

# $\Gamma$ -limit for the extended Fisher–Kolmogorov equation

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(MS received 18 May 2000; accepted 15 March 2001)

We consider the Lyapunov functional,

$$E_\varepsilon^\gamma(u) \stackrel{\text{def}}{=} \int_\Omega \left\{ \frac{1}{2} \varepsilon^3 \gamma (\Delta u)^2 + \frac{1}{2} \varepsilon |\nabla u|^2 + \frac{1}{4\varepsilon} (u^2 - 1)^2 \right\} dx,$$

of the rescaled Extended Fisher–Kolmogorov equation

$$u_t + \varepsilon^2 \gamma \Delta^2 u - \Delta u + \frac{1}{\varepsilon^2} u(u^2 - 1) = 0, \quad \gamma > 0.$$

This is a fourth order generalization of the Fisher–Kolmogorov or Allen–Cahn equation. We prove that if  $\varepsilon \rightarrow 0$ , then  $E_\varepsilon^\gamma$  tends to the area functional in the sense of  $\Gamma$ -limits, where the transition energy is given by the one-dimensional kink of the Extended Fisher–Kolmogorov equation.

## 1. Introduction

In this paper we shall prove a limiting property of the *extended Fisher–Kolmogorov (EFK) equation*

$$u_t + \gamma \Delta^2 u - \Delta u + f(u) = 0, \quad \gamma > 0, \quad (1.1a)$$

in which  $f = F'$  and  $F$  is given by the symmetric double-well potential

$$F(s) = \frac{1}{4} (s^2 - 1)^2. \quad (1.1b)$$

The EFK equation is a higher-order generalization of the *Fisher–Kolmogorov (FK) equation*, or *Allen–Cahn equation*,

$$u_t - \Delta u + f(u) = 0. \quad (1.2)$$

It arises in a variety of situations, such as phase transitions, and was first proposed as a model equation for non-trivial spatio-temporal pattern formation by Dee and van Saarloos [15]. In recent years, it has been the subject of intensive study. We refer to [1, 3, 4, 6, 7, 12, 22, 24, 26, 30–33, 35, 40, 41].

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In the present paper we are interested in (1.1) in the scaled variables

$$x^* = \varepsilon x, \quad t^* = \varepsilon^2 t \quad \text{and} \quad u^*(x^*, t^*) = u(x, t),$$

and particularly in the limit as  $\varepsilon \rightarrow 0$ . Substitution yields the equation

$$u_t + \varepsilon^2 \gamma \Delta^2 u - \Delta u + \frac{1}{\varepsilon^2} f(u) = 0, \quad \gamma > 0, \tag{1.3}$$

where we have dropped the asterisks again.

For the FK equation, it has been proved in several articles that in this limit the flow converges to the mean curvature flow. We mention here the papers by Barles *et al.* [5], Chen [10], de Mottoni and Schatzman [13,14] and Evans *et al.* [18], in the anisotropic case Elliott and Schätzle [17], and also Ilmanen [23], in which the mean curvature flow is considered in the sense of the varifold formulation of Brakke [9].

For the scaled EFK equation (1.3), formal asymptotic methods suggest that in the limit as  $\varepsilon \rightarrow 0$ , the flow also converges to the mean curvature flow. We give a brief sketch of this formal argument. We make the *Ansatz*

$$u(x, t) = U\left(\frac{d(x, t)}{\varepsilon}\right), \quad U(0) = 0, \tag{1.4}$$

where  $d$  is the signed distance function of the limiting interface and  $U : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function to be determined below. When we substitute this Ansatz into (1.3), then after a short calculation we find that

$$\begin{aligned} u_t + \varepsilon^2 \gamma \Delta^2 u - \Delta u + \frac{1}{\varepsilon^2} f(u) &= \frac{1}{\varepsilon^2} \{ \gamma U'''' - U'' + f(U) \} + \frac{1}{\varepsilon} U' \left\{ d_t - \left( 1 - 2\gamma \frac{U'''}{U'} \right) \Delta d \right\} + \mathcal{O}(1), \end{aligned} \tag{1.5}$$

whenever  $U' \neq 0$ . Assuming that the Ansatz (1.4) yields an approximate solution of (1.3), we conclude that  $U$  has to be a solution of the ordinary differential equation

$$\gamma U'''' - U'' + f(U) = 0 \quad \text{on } \mathbb{R}. \tag{1.6 a}$$

Since  $U$  represents a transition across an interface, we suppose that

$$U(x) \rightarrow \pm 1 \quad \text{as } x \rightarrow \pm \infty. \tag{1.6 b}$$

We shall refer to solutions of problem (1.6) as *kinks*. In addition, the signed distance function  $d$  must be a solution of the evolution equation

$$d_t - \left( 1 - 2\gamma \frac{U''''(0)}{U'(0)} \right) \Delta d = 0 \quad \text{on the set } \{(x, t) : d(x, t) = 0\}. \tag{1.7}$$

We conclude (formally) that as  $\varepsilon \rightarrow 0^+$ , solutions of (1.3) converge to a limit function  $\bar{u} = \pm 1$  a.e. in  $\Omega \times \mathbb{R}^+$  and that the interface between the regions where  $\bar{u} = 1$  and those where  $\bar{u} = -1$  moves according to (1.7), i.e. we have motion by mean curvature.

Any solution  $U$  of problem (1.6) must satisfy the identity

$$2\gamma U' U'''' - \gamma (U'')^2 - (U')^2 + 2F(U) = 0 \quad \text{on } \mathbb{R}, \tag{1.8}$$

which is obtained from (1.6a) by multiplying the equation by  $2U'$  and integrating the result over  $(-\infty, x)$ . For odd solutions,  $U(0) = 0$  and  $U''(0) = 0$ , so that (1.8) then implies that

$$2\gamma \frac{U'''(0)}{U'(0)} = 1 - \frac{1}{2(U'(0))^2} < 1. \tag{1.9}$$

It has been shown in [24, 30, 34] that for any  $\gamma > 0$ , problem (1.6) has an odd solution that is positive for  $x > 0$  and negative for  $x < 0$ . In [30, 32], this was done by means of a topological shooting argument, while in [34] and [24] variational methods were used involving the functional

$$e^\gamma(U) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left\{ \frac{1}{2}\gamma(U'')^2 + \frac{1}{2}(U')^2 + F(U) \right\} dx. \tag{1.10}$$

For  $\gamma \leq \frac{1}{8}$ , this solution is the only solution of problem (1.6) [30, 38, 39]. In this case, the kink is odd and strictly increasing. For  $\gamma > \frac{1}{8}$ , there are many odd solutions of problem (1.6), distinguished by the number of local maxima (bumps). However, we conjecture that for all  $\gamma > 0$  the odd kink that is *positive* for  $x > 0$  is unique.

Thus, from (1.9), we conclude that the coefficient of  $\Delta d$  in (1.7) is positive, so that this equation is parabolic. Plainly, it governs the mean curvature flow.

In §2 we shall derive new estimates for the second derivative of solutions  $U$  of problem (1.6). They will enable us to prove that bumps of these solutions need to have a minimal size when  $\gamma < \frac{27}{8}$ , and therefore cannot accumulate.

The main result of this paper is the determination of the  $\Gamma$ -limit of the functional

$$E_\varepsilon^\gamma(u) \stackrel{\text{def}}{=} \int_\Omega \left\{ \frac{1}{2}\varepsilon^3\gamma(\Delta u)^2 + \frac{1}{2}\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}F(U) \right\} dx, \tag{1.11}$$

which is a Lyapunov functional of (1.3). Here,  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ . Throughout, we shall assume that its boundary  $\partial\Omega$  is smooth.

Let us recall the definition of  $\Gamma$ -limits.

DEFINITION 1.1. We say that

$$E_0^\gamma \stackrel{\text{def}}{=} \Gamma - \lim_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma$$

is the  $\Gamma$ -limit of the sequence  $\{E_\varepsilon^\gamma\}$  if it has the following properties.

(a) If  $u_\varepsilon \rightarrow u$  strongly in  $L^1(\Omega)$ , then

$$E_0^\gamma(u) \leq \liminf_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma(u_\varepsilon).$$

(b) For  $u \in L^1(\Omega)$ , there exists a sequence  $\{u_\varepsilon\}$  converging strongly in  $L^1(\Omega)$  to  $u$  such that

$$E_0^\gamma(u) = \lim_{\varepsilon \downarrow 0} E_\varepsilon^\gamma(u_\varepsilon).$$

For the FK equation, when  $\gamma = 0$ , it was proved by Modica and Mortola [28] and Modica [27] that the  $\Gamma$ -limit of  $\{E_\varepsilon^0\}$  is the area functional multiplied by the transition energy for the double well  $F$ . It turns out that the same is true in the presence of the higher-order term, i.e. when  $\gamma > 0$ . Actually, it is not surprising

that the  $\Gamma$ -limit will be the area functional up to a constant factor, as theorem 3.1 in [2] indicates. Specifically, we will determine the factor in front and show that

$$E_0^\gamma(u) = \begin{cases} \frac{1}{2}P^\gamma \int_\Omega |\nabla u| & \text{if } u \in BVC(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus BVC(\Omega), \end{cases}$$

where

$$P^\gamma \stackrel{\text{def}}{=} \inf_{u \text{ odd}} e^\gamma(u),$$

and  $BVC(\Omega) = \{\psi \in BV(\Omega) : \psi(\Omega) \subset \{\pm 1\}\}$ . For one-dimensional domains this result has been proved by Kalies *et al.* [25]. For more general results on  $\Gamma$ -limits in an isotropic setting, see [8, 20, 29].

The approximation part (part (b)) of the  $\Gamma$ -limit carries over from [27] to the case when  $\gamma > 0$ , whereas the main difficulty lies in the proof of the lower semicontinuity of the sequence  $\{E_\varepsilon^\gamma(u_\varepsilon)\}$  (part (a)). We will localize this assertion by using Hausdorff densities and reduce the problem to one where the limit of the interface is a half-plane. Writing this limit interface as  $\partial H = \{x = (y, t) : t = 0, y \in \mathbb{R}^{n-1}\}$ , we see that

$$E_\varepsilon^\gamma(u) = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \left\{ \frac{1}{2}\varepsilon^3\gamma|u_{tt} + \Delta_y u|^2 + \frac{1}{2}\varepsilon(|u_t|^2 + |\nabla_y u|^2) + \frac{1}{\varepsilon}F(u) \right\}, \tag{1.12}$$

where  $\Delta_x u = u_{tt} + \Delta_y u$ . We see from (1.12) that symmetry breaking, leading to variations in the  $y$ -direction, and thus a non-vanishing term  $\Delta_y u$ , could lead to annihilating the effect of the second-order derivative  $u_{tt}$ . In addition, such variations would lead to an increase in the term involving first-order derivatives. Therefore, in higher dimensions, it is crucial to exclude this symmetry breaking, and show that the minimization is performed by one-dimensional minimizers. This then also justifies the Ansatz (1.4) made above in the formal asymptotics, which involves the choice  $u_\varepsilon := U(d/\varepsilon)$ , that is, it justifies the choice of a one-dimensional stationary wave  $U$ , rather than a more-dimensional object.

The paper is organized as follows. We begin, in § 2, with a discussion of the one-dimensional stationary wave  $U$ , and, in particular, of two universal bounds, one for  $U$ , and one for  $U''$ . Then, in § 3, we obtain the  $\Gamma$ -limit of the sequence of functionals  $E_\varepsilon^\gamma$  defined in (1.11) in the absence of a constraint. In § 4 we do the same, but now we impose the constraint of a given average value of  $u$ ,

$$P \int_\Omega u \stackrel{\text{def}}{=} \frac{1}{|\Omega|} \int_\Omega u = m \quad \text{form } m \in [-1, 1]. \tag{1.13}$$

Here we consider the family of functionals

$$G_\varepsilon^\gamma(u) \stackrel{\text{def}}{=} E_\varepsilon^\gamma(u) + I(u), \tag{1.14a}$$

where

$$I(u) = \begin{cases} 0 & \text{if } P \int_\Omega u = m, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.14b}$$

and establish the  $\Gamma$ -limit.

In §5 we conclude with a discussion of the case when the symmetric functional  $F(u)$  defined in (1.1 *b*) is replaced by a non-symmetric and non-autonomous potential function  $F(x, u; \varepsilon)$  corresponding to

$$f = s^3 - s + \varepsilon a(x). \tag{1.15}$$

In future work we shall study the limiting behaviour as  $\varepsilon \rightarrow 0^+$  of solutions of the evolution equation (1.3) and prove that the limit problem involves motion by mean curvature (see (1.7)).

## 2. The one-dimensional equation

In this section we study bounded solutions, and, in particular, transition layers, or *kinks*, between the constant solutions  $u = \pm 1$  of the one-dimensional extended Fisher–Kolmogorov equation

$$\gamma U'''' - U'' + f(U) = 0, \tag{2.1}$$

where  $f = F'$  and  $F$  is the symmetric double-well potential defined in (1.1 *b*). As a further restriction, we consider solutions with finite energy, that is,

$$e^\gamma(U) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left\{ \frac{1}{2} \gamma (U'')^2 + \frac{1}{2} (U')^2 + F(U) \right\} < \infty. \tag{2.2}$$

This implies that  $U$  will belong to the set of functions

$$\mathcal{F} = \{w : \mathbb{R} \rightarrow \mathbb{R} \mid w^2 - 1, w', w'' \in L^2(\mathbb{R})\}. \tag{2.3}$$

Moreover,

$$U(x) \rightarrow \pm 1 \quad \text{as } x \rightarrow \pm\infty \tag{2.4 a}$$

and  $U$  is bounded. As  $f(U) = U(U^2 - 1) \in L^2(\mathbb{R})$ , it follows from (2.1) that  $U'''' \in L^2(\mathbb{R})$ . Multiplying  $U''''$  by  $U''\varphi$ , where  $\varphi$  is a suitable cut-off function and integrating by parts on  $\mathbb{R}$ , we deduce that  $U''' \in L^2(\mathbb{R})$ . Therefore,

$$D^j U(x) \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty \quad \text{for } j = 1, 2, 3, 4. \tag{2.4 b}$$

Regarding existence and uniqueness of a solution of problem (2.1), (2.4) we have the following result.

**PROPOSITION 2.1.** *For every  $\gamma > 0$ , there exists an odd solution  $U \in \mathcal{F}$  of (2.1) such that*

$$(U, U', U'', U''')(x) \rightarrow (\pm 1, 0, 0, 0) \quad \text{as } x \rightarrow \pm\infty,$$

and

$$U(x) > 0 \ (\lt 0) \quad \text{if } x > 0 \ (\lt 0).$$

If  $0 < \gamma \leq \frac{1}{8}$ , the kink is unique (modulo shifts) and strictly monotone on  $\mathbb{R}$ .

The existence of odd kinks was proved in [30] for  $0 < \gamma \leq \frac{1}{8}$  and in [24, 33, 34] for arbitrary  $\gamma > 0$ . The uniqueness for  $\gamma \leq \frac{1}{8}$  was established in [30, 38, 39]. The

critical value  $\gamma = \frac{1}{8}$  stems from the nature of the spectrum at the constant solutions  $u = \pm 1$ , which changes at this value of  $\gamma$ . Putting  $U = 1 - V$ , we can write (2.1) as

$$\gamma V'''' - V'' + 2V = 3V^2 - V^3,$$

which yields the characteristic equation

$$\gamma \lambda^4 - \lambda^2 + 2 = 0.$$

Its roots are given by  $\lambda = \pm \sqrt{\mu_{\pm}}$ , where

$$\mu_{\pm} = \frac{1}{2\gamma} (1 \pm \sqrt{1 - 8\gamma}). \tag{2.5}$$

Thus, for  $\gamma \leq \frac{1}{8}$ , the eigenvalues in the spectrum are all real, while if  $\gamma > \frac{1}{8}$ , they are all complex.

Below, we shall prove a bound for solutions  $U \in \mathcal{F}$  of (2.1). In the proof we shall make use of the fact that such solutions possess the property

$$2\gamma U'U''' - \gamma(U'')^2 - (U')^2 + 2F(U) = 0 \quad \text{on } \mathbb{R}. \tag{2.6}$$

We begin with a universal bound for  $U''$ .

LEMMA 2.2. *Let  $\gamma > 0$ , and let  $U \in \mathcal{F}$  be a solution of (2.1). Then*

$$|U''(x)| \leq \frac{2}{3\sqrt{3}}.$$

*Proof.* Note that the second derivative  $V := U''$  satisfies the elliptic equation

$$-\gamma V'' + V = f(U). \tag{2.7}$$

Because  $V(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ , there exists a point  $x_0 \in \mathbb{R}$  such that

$$V(x_0) = \sup\{|V(x)| : x \in \mathbb{R}\}.$$

Without loss of generality, we may assume that  $V(x_0) > 0$ . At the point  $x_0$ , we have

$$U'''(x_0) = V'(x_0) = 0 \quad \text{and} \quad U''''(x_0) = V''(x_0) \leq 0,$$

and hence, by (2.7),

$$V(x_0) = \gamma V''(x_0) + f(U(x_0)) \leq f(U(x_0)). \tag{2.8}$$

If  $U(x_0) \leq 1$ , then we conclude that

$$V(x_0) \leq \sup\{f(s) : s \leq 1\} = \frac{2}{3\sqrt{3}}. \tag{2.9}$$

Suppose that  $V(x_0) > 2/3\sqrt{3}$ . Then (2.9) implies that  $U(x_0) > 1$ . Let

$$U(x_1) = \sup\{U(x) : x \in \mathbb{R}\}.$$

Because  $U(x) \rightarrow \pm 1$  as  $x \rightarrow \pm\infty$ , such a point  $x_1$  exists. Note that because  $U''(x_0) = V(x_0) > 0$ , we must have

$$U(x_1) > U(x_0) > 1 \quad \text{and} \quad F(U(x_1)) > F(U(x_0)) > 0. \tag{2.10}$$

However, by the integral identity (2.6), we have

$$2F(U(x_1)) = \gamma(U''(x_1))^2 \quad \text{and} \quad 2F(U(x_0)) = \gamma(U''(x_0))^2 + (U'(x_0))^2.$$

Hence, we conclude from (2.10) that

$$(U''(x_1))^2 > (U''(x_0))^2,$$

or

$$|V(x_1)| > |V(x_0)| = V(x_0).$$

This contradicts the definition of  $V(x_0)$ . □

The following bounds are an immediate corollary of lemma 2.2.

LEMMA 2.3. *Suppose that  $0 < \gamma < \frac{27}{8}$ . Let  $U \in \mathcal{F}$  be a solution of (2.1) and let  $a \in \mathbb{R}$  be a critical point of the graph of  $U$ , i.e.  $U'(a) = 0$ . Then*

$$\sqrt{1 - \sqrt{\frac{8}{27}\gamma}} \leq |U(a)| \leq \sqrt{1 + \sqrt{\frac{8}{27}\gamma}}.$$

*Proof.* Since  $U'(a) = 0$ , it follows from the integral identity (2.6) and lemma 2.2 that

$$2F(U(a)) = \gamma\{U''(a)\}^2 \leq \frac{4}{27}\gamma.$$

Thus, in view of the definition (1.1 b) of  $F$ , we conclude that

$$\{U^2(a) - 1\}^2 \leq \frac{8}{27}\gamma,$$

from which the assertion follows. □

Lemma 2.3 states that bumps of  $U$  must have a certain minimum size if  $\gamma < \frac{27}{8}$ .

We conclude this section with a brief analysis of the decay properties of kinks as  $x \rightarrow \pm\infty$ . It is clear from (2.5) that none of the eigenvalues in the spectrum at  $U = 1$  has zero real part. Knowing that  $V = 1 - U$  possesses the asymptotic behaviour,  $(V, V', V'', V''')(x) \rightarrow (0, 0, 0, 0)$  as  $x \rightarrow \infty$ , it follows from standard theory (cf. [11]) that there exist constants  $\alpha > 0$  and  $C > 0$  such that

$$|1 - U(x)| + |U'(x)| + |U''(x)| + |U'''(x)| \leq Ce^{-\alpha x} \quad \text{as } x \rightarrow \infty. \tag{2.11 a}$$

For  $x \rightarrow -\infty$ , we can similarly prove

$$|1 + U(x)| + |U'(x)| + |U''(x)| + |U'''(x)| \leq Ce^{+\alpha x} \quad \text{as } x \rightarrow -\infty. \tag{2.11 b}$$

### 3. Gamma convergence

In this section we shall prove the main result of this paper. We recall the functionals

$$E_\varepsilon^\gamma(u) = \int_\Omega \left\{ \frac{1}{2}\varepsilon^3\gamma(\Delta u)^2 + \frac{1}{2}\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}F(u) \right\} dx, \tag{3.1}$$

the Lyapunov functional of (1.3) and

$$e^\gamma(u) = \int_{\mathbb{R}} \left\{ \frac{1}{2}\gamma(u'')^2 + \frac{1}{2}(u')^2 + F(u) \right\} dx, \tag{3.2}$$

where  $F$  is the symmetric double-well potential given by (1.1 b).

**THEOREM 3.1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$  with boundary  $\partial\Omega \in C^{0,1}$ . Then the  $\Gamma$ -limit*

$$E_0^\gamma = \Gamma - \lim_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma$$

*equals the area functional multiplied by the transition energy for the double well  $F$ , that is,*

$$E_0^\gamma(u) = \begin{cases} \frac{1}{2}P^\gamma \int_\Omega |\nabla u| & \text{if } u \in BVC(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus BVC(\Omega), \end{cases}$$

where

$$P^\gamma = \inf_{u \text{ odd}} e^\gamma(u) \quad \text{and} \quad BVC(\Omega) = \{\psi \in BV(\Omega) : \psi(\Omega) \subset \{\pm 1\}\}.$$

We need to prove two propositions.

**PROPOSITION 3.2.** *Given a sequence  $\{u_\varepsilon\}$  such that  $u_\varepsilon \rightarrow u$  as  $\varepsilon \rightarrow 0$  strongly in  $L^1(\Omega)$ , then*

$$E_0^\gamma(u) \leq \liminf_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma(u_\varepsilon).$$

**PROPOSITION 3.3.** *For any  $u \in L^1(\Omega)$ , there exists a sequence  $\{u_\varepsilon\}$  such that*

- (a)  $u_\varepsilon \rightarrow u$  as  $\varepsilon \rightarrow 0$  strongly in  $L^1(\Omega)$ ,
- (b)  $E_0^\gamma(u) = \liminf_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma(u_\varepsilon)$ .

*Proof of proposition 3.2.* We may assume that there is a constant  $\Lambda \in \mathbb{R}^+$  such that, along a subsequence  $\{u_\varepsilon\}$ ,

$$E_\varepsilon^\gamma(u_\varepsilon) \leq \Lambda. \tag{3.3}$$

This implies that  $u \in BVC(\Omega)$ , and hence that we can write  $u$  as

$$u = \chi_D - \chi_{\Omega \setminus D} = 2\chi_D - 1, \tag{3.4}$$

for some set  $D \subset \Omega$  (cf. [27]).

We first show that proposition 3.2 is true when  $\partial D$  is a hyperplane.

**LEMMA 3.4.** *Suppose that  $\partial D$  is the intersection of a hyperplane with a ball. Then the assertion of proposition 3.2 holds.*

*Proof.* Let  $H$  be the half-space  $H = \{x = (y, t) : y \in \mathbb{R}^{n-1}, t > 0\}$  and  $B_1 = B_1(0)$  be the unit ball centred at the origin, and let

$$u_\varepsilon \rightarrow \chi_{B_1 \cap H} - \chi_{B_1 \setminus H} \quad \text{in } L^1(B_1).$$

Then we wish to show that

$$E_{0,B_1}^\gamma(u) = P^\gamma \mathcal{L}^{n-1}(\partial H \cap B_1) \leq \liminf_{\varepsilon \rightarrow 0} E_{\varepsilon,B_1}^\gamma(u_\varepsilon), \tag{3.5}$$

where  $\mathcal{L}^{n-1}$  denotes the  $n - 1$  dimensional Lebesgue measure, and

$$E_{\varepsilon,A}^\gamma(u) = \int_A \left\{ \frac{1}{2} \varepsilon^3 \gamma |\Delta u|^2 + \frac{1}{2} \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} F(u) \right\} dx \tag{3.6}$$



for any open set  $A$ . In order to keep the computations as simple as possible, we prove (3.5) not for the ball  $B_1$ , but for a rectangular domain

$$\Sigma := W \times (-1, 1) \quad \text{and} \quad x = (y, t),$$

in which  $W$  is a bounded domain of  $\mathbb{R}^{n-1}$  with smooth boundary and

$$u_\varepsilon \rightarrow \chi_{H \cap \Sigma} - \chi_{\Sigma \setminus H} \quad \text{in } L^1(\Sigma).$$

The proofs of (3.5) for the ball  $B_1$  and the rectangle  $\Sigma$  are essentially the same.

We may assume that  $u_\varepsilon \in C^\infty(\bar{\Sigma})$ , and we write

$$\Delta_x u_\varepsilon = (u_\varepsilon)_{tt} + \Delta_y u_\varepsilon, \quad y \in W, \quad t \in (-1, 1).$$

Using this in the expression for  $E_{\varepsilon, \Sigma}^\gamma(u_\varepsilon)$ , and suppressing the subscript  $\varepsilon$  from the function  $u_\varepsilon$  in order to keep the formulae transparent, we obtain

$$\int_{-1}^1 \int_W \left\{ \frac{1}{2} \varepsilon^3 \gamma |u_{tt} + \Delta_y u|^2 + \frac{1}{2} \varepsilon (|u_t|^2 + |\nabla_y u|^2) + \frac{1}{\varepsilon} F(u) \right\} = E_{\varepsilon, \Sigma}^\gamma(u) \leq \Lambda. \quad (3.7)$$

To obtain a lower bound involving  $P^\gamma$ , we estimate the left-hand side from below by a functional that no longer contains partial derivatives with respect to  $y$ . We begin with the first term. For  $\eta \in C_0^\infty(\Sigma)$ , we can write

$$\int_{-1}^1 \int_W |u_{tt} + \Delta_y u|^2 \eta^2 = \int_{-1}^1 \int_W |u_{tt}|^2 \eta^2 + \int_{-1}^1 \int_W (|\Delta_y u|^2 \eta^2 + 2u_{tt} \Delta_y u \eta^2). \quad (3.8)$$

The term involving  $\Delta_y u$  is non-negative, so that we only have to estimate the last integral in (3.8) from below. To that end, we write it as

$$\begin{aligned} & \int_{-1}^1 \int_W u_{tt} \Delta_y u \eta^2 \\ &= - \int_{-1}^1 \int_W u_t (\Delta_y u_t) \eta^2 - 2 \int_{-1}^1 \int_W u_t (\Delta_y u) \eta \eta_t \\ &= \int_{-1}^1 \int_W |\nabla_y u_t|^2 \eta^2 + 2 \int_{-1}^1 \int_W u_t (\nabla_y u_t) \eta \nabla_y \eta - 2 \int_{-1}^1 \int_W u_t (\Delta_y u) \eta \eta_t. \end{aligned} \quad (3.9)$$

We have

$$2 \int_{-1}^1 \int_W u_t (\nabla_y u_t) \eta \nabla_y \eta \leq \int_{-1}^1 \int_W |\nabla_y u_t|^2 \eta^2 + \int_{-1}^1 \int_W (u_t)^2 |\nabla_y \eta|^2$$

and

$$2 \int_{-1}^1 \int_W u_t (\Delta_y u) \eta \eta_t \leq \frac{1}{2} \int_{-1}^1 \int_W |\Delta_y u|^2 \eta^2 + 2 \int_{-1}^1 \int_W (u_t)^2 \eta_t^2.$$

Using these bounds in (3.8), we obtain the lower bound

$$\int_{-1}^1 \int_W (|\Delta_y u|^2 + 2u_{tt} \Delta_y u) \eta^2 \geq - \int_{-1}^1 \int_W (u_t)^2 (4\eta_t^2 + 2|\nabla_y \eta|^2). \quad (3.10)$$

Thus, using (3.8) together with the upper bound for the integral on the right of (3.10) provided by (3.7), we find that

$$\frac{1}{2}\varepsilon^3\gamma \int_{-1}^1 \int_W |u_{tt} + \Delta_y u|^2 \eta^2 \geq \frac{1}{2}\varepsilon^3\gamma \int_{-1}^1 \int_W (u_{tt})^2 \eta^2 - C(\eta)\Lambda\varepsilon^2. \tag{3.11}$$

Therefore, if we assume in addition that  $\|\eta\|_\infty \leq 1$ , we conclude that

$$\begin{aligned} &\int_{-1}^1 \int_W \frac{1}{2}\varepsilon^3\gamma |u_{tt} + \Delta_y u|^2 + \frac{1}{2}\varepsilon(|u_t|^2 + |\nabla_y u|^2) + \frac{1}{\varepsilon}F(u) \\ &\geq \int_{-1}^1 \int_W \left\{ \frac{1}{2}\varepsilon^3\gamma |u_{tt}|^2 + \frac{1}{2}\varepsilon |u_t|^2 + \frac{1}{\varepsilon}F(u) \right\} \eta^2 - C(\eta)\Lambda\varepsilon^2. \end{aligned} \tag{3.12}$$

To prove (3.5) for  $\Sigma$ , it now suffices to show that

$$P^\gamma \leq \liminf_{\varepsilon \rightarrow 0} \int_{-1/2}^{1/2} \left\{ \frac{1}{2}\varepsilon^3\gamma |u_{tt}|^2 + \frac{1}{2}\varepsilon |u_t|^2 + \frac{1}{\varepsilon}F(u) \right\} dt \tag{3.13}$$

for almost every  $y \in W$ .

We know that for almost every fixed  $y \in W$ ,

$$u(y, \cdot) \rightarrow \chi_{(0,1)} - \chi_{(-1,0)} \quad \text{in } L^1(-1, 1).$$

Hence, for small  $\varepsilon$ , there exists a  $t_\varepsilon \in (-\frac{1}{4}, \frac{1}{4})$  such that

$$u(y, t_\varepsilon) = 0.$$

On the interval  $(-1/4\varepsilon, 1/4\varepsilon)$  we define the function

$$v_\varepsilon(s) := u(y, t_\varepsilon + \varepsilon s).$$

Upon substitution into the integral in (3.13), we obtain

$$\Phi\left(v_\varepsilon, \frac{1}{4\varepsilon}\right) \leq \int_{-1/2}^{1/2} \left\{ \frac{1}{2}\varepsilon^3\gamma |u_{tt}|^2 + \frac{1}{2}\varepsilon |u_t|^2 + \frac{1}{\varepsilon}F(u) \right\} dt, \tag{3.14}$$

where we have written

$$\Phi(v, L) := \int_{-L}^L \left\{ \frac{1}{2}\gamma |v''|^2 + \frac{1}{2}|v'|^2 + F(v) \right\} ds.$$

We define  $\tilde{v}_\varepsilon$  to be an odd function that coincides with  $v_\varepsilon$  on either  $[0, \frac{1}{4})$  or  $(-\frac{1}{4}, 0]$ , such that

$$\Phi\left(\tilde{v}_\varepsilon, \frac{1}{4\varepsilon}\right) \leq \Phi\left(v_\varepsilon, \frac{1}{4\varepsilon}\right). \tag{3.15}$$

Along subsequences, the functions  $\tilde{v}_\varepsilon$  converge weakly in  $H^2(\mathbb{R})$  to an odd function  $v$ . Therefore, we deduce from Fatou’s lemma that for each  $R > 0$ ,

$$\Phi(v, R) \leq \liminf_{\varepsilon \rightarrow 0} \Phi(\tilde{v}_\varepsilon, R) \leq \liminf_{\varepsilon \rightarrow 0} \Phi\left(\tilde{v}_\varepsilon, \frac{1}{4\varepsilon}\right),$$

and letting  $R \rightarrow \infty$  we conclude that

$$P^\gamma \leq e^\gamma(v) = \Phi(v, \infty) \leq \liminf_{\varepsilon \rightarrow 0} \Phi\left(\tilde{v}_\varepsilon, \frac{1}{4\varepsilon}\right). \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we deduce (3.13). This completes the proof of (3.5).  $\square$

We are now ready to prove proposition 3.2 when the boundary  $\partial D$  of  $D$  is not a hyperplane. First, we need to introduce some notation. For any Borel set  $A \subset \Omega$ , we define the measure

$$\mu_\varepsilon(A) := \int_A \left\{ \frac{1}{2} \varepsilon^3 \gamma (\Delta u_\varepsilon)^2 + \frac{1}{2} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right\}. \tag{3.17}$$

Thus

$$\mu_\varepsilon(\Omega) = E_\varepsilon^\gamma(u_\varepsilon).$$

It follows from the uniform upper bound (3.3) for  $\mu_\varepsilon(\Omega)$  that there exists a subsequence  $\{\varepsilon_j\}$  tending to zero as  $j \rightarrow \infty$ , such that

$$\int_\Omega \varphi \, d\mu_{\varepsilon_j} \rightarrow \int_\Omega \varphi \, d\mu \quad \text{as } j \rightarrow \infty \tag{3.18 a}$$

for any function  $\varphi \in C(\Omega)$  with compact support in  $\Omega$ , and

$$\mu_{\varepsilon_j}(\Omega) \rightarrow \liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(\Omega) \quad \text{as } j \rightarrow \infty. \tag{3.18 b}$$

In addition, for open sets  $A \subset B \subset \Omega$  such that  $\bar{A} \subset B$ , we have

$$\limsup_{\varepsilon \rightarrow 0} \mu_\varepsilon(A) \leq \mu(B) \leq \liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(B). \tag{3.18 c}$$

DEFINITION 3.5. Let  $A$  be an open subset of  $\Omega$ . Then

$$\int_A |\nabla \chi_D| := \sup \left\{ \int_A (\operatorname{div} \varphi) \chi_D : \varphi \in (C_0^1(A))^n, |\varphi| \leq 1 \right\}.$$

The function  $\nabla \chi_D$  is a vector-valued Radon measure [21, p. 5].

We next define the notion of the *reduced boundary*  $\partial^* D$  of  $D$ . This concept was first introduced by DiGiorgi [16]. For a description we refer to ch. 3 of [21].

DEFINITION 3.6. We say that a point  $y \in \partial D$  is a point of the reduced boundary  $\partial^* D$  of  $D$  if and only if

$$\nu(y) = \lim_{r \rightarrow 0} \frac{\int_{B_r(y)} \nabla \chi_D}{\int_{B_r(y)} |\nabla \chi_D|} \text{ exists and } |\nu(y)| = 1.$$

DEFINITION 3.7. At a point  $y \in \partial^* D$ , we define the *lower*  $(n - 1)$ -dimensional density  $\theta_*^{n-1}(\mu, y)$  by

$$\theta_*^{n-1}(\mu, y) := \liminf_{r \rightarrow 0} \frac{\mu(B_r(y))}{\omega_{n-1} r^{n-1}},$$

where  $\omega_{n-1}$  denotes the Lebesgue measure of the unit ball in  $\mathbb{R}^{n-1}$ .

LEMMA 3.8. *For every  $x_0 \in \partial^*D$ , we have*

$$P^\gamma \leq \theta_*^{n-1}(\mu, x_0). \tag{3.19}$$

Accepting this lemma for the moment, we can readily complete the proof of proposition 3.2. From theorem 4.4 of [21], we know that

$$\int_\Omega |\nabla \chi_D| = \mathcal{H}^{n-1}(\partial^*D), \tag{3.20}$$

where  $\mathcal{H}^{n-1}$  is the  $(n - 1)$ -dimensional Hausdorff measure of  $\partial^*D$ . Furthermore, one can deduce from theorem 3.2 in [37] that (3.19) implies that

$$P^\gamma \mathcal{H}^{n-1}(\partial^*D) \leq \mu(\Omega). \tag{3.21}$$

Hence, remembering the definition of  $E_0^\gamma(u)$ , we find that

$$E_0^\gamma(u) = \frac{1}{2} P^\gamma \int_\Omega |\nabla u| = P^\gamma \int_\Omega |\nabla \chi_D| = P^\gamma \mathcal{H}^{n-1}(\partial^*D). \tag{3.22}$$

Combining (3.18c), (3.21) and (3.22), we conclude that

$$E_0^\gamma(u) \leq \mu(\Omega) \leq \liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(\Omega) = \liminf_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma(u_\varepsilon). \tag{3.23}$$

□

*Proof of lemma 3.8.* Let  $x_0 \in \partial^*D$ . It will be convenient to shift the origin to  $x_0$  and scale the independent variable. Thus, for some positive constant  $\delta$ , we write

$$x = x_0 + \delta y, \quad v_{\varepsilon,\delta}(y) = u_\varepsilon(x) \quad \text{and} \quad v_\delta(y) = u(x).$$

Because  $x_0 \in \partial^*D$ , we know that

$$v_\delta \rightarrow \chi_H - \chi_{\mathbb{R}^n \setminus H} \quad \text{as } \delta \rightarrow 0 \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n), \tag{3.24}$$

where  $H = \{y \in \mathbb{R}^n : y \cdot \nu(x_0) > 0\}$ , where  $\nu(x_0)$  has been defined in definition 3.6 (see also ch. 3 of [21]). Clearly,

$$v_{\varepsilon,\delta} \rightarrow v_\delta \quad \text{as } \varepsilon \rightarrow 0 \quad \text{in } L^1(\delta^{-1}(\Omega - x_0)).$$

Hence there exists a function  $\varepsilon_0 : \mathbb{R} \rightarrow \mathbb{R}^+$  such that  $\varepsilon_0(s) \rightarrow 0$  as  $s \rightarrow 0$ , and

$$\|v_{\varepsilon,\delta} - v_\delta\|_{L^1(B_1(0))} \leq \delta \quad \text{if } 0 < \varepsilon < \varepsilon_0(\delta). \tag{3.25}$$

Combining (3.24) and (3.25), we find that for any sequence  $\{(\varepsilon_j, \delta_j)\}$  such that  $\delta_j \rightarrow 0$  as  $j \rightarrow \infty$  and  $0 < \varepsilon_j < \varepsilon_0(\delta_j)$ , we have

$$v_{\varepsilon_j,\delta_j} \rightarrow \chi_H - \chi_{\mathbb{R}^n \setminus H} \quad \text{as } j \rightarrow \infty \quad \text{in } L^1(B_1(0)).$$

If we assume that  $\varepsilon_0(s)/s \rightarrow 0$  as  $s \rightarrow 0$ , then by lemma 3.4 it follows that

$$\begin{aligned} P^\gamma \mathcal{L}^{n-1}(\partial H \cap B_1(0)) &= \frac{1}{2} P^\gamma \int_{B_1(0)} |\nabla(\chi_H - \chi_{\mathbb{R}^n \setminus H})| \\ &\leq \liminf_{j \rightarrow \infty} E_{\varepsilon_j/\delta_j, B_1(0)}^\gamma(v_{\varepsilon_j,\delta_j}), \end{aligned} \tag{3.26}$$

where  $E_{\varepsilon, B_1(0)}^\gamma(u)$  has been defined in (3.6). Observe that

$$\begin{aligned} E_{\varepsilon/\delta, B_1(0)}^\gamma(v_{\varepsilon, \delta}) &= \int_{B_1(0)} \left\{ \left( \frac{\varepsilon}{\delta} \right)^3 \frac{\gamma}{2} |\Delta v_{\varepsilon, \delta}|^2 + \frac{\varepsilon}{2\delta} |\nabla v_{\varepsilon, \delta}|^2 + \frac{\delta}{\varepsilon} F(v_{\varepsilon, \delta}) \right\} \\ &= \delta^{1-n} \int_{B_\delta(x_0)} \left\{ \frac{1}{2} \varepsilon^3 \gamma |\Delta u_\varepsilon|^2 + \frac{1}{2} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right\} \\ &= \delta^{1-n} \mu_\varepsilon(B_\delta(x_0)). \end{aligned}$$

Therefore, because  $\mathcal{L}^{n-1}(\partial H \cap B_1(0)) = \omega_{n-1}$ , we can write (3.26) as

$$P^\gamma \leq \liminf_{j \rightarrow \infty} \frac{\mu_\varepsilon(B_{\delta_j}(x_0))}{\omega_{n-1} \delta_j^{n-1}}. \tag{3.27}$$

Since we may choose, for  $\varepsilon_j$ , any number in the interval  $(0, \varepsilon_0(\delta_j))$ , we can sharpen (3.27) to

$$\begin{aligned} P^\gamma &\leq \liminf_{j \rightarrow \infty} \inf \left\{ \frac{\mu_\varepsilon(B_{\delta_j}(x_0))}{\omega_{n-1} \delta_j^{n-1}} : 0 < \varepsilon < \varepsilon_0(\delta) \right\} \\ &\leq \liminf_{\delta \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \frac{\mu_\varepsilon(B_\delta(x_0))}{\omega_{n-1} \delta^{n-1}}, \end{aligned} \tag{3.28}$$

where in the last inequality we use the fact that we are free to choose any sequence  $\{\delta_j\}$  that tends to zero. By (3.18c),

$$\liminf_{\varepsilon \rightarrow 0} \mu_\varepsilon(B_\delta(x_0)) \leq \mu(B_{\tau\delta}(x_0)) \quad \text{for any } \tau > 1.$$

Using this in (3.28), we find that for any  $\tau > 1$ ,

$$P^\gamma \leq \liminf_{\delta \rightarrow 0} \tau^{n-1} \frac{\mu(B_{\tau\delta}(x_0))}{\omega_{n-1} (\tau\delta)^{n-1}} = \tau^{n-1} \theta_*^{n-1}(\mu, x_0).$$

Because  $\tau$  may be chosen arbitrary close to 1, we conclude that

$$P^\gamma \leq \theta_*^{n-1}(\mu, x_0),$$

as asserted. This completes the proof of lemma 3.8. □

With the proof of lemma 3.8 we have completed the proof of proposition 3.2, and we are ready to turn to the proof of proposition 3.3.

*Proof of proposition 3.3.* Let  $u \in L^1(\Omega)$ . We need to construct a sequence  $\{u_\varepsilon\}$  such that

$$u_\varepsilon \rightarrow u \quad \text{as } \varepsilon \rightarrow 0 \quad \text{in } u \in L^1(\Omega), \tag{3.29 a}$$

and

$$\limsup_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma(u_\varepsilon) \leq E_0^\gamma(u). \tag{3.29 b}$$

From proposition 3.2 we then conclude that

$$\lim_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma(u_\varepsilon) = E_0^\gamma(u),$$

as required in part (b) of the proposition. We may assume that  $E_0^\gamma(u) < \infty$ . Thus, by definition,  $u \in BVC(\Omega)$  and we can define a set  $D \subset \Omega$  so that we can write  $u$  as

$$u = \chi_D - \chi_{\Omega \setminus D}.$$

Since  $\partial\Omega \in C^{0,1}$ , we may extend  $u$  to  $\mathbb{R}^n$  by a function  $\tilde{u} \in BVC(\mathbb{R}^n)$  such that  $\tilde{u} = u$  on  $\Omega$  and

$$\int_{\partial\Omega} |\nabla \tilde{u}| = 0.$$

Henceforth we shall omit the tilde.

By theorem 1.24 of [21] there exists a sequence  $\{D_j\}$  of bounded open sets, with boundary  $\partial D_j \in C^\infty$  for every  $j \geq 1$ , with the property that

$$u_j := \chi_{D_j} - \chi_{\mathbb{R}^n \setminus D_j} \rightarrow u \quad \text{as } j \rightarrow \infty \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \tag{3.30 a}$$

and

$$E_0^\gamma(u_j) \rightarrow E_0^\gamma(u) \quad \text{as } j \rightarrow \infty. \tag{3.30 b}$$

Since  $\partial D_j$  is smooth, we can ensure that  $\mathcal{H}^{n-1}(\partial D_j \cap \partial\Omega) = 0$  by translating  $\partial D$  slightly in normal direction, whenever necessary. Thus, in view of the limits (3.30 a) and (3.30 b), we may assume from now on that

$$u = \chi_D - \chi_{\mathbb{R}^n \setminus D}, \tag{3.31 a}$$

in which

$$D \text{ is bounded, } \partial D \in C^\infty \quad \text{and} \quad \mathcal{H}^{n-1}(\partial D \cap \partial\Omega) = 0. \tag{3.31 b}$$

A more detailed discussion of this construction, as well as a proof, can be found in lemma 1 of [27].

Let  $d$  be the signed distance function of  $\partial D$ , that is,

$$d(x) = \begin{cases} + \inf_{y \in \partial D} |x - y| & \text{if } x \in D, \\ - \inf_{y \in \partial D} |x - y| & \text{if } x \notin D. \end{cases}$$

Because  $\partial D \in C^\infty$ , and  $D$  is compact, there exists a constant  $h > 0$  [36] such that  $d \in C^2(N_h)$ , where  $N_h$  is the neighbourhood of  $\partial D$  defined by

$$N_h(\partial D) = \bigcup_{y \in \partial D} B_h(y).$$

Let  $\eta$  be a smooth function defined on  $\bar{\Omega}$ , say  $\eta \in C^2(\bar{\Omega})$ , such that

$$\eta(x) = d(x) \quad \text{for } x \in N_h \quad \text{and} \quad |\eta(x)| \geq h \quad \text{for } x \notin N_h.$$

We are now ready to introduce an appropriate test function, which approximates the transition from  $u = -1$  to  $u = +1$  near  $\partial D$ . Let  $U(z)$  be an odd minimizer of the variational problem,

$$P^\gamma = \min\{e^\gamma(w) : w \in \mathcal{F}\},$$

as discussed in § 2. Then, by (3.11),  $P^\gamma = e^\gamma(U)$ . We choose

$$u_\varepsilon(x) = U\left(\frac{\eta(x)}{\varepsilon}\right), \quad x \in \Omega. \tag{3.32}$$

Plainly,  $u_\varepsilon \in C^2(\bar{\Omega})$  and

$$u_\varepsilon \rightarrow u \quad \text{as } \varepsilon \rightarrow 0 \quad \text{in } L^1(\Omega),$$

as required in (3.29 a). To prove (3.29 b), we compute  $E_\varepsilon^\gamma(u_\varepsilon)$ . An easy computation shows that

$$\nabla u_\varepsilon = \frac{1}{\varepsilon}U'\left(\frac{\eta(x)}{\varepsilon}\right)\nabla\eta \quad \text{and} \quad \Delta u_\varepsilon = \frac{1}{\varepsilon^2}U''\left(\frac{\eta(x)}{\varepsilon}\right)|\nabla\eta|^2 + \frac{1}{\varepsilon}U'\left(\frac{\eta(x)}{\varepsilon}\right)\Delta\eta.$$

Hence

$$\begin{aligned} E_\varepsilon^\gamma(u_\varepsilon) &= \frac{1}{\varepsilon} \left( \int_{\Omega \cap N_h} + \int_{\Omega \setminus N_h} \right) \left\{ \frac{1}{2}\gamma \left| U''\left(\frac{\eta(x)}{\varepsilon}\right) |\nabla\eta|^2 + \varepsilon U'\left(\frac{\eta(x)}{\varepsilon}\right) \Delta\eta \right|^2 \right. \\ &\quad \left. + \frac{1}{2} \left| U'\left(\frac{\eta(x)}{\varepsilon}\right) \right|^2 |\nabla\eta|^2 + F\left(U\left(\frac{\eta(x)}{\varepsilon}\right)\right) \right\} \\ &= J_1(\varepsilon) + J_2(\varepsilon), \end{aligned} \tag{3.33}$$

where  $J_1$  is the integral over  $\Omega \cap N_h$  and  $J_2$  is the integral over  $\Omega \setminus N_h$ . Since  $\eta > h$  in  $\Omega \setminus N_h$ , and, by (2.11 a) and (2.11 b),

$$F(U(z)), U'(z), U''(z) = \mathcal{O}(e^{-\alpha|z|}) \quad \text{as } z \rightarrow \pm\infty$$

for some constant  $\alpha > 0$ , it follows at once that

$$J_2(\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{3.34}$$

In  $N_h$  we have  $|\nabla\eta| = 1$ , so that we can write  $J_1(\varepsilon)$  as

$$J_1(\varepsilon) = \frac{1}{\varepsilon} \int_{\Omega \cap N_h} \left\{ \frac{1}{2}\gamma \left| U''\left(\frac{\eta(x)}{\varepsilon}\right) \right|^2 + \frac{1}{2} \left| U'\left(\frac{\eta(x)}{\varepsilon}\right) \right|^2 + F\left(U\left(\frac{\eta(x)}{\varepsilon}\right)\right) \right\} + R(\varepsilon),$$

where

$$\begin{aligned} R(\varepsilon) &\leq \frac{\gamma}{2\varepsilon} \int_{\Omega \cap N_h} \left\{ 2\varepsilon \left| U'\left(\frac{\eta(x)}{\varepsilon}\right) U''\left(\frac{\eta(x)}{\varepsilon}\right) \right| |\Delta\eta| + \varepsilon^2 \left| U'\left(\frac{\eta(x)}{\varepsilon}\right) \right|^2 |\Delta\eta|^2 \right\} \\ &\leq \frac{\nu}{\varepsilon} \int_{\Omega \cap N_h} \frac{1}{2}\gamma \left| U''\left(\frac{\eta(x)}{\varepsilon}\right) \right|^2 + C(\nu)\varepsilon. \end{aligned}$$

Here,  $\nu$  is a positive constant, which may be chosen arbitrary small, once  $\varepsilon$  has been chosen sufficiently small. Thus

$$\begin{aligned} J_1(\varepsilon) &\leq \frac{1+\nu}{\varepsilon} \int_{\Omega \cap N_h} \left\{ \frac{1}{2}\gamma \left| U''\left(\frac{\eta(x)}{\varepsilon}\right) \right|^2 + \frac{1}{2} \left| U'\left(\frac{\eta(x)}{\varepsilon}\right) \right|^2 + F\left(U\left(\frac{\eta(x)}{\varepsilon}\right)\right) \right\} \\ &\quad + C(\nu)\varepsilon \\ &= I(\varepsilon) + C(\nu)\varepsilon. \end{aligned} \tag{3.35}$$

We can write the integral  $I(\varepsilon)$  in (3.35) as

$$I(\varepsilon) = (1 + \nu) \int_{\Omega \cap N_h} \left\{ \frac{1}{2} \gamma |U''(\omega(x))|^2 + \frac{1}{2} |U'(\omega(x))|^2 + F(U(\omega(x))) \right\} |\nabla \omega(x)| \, dx,$$

where  $\omega(x) = d(x)/\varepsilon$ , and hence  $|\nabla \omega(x)| = 1/\varepsilon$ . Using the co-area formula (see [37, § 10] or [19, theorem 3.2.12]), we decide that

$$I(\varepsilon) = (1 + \nu) \int_R \int_{\omega^{-1}(t) \cap \Omega \cap N_h} \left\{ \frac{1}{2} \gamma |U''(\omega(x))|^2 + \frac{1}{2} |U'(\omega(x))|^2 + F(U(\omega(x))) \right\} d\mathcal{H}^{n-1}(x) \, dt.$$

Next we remark that

$$\omega^{-1}(t) = \{x \mid d(x) = \varepsilon t\},$$

so that if

$$t > h/\varepsilon,$$

then

$$\omega^{-1}(t) \cap \Omega \cap \{x \mid d(x) \leq h\} = \emptyset$$

and

$$I(\varepsilon) = (1 + \nu) \int_{-h/\varepsilon}^{h/\varepsilon} \int_{d(x)=\varepsilon t} \left\{ \frac{1}{2} \gamma |U''(t)|^2 + \frac{1}{2} |U'(t)|^2 + F(U(t)) \right\} d\mathcal{H}^{n-1}(x) \, dt.$$

Next we write  $I(\varepsilon)$  in the form

$$I(\varepsilon) = (1 + \nu) \int_{-h/\varepsilon}^{h/\varepsilon} \left\{ \frac{1}{2} \gamma |U''(t)|^2 + \frac{1}{2} |U'(t)|^2 + F(U(t)) \right\} \mathcal{H}^{n-1} \{x \mid d(x) = \varepsilon t\} \, dt,$$

which converges to

$$(1 + \nu) \int_{-\infty}^{+\infty} \left\{ \frac{1}{2} \gamma |U''(t)|^2 + \frac{1}{2} |U'(t)|^2 + F(U(t)) \right\} dt \mathcal{H}^{n-1}(\{x \mid d(x) = 0\})$$

as  $\varepsilon \rightarrow 0$ . Thus, letting  $\varepsilon \rightarrow 0$ , we obtain

$$I(\varepsilon) \rightarrow (1 + \nu) e^\gamma(U) \mathcal{H}^{n-1}(\partial D \cap \Omega). \tag{3.36}$$

Returning to the upper bound (3.32) for  $E_\varepsilon^\gamma(u_\varepsilon)$ , and letting  $\varepsilon \rightarrow 0$ , we find, in view of (3.33), (3.34), (3.35), (3.36) and (3.21), that

$$\limsup_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma(u_\varepsilon) \leq (1 + \nu) P^\gamma \mathcal{H}^{n-1}(\partial D \cap \Omega) = (1 + \nu) E_0^\gamma(u).$$

Since  $\nu$  may be chosen arbitrary small, the assertion follows. □

#### 4. A mass constraint

In this section we prove the  $\Gamma$ -convergence of  $E_\varepsilon^\gamma(u)$  subject to the integral constraint

$$P \int_\Omega u \stackrel{\text{def}}{=} \frac{1}{|\Omega|} \int_\Omega u = m \quad \text{for } m \in [-1, 1]. \tag{4.1}$$



To do so, we define the functional

$$G_\varepsilon^\gamma(u) = E_\varepsilon^\gamma(u) + I(u), \tag{4.2}$$

where

$$I(u) = \begin{cases} 0 & \text{if } P \int_\Omega u = m, \\ +\infty & \text{otherwise.} \end{cases}$$

We shall prove the following theorem.

**THEOREM 4.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with boundary  $\partial\Omega \in C^{0,1}$ . Then the  $\Gamma$ -limit*

$$G_0^\gamma = \Gamma - \lim_{\varepsilon \rightarrow 0} G_\varepsilon^\gamma$$

*equals the area functional multiplied by the transition energy for the double well  $F$ , that is,*

$$G_0^\gamma(u) = \begin{cases} \frac{1}{2}P^\gamma \int_\Omega |\nabla u| & \text{if } u \in BVC(\Omega) \text{ and } \frac{1}{|\Omega|} \int_\Omega u = m, \\ +\infty & \text{otherwise,} \end{cases}$$

where  $P^\gamma$  and  $BVC(\Omega)$  have been defined in theorem 3.1.

The proof of theorem 4.1 proceeds along the lines of the proof of theorem 3.1. The lower semicontinuity of  $G_\varepsilon^\gamma(u_\varepsilon)$  is an easy corollary of proposition 3.2. Thus it remains to show that there exists a sequence with properties comparable to those listed in proposition 3.3. This is the content of the following proposition.

**PROPOSITION 4.2.** *For any  $u \in L^1(\Omega)$ , there exists a sequence  $\{u_\varepsilon\}$  such that*

- (a)  $u_\varepsilon \rightarrow u$  as  $\varepsilon \rightarrow 0$  strongly in  $L^1(\Omega)$ ,
- (b)  $G_0^\gamma(u) = \liminf_{\varepsilon \rightarrow 0} G_\varepsilon^\gamma(u_\varepsilon)$ .

*Proof.* Let  $u \in L^1(\Omega)$ . We need to construct a sequence  $\{u_\varepsilon\}$  such that

$$u_\varepsilon \rightarrow u \quad \text{as } \varepsilon \rightarrow 0 \quad \text{in } u \in L^1(\Omega) \tag{4.3 a}$$

and

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon^\gamma(u_\varepsilon) \leq G_0^\gamma(u). \tag{4.3 b}$$

From the lower semicontinuity we then conclude that

$$\lim_{\varepsilon \rightarrow 0} G_\varepsilon^\gamma(u_\varepsilon) = G_0^\gamma(u),$$

as required in part (b) of the proposition. We may assume that  $G_0^\gamma(u) < \infty$ . Thus, by definition,  $u \in BVC(\Omega)$  and we can define a set  $D \subset \Omega$  so that we can write  $u$  as

$$u = \chi_D - \chi_{\Omega \setminus D} = 2\chi_D - 1. \tag{4.4}$$

If

$$\int_\Omega |\nabla \chi_D| = 0,$$

then we can take  $u_\varepsilon = 1$  and  $m = 1$  or  $u_\varepsilon = -1$  and  $m = -1$ . Suppose that  $m \in (-1, 1)$ . Then

$$\int_{\Omega} |\nabla \chi_D| \neq 0.$$

To begin with, we construct sequences  $\{u_j\}$  and  $\{D_j\}$  with

$$u_j = \chi_{D_j} - \chi_{\Omega \setminus D_j} \quad \text{and} \quad P \int_{\Omega} u_j = m,$$

such that the boundaries  $\partial D_j$  are smooth. According to theorem 3.1 of [21], there exist two points  $x_1$  and  $x_2$  in  $\Omega$  and a constant  $\delta_0 > 0$  with the property

$$0 < |D \cap B_\delta(x_i)| < |B_\delta(x_i)| \quad \text{for all } \delta \in (0, \delta_0) \quad \text{and} \quad i \in \{1, 2\}.$$

We write  $D_{\delta_1, \delta_2} = (D \cup B_{\delta_1}(x_1)) \setminus B_{\delta_2}(x_2)$ , where we assume that  $\delta_1 + \delta_2 < |x_1 - x_2|$ . Plainly, we have

$$P \int_{\Omega} (2\chi_{D(0, \delta_0)} - 1) < m$$

for any  $\delta_0 > 0$  small. This implies that for  $\delta > 0$  small enough,

$$P \int_{\Omega} (2\chi_{D(\delta, \delta_0)} - 1) < m$$

as well. Since we also have that

$$P \int_{\Omega} (2\chi_{D(\delta, 0)} - 1) > m,$$

we deduce that there exists a  $\rho \in (0, \delta_0)$  such that

$$P \int_{\Omega} (2\chi_{D(\delta, \rho)} - 1) = m.$$

In addition,

$$\mathcal{H}^{n-1}(\partial D_{(\delta, \rho)}) \leq \mathcal{H}^{n-1}(\partial D) + n\omega_n(\delta^{n-1} + \rho^{n-1}).$$

Thus, without loss of generality, we may assume that  $D$ , as well as  $\Omega \setminus D$ , contains a ball.

Again, we extend  $u$  to the whole of  $\mathbb{R}^n$ . Following theorem 1.24 of [21], we construct a sequence  $\{D_j\}$  of bounded open sets with boundaries  $\partial D_j \in C^\infty$  for every  $j \geq 1$  endowed with the properties

$$u_j := \chi_{D_j} - \chi_{\mathbb{R}^n \setminus D_j} \rightarrow u \quad \text{as } j \rightarrow \infty \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \tag{4.5 a}$$

and

$$E_0^\gamma(u_j) \rightarrow E_0^\gamma(u) \quad \text{as } j \rightarrow \infty. \tag{4.5 b}$$

By translating  $\partial D_j$  slightly in normal direction, whenever necessary, we can ensure that  $\mathcal{H}^{n-1}(\partial D_j \cap \partial \Omega) = 0$ . Since the sets  $D_j$  are obtained by a convolution argument, both  $D_j$  and  $\Omega \setminus D_j$  contain a ball of fixed radius. The idea is then to *add* to  $D_j$  a small ball whose diameter depends on  $j$  to the interior of  $\Omega \setminus D_j$  if we wish to increase the integral of  $u_j$ , and to *remove* a small ball whose diameter depends

on  $j$  from the interior of  $D_j$  if we wish to decrease the integral. This enables us to ensure that for  $j$  large enough,

$$P \int_{\Omega} (2\chi_{D_j} - 1) = m.$$

Then  $G_0^\gamma(u_j) = E_0^\gamma(u_j)$  for  $j$  large enough. Remembering that  $G_0^\gamma(u) = E_0^\gamma(u)$ , we conclude that

$$G_0^\gamma(u_j) \rightarrow G_0^\gamma(u) \quad \text{as } j \rightarrow \infty. \tag{4.5 c}$$

Thus, in view of the limits (4.5 a) and (4.5 c), we may assume from now on that

$$u = \chi_D - \chi_{\mathbb{R}^n \setminus D}, \tag{4.6 a}$$

in which

$$D \text{ is bounded, } \partial D \in C^\infty \text{ and } \mathcal{H}^{n-1}(\partial D \cap \partial\Omega) = 0. \tag{4.6 b}$$

As in §3, we define the signed distance function  $d(x)$  and the neighbourhood  $N_h(\partial D)$  of the boundary  $\partial D$  of  $D$ . We now define the family of functions

$$u_{\varepsilon,\rho}(x) = U\left(\frac{\eta(x) - \rho}{\varepsilon}\right) \quad \text{for } \rho \in (-h, h) \quad \text{and } \varepsilon > 0.$$

For  $\rho$  fixed, we obtain

$$u_{\varepsilon,\rho} \rightarrow u_\rho = \chi_{\{d>\rho\}} - \chi_{\{d<-\rho\}}.$$

Observe that

$$P \int_{\Omega} u_\rho \begin{cases} < m & \text{if } \rho > 0, \\ > m & \text{if } \rho < 0. \end{cases} \tag{4.7}$$

Since, for any fixed  $\rho_0 > 0$ ,

$$P \int_{\Omega} u_{\varepsilon,\rho_0} \rightarrow P \int_{\Omega} u_{\rho_0} < m$$

and

$$P \int_{\Omega} u_{\varepsilon,-\rho_0} \rightarrow P \int_{\Omega} u_{-\rho_0} > m,$$

it follows that

$$P \int_{\Omega} u_{\varepsilon,\rho_0} < m < P \int_{\Omega} u_{\varepsilon,-\rho_0}$$

if  $\varepsilon$  is small enough. Therefore, there exists  $\rho(\varepsilon) \in (-\rho_0, +\rho_0)$  such that

$$P \int_{\Omega} u_{\varepsilon,\rho(\varepsilon)} = m. \tag{4.8}$$

We set

$$u_\varepsilon = u_{\varepsilon,\rho(\varepsilon)}.$$

Then, for every  $\varepsilon > 0$  small enough,

$$P \int u_\varepsilon = m,$$

as required. We deduce from (4.7) and (4.8) that  $\rho(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and

$$u_\varepsilon \rightarrow u \quad \text{as } \varepsilon \rightarrow 0 \quad \text{in } L^1(\Omega).$$

This proves (4.3a).

The proof of (4.3b) proceeds as in §3, the only difference occurring in (3.35), where we now set  $\eta = \varepsilon t + \rho(\varepsilon)$ . □

### 5. A non-symmetric potential

In this section we introduce a slight asymmetry and inhomogeneity in the source function  $f$  and set

$$f = f_0(s) + \varepsilon a(x), \quad f_0(s) = s^3 - s, \tag{5.1}$$

where  $a \in C^2(\bar{\Omega})$ . Equation (1.3) then becomes

$$\mathcal{L}_\varepsilon(u) \stackrel{\text{def}}{=} u_t + \varepsilon^2 \gamma \Delta^2 u - \Delta u + \frac{1}{\varepsilon^2} \{f_0(u) + \varepsilon a(x)\} = 0. \tag{5.2}$$

For the corresponding potential, we then use the function

$$F = F_0(s) + \varepsilon a(x)s, \quad F_0(s) = \frac{1}{4}(s^2 - 1)^2. \tag{5.3}$$

In analogy with the functional defined in (1.11), we now define

$$E_\varepsilon^{\gamma,a}(u) \stackrel{\text{def}}{=} \int_\Omega \left\{ \frac{1}{2} \varepsilon^3 \gamma (\Delta u)^2 + \frac{1}{2} \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} F_0(u) \right\} dx + \int_\Omega a(x)u. \tag{5.4}$$

Because the last term in (5.4) is continuous, the  $\Gamma$ -limit as  $\varepsilon \rightarrow 0$  now becomes

$$E_\varepsilon^{\gamma,a} \rightarrow E_0^\gamma + \int_{\Omega^+} a - \int_{\Omega^-} a, \tag{5.5}$$

where  $E_0^\gamma$  is defined in theorem 3.1, and  $\Omega^\pm = \{x \in \Omega : u(x) = \pm 1\}$ .

To study the motion of the interface in the limit, we make the *Ansatz*

$$u(x, t) = V \left( \frac{d(x, t)}{\varepsilon}, \varepsilon a(x) \right), \tag{5.6}$$

where, as in §1,  $d$  is the signed distance function of the limiting interface and  $V : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function to be determined below. When we substitute (5.6) into (5.2), expand and then collect terms with equal powers of  $\varepsilon$ , we obtain

$$\mathcal{L}_\varepsilon(u) = \frac{1}{\varepsilon^2} \{ \gamma V'''' - V'' + f_0(V) \} + \frac{1}{\varepsilon} V' \left\{ d_t - \left( 1 - 2\gamma \frac{V'''}{V'} \right) \Delta d \right\} + \frac{1}{\varepsilon} a(x) + \mathcal{O}(1), \tag{5.7}$$

whenever  $V' \neq 0$ . To insert the term involving  $a(x)$  into the equation for the distance function  $d$ , we add and subtract the term  $\varepsilon^{-2}c(\varepsilon a(x))V'$ , where  $c(\varepsilon a(x))$  is a constant yet to be determined. We reorder and arrive at the equation

$$\begin{aligned} \mathcal{L}_\varepsilon(u) = \frac{1}{\varepsilon^2} \{ & \gamma V'''' - V'' - c(\varepsilon a(x))V' + f_0(V) + \varepsilon a(x) \} \\ & + \frac{1}{\varepsilon} V' \left\{ d_t - \left( 1 - 2\gamma \frac{V'''}{V'} \right) \Delta d + \frac{c(\varepsilon a(x))}{\varepsilon} \right\} + \mathcal{O}(1). \end{aligned} \quad (5.8)$$

From [1], we know that for each  $\sigma \in \mathbb{R}$  sufficiently small, there exists a unique solution  $V$  and a unique wave speed  $c(\sigma)$ , such that

$$\gamma V'''' - V'' - c(\sigma)V' + f_0(V) + \sigma = 0.$$

Since  $c(0) = 0$ , this yields for  $d$  the parabolic equation on the interface

$$d_t - \left( 1 - 2\gamma \frac{V'''(0)}{V'(0)} \right) \Delta d + c'(0)a(x) = 0 \quad \text{for } (x, t) \in \Gamma_t. \quad (5.9)$$

For the velocity  $V_n$  of the interface, we thus obtain

$$V_n = - \left( 1 - 2\gamma \frac{V'''(0)}{V'(0)} \right) \kappa + c'(0)a \Big|_{\Gamma_t}.$$

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(Issued 15 February 2002)