was found in MR2 that the relative motion of the trajectory $\delta(S)$ and of the barrier is a stochastic motion with diffusion constant $(1 + D_B)$ so that, for instance, in the Markovian case the evolution of the probability distribution is governed by a Fokker-Planck equation of the form

$$\frac{\partial \Pi_{\rm mb}(\delta_n; S_n)}{\partial S_n} = \frac{(1 + D_B)}{2} \frac{\partial^2 \Pi_{\rm mb}(\delta_n; S_n)}{\partial \delta^2} . \tag{59}$$

The result for this diffusive barrier can therefore be obtained from the result for the spherical collapse barrier, or from the result for the barrier (58), by formally rescaling $S \to (1/a)S$, where $a = 1/(1 + D_B)$. For the halo mass function, which depends only on the combination $\nu = \delta_c/\sigma = \delta_c/\sqrt{S}$, the rescaling $S_n \to (1/a)S_n$ is equivalent to the rescaling $\delta_c \to \sqrt{a}\delta_c$, and therefore the diffusive barrier model produces the same result as that obtained with the SMT barrier (2).

This equivalence, however, does not extend to the halo bias. In fact, in the large mass limit, the halo bias deduced using the barrier (58) in equation (51) is $b_1 \simeq (\nu^2/\delta_c) = \delta_c/S_n$, and rescaling $S_n \to (1/a)S_n$ gives

$$b_1 \simeq a \, \frac{\nu^2}{\delta_c} , \tag{60}$$

which differs from equation (57) by a factor $\sqrt{a}$ and agrees with Sheth et al. (2001).

3.2 Adding non-Markovianity from a top-hat window function in real space, in the high mass limit

Furthermore, one should add to this result the non-Markovian corrections due to the use of a top-hat filter function in coordinate space. In the large mass limit, where equation (58) reduces the constant barrier, the computation of the bias was performed in Ma et al. (2010), where it was found that b_1 gets multiplied by a factor $1/(1 - a\kappa)$, so for the diffusive barrier model with Markovian corrections due to the filter we get, in the large mass limit,

$$b_1 \simeq \frac{a}{1 - a\kappa} \, \frac{\nu^2}{\delta_c} = \left(\frac{\sqrt{a}}{1 - a\kappa} \right) \sqrt{a} \, \frac{\nu^2}{\delta_c}, \tag{61}$$

where κ is a parameter that controls the non-Markovian effects of the filter function, whose numerical value can be estimated as in MR1, but which for accurate fitting is better treated as a free parameter, see Ma et al. (2010). Observe that in Ma et al. (2010) the spherical collapse result was rescaled according to $\delta_c \to \sqrt{a}\delta_c$, which actually corresponds to the replacement in the SMT barrier model and not to the diffusive barrier model, so the resulting value for the bias was incorrectly obtained to be $b_1 \simeq [\sqrt{a}/(1 - a\kappa)](\nu^2/\delta_c)$.

In particular, consider that the rescaling $\kappa \to a\kappa$, which leads to the factor $(1 - a\kappa)$ in the denominator of equation (61), is obtained assuming that the barrier performs a simple diffusive Markovian motion. In a realistic description, the actual stochastic motion can be more complicated and the reader should be aware of the limitations of this procedure. Unfortunately, we have not been able to find an analogue of the expression (61) for small masses.

3.3 Conditional probability: the moving barrier case and non-Gaussian initial conditions

Deviations from Gaussianity are encoded, e.g., in the connected three- and four-point correlation functions which are dubbed the bispectrum and the trispectrum, respectively. A phenomenological way of parametrizing the level of NG is to expand the fully non-linear primordial Bardeen gravitational potential Φ in powers of the linear gravitational potential $\Phi_{\rm L}$

$$\Phi = \Phi_{\rm L} + f_{\rm NL} \left(\Phi_{\rm L}^2 - \left\langle \Phi_{\rm L}^2 \right\rangle \right) . \tag{62}$$

The dimensionless quantity $f_{\rm NL}$ sets the magnitude of the three-point correlation function (Bartolo et al. 2004). If the process generating the primordial NG is local in space, the parameter $f_{\rm NL}$ in Fourier space is independent of the momenta entering the corresponding correlation functions; if instead the process which generates the primordial cosmological perturbations is non-local in space, like in models of inflation with non-canonical kinetic terms, $f_{\rm NL}$ acquires a dependence on the momenta. The strongest current limits on the strength of local NG set the $f_{\rm NL}$ parameter to be in the range $-4 < f_{\rm NL} < 80$ at 95 per cent confidence level (Smith, Senatore & Zaldarriaga 2010). The goal of this subsection is to compute the halo bias parameters in the presence of NG and for the ellipsoidal collapse, using the technique developed in MR3, which generalizes excursion set theory to deal with non-Gaussianities in the primordial density field. We will also work with a top-hat window function in real space.

Similarly to the Gaussian case, the probability of arriving in δ_n in a 'time' S_n, starting from the initial value $\delta_0 = 0$, without ever going above the threshold, in the presence of NG is given by

$$\Pi_{\rm mb}(\delta_n; S_n) \equiv \int_{-\infty}^{B_1} {\rm d}\delta_1 \cdots \int_{-\infty}^{B_{n-1}} {\rm d}\delta_{n-1} \times W_{\rm NG}(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n). \tag{63}$$

where

$$W_{\rm NG}(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \int \mathcal{D}\lambda \times \exp\left\{ i \sum_{i=1}^{n} \lambda_i \delta_i - \frac{1}{2} \sum_{i,j=1}^{n} \lambda_i \lambda_j \min(S_i, S_j) \right\} \times \exp\left\{ \frac{(-i)^3}{6} \sum_{i,j,k=1}^{n} \langle \delta_i \delta_j \delta_k \rangle_c \lambda_i \lambda_j \lambda_k \right\} , \tag{64}$$

and we have retained only the three-point connected correlator $\langle \delta_i \delta_j \delta_k \rangle_c$ as a signal of NG. It is now clear where the non-Markovianity is coming from when non-Gaussian initial conditions are present: expanding $W_{\rm NG}$ in powers of $\sum_{i,j,k=1}^{n} \langle \delta_i \delta_j \delta_k \rangle_c \lambda_i \lambda_j \lambda_k$ and going to the continuum, one obtains integrals over the intermediate times which introduce memory effects.

As in equation (44), the conditional probability relevant to the halo bias is obtained keeping δ_m fixed, rather than treating it as an integration variables. In principle the non-Markovian contribution to the halo probability from NG should be computed separating the various contributions to the sum according to whether an index is smaller than or equal to m, larger than m and smaller than or equal to n. Fortunately, to compute the halo bias parameters, at the end one needs to take the limit $S_m \ll S_n$ and $\delta_m \ll \delta_n$. In this limit we can the safely neglect the contribution to the sum from all indices running from 1 to m, since in general $\langle \delta_m^p \delta_n{}^q \rangle$ scales like $(S_m^{p/2} S_n{}^{q/2})$, where $p, q \geq 0$ and $p + q = 3$, and therefore vanishes for $S_m \to 0$. In other words, in equation (64) we can replace $\sum_{i,j,k=1}^{n}$ with

$\sum_{i,j,k=m+1}^{n}$. As usual, we then expand the exponential and use the fact that $i\lambda_k \exp\{i\sum_i \lambda_i\delta_i\} = \partial_k \exp\{i\sum_i \lambda_i\delta_i\}$, and we use the factorization property (30) (Maggiore & Riotto 2010c). This gives

$$
\begin{aligned}
&W_{\rm NG}(\delta_0;\delta_1,\ldots,\delta_n;S_n)\\
&\simeq\left[1-\frac{1}{6}\sum_{i,j,k=m+1}^{n}\langle\delta_i\delta_j\delta_k\rangle_c\partial_i\partial_j\partial_k\right]\\
&\quad\times W^{\rm gm}(\delta_0;\delta_1,\ldots,\delta_m;S_m)\\
&\quad\times W^{\rm gm}(\delta_m;\delta_{m+1},\ldots,\delta_n;S_n-S_m)\,.
\end{aligned}
\tag{65}
$$

Since now the derivatives act only on the second W factor, the first W factor factorizes and, after integration over $\delta_1,\ldots,\delta_{m-1}$, cancels the denominator in the conditional probability (44).

Next, as in MR3, we introduce the notation

$$
\begin{aligned}
&G_3^{(p,q,r)}(S_n)\\
&\equiv\left[\frac{{\rm d}^p}{{\rm d}S_i^p}\frac{{\rm d}^q}{{\rm d}S_j^q}\frac{{\rm d}^r}{{\rm d}S_k^r}\langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle_c\right]_{S_i=S_j=S_k=S_n}.
\end{aligned}
\tag{66}
$$

and we expand the correlator as

$$
\begin{aligned}
\langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle &= \sum_{p,q,r=0}^{\infty}\frac{(-1)^{p+q+r}}{p!q!r!}(S_n-S_i)^p\\
&\quad\times(S_n-S_j)^q(S_n-S_k)^r G_3^{(p,q,r)}(S_n)\,.
\end{aligned}
\tag{67}
$$

As shown in MR3, in the large mass limit (which is the most interesting regime for observing the non-Gaussianities) the leading contribution to the halo bias probability is given by the term in equation (67) with $p=q=r=0$. We neglect subleading contributions, which can be computed with the same technique developed in MR3. The discrete sum then reduces to $\langle\delta_n^3\rangle_c\sum_{i,j,k=m+1}^{n}\partial_i\partial_j\partial_k$. We can now proceed as in the previous section and perform the shift of variables (19) on δ_i, for with $i=m+1,\ldots,n-1$. The halo probability becomes

$$
\begin{aligned}
&\Pi_{\rm halo\,NG}(\delta_n,S_n|\delta_m,S_m)\\
&=\int_{-\infty}^{B_n}{\rm d}\delta_{m+1}\cdots\int_{-\infty}^{B_n}{\rm d}\delta_{n-1}\\
&\quad\times{\rm e}^{-\frac{1}{6}\langle\delta_n^3\rangle_c\sum_{i,j,k=m+1}^{n}\partial_i\partial_j\partial_k}W_{\rm mb}(\delta_m;,\ldots,\delta_{n-1},\delta_n;S_n)\,,
\end{aligned}
\tag{68}
$$

where $W_{\rm mb}$ is the probability density in the space of trajectories with a moving barrier, so that, in general,

$$
\begin{aligned}
&\int_{-\infty}^{B_n}d\delta_{m+1}\cdots\int_{-\infty}^{B_n}d\delta_{n-1}\,W_{\rm mb}(\delta_m;\delta_{m+1},\ldots,\delta_{n-1},\delta_n;S_n)\\
&=\Pi_{\rm mb}(\delta_m;\delta_n;S_n)=\Pi_{\rm mb}^{(0)}+\Pi_{\rm mb}^{(1)}+\Pi_{\rm mb}^{(2)}+\cdots,
\end{aligned}
\tag{69}
$$

where, as in equation (23), the terms $\Pi_{\rm mb}^{(0)}$, $\Pi_{\rm mb}^{(1)}$, $\Pi_{\rm mb}^{(2)}$, etc. correspond to the different orders in the expansion of the exponential in equation (22), but now for trajectories that start at δ_m, with δ_m small but finite, rather than at $\delta_0=0$.

To compute these expressions we can now use the following identity, proven in MR3,

$$
\begin{aligned}
&\sum_{i,j,k=1}^{n}\int_{-\infty}^{B_n}{\rm d}\delta_1\cdots{\rm d}\delta_n\,\partial_i\partial_j\partial_k\,W_{\rm mb}\\
&=\frac{\partial^3}{\partial B_n^3}\int_{-\infty}^{B_n}{\rm d}\delta_n\,\Pi_{\rm halo}(\delta_n,S_n|\delta_m,S_m)\,,
\end{aligned}
\tag{70}
$$

where $\Pi_{\rm halo}(\delta_n,S_n|\delta_m,S_m)$ is given in equation (48). The calculation of the conditional first-crossing rate $\mathcal{F}_{\rm mb\,NG}(S_n|\delta_m,S_m)$ proceeds by finally taking the derivative with respect to S_n:

$$
\begin{aligned}
&\mathcal{F}_{\rm mb\,NG}(S_n|\delta_m,S_m)=\frac{(B_n-\delta_m+\mathcal{P}_{mn})}{\sqrt{2\pi}(S_n-S_m)^{3/2}}{\rm e}^{-\frac{(B_n-\delta_m)^2}{2(S_n-S_m)}}\\
&+\frac{\mathcal{S}_3}{6\sqrt{2\pi}(S_n-S_m)^{5/2}}\left[(B_n-\delta_m)^4\right.\\
&-(B_n-\delta_m)^3(\mathcal{P}_{mn}+2(S_n-S_m)B_n')\\
&+2(B_n-\delta_m)^2\left(-(S_n-S_m)+\mathcal{P}_{mn}^2+(S_n-S_m)\mathcal{P}_{mn}B_n'\right)\\
&+(S_n-S_m)(B_n-\delta_m)(\mathcal{P}_{mn}+6(S_n-S_m)B_n'\\
&-4\mathcal{P}_{mn}^2B_n'-2(S_n-S_m)\mathcal{P}_{mn}')-(S_n-S_m)^2\\
&-2(S_n-S_m)\mathcal{P}_{mn}(\mathcal{P}_{mn}+(S_n-S_m)B_n'\\
&\left.-\,4(S_n-S_m)\mathcal{P}_{mn}')\right]{\rm e}^{-\frac{(B_n-\delta_m)^2}{2(S_n-S_m)}}\\
&+\frac{(S_n-S_m)^2\mathcal{S}_3'}{3\sqrt{2\pi}(S_n-S_m)^{5/2}}\left[(B_n-\delta_m)^2-(B_n-\delta_m)\mathcal{P}_{mn}\right.\\
&\left.-\,(S_n-S_m)+2\mathcal{P}_{mn}^2\right]{\rm e}^{-\frac{(B_n-\delta_m)^2}{2(S_n-S_m)}}\,,
\end{aligned}
\tag{71}
$$

where the prime stands for derivative with respect to S_n. Note that in the limit of vanishing $f_{\rm NL}$ only the first line of the above equation survives. It should also be observed that terms proportional to derivatives of the cumulant such as $\mathcal{S}_3'$ (which are subleading compared to the terms proportional to $\mathcal{S}_3$) can also come from terms with $p+q+r\geq 1$ in equation (67), that we have neglected, but which in principle can be computed as in MR3. Expanding again in powers of δ_m up to δ_m^2, we find the scale-independent NG contribution to the Eulerian bias parameters. Normalizing the bispectrum as

$$
\mathcal{S}_3(S_n)\equiv\frac{1}{S_n^2}\left\langle\delta^3(S_n)\right\rangle,
\tag{72}
$$

we find the corrections to the halo bias parameters from NG:

$$
\begin{aligned}
\Delta b_{1\,\rm NG}&\simeq-\frac{\mathcal{S}_3(S_n)}{6S_n(B_n+\mathcal{P}_n)^2}\left[3B_n^4+2B_n^3(\mathcal{P}_n-2S_nB_n')\right.\\
&-B_n^2\left(2S_n+\mathcal{P}_n^2+4S_n\mathcal{P}_nB_n'\right)\\
&+4B_n\mathcal{P}_n\left(-S_n+\mathcal{P}_n^2+S_n\mathcal{P}_nB_n'\right)\\
&+\left.S_n(S_n+\mathcal{P}_n\left(3\mathcal{P}_n+8S_nB_n'-4\mathcal{P}_n^2B_n'-10S_n\mathcal{P}_n'\right)\right]\\
&-\frac{\mathcal{S}_3'(S_n)S_n}{3(B_n+\mathcal{P}_n)^2}\left((B_n+3\mathcal{P}_n)(B_n-\mathcal{P}_n)+S_n\right),
\end{aligned}
\tag{73}
$$

and

$$
\begin{aligned}
\Delta b_{2\,\rm NG}&\simeq\frac{\mathcal{S}_3(S_n)}{3S_n^2(B_n+\mathcal{P}_n)^2}\left[-3B_n^5+B_n^4(-2\mathcal{P}_n+4B_n'S_n)\right.\\
&+B_n^3\left(\mathcal{P}_n^2+4S_n\mathcal{P}_nB_n'+8S_n\right)\\
&+B_n^2\left(7\mathcal{P}_nS_n-4\mathcal{P}_n^3-6S_n^2B_n'-4S_n\mathcal{P}_n^2B_n'\right)\\
&+B_n\left(4S_n\mathcal{P}_n^3B_n'-12S_n^2\mathcal{P}_nB_n'+10S_n^2\mathcal{P}_n\mathcal{P}_n'\right.\\
&-\left.S_n\left(3S_n+4\mathcal{P}_n^2\right)\right)+2\mathcal{P}_nS_n\left(\mathcal{P}_n^2-S_n\right)+2\mathcal{P}_n^2S_n^2B_n'\Big]\\
&-\frac{2\mathcal{S}_3'(S_n)}{3(B_n+\mathcal{P}_n)^2}\left(B_n(B_n+3\mathcal{P}_n)(B_n-\mathcal{P}_n)-S_n\mathcal{P}_n\right),
\end{aligned}
\tag{74}
$$

where $\mathcal{P}_n=\mathcal{P}(0,S_n)$. As shown in Fig. 2, these terms correct the bias calculated with Gaussian initial conditions by a few percents. In the last term we have expanded for large masses and again assumed that the barrier as well as $\mathcal{S}_3$ are slowly varying. At first one

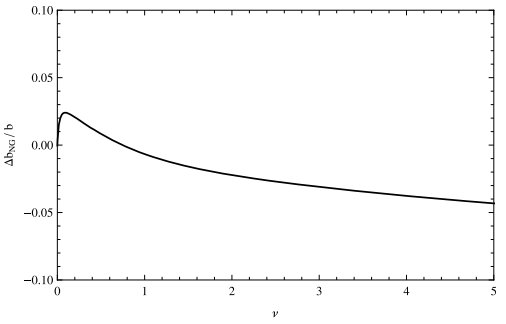

Figure 2. The magnitude of the NG correction (with $f_{\rm NL} = 100$) to the first bias parameter (equation 73), normalized to the halo bias without NG (equation 51), as a function of $\nu = \delta_c/\sigma$.

might think that such an approximation, although useful in some cases, would not apply to the barrier which corresponds to the the ellipsoidal collapse, equation (2). In this case in fact $B_{\rm ST}(S_n)$ is given by a constant plus a term proportional to S_n^γ with $\gamma \simeq 0.6 < 1$, and therefore already its first derivative, which is proportional to $S_n^{\gamma-1}$ is large at sufficiently small S_n, and formally even diverges as $S_n \to 0$. However one should not forget that, in practice, even the largest galaxy clusters than one finds in observations, as well as in large-scale N-body simulations, have typical masses smaller than about $10^{15}\, h^{-1}\, {\rm M}_\odot$ which, in the standard ΛCDM cosmology, corresponds to values of $S_n \gtrsim 0.35$, see e.g. fig. 1 of Zentner (2007). Even for such a value, which is the smallest we are interested in, the value of $B'_{\rm ST}(S_n)$ is just of order 0.3 which means that, in the range of masses of interest, the barrier of ellipsoidal collapse can be considered as slowly varying. Note that we have assumed a barrier shape which is the same for Gaussian initial conditions. A justification of this assumption can be found in Lam, Sheth & Desjacques (2009).

For high mass haloes, $\nu^2 \gg 1$, the scale-independent NG contribution to the Eulerian bias parameters are

$$\Delta b_{1\,\rm NG} \simeq -\frac{1}{6}\mathcal{S}_3\left[3a\,\nu^2 - 1.6\,(a\nu^2)^{0.4}\right],$$
$$\Delta b_{2\,\rm NG} \simeq -\frac{1}{3}\mathcal{S}_3\left[3a^{3/2}\frac{\nu^4}{\delta_c} - 1.6\,a^{0.9}\frac{\nu^{2.8}}{\delta_c}\right]. \qquad (75)$$

Note that the leading term of the first bias coefficient does not agree with what found by Smith, Desjaques & Marian (2010), who found $\Delta b_{1\,\rm NG} \simeq -(2/6)\mathcal{S}_3\nu^2$ for high mass haloes, even in the limit in which we reduce artificially ourselves to the spherical collapse case ($a = 1$) adopted to produce their formula (39). It might well be that the discrepancy arises from the fact that Smith et al. (2010) deduce the halo bias coefficient from the NG halo mass function defined artificially as the Gaussian Sheth–Tormen mass function multiplied by the ratio of the Press–Schecter NG and the Gaussian mass functions. On the other hand, had we also considered the variance S_n and the skewness $\mathcal{S}_3$ as a function of scale through the long wavelength model $\delta_m(\boldsymbol{k})$, we could have obtained as well the standard scale-dependent part of the NG bias coefficient, see Desjacques et al. (2011).

4 THE TWO-BARRIER FIRST-CROSSING RATE

Following Lacey & Cole (1993), we now wish to calculate through the excursion set method the two-barrier first-crossing rate. Before launching ourselves into the computation of the conditional probability, we pause to make some preliminary considerations. In particular, Section 4.2 will be devoted to the case of Gaussian initial conditions and our aim is there to provide an analytical understanding of the problem previously discussed by Bond et al. (1991); the case of NG initial conditions will be discussed in Section 4.3. Bond et al. (1991) showed that the uncorrelated steps solution for the two-barrier problem appears to be in better agreement with halo conditional mass functions than did the solution for correlated steps. Given this, it is natural to ask if going from constant to moving barriers alleviates the discrepancy. This is excellent motivation for studying the two-moving barriers problem. In this case, the uncorrelated steps problem was solved numerically by Sheth & Tormen (2002) who also provided an analytic approximation, which will will discuss below. Moreover, Sheth & Tormen (2002) also showed that these walks (with uncorrelated steps) and their associated formulae for the first crossing distribution, were able to provide a rather good description of the conditional halo mass function. So the most interesting question would be if accounting for correlated steps helps in the computation of the two-barrier problem with moving barriers. We will not discuss this problem combining Gaussian initial conditions with non-Markovian evolution of the noise for moving barriers. However, we will include the non-Markovianity induced by the NG initial conditions.

4.1 Some preliminaries

Let us first analyse the problem with only one barrier, B_n. An instructive alternative way of understanding why the first crossing rate is obtained by taking (minus) the derivative with respect to S_n is the following. In order to impose that the trajectory makes its first barrier crossing at 'time' S_n, we require that for all times $S_1, \ldots, S_{n-1}$ the trajectory stays below B_n, while at S_n it must be above. Therefore the quantity that we need is

$$\int_{-\infty}^{B_1} {\rm d}\delta_1 \cdots \int_{-\infty}^{B_{n-1}} {\rm d}\delta_{n-1} \int_{B_n}^{\infty} {\rm d}\delta_n\, W(\delta_0; \delta_1, \ldots, \delta_n; S_n). \qquad (76)$$

We now write

$$\int_{B_n}^{\infty} {\rm d}\delta_n = \int_{-\infty}^{\infty} {\rm d}\delta_n - \int_{-\infty}^{B_n} {\rm d}\delta_n \qquad (77)$$

and use the fact that

$$\begin{aligned}&\int_{-\infty}^{B_1} {\rm d}\delta_1 \cdots \int_{-\infty}^{B_{n-1}} {\rm d}\delta_{n-1} \int_{-\infty}^{\infty} {\rm d}\delta_n\, W(\delta_0; \delta_1, \ldots, \delta_n; S_n)\\ &= \int_{-\infty}^{B_n} {\rm d}\delta_1 \cdots \int_{-\infty}^{B_{n-1}} {\rm d}\delta_{n-1} W(\delta_0; \delta_1, \ldots, \delta_{n-1}; S_{n-1})\\ &= \int_{-\infty}^{B_{n-1}} {\rm d}\delta_{n-1}\, \Pi_{\rm mb}(\delta_0; \delta_{n-1}; S_{n-1}). \qquad (78)\end{aligned}$$

The second contribution obtained from equation (77) is

$$\begin{aligned}&\int_{-\infty}^{B_1} {\rm d}\delta_1 \cdots \int_{-\infty}^{B_n} {\rm d}\delta_n\, W(\delta_0; \delta_1, \ldots, \delta_n; S_n)\\ &= \int_{-\infty}^{B_n} {\rm d}\delta_n\, \Pi_{\rm mb}(\delta_0; \delta_n; S_n). \qquad (79)\end{aligned}$$

Then we get

$$\begin{aligned}
&\int_{-\infty}^{B_1} \mathrm{d}\delta_1 \cdots \int_{-\infty}^{B_{n-1}} \mathrm{d}\delta_{n-1} \int_{B_n}^{\infty} \mathrm{d}\delta_n \; W(\delta_0; \delta_1, \ldots, \delta_n; S_n) \\
&\quad = \int_{-\infty}^{B_{n-1}} \mathrm{d}\delta_{n-1} \Pi_{\rm mb}(\delta_0; \delta_{n-1}; S_{n-1}) \\
&\qquad - \int_{-\infty}^{B_n} \mathrm{d}\delta_n \Pi_{\rm mb}(\delta_0; \delta_n; S_n) \qquad , \\
&\quad = -\epsilon \frac{\partial}{\partial S_n} \int_{-\infty}^{B_n} d\delta_n \Pi_{\rm mb}(\delta_0; \delta_n; S_n) \\
&\qquad - \int_{B_{n-1}}^{B_n} \mathrm{d}\delta_{n-1} \; \Pi_{\rm mb}(\delta_0; \delta_{n-1}; S_{n-1}) + \mathcal{O}(\epsilon^2) \qquad (80)
\end{aligned}$$

where $\epsilon = (S_n - S_{n-1})$. The integral in the last line in the continuum limit becomes $(B_n - B_{n-1})\Pi_{\rm mb}(\delta_0; B_n; S_n)$ and, since $(B_n - B_{n-1}) = O(\epsilon)$ while $\Pi_{\rm mb}(\delta_0; \delta_n = B_n; S_n)$ vanishes as $\sqrt{\epsilon}$, this term is overall $\mathcal{O}(\epsilon^{3/2})$, while the term proportional to $\partial/\partial S_n$ is $\mathcal{O}(\epsilon)$. Therefore the transition rate per unit time-step ϵ is

$$\mathcal{F}_{\rm mb}(S_n) = -\frac{\partial}{\partial S_n} \int_{-\infty}^{B_n} \mathrm{d}\delta_n \; \Pi_{\rm mb}(\delta_0; \delta_n; S_n) \, , \qquad (81)$$

which is the standard result.

In the two-barrier problem, denoting by $B^a(S_m)$ and $B^b(S_n)$ the two barriers corresponding to redshift z_a and halo mass M_m and z_b and M_n, respectively, the relevant quantity is the conditional probability given by the ratio between

$$\begin{aligned}
\mathcal{N} \equiv &\int_{-\infty}^{B_1^a} \mathrm{d}\delta_1 \cdots \int_{-\infty}^{B_{m-1}^a} \mathrm{d}\delta_{m-1} \int_{B_m^a}^{B_m^b} \mathrm{d}\delta_m \\
&\times \int_{-\infty}^{B_{m+1}^b} \mathrm{d}\delta_{m+1} \cdots \int_{-\infty}^{B_{n-1}^b} \mathrm{d}\delta_{n-1} \\
&\times \int_{B_n^b}^{\infty} \mathrm{d}\delta_n \; W(\delta_0; \delta_1, \ldots, \delta_n; S_n) \, , \qquad (82)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D} \equiv &\int_{-\infty}^{B_1^a} \mathrm{d}\delta_1 \cdots \int_{-\infty}^{B_{m-1}^a} \mathrm{d}\delta_{m-1} \\
&\times \int_{B_m^a}^{B_m^b} \mathrm{d}\delta_m \; W(\delta_0; \delta_1, \ldots, \delta_m; S_m) \, . \qquad (83)
\end{aligned}$$

In equation (82) the integral over $\mathrm{d}\delta_m$ has a lower limit B_m^a because we require that the trajectory crosses above this barrier. The subsequent evolution can bring it below this barrier again, so the integrals over $\mathrm{d}\delta_{m+1}, \ldots, \mathrm{d}\delta_{n-1}$ have as lower limit $-\infty$. Furthermore, the upper integration limit for the integrals over $\mathrm{d}\delta_m, \ldots, \mathrm{d}\delta_{n-1}$ is given by the upper barrier B^b because we want to sum only over trajectories that never crossed the second barrier δ_n at times smaller than S_n. Observe that this must be imposed even in the integral over $\mathrm{d}\delta_m$. Finally, at S_n, the trajectory crosses for the first time above B_n, so the corresponding integral runs from B_n^b to $+\infty$. The denominator (83) gives the appropriate normalization to the conditional probability, so that in the Markovian case, where W factorizes, it cancels against the integrations over $\mathrm{d}\delta_1, \ldots, \mathrm{d}\delta_m$ in the numerator. Observe also that the integral over $\mathrm{d}\delta_m$ (both in the numerator and in the denominator) in the continuum limit can equivalently written as in integral from B_m^a and $+\infty$, since at time S_{m-1} the trajectory was below the lower barrier B^a, and the probability that in an infinitesimal time step ϵ it jumps above the upper B^b vanishes as $\exp\{-(B_m^b - B^a{}_{m-1})^2/(2\epsilon)\}$, so it does not contribute to the continnum limit (at least, as long as there is a finite separation between the two barriers).

We now wish to derive a result analogous to equation (81) for the conditional two-barrier first-crossing rate, i.e. for the rate at which trajectories first cross the upper barrier B^b at $S = S_n$, under the condition that they first crossed the lower barrier B^a at $S = S_m$. In the Markovian case we can repeat the derivation of equations (76)–(81); in this case, in fact, W factorizes as $W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n) = W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_m; S_m) W^{\rm gm}(\delta_m; \delta_{m+1}, \ldots, \delta_n; S_n - S_m)$. The fact that the integral over δ_{m-1} runs over $\delta_{m-1} \leq B^{\rm a}_{m-1}$ while the integral over δ_m runs over $\delta_m \geq B_m^a$ implies that the factor $(2\pi\epsilon)^{-1/2} \exp\{-(\delta_m - \delta_{m-1})^2/(2\epsilon)\}$ which appears in the first W factor (see equation (27)) becomes, in the continuum limit, a Dirac delta which forces δ_m to become equal to B_m^a in the second factor $W^{\rm gm}$, so inside the integral we can write

$$\begin{aligned}
&W^{\rm gm}(\delta_0; \ldots, \delta_n; S_n) \\
&\quad = W^{\rm gm}(\delta_0; \ldots, \delta_m; S_m) W^{\rm gm}\Big(B_m^a; \delta_{m+1}, \ldots, \delta_n; S_n - S_m\Big) \, . \qquad (84)
\end{aligned}$$

The first W factor, integrated over $\mathrm{d}\delta_1, \ldots, \mathrm{d}\delta_m$ in equation (82), cancels by construction the denominator $\mathcal{D}$, and the integral over $\mathrm{d}\delta_n$ in equation (82) can be treated as in equation (77). As a result, we get again equation (81), except that δ_0 is now replaced by B_m^b and S_n by $S_n - S_m$. The same argument can be repeated in the non-Markovian case. Simply, all derivatives ∂_i, when acting on the second factor $W^{\rm gm}(\delta_m; \delta_{m+1}, \ldots, \delta_n; S_n)$, must be evaluated at $\delta_m = B_m^a$ (including derivatives ∂_i with $i = m$). In this way the dependence on δ_m remains only in the first $W^{\rm gm}$ factor, and the derivation of equations (76)–(80) goes through.

After all these considerations, we may write the numerator (82) in general as

$$\begin{aligned}
\mathcal{N} = &\frac{\epsilon^2 \partial^2}{\partial S_m' \partial S_n} \int_{-\infty}^{B_1^a} \mathrm{d}\delta_1 \cdots \int_{-\infty}^{B_m^a} \mathrm{d}\delta_m \\
&\times \int_{-\infty}^{B_{m+1}^b} \mathrm{d}\delta_{m+1} \cdots \int_{-\infty}^{B_n^b} \mathrm{d}\delta_n \\
&\times \left[1 + \hat{f}\right] \; W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_m; S_m')\big|_{S_m' = S_m} \\
&\times \; W^{\rm gm}(\delta_m; \delta_{m+1}, \ldots, \delta_n; S_n - S_m)|_{\delta_m = B_m^a} \, , \qquad (85)
\end{aligned}$$

where the differential operator $\hat{f}$, which acts on both W factors, has the general form

$$\begin{aligned}
\hat{f} = &\sum_{i=1}^{n} a(S_i) \partial_i + \sum_{i,j=1}^{n} b(S_i, S_j) \partial_i \partial_j \\
&+ \sum_{i,j,k=1}^{n} c(S_i, S_j, S_k) \partial_i \partial_j \partial_k + \cdots \qquad (86)
\end{aligned}$$

and takes into account the effects of the the moving barrier and/or the non-Gaussian contributions. The denominator (83), as we saw, is given by

$$\begin{aligned}
\mathcal{D} = &-\epsilon \frac{\partial}{\partial S_m} \int_{-\infty}^{B_1^a} \mathrm{d}\delta_1 \cdots \int_{-\infty}^{B_m^a} \mathrm{d}\delta_m \\
&\times \left[1 + \hat{f}\right] W^{\rm gm}(\delta_0; \delta_1, \ldots, \delta_m; S_m) \, . \qquad (87)
\end{aligned}$$

The ratio $\mathcal{N}/\mathcal{D}$ defining the conditional two-barrier probability is therefore proportional to ϵ and the flux $\mathcal{F}_{\rm mb}$, which is obtained dividing further by ϵ, is therefore equal to the ratio of equation (85) and equation (87), i.e.

$$\mathcal{F}_{\rm mb}\Big(B_n^b, S_n | B_m^a, S_m\Big) = \lim_{\epsilon \to 0^+} \left(\frac{\mathcal{N}}{\epsilon \, \mathcal{D}}\right) \, . \qquad (88)$$

Note that in the fully Markovian case and constant spherical collapse barrier the operator $\hat{f}$ vanishes. Then factorization applies

and one recovers the standard result of Lacey & Cole (1993) that the two-barrier conditional probability with constant barriers $\delta_b(z_b)$ and $\delta_c(z_c)$ is given by the usual first-crossing rate where the initial conditions are such that the smoothed density contrast is $\delta_m = \delta_b$ at time S_m

$$\mathcal{F}_{\rm sph}(\delta_c, S_n|\delta_b, S_m) = \frac{(\delta_c - \delta_b)\mathrm{e}^{-(\delta_c-\delta_b)^2/(2(S_n-S_m))}}{\sqrt{2\pi}(S_n - S_m)^{3/2}}\,. \tag{89}$$

4.2 The two-barrier conditional probability: the moving barrier case and Gaussian initial conditions

In the case in which the barrier threshold is moving, as for the Sheth–Tormen barrier, and the initial conditions are Gaussian, the computation of the two-barrier conditional probability is very similar to the one we have performed in the previous section for the halo bias. The key point is to perform, in equations (82) and (83), the following shift of the integration variables δ_i with $i \neq m$ and $i \neq n$:

$$\begin{aligned}
\delta_i &\to \delta_i - \sum_{p=1}^{\infty} \frac{B_m^{a,(p)}}{p!}\,(S_i - S_m)^p\,, \quad i = 1, \ldots, m-1\,,\\
\delta_i &\to \delta_i - \sum_{p=1}^{\infty} \frac{B_n^{b,(p)}}{p!}\,(S_i - S_n)^p\,, \quad i = m+1, \ldots, n-1\,,
\end{aligned} \tag{90}$$

where $B_m^{a,(p)}$ denotes the pth derivative with respect to S of the barrier $B^a(S)$, evaluated in $S = S_m$, and similarly $B_m^{b,(p)}$ is the pth derivative of the barrier $B^b(S)$, while no shift is made for the variables δ_m and δ_n. It is easy to convince oneself that, exploiting the factorization property (46), the contributions to the two-barrier conditional probability coming from the random walks starting at $\delta_0 = 0$ at $S = 0$ and crossing the barrier B_m for the first time at S_m cancel and one is left with

$$\begin{aligned}
\mathcal{F}_{\rm mb}\left(B_n^b, S_n|B_m^a, S_m\right) &= -\frac{\partial}{\partial S_n}\int_{-\infty}^{B_{m+1}^b} \mathrm{d}\delta_{m+1}\cdots\int_{-\infty}^{B_n^b}\mathrm{d}\delta_n\\
&\qquad W(B_m; \delta_{m+1}, \ldots, \delta_n; S_n - S_m)\\
&= \frac{\mathrm{e}^{-(B_n^b - B_m^a)^2/(2(S_n-S_m))}}{\sqrt{2\pi}(S_n - S_m)^{3/2}}\left[B_n^b - B_m^a + \mathcal{P}(S_m, S_n)\right]\,.
\end{aligned} \tag{91}$$

This finding coincides with the result obtained by Sheth & Tormen (2002); in fact it is valid for any generic moving barrier (and therefore also for the ST barrier with a different from unity) and justifies what found in Sheth & Tormen (2002) on more rigorous grounds. As was the case for the unconditional mass function, the height of the barrier diverges for $(S_n - S_m) \to \infty$, so not all trajectories intersect the second barrier. It seems reasonable (Sheth & Tormen 2002) to associate the fraction of random walks that do not with the fraction of the parent halo mass that is not associated with bound subclumps.

Once the two-barrier first-crossing rate is known, we may compute the conditional mass function, equation (3), to study how many progenitors at z_b are associated with a descendent halo mass M_0 at z_a. In Fig. 3 we show the conditional mass functions for the progenitor haloes of a descendant halo of mass $M_0 = 10^{15}\,h^{-1}\,\mathrm{M}_\odot$ at $z_a = 0$ and look-back times $\Delta z = (z_b - z_a) = 0.1$ and 0.01. We plot the moving barrier result equation (91) for the ellipsoidal barrier in equation (2) with $a = 0.707$, and the spherical collapse result equation (89) (both with $a = 0.707$ and $a = 1$).

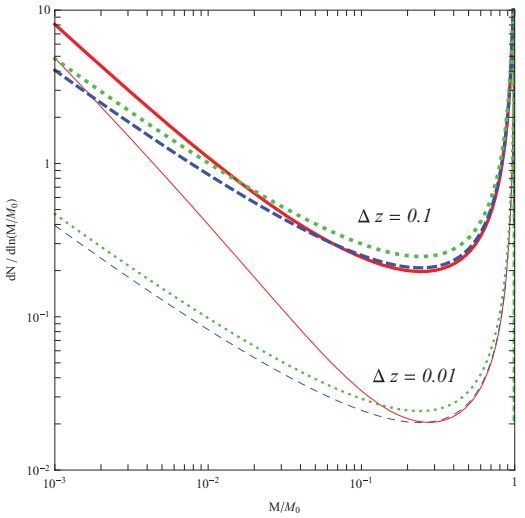

Figure 3. The conditional mass functions for the progenitor haloes of a descendant halo of mass $M_0 = 10^{15}\,h^{-1}\,\mathrm{M}_\odot$ at $z = 0$, according to equation (91) with $a = 0.707$ (solid red) and to the spherical collapse model with $a = 0.707$ (dashed blue) and with $a = 1$ (dotted green). Two different look-back times are shown: $\Delta z = 0.1$ (thick lines) and $\Delta z = 0.01$ (thin lines).

4.3 The two-barrier conditional probability: the moving barrier case and non-Gaussian initial conditions

In the case in which non-Gaussian initial conditions are present, we can deal with the problem in the same way we have been treating the computation of the halo bias in the case of a non-Gaussian theory. Expanding for weak non-Gaussianities in the expression (85) brings down the sum

$$\begin{aligned}
\sum_{i,j,k=1}^{n}\langle\delta_i\delta_j\delta_k\rangle_c\partial_i\partial_j\partial_k &= \sum_{i,j,k=1}^{m}\langle\delta_i\delta_j\delta_k\rangle_c\partial_i\partial_j\partial_k\\
&\quad+\sum_{i,j,k=m+1}^{n}\langle\delta_i\delta_j\delta_k\rangle_c\partial_i\partial_j\partial_k\\
&\quad+3\sum_{i=1}^{m}\sum_{j,k=m+1}^{n}\langle\delta_i\delta_j\delta_k\rangle_c\partial_i\partial_j\partial_k\\
&\quad+3\sum_{i,j=1}^{m}\sum_{k=m+1}^{n}\langle\delta_i\delta_j\delta_k\rangle_c\partial_i\partial_j\partial_k\,.
\end{aligned} \tag{92}$$

We now perform the Taylor expansion (67) and retain only the leading terms

$$\begin{aligned}
\sum_{i,j,k=1}^{n}\langle\delta_i\delta_j\delta_k\rangle_c\partial_i\partial_j\partial_k &\simeq \sum_{i,j,k=1}^{m}\left\langle\delta_m^3\right\rangle_c\partial_i\partial_j\partial_k\\
&\quad+\sum_{i,j,k=m+1}^{n}\left\langle\delta_n^3\right\rangle_c\partial_i\partial_j\partial_k\\
&\quad+3\sum_{i=1}^{m}\sum_{j,k=m+1}^{n}\left\langle\delta_m\delta_n^2\right\rangle_c\partial_i\partial_j\partial_k\\
&\quad+3\sum_{i,j=1}^{m}\sum_{k=m+1}^{n}\left\langle\delta_m^2\delta_n\right\rangle_c\partial_i\partial_j\partial_k\,.
\end{aligned} \tag{93}$$

As $B_m^a < B_n{}^b$ we may further approximate this sum ignoring the terms proportional to $\langle\delta_m^3\rangle_c$ and $\langle\delta_m^2\delta_n\rangle_c$ with respect to the others

$$\sum_{i,j,k=1}^{n}\langle\delta_i\delta_j\delta_k\rangle_c\partial_i\partial_j\partial_k \simeq \sum_{i,j,k=m+1}^{n}\left\langle\delta_n^3\right\rangle_c\partial_i\partial_j\partial_k + 3\sum_{i=1}^{m}\sum_{j,k=m+1}^{n}\left\langle\delta_m\delta_n^2\right\rangle_c\partial_i\partial_j\partial_k\,. \tag{94}$$

Again, the crucial point is that, when considering a moving barrier, the shift of variables (90) does not involve neither δ_m nor δ_n. This allows factorization and cancellations between the numerator (85) and the denominator (87). The non-Gaussian two-barrier conditional probability $\mathcal{F}_{\rm mb\,NG}$ then becomes

$$\begin{aligned}
&\mathcal{F}_{\rm mb\,NG}\left(B_n^b, S_n|B_m^a, S_m\right)\\
&= -\frac{\partial}{\partial S_n}\int_{-\infty}^{B_n^b}{\rm d}\delta_n\,\Pi_{\rm two}\left(\delta_n, S_n|B_m^a, S_m\right)\\
&+\frac{1}{6}\frac{\partial^4}{\partial S_n\partial\left(B_n^b\right)^3}\left(\left\langle\delta_n^3\right\rangle_c\int_{-\infty}^{B_n^b}{\rm d}\delta_n\,\Pi_{\rm two}\left(\delta_n, S_n|B_m^a, S_m\right)\right)\\
&-\frac{3}{6}\frac{\partial^4}{\partial S_m'\partial S_n\partial\left(B_n^b\right)^2}\left(\left\langle\delta_m\delta_n^2\right\rangle_c\int_{-\infty}^{B_m^a}{\rm d}\delta_m\frac{\partial\Pi_{\rm mb}(\delta_m;S_m')}{\partial B_m^a}\right|_{S_m'=S_m}\\
&\times\int_{-\infty}^{B_n^b}{\rm d}\delta_n\,\Pi_{\rm two}\left(\delta_n, S_n|B_m^a, S_m\right)\Bigg)\frac{1}{\mathcal{F}_{\rm mb}(S_m)}\\
&-\frac{3}{6}\frac{\partial^4}{\partial S_m'\partial S_n\partial\left(B_n^b\right)^2}\left(\left\langle\delta_m\delta_n^2\right\rangle_c\int_{-\infty}^{B_m^a}{\rm d}\delta_m\Pi_{\rm mb}\left(\delta_m;B_m^a,S_m'\right)\right|_{S_m'=S_m}\\
&\times\int_{-\infty}^{B_n^b}{\rm d}\delta_n\partial_m\,\Pi_{\rm two}(\delta_n, S_n|\delta_m, S_m)|_{\delta_m=B_m^a}\Bigg)\frac{1}{\mathcal{F}_{\rm mb}(S_m)}\\
&\equiv\mathcal{F}^{\rm (I)}_{\rm mb\,NG}+\mathcal{F}^{\rm (II)}_{\rm mb\,NG}+\mathcal{F}^{\rm (III)}_{\rm mb\,NG}+\mathcal{F}^{\rm (IV)}_{\rm mb\,NG}\,,
\end{aligned}\tag{95}$$

where $\Pi_{\rm mb}$ can be read from equation (69), $\mathcal{F}_{\rm mb}(S_m)$ from equation (40) and

$$\begin{aligned}
&\Pi_{\rm two}(\delta_n, S_n|\delta_m, S_m)\\
&=\frac{1}{\sqrt{2\pi(S_n-S_m)}}\left(\mathrm{e}^{-(\delta_n-\delta_m)^2/(2(S_n-S_m))}\right.\\
&\left.-\,\mathrm{e}^{-(2B_n^b-\delta_n-\delta_m)^2/(2(S_n-S_m))}\right)\\
&+\frac{2\left(B_n^b-\delta_n\right)}{\sqrt{2\pi}(S_n-S_m)^{3/2}}\mathrm{e}^{-(2B_n^b-\delta_n-\delta_m)^2/(2(S_n-S_m))}\mathcal{P}_{mn}\\
&-\frac{2\left(B_n^b-\delta_n\right)^2}{\sqrt{2\pi}(S_n-S_m)^{5/2}}\mathrm{e}^{-(2B_n^b-\delta_n-\delta_m)^2/(2(S_n-S_m))}\mathcal{P}_{mn}^2\,.
\end{aligned}\tag{96}$$

Note that repeating with care the steps described in Section 4.1, in equation (95) we do not have to differentiate the cumulant $\langle\delta_m\delta_n^2\rangle_c$ with respect to S_m' and S_n.

Let us set $\langle\delta_n^2\delta_m\rangle_c = \mathcal{S}_{2,1}(S_n, S_m)S_nS_m^{1/2}$, where $\mathcal{S}_{2,1}(S_n, S_m)$ is slowly changing with S_n and S_m (see Appendix A for further details and useful fitting functions). Then, the first two terms of (95) read

$$\begin{aligned}
&\mathcal{F}^{\rm (I)}_{\rm mb\,NG}\left(B_n^b, S_n|B_m^a, S_m\right)+\mathcal{F}^{\rm (II)}_{\rm mb\,NG}\left(B_n^b, S_n|B_m^a, S_m\right)\\
&=\frac{\left(B_n^b-B_m^a+\mathcal{P}_{mn}\right)\mathrm{e}^{-\frac{(B_n^b-B_m^a)^2}{2(S_n-S_m)}}}{\sqrt{2\pi}(S_n-S_m)^{3/2}}\\
&+\frac{S_n\mathcal{S}_3(S_n)}{6\sqrt{2\pi}(S_n-S_m)^{9/2}}\mathrm{e}^{-\frac{(B_n^b-B_m^a)^2}{2(S_n-S_m)}}\left[\left(B_n^b-B_m^a\right)^4S_n\right.\\
&-\left(B_n^b-B_m^a\right)^3S_n\left(\mathcal{P}_{mn}+2B_n'^b(S_n-S_m)\right)\\
&+2\left(B_n^b-B_m^a\right)^2\left(S_n\mathcal{P}_{mn}^2+S_n(S_n-S_m)\mathcal{P}_{mn}B_n'^b\right.\\
&\left.-S_n^2+2S_m^2-S_nS_m\right)\\
&+\left(B_n^b-B_m^a\right)(S_n-S_m)\left(\mathcal{P}_{mn}(S_n+4S_m)\right.\\
&+6S_n(S_n-S_m)B_n'^b-4\mathcal{P}_{mn}^2S_nB_n'^b-2S_n(S_n-S_m)\mathcal{P}_{mn}'\Big)\\
&-2(S_n-S_m)\mathcal{P}_{mn}(\mathcal{P}_{mn}(S_n+4S_m)+(S_n-S_m)S_nB_n'^b\\
&\left.-4S_n(S_n-S_m)\mathcal{P}_{mn}')-S_n^3+4S_m^3+6S_n^2S_m-9S_nS_m^2\right]\\
&+\frac{S_n^2\mathcal{S}_3'(S_n)}{3\sqrt{2\pi}(S_n-S_m)^{5/2}}\mathrm{e}^{-\frac{(B_n^b-B_m^a)^2}{2(S_n-S_m)}}\left[\left(B_n^b-B_m^a\right)^2\right.\\
&\left.-(B_n^b-B_m^a)\mathcal{P}_{mn}-(S_n-S_m)+2\mathcal{P}_{mn}^2\right]\,,
\end{aligned}\tag{97}$$

where the prime stands for derivative with respect to S_n. As for the third and fourth terms of equation (95), they are

$$\begin{aligned}
\mathcal{F}^{\rm (III)}_{\rm mb\,NG}\left(B_n^b, S_n|B_m^a, S_m\right) &= -\frac{3}{6}\mathcal{S}_{2,1}(S_n, S_m)S_nS_m^{1/2}\\
&\times\frac{\partial^2}{\partial\left(B_n^b\right)^2}\mathcal{F}_{\rm mb}\left(B_n^b, S_n|B_m^a, S_m\right)\frac{\partial}{\partial B_m^a}\ln\mathcal{F}_{\rm mb}(S_m)\,,
\end{aligned}\tag{98}$$

where is $\mathcal{F}_{\rm mb}(S_m)$ is given by equation (40) and

$$\begin{aligned}
\mathcal{F}^{\rm (IV)}_{\rm mb\,NG}(B_n^b, S_n|B_m^a, S_m) &= -\frac{3}{6}\mathcal{S}_{2,1}(S_n, S_m)S_nS_m^{1/2}\\
&\times\frac{\partial^3}{\partial\left(B_n^b\right)^2\partial B_m^a}\mathcal{F}_{\rm mb}\left(B_n^b, S_n|B_m^a, S_m\right)\,.
\end{aligned}\tag{99}$$

Note that the total NG conditional probability is no longer a function of the variables $(S_n - S_m)$ and $(B_n^b - B_{\rm m}^a)$ as in the Markovian case. In Fig. 4 we show the ratio of the conditional mass functions with and without NG, for the progenitor haloes of a descendant halo of mass $M_0 = 10^{15}\,h^{-1}\,{\rm M}_\odot$ at $z_a = 0$, look-back time $\Delta z = 0.3$, $f_{\rm NL} = 50$ and $a = 0.707$.

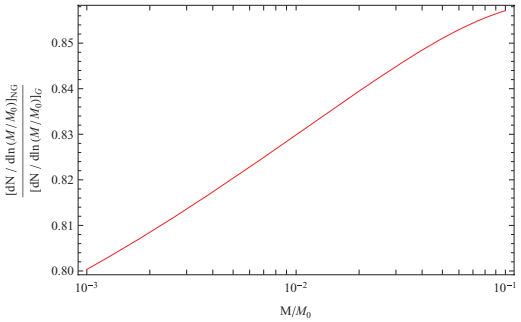

Figure 4. Ratio of the conditional mass functions with and without NG, for the progenitor haloes of a descendant halo of mass $M_0 = 10^{15}\,h^{-1}\,{\rm M}_\odot$ at $z = 0$, look-back time $\Delta z = 0.3, f_{\rm NL} = 50$ and $a = 0.707$

5 HALO FORMATION TIME PROBABILITY WITH NON-GAUSSIANITIES

In this section we present the results for the probability distribution of halo formation redshifts with the inclusion of non-Gaussianities.

We follow the convention to define the epoch of formation of a halo as the time when the halo contains half of its final mass, although the generalization to an arbitrary fraction between 1/2 and 1 is straightforward. Let us fix the mass M_0 and the redshift z_a of the descendent halo. Then, the probability that such halo had a progenitor at redshift $z_b > z_a$ with mass between $M_0/2$ and M_0 (or, equivalently, the probability that the formation redshift is bigger than z_b) is given by equation (3) integrated from $M_0/2$ to M_0:

$$P(z_b; M_0, z_a) = \int_{S_0}^{S_h} \mathrm{d}S_n \frac{M_0}{M(S_n)} \times \mathcal{F}_{\mathrm{mb}}(B(z_b; S_n), S_n | B(z_a; S_0), S_0)\,, \tag{100}$$

where $S_0 \equiv S(M_0)$, $S_h \equiv S(M_0/2)$. From this, it is possible to find the probability distribution that the halo of mass M_0 at z_a would have formed between z_b and $z_b + \mathrm{d}z_b$:

$$p(z_b)\mathrm{d}\,z_b = \left|\frac{\mathrm{d}P(z_b; M_0, z_a)}{\mathrm{d}z_b}\right| \mathrm{d}z_b\,. \tag{101}$$

In the simple case of constant barrier $\delta_{\mathrm{c}}(z)$ and Gaussian initial conditions, the distribution of halo formation redshifts is simply given by (see Lacey & Cole 1993)

$$p(z_b) = 2\omega(z_b)\mathrm{Erfc}\left[\frac{\omega(z_b)}{\sqrt{2}}\right]\frac{\mathrm{d}\omega(z_b)}{\mathrm{d}z_b}\,, \tag{102}$$

where

$$\omega(z_b) \equiv \frac{\delta_c(z_b) - \delta_c(z_a)}{\sqrt{S(M_0/2) - S(M_0)}}\,, \tag{103}$$

if it is assumed a white-noise power spectrum, leading to $S(M) \propto M^{-1}$ (Lacey & Cole 1993).

Now let us introduce the contribution of non-Gaussianities. The two barrier conditional probability is given by equation (95). In order to simplify the calculation and reach a compact result we make the following assumptions: we ignore the cumulant $\langle\delta_m\delta_n^2\rangle$, keeping only the leading one $\langle\delta_n^3\rangle$ (which means we only retain the terms in equation 97); we consider $\mathcal{S}_3$ as a constant and we assume that $S(M)$ scales like M^{-1}, In these approximations, the (normalized) probability distribution of formation redshifts in presence of non-Gaussianities becomes

$$\begin{aligned} p_{\mathrm{NG}}(z_b) = &\left\{2\omega(z_b)\mathrm{Erfc}\left[\frac{\omega(z_b)}{\sqrt{2}}\right]\right. \\ &+ \frac{4}{3\sqrt{\pi}}\mathcal{S}_3\sqrt{S(M_0)}\left[\sqrt{2}\omega(z_b)\left(\omega^2(z_b) - 3\right)\mathrm{e}^{-\omega(z_b)^2/2}\right. \\ &\left.\left. + \sqrt{\pi}\left(1 - \omega^2(z_b)\right)\mathrm{Erfc}\left[\frac{\omega(z_b)}{\sqrt{2}}\right]\right]\right\} \\ &\times \left[1 - \frac{8}{9}\sqrt{\frac{2}{\pi}}\mathcal{S}_3\sqrt{S(M_0)}\right]^{-1}\frac{\mathrm{d}\omega(z_b)}{\mathrm{d}z_b}\,. \end{aligned} \tag{104}$$

This expression is also valid for a spherical collapse model with barrier $\sqrt{a}\delta_c(z)$ which may be considered as an approximation to the ellipsoidal model (Giocoli et al. 2007). In order to gain an intuition of the impact of the non-Gaussian correction, for $M_0 = 10^{15}\,\mathrm{M}_\odot\,h^{-1}$ and $f_{\mathrm{NL}} = 50$, in the regions of z_b where the distribution is greater than 0.1, the NG term contributes typically less than 10 per cent with respect to the Gaussian one.

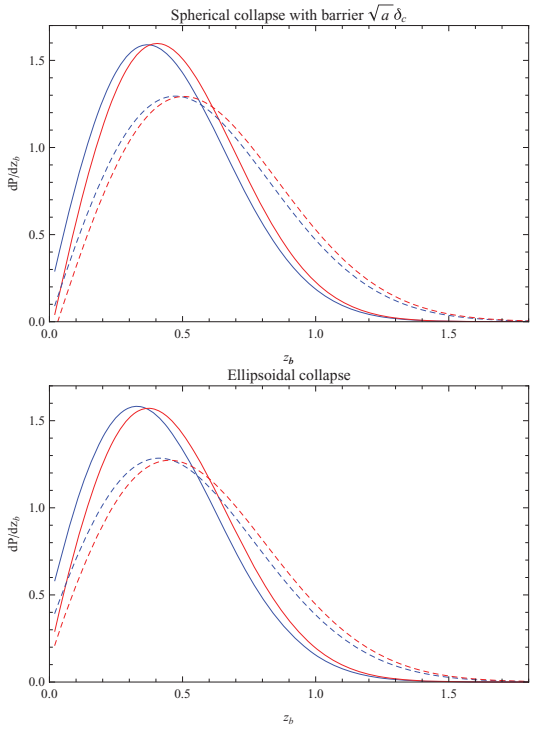

Figure 5. Probability distributions for the formation redshift z_b of a halo of mass $M_0 = 10^{15}\,h^{-1}\,\mathrm{M}_\odot$ at $z_a = 0$, in the spherical collapse model with barrier $\sqrt{a}\delta_c$ (top panel) and in the ellipsoidal collapse model (bottom panel). The Gaussian case is shown in blue, while the non-Gaussian one, with $f_{\mathrm{NL}} = 50$, is in red. Solid lines correspond to the numerical fit to $S(M)$ in equation (A1) while for the dashed lines the simple approximation $S(M) \propto M^{-1}$ is used.

In Fig. 5 we show the probability distributions of formation redshifts of a halo of mass $M_0 = 10^{15}\,h^{-1}\,\mathrm{M}_\odot$ at $z_a = 0$, both for the spherical collapse model with constant barrier $\sqrt{a}\delta_c$ and for the ellipsoidal collapse model with barrier B_{ST} in equation (2). These results are obtained by integrating numerically equation (100) using the first-crossing rates in equation (91) and in equations (97)–(99) for Gaussian and non-Gaussian initial conditions, respectively. The inclusion of non-Gaussianities tends to shift slightly the distributions towards higher redshifts. Furthermore, the results for the spherical collapse with barrier $\sqrt{a}\delta_c$ and ellipsoidal barrier are quite close to each other, confirming the suggestions of Giocoli et al. (2007).

The mass dependence of the variance $S(M)$ enters into equation (100) in an important way. We make use of the numerical fit (A1) (solid lines in Fig. 5), but we have verified that using a different fit to $S(M)$, like the one of Neistein & Dekel (2008), does not change the solid curves appreciably. For comparison, we also show the distributions (dashed lines) one would obtain by using the simple scaling $S(M) \propto M^{-1}$, which is very useful to carry out analytical calculations and it is commonly used in the literature. However, as shown in the figure, the use of this simple scaling leads to non-negligible differences with respect to a more accurate numerical fit. Note also that for $z_b \to z_a$ the numerical fit $S(M)$ gives values of $\mathrm{d}p/\mathrm{d}z_b$ systematically higher than for the case $S(M) \propto M^{-1}$. This point was already noted in Lacey & Cole (1993) where it was stressed that using a variance with a scaling different from M^{-1} gives rise to probabilities $\mathrm{d}p/\mathrm{d}z_b$ which do not vanish for $z_b \to z_a$. This is probably due to the fact that the particles that the analysis of trajectories tags as having a given mass M are not really grouped into objects of mass M (Lacey & Cole 1993).

The analytical prediction (104), which has been found for a constant barrier under the assumption $S(M) \propto M^{-1}$ and leading NG term, turns out to be in very good agreement with the numerical result obtained by integrating equation (100) numerically, under the same assumption for $S(M)$.

6 CONCLUSIONS

In the excursion set theory the density perturbations depend stochastically with the smoothing scale and the computation of the halo mass function is mapped into the so-called first-passage time problem in the presence of a barrier. Other properties of dark matter haloes, such as halo bias, accretion rate, formation time, merging rate and the formation history of haloes, can be studied using the excursion set formalism by computing the conditional probability with non-trivial initial conditions and the conditional two-barrier crossing rate.

In this paper we have performed the calculations of such conditional probabilities in the presence of a generic moving barrier and for both Gaussian and non-Gaussian initial conditions. Our generic results can therefore be applied to the case of the ellipsoidal collapse where the barrier is moving, given by expression (2), and to the case of the diffusive barrier discussed in MR2. Our findings include the non-Markovianity of the random walks induced by NG.

Let us summarize the main results of this paper:

(i) assuming a sharp filter in momentum space, the first two halo bias coefficients for a generic moving barrier (equations 51 and 52) and their corrections due to non-Gaussianities (equations 73 and 74);

(ii) the conditional mass function (equation 3) for a generic moving barrier with Gaussian initial conditions (equation 91) which reproduces what found in Sheth & Tormen (2002) and in presence of non-Gaussianities (equations 97, 98 and 99), again for a sharp filter in momentum space;

(iii) the probability distribution of the halo formation time, including non-Gaussianities; we have provided numerical results for the spherical and the ellipsoidal collapse models (Fig. 5) and an analytical approximation (equation 104) valid for constant barrier, again for a sharp filter in momentum space.

Our calculations have revealed a different scaling of the Gaussian halo bias parameter at high halo masses from what found in Sheth et al. (2001), our prediction being about 20 per cent higher. This seems to go in the right direction to fit better the N-body data by Tinker et al. (2010) when such a comparison is possible.

As an application of our findings, it would be interesting to investigate e.g. the NG halo assembly bias, recently discussed in Sheth & Tormen (2004), Gao, Springel & White (2005) and Reid et al. (2010).

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APPENDIX A: NUMERICAL FITS TO CUMULANTS

We report here a collection of numerical results we have found and used throughout the paper. For the window function, we have assumed a top-hat in wavenumber space, but with a mass-to-smoothing scale relation of a real space top-hat filter. The following cosmological parameters from Komatsu et al. (2010) are used: $h = 0.703,\ \Omega_\Lambda = 0.729,\ \Omega_{\rm m} = 0.271,\ \Omega_{\rm b} = 0.0451,\ \sigma_8 = 0.809$.

The variance $S \equiv \langle\delta_R^2\rangle$ depends on the smoothing scale R and, in turn, on the halo mass M (in units of $\mathrm{M}_\odot\, h^{-1}$) according to the fitting function

$$S(M) = c_0\left[1 + \frac{c_1}{10}M^{1/10} + \frac{c_2}{10^2}M^{2/15} + \frac{c_3}{10^3}M^{1/6} + \frac{c_4}{10^4}M^{1/5}\right]^{-10} \tag{A1}$$

with $c_0 = 7.2\times10^2, c_1 = 1.2, c_2 = -5.8, c_3 = 9.5, c_4 = 2.6$, which agrees rather well with the one reported in Neistein & Dekel (2008), who instead make use of a top-hat filter in real space.

For the scale dependence of $\mathcal{S}_3 \equiv \langle\delta^3(S)\rangle/S^2$ we have found the following simple fitting formula, computed along the lines of Matarrese et al. (2000), with updated cosmological parameters

$$\mathcal{S}_3(S) \simeq \frac{2.9\times10^{-4}}{S^{0.3}}f_{\rm NL}. \tag{A2}$$

The time variation of $\mathcal{S}_3$ is such that $S\mathcal{S}_3'/\mathcal{S}_3 \sim 0.3$. Alternatively, one may define the quantity

$$\epsilon_1(S) \equiv \frac{\langle\delta^3(S)\rangle}{S^{3/2}}, \tag{A3}$$

which varies more slowly in S, since $S\epsilon_1'/\epsilon_1 \sim 0.2$. This quantity is well fitted by the formula

$$\epsilon_1(S) \simeq 2.9\times10^{-4}\, S^{0.2} f_{\rm NL}. \tag{A4}$$

For the cumulant $\langle\delta^2(S_1)\delta(S_2)\rangle$, we have found that it scales approximately like $S_1\sqrt{S_2}$. Therefore, it is convenient to define a slowly varying $\mathcal{S}_{2,1}(S_1, S_2)$ as

$$\mathcal{S}_{2,1}(S_1, S_2) = \frac{\langle\delta^2(S_1)\delta(S_2)\rangle}{S_1\sqrt{S_2}}. \tag{A5}$$

Having fixed $S_2 \equiv S_{15} \simeq 0.2$, the variance corresponding to a halo mass of $10^{15}\,\mathrm{M}_\odot\, h^{-1}$, we have found the fitting formula (for $S > S_{15}$)

$$\mathcal{S}_{2,1}(S, S_{15}) \simeq \frac{2.4\times10^{-4}}{S^{0.02}}f_{\rm NL}. \tag{A6}$$

Its very mild dependence on S justifies the assumption made in the text, where we consider it a constant.

APPENDIX B: THE TWO-BARRIER FIRST CROSSING RATE WITH A TOP-HAT WINDOW FUNCTION IN REAL SPACE

In this appendix we would like to extend the computation of the two-barrier first crossing rate to the case in which the window function for the smoothed density contrast is a top-hat in real space. An analogous computation has been performed by Ma et al. (2010) for the halo bias and we are going to use many of the results of the appendix of that paper. The choice of a top-hat filter in real space introduces by itself a level of non-Markovianity. The latter is manifest in the two-point correlator of the smoothed density contrast (MR1)

$$\langle\delta(S_i)\delta(S_j)\rangle = \min(S_i, S_j) + \Delta(S_i, S_j), \tag{B1}$$

where $\Delta(S_i, S_j) = \Delta(S_j, S_i)$ and, for $S_i \le S_j$, the function $\Delta(S_i, S_j)$ is well approximated by

$$\Delta(S_i, S_j) \equiv \Delta_{ij} \simeq \kappa\,\frac{S_i(S_j - S_i)}{S_j}, \quad S_i < S_j, \tag{B2}$$

with $\kappa \approx 0.44$ (0.35) for a top-hat (Gaussian) filter in coordinate space. The parameter κ gives a measure of the non-Markovianity of the stochastic process.

We perform the computation of the two-barrier crossing rate assuming a spherical collapse and we will extend them to the diffusive barrier model introduced by Maggiore & Riotto (2010b) at the end. The barriers are called δ_b and δ_c.

To perform the computation we use the technique discussed in detail in MR1. We consider first the numerator in equation (44). The first step is to express the non-Markovian W in terms of $W^{\rm gm}$,

$$\begin{aligned} W(\delta_0;\dots,\delta_n;S_n) &= \int \mathcal{D}\lambda\ \mathrm{e}^{i\sum_{i=1}^n\lambda_i\delta_i - \frac{1}{2}\sum_{i,j=1}^n\lambda_i\lambda_j(\min(S_i,S_j)+\Delta_{ij})} \\ &\simeq W^{\rm gm}(\delta_0;\dots,\delta_n;S_n) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^n \Delta_{ij}\partial_i\partial_j W^{\rm gm}(\delta_0;\dots,\delta_n;S_n). \end{aligned} \tag{B3}$$

As usual it is convenient to split the sum into various pieces:

$$\begin{aligned} \frac{1}{2}\sum_{i,j=1}^n \Delta_{ij}\partial_i\partial_j &= \frac{1}{2}\sum_{i,j=1}^{m-1}\Delta_{ij}\partial_i\partial_j + \sum_{i=1}^{m-1}\Delta_{im}\partial_i\partial_m \\ &\quad + \frac{1}{2}\sum_{i,j=m+1}^{n-1}\Delta_{ij}\partial_i\partial_j + \sum_{i=m+1}^{n-1}\Delta_{in}\partial_i\partial_n \\ &\quad + \sum_{i=1}^{m-1}\Delta_{in}\partial_i\partial_n + \Delta_{mn}\partial_m\partial_n \\ &\quad + \sum_{i=1}^{m-1}\sum_{j=m+1}^{n-1}\Delta_{ij}\partial_i\partial_j + \sum_{j=m+1}^{n-1}\Delta_{mj}\partial_j\partial_m, \end{aligned} \tag{B4}$$

where when sums are from 1 to m the non-Markovian kernel has to be thought as a function of S_m'. The goal is to compute the numerator (85) and the denominator (87). Consider first the contribution from the first line of (B4). Its contribution to the numerator in (85) (a part from the time differentiation with respect to S_m' and S_n) can be

written as

$$\int_{-\infty}^{\delta_b} \mathrm{d}\delta_1 \cdots \int_{-\infty}^{\delta_b} \mathrm{d}\delta_m \int_{-\infty}^{\delta_c} \mathrm{d}\delta_{m+1} \cdots \int_{-\infty}^{\delta_c} \mathrm{d}\delta_n \left[\frac{1}{2}\sum_{i,j=1}^{m-1} \Delta_{ij}\partial_i\partial_j + \sum_{i=1}^{m-1}\Delta_{im}\partial_i\partial_m\right] \times W^{\rm gm}(\delta_0;\ldots,\delta_m;S'_m)W^{\rm gm}(\delta_m;\ldots,\delta_n;S_n-S_m)$$
$$= \int_{-\infty}^{\delta_b} \mathrm{d}\delta_1\cdots\mathrm{d}\delta_m \left[\frac{1}{2}\sum_{i,j=1}^{m-1} \Delta_{ij}\partial_i\partial_j + \sum_{i=1}^{m-1}\Delta_{im}\partial_i\partial_m\right] \times W^{\rm gm}(\delta_0;\ldots,\delta_m;S'_m) \times \int_{-\infty}^{\delta_c} \mathrm{d}\delta_{m+1}\cdots\mathrm{d}\delta_n W^{\rm gm}(\delta_m;\ldots,\delta_n;S_n-S_m)$$
$$+ \int_{-\infty}^{\delta_b} \mathrm{d}\delta_1\cdots\mathrm{d}\delta_m \sum_{i=1}^{m-1}\Delta_{im}\partial_i W^{\rm gm}(\delta_0;\ldots,\delta_m;S'_m) \times \int_{-\infty}^{\delta_c} \mathrm{d}\delta_{m+1}\cdots\mathrm{d}\delta_n\partial_m W^{\rm gm}(\delta_m;\ldots,\delta_n;S_n-S_m). \quad \text{(B5)}$$

The first term is easily dealt with by observing that

$$\int_{-\infty}^{\delta_c} \mathrm{d}\delta_{m+1}\cdots\mathrm{d}\delta_n W^{\rm gm}(\delta_m;\ldots,\delta_n;S_n-S_m) = \int_{-\infty}^{\delta_c}\mathrm{d}\delta_n \Pi^{\rm gm}(\delta_m;\delta_n;S_n-S_m). \quad \text{(B6)}$$

Combining this with the contribution coming from the zeroth order term $W^{\rm gm}(\delta_0;\ldots,\delta_n;S_n)$ in equation (B3) and using again the factorization property of $W^{\rm gm}$, we therefore get

$$\int_{-\infty}^{\delta_c}\mathrm{d}\delta_n\Pi^{\rm gm}(\delta_m;\delta_n;S_n-S_m)\int_{-\infty}^{\delta_b}\mathrm{d}\delta_1\cdots\mathrm{d}\delta_m \times\left[1+\frac{1}{2}\sum_{i,j=1}^{m-1}\Delta_{ij}\partial_i\partial_j+\sum_{i=1}^{m-1}\Delta_{im}\partial_i\partial_m\right] \times W^{\rm gm}(\delta_0;\ldots,\delta_m;S'_m) + \int_{-\infty}^{\delta_b}\mathrm{d}\delta_1\cdots\mathrm{d}\delta_m\sum_{i=1}^{m-1}\Delta_{im}\partial_i W^{\rm gm}(\delta_0;\ldots,\delta_m;S'_m) \times\int_{-\infty}^{\delta_c}\mathrm{d}\delta_{m+1}\cdots\mathrm{d}\delta_n\partial_m W^{\rm gm}(\delta_m;\ldots,\delta_n;S_n-S_m), \quad \text{(B7)}$$

where one has to recall that the derivatives ∂_m acting on $W^{\rm gm}(\delta_m;\ldots,\delta_n;S_n-S_m)$ have to be evaluated at $\delta_m=\delta_c$. Note that the term in brackets gives just the expansion to $\mathcal{O}(\kappa)$ of the denominator (87) and therefore it will provide the usual Markovian tow-barrier crossing rate (89). The other terms contribute to the numerator (85) as

$$\frac{\partial^2}{\partial S'_m\partial S_n}\int_{-\infty}^{\delta_b}\mathrm{d}\delta_m\int_{-\infty}^{\delta_c}\mathrm{d}\delta_n(N_a+N_b+N_c+N_d)\Big|_{S'_m=S_m}, \quad \text{(B8)}$$

where

$$N_a = \int_{-\infty}^{\delta_b}\mathrm{d}\delta_1\cdots\mathrm{d}\delta_{m-1}\sum_{i=1}^{m-1}\Delta_{im}\partial_i W^{\rm gm}(\delta_0;\ldots,\delta_m;S'_m) \times\int_{-\infty}^{\delta_c}\mathrm{d}\delta_{m+1}\cdots\mathrm{d}\delta_{n-1}\ \partial_m W^{\rm gm}(\delta_m;\ldots,\delta_n;S_n-S_m)|_{\delta_m=\delta_b}, \quad \text{(B9)}$$

$$N_b = \int_{-\infty}^{\delta_b}\mathrm{d}\delta_1\cdots\mathrm{d}\delta_{m-1}\int_{-\infty}^{\delta_c}\mathrm{d}\delta_{m+1}\cdots\mathrm{d}\delta_{n-1} \left[\frac{1}{2}\sum_{i,j=m+1}^{n-1}\Delta_{ij}\partial_i\partial_j+\sum_{i=m+1}^{n-1}\Delta_{in}\partial_i\partial_n\right] \times W^{\rm gm}(\delta_0;\ldots,\delta_m;S'_m)W^{\rm gm}(\delta_b;\ldots,\delta_n;S_n-S_m), \quad \text{(B10)}$$

$$N_c = \int_{-\infty}^{\delta_b}\mathrm{d}\delta_1\cdots\mathrm{d}\delta_{m-1}\int_{-\infty}^{\delta_c}\mathrm{d}\delta_{m+1}\cdots\mathrm{d}\delta_{n-1} \left[\sum_{i=1}^{m-1}\Delta_{in}\partial_i\partial_n+\Delta_{mn}\partial_m\partial_n\right] \times W^{\rm gm}(\delta_0;\ldots,\delta_m;S'_m)W^{\rm gm}(\delta_b;\ldots,\delta_n;S_n-S_m), \quad \text{(B11)}$$

$$N_d = \int_{-\infty}^{\delta_b}\mathrm{d}\delta_1\cdots\mathrm{d}\delta_{m-1}\int_{-\infty}^{\delta_c}\mathrm{d}\delta_{m+1}\cdots\mathrm{d}\delta_{n-1} \left[\sum_{i=1}^{m-1}\sum_{j=m+1}^{n-1}\Delta_{ij}\partial_i\partial_j+\sum_{j=m+1}^{n-1}\Delta_{mj}\partial_j\partial_m\right] \times W^{\rm gm}(\delta_0;\ldots,\delta_m;S'_m)W^{\rm gm}(\delta_b;\ldots,\delta_n;S_n-S_m), \quad \text{(B12)}$$

Note that in the last term N_d particular attention has to be paid on how to reconstruct the derivative with respect to S'_m that appears in the numerator (85). Again, we reiterate that the second $W^{\rm gm}$ has to evaluated at $\delta_m=\delta_b$ as well as its derivatives with respect to δ_m.

The contribution from N_c vanishes because it contains a total derivative ∂_n of a quantity that vanishes at $\delta_n=\delta_c$.

The term N_a is immediately obtained using equations (108) and (109) of MR1, and is given by

$$N_a = \frac{\kappa}{\pi}\left[\sqrt{2\pi}\frac{\delta_b}{\sqrt{S'_m}}\mathrm{e}^{-\frac{(2\delta_b-\delta_m)^2}{2S'_m}} - \sqrt{2\pi}\frac{\delta_b\sqrt{S'_m}}{S_m}\mathrm{e}^{-\frac{(2\delta_b-\delta_m)^2}{2S_m}} + \pi\frac{\delta_b(\delta_b-\delta_m)}{S_m}\mathrm{Erfc}\left[\frac{2\delta_b-\delta_m}{\sqrt{2S'_m}}\right]\right] \times\ \partial_m\Pi^{\rm gm}(\delta_m;\delta_n;S_n-S_m)|_{\delta_m=\delta_b}. \quad \text{(B13)}$$

The corresponding flux rate is given by

$$\mathcal{F}^{(a)}_{\rm sph}(\delta_c,S_n|\delta_b,S_m) = \frac{\kappa}{2}\mathrm{e}^{-\frac{(\delta_b-\delta_c)^2}{2(S_n-S_m)}}\frac{\sqrt{S_m}}{(S_n-S_m)^{5/2}} \times\left(S_m-S_n+(\delta_b-\delta_c)^2\right)\mathrm{e}^{\frac{\delta_b^2}{2S_m}}\mathrm{Erfc}\left[\frac{\delta_b}{\sqrt{2S_m}}\right]. \quad \text{(B14)}$$

The term N_b is given by

$$N_b = \Pi^{\rm gm}(\delta_0;\delta_m;S'_m) \times\left[\Pi^{b1}(\delta_b,S_m;\delta_n,S_n)+\Pi^{b2}(\delta_b,S_m;\delta_n,S_n)\right], \quad \text{(B15)}$$

where

$$\Pi^{b1}(\delta_b,S_m;\delta_n,S_n) \equiv \int_{-\infty}^{\delta_c}\mathrm{d}\delta_{m+1}\cdots\mathrm{d}\delta_{n-1} \times\sum_{i=m+1}^{n-1}\Delta_{in}\partial_i\partial_n W^{\rm gm}(\delta_b;\ldots,\delta_n;S_n-S_m), \quad \text{(B16)}$$

and

$$\Pi^{b2}(\delta_b, S_m; \delta_n, S_n) \equiv \int_{-\infty}^{\delta_c} \mathrm{d}\delta_{m+1} \cdots \mathrm{d}\delta_{n-1} \times \frac{1}{2} \sum_{i,j=m+1}^{n-1} \Delta_{ij} \partial_i \partial_j W^{\mathrm{gm}}(\delta_b; \ldots, \delta_n; S_n - S_m). \tag{B17}$$

The computation of Π^{b1} and Π^{b2} is quite similar to the computation of the terms called Π^{mem} and $\Pi^{\mathrm{mem-mem}}$ in MR1, and in the continuum limit $\epsilon \to 0$ we get

$$\begin{aligned}\Pi^{b1}(\delta_b, S_m; \delta_n, S_n) &= \partial_n \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{S_m}^{S_n} \mathrm{d}S_i \\ &\times \Delta(S_i, S_n) \Pi_\epsilon^{\mathrm{gm}}(\delta_b; \delta_c; S_i - S_m) \Pi_\epsilon^{\mathrm{gm}}(\delta_c; \delta_n; S_n - S_i) \\ &= \frac{\kappa}{\pi}(\delta_c - \delta_b)\partial_n \left\{ (\delta_c - \delta_n) \int_{S_m}^{S_n} \mathrm{d}S_i \right. \\ &\times \frac{S_i}{S_n (S_i - S_m)^{3/2} (S_n - S_i)^{1/2}} \\ &\left. \times \exp\left[-\frac{(\delta_c - \delta_b)^2}{2(S_i - S_m)} - \frac{(\delta_c - \delta_n)^2}{2(S_n - S_i)} \right] \right\}\end{aligned} \tag{B18}$$

and

$$\begin{aligned}\Pi^{b2}(\delta_b, S_m; \delta_n, S_n) &= \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} \int_{S_m}^{S_n} \mathrm{d}S_i \int_{S_i}^{S_n} \mathrm{d}S_j \\ &\times \Delta(S_i, S_j) \Pi_\epsilon^{\mathrm{gm}}(\delta_b; \delta_c; S_i - S_m) \\ &\times \Pi_\epsilon^{\mathrm{gm}}(\delta_c; \delta_c; S_j - S_i) \Pi_\epsilon^{\mathrm{gm}}(\delta_c; \delta_n; S_n - S_j) \\ &= \frac{\kappa}{\pi\sqrt{2\pi}} (\delta_c - \delta_b)(\delta_c - \delta_n) \\ &\times \int_{S_m}^{S_n} \mathrm{d}S_i \, \frac{S_i}{(S_i - S_m)^{3/2}} \mathrm{e}^{-(\delta_c - \delta_b)^2/[2(S_i - S_m)]} \\ &\times \int_{S_i}^{S_n} \mathrm{d}S_j \, \frac{\mathrm{e}^{-(\delta_c - \delta_n)^2/[2(S_n - S_j)]}}{S_j (S_j - S_i)^{1/2} (S_n - S_j)^{3/2}} .\end{aligned} \tag{B19}$$

This can be rewritten as a total derivative with respect to δ_n, as

$$\begin{aligned}\Pi^{b2}(\delta_b, S_m; \delta_n, S_n) &= \frac{\kappa}{\pi\sqrt{2\pi}} (\delta_c - \delta_b)\partial_n \\ &\times \int_{S_m}^{S_n} \mathrm{d}S_i \, \frac{S_i}{(S_i - S_m)^{3/2}} \mathrm{e}^{-(\delta_c - \delta_b)^2/[2(S_i - S_m)]} \\ &\times \int_{S_i}^{S_n} \mathrm{d}S_j \, \frac{\mathrm{e}^{-(\delta_c - \delta_n)^2/[2(S_n - S_j)]}}{S_j (S_j - S_i)^{1/2} (S_n - S_j)^{1/2}} .\end{aligned} \tag{B20}$$

The fact that both Π^{b1} and Π^{b2} can be written as a derivative with respect to δ_n simplifies considerably the computation of the contribution of N_b to the numerator (85), since we can integrate ∂_n by parts, and then we only need to evaluate the integrals in equations (B18) and (B20) in $\delta_n = \delta_c$, which can be done analytically, as discussed in MR1. In particular the contribution to the numerator from Π^{b1} vanishes because it is a derivative ∂_n of a quantity that vanishes at $\delta_n = \delta_c$. Following the appendix of Ma et al. (2010), the contribution to the two-barrier first crossing rate from N_b can be easily computed to be

$$\begin{aligned}\mathcal{F}_{\mathrm{sph}}^{(b)}(\delta_c, S_n|\delta_b, S_m) &= -\frac{\partial}{\partial S_n} \left[\frac{\kappa(\delta_c - \delta_b)}{\sqrt{2\pi S_n}} \right. \\ &\left. \times \int_{S_m}^{S_n} \mathrm{d}S_i \, \frac{S_i^{1/2}}{(S_i - S_m)^{3/2}} \mathrm{e}^{-\frac{(\delta_c - \delta_b)^2}{2(S_i - S_m)}} \right] .\end{aligned} \tag{B21}$$

The most complicated term is N_d. We get

$$\begin{aligned}N_d &= \frac{\kappa}{\pi} \left[\sqrt{2\pi} \frac{\delta_b}{\sqrt{S_m'}} \mathrm{e}^{-\frac{(2\delta_b - \delta_m)^2}{2S_m'}} \frac{\mathcal{I}_2}{\pi} \right. \\ &- \left(\sqrt{2\pi}\delta_b \sqrt{S_m'} \mathrm{e}^{-\frac{(2\delta_b - \delta_m)^2}{2S_m'}} \right. \\ &\left. \left. - \pi\delta_b(\delta_b - \delta_m) \mathrm{Erfc}\left[\frac{2\delta_b - \delta_m}{\sqrt{2S_m'}} \right] \right) \frac{\mathcal{I}_1}{\pi} \right] \\ &+ \frac{\kappa}{\pi} \Pi^{\mathrm{gm}}(\delta_m; S_m') \partial_m \left[S_m \mathcal{I}_2 - S_m^2 \mathcal{I}_1 \right]\big|_{\delta_m = \delta_b} \\ &+ \frac{\kappa}{\pi} \partial_m \Pi^{\mathrm{gm}}(\delta_m; S_m') \left[S_m \mathcal{I}_2 - S_m^2 \mathcal{I}_1 \right]\big|_{\delta_m = \delta_b} ,\end{aligned} \tag{B22}$$

where

$$\mathcal{I}_1 = \partial_n \partial_m \mathcal{J}(\delta_m, \delta_n)|_{\delta_m = \delta_b} , \tag{B23}$$

$$\begin{aligned}\mathcal{I}_2 &= \int_{S_m}^{S_n} \mathrm{d}S_j \, \frac{(\delta_c - \delta_m)(\delta_c - \delta_n)}{(S_j - S_m)^{3/2} (S_n - S_j)^{3/2}} \\ &\times \exp\left\{ -\frac{(\delta_c - \delta_m)^2}{2(S_j - S_m)} - \frac{(\delta_c - \delta_n)^2}{2(S_n - S_j)} \right\} \\ &= \sqrt{\frac{2\pi}{S_n - S_m}} \partial_n \mathrm{e}^{-(2\delta_c - \delta_m - \delta_n)^2/[2(S_n - S_m)]},\end{aligned} \tag{B24}$$

$$\begin{aligned}\mathcal{J} &\equiv \int_{S_m}^{S_n} \mathrm{d}S_j \, \frac{1}{S_j (S_j - S_m)^{1/2} (S_n - S_j)^{1/2}} \\ &\times \exp\left\{ -\frac{(\delta_c - \delta_m)^2}{2(S_j - S_m)} - \frac{(\delta_c - \delta_n)^2}{2(S_n - S_j)} \right\} .\end{aligned} \tag{B25}$$

We are only interested in its value for $\delta_n = \delta_c$ as N_d contains a total derivative with respect to δ_n

$$\begin{aligned}\mathcal{J}(\delta_m, \delta_n = \delta_c) &= \frac{\pi}{(S_m S_n)^{1/2}} \mathrm{e}^{+(\delta_c - \delta_m)^2/(2S_m)} \\ &\times \mathrm{Erfc}\left[(\delta_c - \delta_m) \sqrt{\frac{S_n}{2S_m(S_n - S_m)}} \right] .\end{aligned} \tag{B26}$$

The corresponding flux rate is given by

$$\begin{aligned}&\mathcal{F}_{\mathrm{sph}}^{(d)}(\delta_c, S_n|\delta_b, S_m) = \tfrac{\kappa}{\sqrt{2\pi}} \tfrac{\mathrm{e}^{-\frac{(\delta_c - \delta_b)^2}{2(S_n - S_m)}}}{(S_n - S_m)^{5/2}} \Bigg\{ \\ &\delta_b \left(S_m - S_n + (\delta_c - \delta_b)^2 \right) + (\delta_c - \delta_b) S_m \left(3 - \tfrac{(\delta_c - \delta_b)^2}{S_n - S_m} \right) \\ &- \tfrac{\sqrt{\pi S_m}}{S_n^2} \left[\tfrac{\delta_b}{\sqrt{2\pi S_m}} + \tfrac{1}{2} \mathrm{e}^{\frac{\delta_b^2}{2S_m}} \mathrm{Erfc}\left[\tfrac{\delta_b}{\sqrt{2S_m}} \right] \right] \\ &\times \Bigg[\sqrt{2} S_n \left(-S_n^2 + S_m \left(S_n + (\delta_c - \delta_b)^2 \right) \right) \\ &+ \sqrt{\pi} \mathrm{e}^{\frac{(\delta_c - \delta_b)^2 S_n}{2S_m(S_n - S_m)}} \sqrt{\tfrac{S_n}{S_m}} (S_n - S_m)^{5/2} (\delta_c - \delta_b) \\ &\times \mathrm{Erfc}\left[\sqrt{\tfrac{S_n}{S_m}} \tfrac{\delta_c - \delta_b}{\sqrt{2(S_n - S_m)}} \right] \Bigg] \\ &+ (\delta_c - \delta_b) \tfrac{\left(S_m^3 - 4S_m S_n^2 + S_n^3 + S_m^2 (2S_n + (\delta_c - \delta_b)^2) \right)}{S_n (S_n - S_m)} \\ &- \sqrt{\tfrac{\pi}{2}} \mathrm{e}^{\frac{(\delta_c - \delta_b)^2 S_n}{2S_m(S_n - S_m)}} \tfrac{(S_n - S_m)^{5/2}}{S_m^{1/2} S_n^{3/2}} \left(S_m + (\delta_c - \delta_b)^2 \right) \\ &\times \mathrm{Erfc}\left[\sqrt{\tfrac{S_n}{S_m}} \tfrac{\delta_c - \delta_b}{\sqrt{2(S_n - S_m)}} \right] \Bigg\} .\end{aligned} \tag{B27}$$

The total two-barrier first-crossing rate is finally obtained by adding to the usual rate the corrections given by equations (B14), (B21) and (B27):

$$\begin{aligned}\mathcal{F}_{\rm sph}(\delta_c, S_n|\delta_b, S_m) = {} & \frac{(\delta_c-\delta_b)\mathrm{e}^{-\frac{(\delta_c-\delta_b)^2}{2(S_n-S_m)}}}{\sqrt{2\pi}(S_n-S_m)^{3/2}} \\ & + \mathcal{F}^{(a)}_{\rm sph}(\delta_c, S_n|\delta_b, S_m) + \mathcal{F}^{(b)}_{\rm sph}(\delta_c, S_n|\delta_b, S_m) \\ & + \mathcal{F}^{(d)}_{\rm sph}(\delta_c, S_n|\delta_b, S_m)\,. \end{aligned} \tag{B28}$$

In the limit $S_m \ll S_n$, the total rate reduces to

$$\begin{aligned}\mathcal{F}_{\rm sph}(\delta_c, S_n|\delta_b) = {} & \frac{1-\kappa}{\sqrt{2\pi}}\frac{(\delta_c-\delta_b)\mathrm{e}^{-\frac{(\delta_c-\delta_b)^2}{2S_n}}}{S_n^{3/2}} \\ & + \frac{\kappa}{2\sqrt{2\pi}}\frac{\delta_c-\delta_b}{S_n^{3/2}}\,\Gamma\left(0, \frac{(\delta_c-\delta_b)^2}{2S_n}\right) \\ & - \frac{\kappa}{\sqrt{2\pi}}\frac{\delta_b}{S_n^{3/2}}\left[1-\frac{(\delta_c-\delta_b)^2}{S_n}\right]\mathrm{e}^{-\frac{(\delta_c-\delta_b)^2}{2S_n}}, \end{aligned} \tag{B29}$$

which reproduces the result of Ma et al. (2010). In the case of a moving barrier, we expect that it is a good approximation (Giocoli et al. 2007) to simply replace the constant barrier δ_c with $\sqrt{a}\delta_c$, and κ with $a\kappa$ (Maggiore & Riotto 2010b; Ma et al. 2010). If so, we obtain

$$\begin{aligned}\mathcal{F}_{\rm mb}(\delta_c, S_n|\delta_b) = {} & \frac{1-a\kappa}{\sqrt{2\pi}}\frac{\sqrt{a}(\delta_c-\delta_b)\mathrm{e}^{-\frac{a(\delta_c-\delta_b)^2}{2S_n}}}{S_n^{3/2}} \\ & + a^{3/2}\frac{\kappa}{2\sqrt{2\pi}}\frac{(\delta_c-\delta_b)}{S_n^{3/2}}\,\Gamma\left(0, \frac{a(\delta_c-\delta_b)^2}{2S_n}\right) \\ & - a^{3/2}\frac{\kappa}{\sqrt{2\pi}}\frac{\delta_b}{S_n^{3/2}}\left[1-\frac{a(\delta_c-\delta_b)^2}{S_n}\right]\mathrm{e}^{-\frac{a(\delta_c-\delta_b)^2}{2S_n}}. \end{aligned} \tag{B30}$$

Instead, the case of the diffusive barrier is obtained from equation (B29) by simply sending S_n into S_n/a and κ into $a\kappa$, that is

$$\begin{aligned}\mathcal{F}_{\rm dif}(\delta_c, S_n|\delta_b) = {} & \frac{1-a\kappa}{\sqrt{2\pi}}a^{3/2}\frac{(\delta_c-\delta_b)\mathrm{e}^{-\frac{a(\delta_c-\delta_b)^2}{2S_n}}}{S_n^{3/2}} \\ & + \frac{a^{5/2}\kappa}{2\sqrt{2\pi}}\frac{\delta_c-\delta_b}{S_n^{3/2}}\,\Gamma\left(0, \frac{a(\delta_c-\delta_b)^2}{2S_n}\right) \\ & - \frac{a^{5/2}\kappa}{\sqrt{2\pi}}\frac{\delta_b}{S_n^{3/2}}\left[1-\frac{a(\delta_c-\delta_b)^2}{S_n}\right]\mathrm{e}^{-\frac{a(\delta_c-\delta_b)^2}{2S_n}}. \end{aligned} \tag{B31}$$

We expect these results to be correct in the high mass limit.

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