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GENERALIZED PRIME MODELS

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Introduction. A prime model O of some complete theory T is a model which can be elementarily imbedded into any model of T (cf. Vaught [7, Introduction]). We are going to replace the assumption that T is complete and that the maps between the models of T are elementary imbeddings (elementary extensions) by more general conditions. T will always be a first order theory with identity and may have function symbols. The language $L(T)$ of T will be denumerable. The maps between models will be so called F-maps, i.e. maps which preserve a certain set F of formulas of $L(T)$ (cf. I.1, 2). Roughly speaking a generalized prime model of T is a denumerable model O which permits an F-map $O \rightarrow M$ into any model M of T. Furthermore O has to be "generated" by formulas which belong to a certain subset G of F. Such a model O will be called an F_G-prime model of T (cf. I.19). For a complete theory T the set F is usually the set of all formulas i.e. $F = L(T)$. Thus the F-maps are the elementary extensions (cf. I.3). Here the set G coincides with F (cf. I.15) and the F_G-prime models are the usual prime models (cf. I.20). In universal algebra or for a universal theory Ω_n (cf. I.16) F and G are chosen so that the F-maps are the homomorphisms and the F_G-prime models are the models which are freely generated by n elements $a_1, \cdots, a_n$ (cf. I.21). A generalization of this is Grätzer's so called Σ-structures (cf. [4, 3.1]). The F_G-prime models then are the free Σ-structures generated by n elements (cf. I.5 and 22).

In Part I we introduce different modifications of the model theoretic concept of a "type" of elements (cf. Vaught [6, p. 304]) called F-type, F_G-type etc. (cf. I.6, 8, 11). Thus we can characterize the F_G-prime models of T by the F-types of their elements (cf. Proposition I.25) if the theory T fulfills certain conditions (cf. Definition I.14 of an F_G-complete theory). This is a generalization of Vaught's characterization of prime models (cf. I.26). It also applies to universal theories as well as Σ-structures because the corresponding theories turn out to be F_G-complete too (cf. I.15, 16, 17).

In Part II we generalize the concept of an atom (cf. Vaught [7, 1.2.1]) and of an atomic model (cf. II.1, 2). Similar to Vaught [7, 3.2 and 3.3] we get the uniqueness of F_G-atomic models up to isomorphism (cf. I.3, 4) for F_G-complete theories as well as the F_G-homogeneity of F_G-atomic models (cf. I.6, 7). The concept of a complete atomistic theory (cf. Vaught [7, 1.2.3]) is generalized in I.15. The existence of an F_G-prime model then implies that T is F_G-atomistic (cf. I.1.16) if T is a so called F_G-elementary theory (cf. Definition II.9). The latter condition is fulfilled in all the examples mentioned above (cf. II.10, 11, 12). For a universal theory Ω_n to be F_G-atomistic means that Ω_n has Grätzer's property P_n (cf. [3, §4], see also II.18).

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In Part III we assume that T is F_G-atomistic in order to construct an extension T_∞ of T (cf. III.6). If T is F_G-complete, T_∞ turns out to be a strictly conservative extension of T (cf. I.12 and III.10). The language of T_∞ contains the language of T and an additional denumerable set of new individual constants.

In Part IV we give an explicit construction of an F_G-atomic model O of T. Its underlying set is the set of new individual constants in T_∞, modulo some equivalence relation (cf. IV.15, 16). For this purpose we have to make some assumptions about T and F (cf. IV.1 and 8 for the definitions of an F-theory and an F_G-theory respectively) which one needs to carry out certain proofs by induction with respect to the structure of formulas (cf. IV.7, 15). We also show the equivalence between T being F_G-atomistic, T having an F_G-atomic model and T having an F_G-prime model (cf. VI.17). This generalizes Vaught's theorem [7, 3.5]. It can also be applied to certain Σ-structures (cf. IV.20) for example to the m-complemented lattices (cf. IV.5, 12, 21) of Chen and Grätzer (cf. [1]). An example, not carried out here, would be the algebraic field extensions.

Another application concerns the F_G-initial models, i.e. the F_G-prime models which permit precisely one F-map into any model of T (cf. IV.22). Proposition IV.25 characterizes those F_G-complete, F_G-elementary F_G-theories which have an F_G-initial model (cf. also Fittler [2]).

It is interesting to notice that there actually are theories T which have F_G-prime models that are not F_G-atomic models. Thus T has nonisomorphic F_G-prime models (cf. IV.18).

In this paper we will use Gödel's completeness theorem as well as many other standard methods of model theory without always referring to them explicitly. The denotations hopefully are self-explanatory.

§I. F_G-prime models.

I.1. By T and $L(T)$ we denote a first order theory and its language respectively. By an n-ary formula we understand a formula with at most n different free variables. We assume that $L(T)$ contains equality and that $L(T)$ is a denumerable set. $L(T)$ may also contain function symbols. We will consider the maps $h: M \to N$ between models M, N, which preserve a certain given set F of formulas in $L(T)$. They are called F-maps. Thus, a map $h: M \to N$ is an F-map if for all n-ary formulas $\mu \in F$ and any n-tuple $\vec{m}$ of elements in M which fulfills $M \vdash \mu(\vec{m})$ we also have $N \vdash \mu(h(\vec{m}))$ (for all integers $n \geq 1$).

I.2. Furthermore we assume that F

(a) contains the formulas $x = y$ for any pair (x, y) of individual variables in $L(T)$,

(b) is closed under substitution of individual variables by individual variables and individual constants,

(c) is closed under conjunction,

(d) is closed under disjunction,

(e) is closed under existential quantification,

(f) is closed under equivalence of formulas (with respect to the theory with language $L(T)$ of T, but without the nonlogical axioms of T).

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Notice that any set E of formulas of $L(T)$ induces a set F by closing E according to (a)–(f). The following special cases will be referred to repeatedly.

I.3. *Complete theories.* If T is a complete theory we choose F to be the set $L(T)$ of all formulas. Thus the F-maps are the elementary imbeddings.

I.4. *Universal theories.* Let Ω be a universal theory, its language containing the predicate constants $r, s, \cdots$ and the function symbols $f, g, \cdots$.

We assume that the axioms are stated in the prenex normal form, i.e. they look like $\forall\vec{x}\alpha(\vec{x})$, where the matrix α is quantifier free. The set F is induced (according to I.2) by the set of formulas containing

$$r(x_1, \cdots, x_m), \qquad s(x_1, \cdots, x_n), \cdots$$
$$x_0 = f(x_1, \cdots, x_p), \qquad y_0 = g(y_1, \cdots, y_q), \cdots$$

and all the formulas α which are the matrices of the axioms of Ω.

The F-maps in this case are usually called homomorphisms.

I.5. *Σ-structures* (cf. [4]). This is a generalization of I.4. We consider a theory Σ with predicate constants $r, s, \cdots$ and function symbols $f, g, \cdots$.

The axioms of Σ are in the following (prenex normal) form

$$\forall\vec{x}_0\exists y_0\forall\vec{x}_1\exists y_1\cdots\exists y_k\forall\vec{x}_{k+1}\psi(\vec{x}_0, y_0, \vec{x}_1, y_1, \cdots, y_k, \vec{x}_{k+1})$$

where ψ is quantifier free.

Following G. Grätzer (cf. [4, Definition 1.3 and 4]) one can introduce the so called m-ary Σ-polynomials $p(x_1, \cdots, x_m)$ ($m \geq 0$), their interpretation being m-ary multivalued functions.

The set F of formulas will be induced (according to I.2) by the formulas

(i) $r(x_1, \cdots, x_m), s(x_1, \cdots, x_n)\cdots,$

(ii) $x_0 = f(x_1, \cdots, x_p), y_0 = g(x_1, \cdots, x_q), \cdots$

and the formulas expressing

(iii) $x_0 \in P(x_1, \cdots, x_m), y_0 \in Q(x_1, \cdots, x_n), \cdots,$

(iv) $\forall y_e(y_e \in P(x_1, \cdots, x_t) \Rightarrow y_e = y_1 \vee y_e = y_2 \vee \cdots \vee y_e = y_{e-1})$, for all Σ-polynomials P, Q and all $e \geq 1$.

Because of (iv) any F-map $h: M \to N$ maps $P(m_1, \cdots, m_t)$ surjectively onto $P(h(m_1), \cdots, h(m_t))$ if $P(m_1, \cdots, m_t)$ is finite ($m_1, \cdots, m_t \in M$). In case all Σ-polynomials in M are finite the set of F-maps $h: M \to N$ into any Σ-structure N coincides with the set of the so-called Σ-homomorphisms $M \to N$ (cf. Grätzer [4, Definition 2.1]).

I.6. DEFINITION. An m-ary F-type I of T is a consistent set of formulas of F having at most m distinct free variables, which is closed with respect to conjunction.

I.7. *Remark.* An m-ary F-type I with free variables $x_1, \cdots, x_m$ or $\vec{x}$ we will sometimes denote by $I\langle x_1, \cdots, x_m\rangle$ or $I\langle\vec{x}\rangle$ respectively.

I.8. DEFINITION. An m-ary F-prime type $I\langle\vec{x}\rangle$ is an m-ary F-type which contains $\mu(\vec{x})$ or $\neg\mu(\vec{x})$ in case both $\mu(\vec{x})$ and $\neg\mu(\vec{x})$ are contained in F.

I.9. DEFINITION. Let G be a subset of the set F of formulas. An m-ary F_G-(prime) type I is an m-ary F-(prime) type which contains a formula of G having as free variables all free variables of any formula in I.

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I.10. *Remark*. The set I of all m-ary formulas of F which hold for a given m-tuple $\vec{c}$ in some model M is an F-prime type. It is called the F-prime type of $\vec{c}$.

We say that $\vec{c}$ realizes a given $F_{(G)}$-type J if all formulas of J hold for $\vec{c}$. In short $M \vdash J\langle\vec{c}\rangle$.

I.11. DEFINITION. An $F_{(G)}$-(prime) type which is realized in every model of T is called an $F_{(G)}$-(prime) character.

I.12. *Remark*. If M is a model of T and I is an m-ary F-character then M is a reduct of a model of the theory T_I which consists of the theory T together with new individual constants $c_1, \cdots, c_m$ and the new axioms $\mu(c_1, \cdots, c_m)$ $(\mu \in I)$.

In general a theory T' which is a conservative extension of a theory T will be called a strictly conservative extension of T if any model of T is a reduct of a model of T'.

In general a conservative extension T' of T is not a strictly conservative extension of T.

I.13. *Remark*. If I is an m-ary F-character in T which in each model M is realized by precisely one m-tuple of elements $\vec{c}$, then I is equivalent to some m-ary formula μ, i.e. for all M, $M \vdash I\langle\vec{c}\rangle$ if and only if $M \vdash \mu(\vec{c})$. The proof is a straight forward application of Beth's definition theorem.

I.14. DEFINITION. The theory T is called F_G-complete if for each model M and each F_G-prime character I there is a formula $\mu_M \in I$ which fulfills $M \vdash \mu_M(\vec{c})$ if and only if $M \vdash I\langle\vec{c}\rangle$.

I.15. *Complete theories*. We set $G = F = L(T)$. The m-ary F_G-prime types I are usually called (dual) prime ideals of m-ary formulas (cf. Vaught [7, 1.2.4]). It can be proved (cf. [7, 2.1]) that such a prime ideal I is realized in all models if and only if it is a principal prime ideal, i.e. I consists of all m-ary formulas with $T \vdash (\mu \Rightarrow \gamma)$ for a so called atom $\mu \in I$. Thus $M \vdash I\langle\vec{c}\rangle$ if and only if $M \vdash \mu(\vec{c})$, for any M and $\vec{c}$ in M. Hence the complete theory T is $L(T)_{L(T)}$-complete.

I.16. *Universal theories*. Let Ω_n be the theory arising from Ω (cf. I.4) by adjoining the new individual constants $a_1, \cdots, a_n$. The new set F is supposed to be induced by the same set E as in the case of Ω. The subset $G \subset F$ shall consist of the conjunctions of the formulas expressing

$$x = \tau(a_1', \cdots, a_n)$$

where the τ's are terms "fed" by the new individual constants $a_1, \cdots, a_n$. Notice that the m-tuples of elements in some model M of Ω_n which have an F_G-type are the ones which are contained in the submodel in M generated by $a_1, \cdots, a_m$.

I.17. Σ-*structures*. In analogy to I.16 we adjoin new individual constants $a_1, \cdots, a_n$ to the theory Σ (cf. I.5) and get the new theory Σ_n. The new set F is supposed to be induced by the old set E (cf. I.5 (i)–(iv)). The formulas in G are supposed to be the conjunctions of the formulas expressing

$$x = P(a_1, \cdots, a_n)$$

where the P's are n-ary Σ-polynomials.

THEOREM. *Σ_n is F_G-complete if the n-ary Σ-polynomials are bounded.*

PROOF. The n-ary Σ-polynomials being bounded means that for any model M

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of Σ_n, the n-ary polynomials P have a restricted cardinality $\|P(c_1, \cdots, c_m)\| \leq K_P$ where K_P is some positive integer (cf. Grätzer [4, Theorem 3.1]). Thus any m-ary F_G-type I is realized in any M by only finitely many m-tuples. It follows readily that there is a formula $\mu \in I$ with $M \vdash \mu(\vec{c})$ if and only if $M \vdash I\langle \vec{c} \rangle$. Q.E.D.

Notice that the elements which realize some F_G-type in some model M of Σ_n are the ones which constitute the submodel of M generated by $a_1, \cdots, a_n$.

I.18. LEMMA. *Let T be an F_G-complete theory and I an $(m + n)$-ary F_G-prime character. For any model M of T and any m-tuple $\vec{c}$ in M there is an n-tuple $\vec{d}$ in M such that $M \vdash I\langle \vec{c}, \vec{d} \rangle$ if and only if $M \vdash \exists \vec{x} \mu(\vec{c}, \vec{x})$ for all $\mu \in I$.*

PROOF. Clearly $M \vdash I\langle \vec{c}, \vec{d} \rangle$ implies $M \vdash \exists \vec{x} \mu(\vec{c}, \vec{x})$ for all $\mu \in I$. Conversely assume $M \vdash \exists \vec{x} \mu(\vec{c}, \vec{x})$ for all $\mu \in I$. As T is F_G-complete there is a formula $\mu_M \in I$ which fulfills $M \vdash \mu_M(\vec{c}, \vec{d})$ if and only if $M \vdash I\langle \vec{c}, \vec{d} \rangle$. But $M \vdash \exists \vec{x} \mu_M(\vec{c}, \vec{x})$ implies $M \vdash \mu_M(\vec{c}, \vec{d})$ for some $\vec{d}$, hence $M \vdash I\langle \vec{c}, \vec{d} \rangle$. Q.E.D.

I.19. DEFINITION. An F_G-prime model O of T is a denumerable model of T which permits an F-map $O \to M$ into any model M of T, and is such that the F-type of any m-tuple of elements in O is an F_G-prime type.

I.20. *Complete theories.* The $L(T)$-prime models of a complete theory are usually called prime models (Vaught [7, Introduction]).

I.21. *Universal theories.* The F_G-prime models of the universal theory Ω_n are the structures $F_\Omega(n)$ which are freely generated (by the n elements $a_1, \cdots, a_n$) in the category of all models of Ω and their homomorphisms. Here the F_G-prime models O of Ω_n permit precisely one F-map $O \to M$ into any model M of Ω_n.

I.22. *Σ-structures.* Here the F_G-prime models are what Grätzer calls the free Σ-structures generated by the n elements $a_1, \cdots, a_n$ (cf. [4, §3, Definition 1]), provided that the Σ-polynomials are bounded.

I.23. DEFINITION. A denumerable model O of an F_G-complete theory is called an F_G-model if the F-type of each m-tuple of elements in O is an F_G-prime character.

I.24. *Complete theories.* An $L(T)_{L(T)}$-model O of a complete theory T is, according to I.15, a model where each m-tuple of elements fulfills an atom. Thus O is a so called atomic model of T (cf. Vaught [7, §3]).

I.25. PROPOSITION. *A model O of an F_G-complete theory T is an F_G-model if and only if it is an F_G-prime model.*

PROOF. Let O be an F_G-prime model. The F-type I of any m-tuple $\vec{c}$ of elements in O is an F_G-prime type. I is realized in any model M of T by the m-tuple $h(\vec{c})$ in M, where $h: O \to M$ is one of the F-maps from O into M. Hence O is an F_G-model. Conversely let O be an F_G-model and $c_1, c_2, \cdots, c_n, \cdots$ an enumeration of its underlying set. We show that there is an F-map $O \to M$ into any model M of T by generalizing Vaught's proof [7, 3.1]. Proceeding by induction we assume that the first n elements $c_1, \cdots, c_n$ are already mapped into $d_1, \cdots, d_n$ in M respectively. Furthermore, if I is the F-type of $(c_1, \cdots, c_n)$ in O, the n-tuple $(d_1, \cdots, d_n)$ realizes I in M. Using the $(n + 1)$-ary F-type J of $(c_1, \cdots, c_n, c_{n+1})$ in O which is an F_G-prime character we can define $c_{n+1} \mapsto d_{n+1}$ in the following way. As T is F_G-complete there is a $(n + 1)$-ary formula $\mu_M \in J$ "representing" J in M.

$$\exists x \mu_M(c_1, \cdots, c_n, x)$$

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is in I, as all formulas $\exists x \mu(c_1, \cdots, c_n, x)$, $\mu \in J$. Hence $M \vdash \exists x \mu_M(d_1, \cdots, d_n, x)$. Choose d_{n+1} to be one of the elements m in M which fulfill $\mu_M(d_1, \cdots, d_n, m)$. Thus we get an F-map: $O \to M$. Q.E.D.

I.26. *Complete theories.* Proposition I.25 is a straightforward generalization of Vaught's theorem saying that a denumerable model of a complete theory T is a prime model if and only if it is an atomic model.

I.27. *Σ-structures and universal theories.* The Σ-structures (resp. models of Ω) freely generated by n elements $a_1, \cdots, a_n$ are F_G-models of Σ_n (resp. Ω_n) by Proposition I.25, provided that the Σ-polynomials are bounded.

§II. F_G-atomic models.

II.1. Definition. An F_G-atom I is an F_G-prime character which is not contained in any other F_G-prime character, and is such that for any $\alpha, \beta \in F$, $\alpha \vee \beta \in I$ implies that $\alpha \in I$ or $\beta \in I$.

II.2. Definition. An F_G-atomic model O of T is a denumerable model of T such that the F-type of any m-tuple of elements of O is an F_G-atom. Notice that an F_G-atomic model of T is a special case of an F_G-model of T.

II.3. Theorem. *Any two F_G-atomic models O and O' of an F_G-complete theory T are isomorphic.*

Proof. Generalizing Vaught's proof of [7, 3.2], let $c_1, c_2, \cdots$ and $c_1', c_2', \cdots$ be enumerations of the elements of O and O' respectively. Going forward and backward we will define, by induction, a correspondence between the elements of O and O' which is easily seen to be an isomorphism. Assume that for $c_{i_1}, \cdots, c_{i_n}$ in O and $c_{j_1}', \cdots, c_{j_n}'$ in O' we already have a correspondence, c_{i_k} corresponding to c_{j_k}' and the F-types of $(c_{i_1}, \cdots, c_{i_n})$ in O and $(c_{j_1}', \cdots, c_{j_n}')$ in O' respectively coincide. In the case that n is odd choose i_{n+1} to be the smallest positive integer which does not occur in $i_1, \cdots, i_n$. We determine the corresponding $c_{j_{n+1}}'$ in the way the d_{n+1} was determined in the proof of Proposition I.25. The $(n+1)$-tuple $(c_{i_1}, \cdots, c_{i_n}, c_{i_{n+1}})$ in O has the same F-type J as $(c_{j_1}', \cdots, c_{j_n}', c_{j_{n+1}}')$, because J is an F_G-atom. The latter also guarantees that $c_{j_{n+1}}'$ is different from the $c_{j_1}, \cdots, c_{j_n}$.

If n is even the roles of O and O' are reversed. Q.E.D.

II.4. *Complete theories.* The m-ary $L(T)_{L(T)}$-atoms coincide with the $L(T)_{L(T)}$-prime characters which are the m-ary principal prime ideals (cf. I.15). Theorem II.3 generalizes Vaught's theorem [7, 3.2] which states that two atomic denumerable models of a complete theory T are isomorphic.

II.5. *Universal theories.* As any m-ary F_G-type is realized by at most one m-tuple of elements in each model of Ω_n, the F_G-models coincide with the F_G-atomic models.

II.6. Definition. A model M of T is called F_G-homogenous if for any two m-tuples $\vec{c}$ and $\vec{d}$ in M which have the same F_G-type there is an automorphism of M carrying $\vec{c}$ into $\vec{d}$.

II.7. Theorem. *An F_G-atomic model of an F_G-complete theory is F_G-homogeneous (cf. also Vaught* [7, 3.3]).

The proof is an inessential variation of the proof of Theorem II.3.

II.8. Definiton. Let M be any model of T. Two m-ary F_G-types are called M-equivalent if they are realized in M by the same set of m-tuples.

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II.9. DEFINITION. The theory T is called F_G-elementary if for any model M of T and any F_G-prime character I there are only finitely many M-inequivalent F_G-types which contain I.

II.10. *Complete theories.* As any principal prime ideal of m-ary formulas of a complete theory T is not contained in any other $L(T)_{L(T)}$-type, it follows that T is $L(T)_{L(T)}$-elementary.

II.11. *Universal theories.* The universal theory Ω_n is an F_G-elementary theory because any m-ary F_G-prime character is realized in any model M by precisely one m-tuple of elements.

II.12. *Σ-structures.* Assuming that the n-ary Σ-polynomials are bounded (cf. I. 17) any m-ary F_G-prime character I is realized in any model M by only finitely many m-tuples of elements. Thus there can be only finitely many M-inequivalent F_G-types which contain I.

II.13. DEFINITION. Let M be any model of T. An m-ary F_G-prime type is called M-maximal if it is realized in M and if it is not contained in any other F_G-type which is realized in M.

II.14. LEMMA. *Any m-ary F_G-prime character I of an F_G-elementary theory T is contained in an M-maximal m-ary F_G-prime type for every model M of T.*

PROOF. The M-equivalence classes of m-ary F_G-prime types can be partially ordered by setting $J \leq K$ if the set M_J of m-tuples which realize J is contained in M_K.

There are "minimal" elements L in this partial ordering, because it is finite. Define an F_G-type $\bar{I}$ by setting $\mu \in \bar{I}$ if μ is contained in some F_G-type of L. Thus $\bar{I}$ is an M-maximal F_G-prime type which contains I. Q.E.D.

II.15. DEFINITION. A theory T is called F_G-atomistic if every F-character is contained in some F_G-atom.

II.16. THEOREM. *An F_G-complete, F_G-elementary theory T which has an F_G-prime model O is F_G-atomistic.*

PROOF. Any m-ary F-character I is realized in O by an m-tuple which has an F_G-prime type J. Hence $I \subseteq J$. J is even an F_G-prime character, by Proposition I.25. According to Lemma II.14 J is contained in some O-maximal F_G-prime type L. But L is an F_G-atom. Q.E.D.

II.17. *Complete theories.* An $L(T)_{L(T)}$-atomistic theory is what Vaught calls an atomistic theory (cf. [7, 1.2.3]).

II.18. *Universal theories.* If the axioms of Ω_n are reduced in the sense of Grätzer [3, 3], and Ω_n is F_G-atomistic, it follows easily that Ω_n has the property (P_n) of [3, 4].

It is also possible though harder to show that the property (P_n) implies that Ω_n is F_G-atomistic. All this will also be a consequence of the forthcoming proposition IV. 17.

II.19. *Σ-structures.* Assuming that the n-ary Σ-polynomials are bounded the following statement is a consequence of a theorem of Grätzer (cf. [4, Theorem 5.1]). For any model A of Σ_n which is generated by $a_1, \cdots, a_n$ (i.e. all m-tuples have an F_G-type, $m \geq 1$) and any model B of Σ_n there is some model C of Σ_n which permits F-maps $C \to A$ and $C \to B$ if and only if Σ_n has an F_G-prime model O. Combining this with Theorem II.16 we conclude: If for any model A of Σ_n, generated by

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$a_1, \cdots, a_n$, and any model B there is a model C which permits F-maps $C \to A$ and $C \to B$, then Σ_n is F_G-atomistic.

§III. **Some strictly conservative extensions.** We are going to construct by induction for an F_G-atomistic F_G-complete theory T, a sequence of theories T_n, $n \geq 0$. This construction is a partial generalization of Vaught's construction of the theory T_2 in the proof of [7, 2.1].

III.1. DEFINITION OF T_0. The language $L(T_0)$ of T_0 consists of the language $L(T)$ together with an enumerable set $c_1, \cdots, c_n, \cdots$ of new individual constants. The axioms of T_0 are the same ones as in T. For the forthcoming definition we fix an enumeration of all finitary tuples $(c_{k_0}, \cdots, c_{k_q})$ of the new individual constants of $L(T_0)$. Furthermore we enumerate by $\varphi_m(x)$, $m \geq 1$ the unary formulas of $L(T_0)$ one gets by replacing some free variables of formulas in F by new individual constants c_i.

III.2. DEFINITION OF T_{n+1}. We assume that T_n is already defined such that it is a strictly conservative extension of T. We set $L(T_{n+1}) = L(T_n)$ and add the following new axioms to T_n.

(a) For all i, $1 \leq i \leq n$, for which $T_n \vdash \exists x \varphi_i(x)$ holds, we add the axioms $\varphi_i(c_{k_i})$, where k_i is the smallest number such that c_{k_i} occurs in none of the axioms of T_n and in none of the new axioms $\varphi_j(c_{k_j})$ for $j < i$, nor in $\varphi_i(x)$.

(b) Let $(d_1, \cdots, d_q)$ denote the $(n + 1)$st finitary tuple of new constants c_i, and $e_1, \cdots, e_r$ be the distinct c_j's occurring in the axioms of T_n and in the newly added axioms $\varphi_j(c_{k_j})$, which are different from the $d_1, \cdots, d_q$. Let I be the set of $(q + r)$-ary formulas $\gamma \in F$ such that $\gamma(d_1, \cdots, d_q, e_1, \cdots, e_r)$ is provable in T_n together with the new axioms $\varphi_j(c_{k_j})$. Then I is an F-character. As T is F_G-atomistic I is contained in an F_G-atom J. The formulas $\exists y_1 \exists y_2 \cdots \exists y_r \gamma(x_1, \cdots, x_q, y_1, \cdots, y_r)$, $\gamma \in J$ constitute an F-character L. As T is F_G-atomistic there is an F_G-atom I_{n+1} containing L. The new axioms we are going to add now are the formulas $\alpha(d_1, \cdots, d_q)$, where $\alpha \in I_{n+1}$.

III.3. *Remark.* The assumption that T_n is a strictly conservative extension of T implies that T_{n+1} is so too. This can be shown step by step: Obviously T_n together with $I\langle d_1, \cdots, d_q, e_1, \cdots, e_r\rangle$ is a strictly conservative extension of T. The same then holds for T_n together with $J\langle d_1, \cdots, d_q, e_1, \cdots, e_r\rangle$. It carries over to T_n together with $L\langle d_1, \cdots, d_q\rangle$ (use Lemma I.18), hence to T_n together with $I_{n+1}\langle d_1, \cdots, d_q\rangle$.

As T_0 is obviously a strictly conservative extension of T it follows that all T_n's, $n \geq 0$, are strictly conservative extensions of T.

III.4. COROLLARY. *If T is consistent then so are the T_n's, $n \geq 0$.*

III.5. *Remarks.* (a) The axioms of T_{n+1}, $n \geq 0$, which are not axioms of T, constitute an F-character $K_{n+1}\langle d_1, \cdots, d_q, e_1, \cdots, e_r\rangle$.

(b) If $n_1 > n_2$ then $K_{n_1} \supseteq K_{n_2}$.

III.6. DEFINITION OF T_∞. T_∞ has the same language as T_0. The axioms of T_∞ are all the axioms of all T_n's, $n \geq 0$.

III.7. *Remarks.* T_∞ is consistent if T is an F_G-complete, F_G-atomistic consistent theory, because any contradiction in T_∞ would already be provable in some T_n.

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The axioms of T_∞ which are not axioms of T are of the form $\gamma(\vec{c})$, where $\gamma \in F$ and $\vec{c}$ is a finitary tuple of individual constants c_i.

III.8. LEMMA. *The subset I of m-ary formulas $\gamma \in F$ which, for some fixed m-tuple $\vec{c}$ of $L(T_0)$, fulfill $T_\infty \vdash \gamma(\vec{c})$ is an F_G-atom* ($m \geq 1$).

PROOF. Let $\vec{c}$ be the nth finitary tuple in our enumeration. By Definition III.2 we know $T_n \vdash I_n\langle\vec{c}\rangle$, where I_n is an F_G-atom. Hence $I_n \subseteq I$. Conversely if $\gamma \in I$, i.e. $T_\infty \vdash \gamma(\vec{c})$ it follows that for some $n_1 \geq 0$ $T_{n_1} \vdash \gamma(\vec{c})$, thus both $I_n\langle\vec{c}\rangle$ and $\gamma\langle\vec{c}\rangle$ are provable in T_{n+n_1}. As T_{n+n_1} is a strictly conservative extension of T and I_n is an F_G-atom, it follows that $\gamma \in I_n$. Q.E.D.

III.9. LEMMA. *If $T_\infty \vdash \exists x\varphi_n(x)$ (cf.* III.1 *and* 2) *then there is an individual constant $c_k \in L(T_0)$ such that $T_\infty \vdash \varphi_n(c_k)$.*

PROOF. $T_\infty \vdash \exists x\varphi_n(x)$ implies $T_m \vdash \exists x\varphi_n(x)$ for some $m \geq 0$. Definition III.2 implies that $T_{m+n+1} \vdash \varphi_n(c_k)$ for a certain c_k. Hence $T_\infty \vdash \varphi_n(c_k)$. Q.E.D.

III.10. THEOREM. *The theory T_∞ is a strictly conservative extension of the theory T, if T is F_G-atomistic and F_G-complete.*

PROOF. For any model M of T we determine the interpretations of the individual constants $c_1, c_2, \cdots, c_n, \cdots$ by induction with respect to n. Assume $m_1, \cdots, m_n$ in M are already determined to be the interpretations of $c_1, \cdots, c_n$ respectively, fulfilling $M \vdash I\langle m_1, \cdots, m_n\rangle$ where I is the F_G-atom of all formulas $\gamma \in F$ such that $T_\infty \vdash \gamma(c_1, \cdots, c_n)$ (cf. III. 8). Let J be the F_G-atom corresponding to $(c_1, \cdots, c_n, c_{n+1})$. The formulas $\exists x\gamma(c_1, \cdots, c_n, x)$, $\gamma \in J$, are contained in I. Because of Lemma I.18 there is an element $m_{n+1} \in M$ such that $M \vdash J\langle m_1, \cdots, m_n, m_{n+1}\rangle$. We set the element m_{n+1} to be the interpretation of c_{n+1}. Q.E.D.

III.11. *Remark.* The proof of Theorem III.10 shows that the theory T_∞ is a strictly conservative extension of the theory T together with the axioms $\gamma(c_1, \cdots, c_n)$, where γ is contained in the F_G-atom corresponding to $(c_1, \cdots, c_n)$ in T_∞.

III.12. DEFINITION OF $\bar{T}_\infty$. The language $L(\bar{T}_\infty)$ is the same as the language of T_∞. For each m-tuple $\vec{c}$ of constants from $c_1, c_2, \cdots$ and each m-ary formula $\alpha \in F$, for which $\alpha(\vec{c})$ is not provable in T_∞ we add a new axiom $\neg\alpha(\vec{c})$.

III.13. LEMMA. *$\bar{T}_\infty$ is consistent if T is a consistent F_G-atomistic and F_G-complete theory.*

PROOF. Assume it were not consistent. This implies that for some $m \geq 0$, T_m, together with some formulas $\neg\alpha(\vec{a})$, $\neg\beta(\vec{b}), \cdots, \neg\gamma(\vec{c})$ (each one being consistent with T_∞) is inconsistent. Hence $T_m \vdash \alpha(\vec{a}) \vee \beta(\vec{b}) \vee \cdots \vee \gamma(\vec{c})$. This implies that $\alpha(\vec{x}) \vee \beta(\vec{y}) \vee \cdots \vee \gamma(\vec{z})$ is contained in the F_G-atom corresponding to $(\vec{a}, \vec{b}, \cdots, \vec{c})$ (cf. III. 8). Hence, so is one of the formulas $\alpha(\vec{x}), \beta(\vec{y}), \cdots, \gamma(\vec{z})$, say $\alpha(\vec{x})$, i.e. $\alpha(\vec{a})$ is provable in T_∞. This contradicts our assumption on $\alpha(\vec{a})$ (cf. also III.7). Q.E.D.

§IV. F_G-theories.

IV.1. DEFINITION. The theory T is called an F-theory if for any formula (written in prenex normal form)

$$\forall\vec{x}_0\exists\vec{y}_0\forall\vec{x}_1\exists\vec{y}_1\cdots\forall\vec{x}_k\exists\vec{y}_k\psi(\vec{x}_0, \vec{y}_0, \vec{x}_1, \vec{y}_1, \cdots, \vec{x}_k, \vec{y}_k)$$

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which is either an axiom (without any free variables) or which belongs to F (and may have free variables), all the formulas

$$\forall \vec{x}_{i+1} \exists \vec{y}_{i+1} \cdots \forall \vec{x}_k \exists \vec{y}_k \psi(\vec{x}_0, \vec{y}_0, \cdots, \vec{y}_i, \vec{x}_{i+1}, \vec{y}_{i+1}, \cdots, \vec{x}_k, \vec{y}_k)$$

belong to F $(0 \leq i \leq k)$.

Notice that in this denotation it is assumed that the $\vec{y}_i$'s for $0 \leq i \leq K - 1$ actually occur.

IV.2. *Remark.* Definition IV.1 could be weakened for our purpose by replacing the passage "or which belongs to F" by "or which belong to some F-character". But in the applications we will always verify the first version.

IV.3. *Complete theories* are $L(T)$-theories, because any theory T is an $L(T)$-theory.

IV.4. *Universal theories.* Ω_n is an F-theory by definition of its set F. F was chosen the way that any $\alpha \in F$ has only existential quantifiers in prenex normal form (cf. I. 4).

IV.5. *Complemented lattices.* The m-complemented lattices are examples of Σ-structures (cf. Chen and Grätzer [1], and Grätzer [4, §3, Remark]). The language of their theory $\Sigma^{(m)}$ contains the binary function symbols $\cup$ and $\cap$ and the individual constants 0, 1 for zero and unit.

The axioms are the idempotent, commutative, associative and absorption laws (cf. [5, 0. §3]), as well as the formulas

$$\text{(a)} \quad \begin{aligned} &\forall x \exists y_1 \cdots \exists y_m (x = 0 \vee x = 1 \\ &\quad \vee [x \cup y_1 = \cdots = x \cup y_m = 1 \wedge x \cap y_1 = \cdots = x \cap y_m = 0 \\ &\quad \wedge y_1 \neq y_2 \wedge \cdots \wedge y_i \neq y_{i+k} \wedge \cdots \wedge y_{m-1} \neq y_m]) \end{aligned}$$

and

$$\text{(b)} \quad \begin{aligned} &\forall x \forall y_1 \cdots \forall y_{m+1} ([x \neq 0 \wedge x \neq 1 \\ &\quad \wedge x \cup y_1 = \cdots = x \cup y_{m+1} = 1 \\ &\quad \wedge x \cap y_1 = \cdots = x \cap y_{m+1} = 0] \\ &\quad \Rightarrow [y_1 = y_2 \vee \cdots \vee y_i = y_{i+k} \vee \cdots \vee y_m = y_{m+1}]) \end{aligned}$$

expressing that each element x which is not 0 or 1 has at least (resp. at most) m different complements $y_1, \cdots, y_m$.

The set F is induced (according to I.2) by the formulas $x = y$, $x = y \cup z$, $x = y \cap z$ and the two formulas which are the matrices of the above mentioned axioms (a) and (b). This also guarantees that any F-map f sends the set of all complements of any element x surjectively onto the set of all complements of $f(x)$.

It follows easily that the F-maps are the Σ-homomorphisms of Grätzer (cf. [4, §2, Definition 1)] which preserve the constants $a_1, \cdots, a_n$. It is also easy to see that $\Sigma_n^{(m)}$ is an F-theory.

IV.6. Definition. Let M be any model of T_∞. By M_c we denote the substructure of M which consists of all elements m_i in M which correspond to the new individual constant c_i of $L(T_\infty)$.

IV.7. Theorem. *M_c is a model of T_∞ if M is a model of T_∞ and T is an F_G-atomistic and F_G-complete F-theory.*

Proof. According to Remark III.7 we are to show that $T_\infty \vdash \alpha(\vec{c})$ implies $M_c \vdash \alpha(\vec{m})$, where $\vec{m}$ is the finitary tuple of elements in M corresponding to $\vec{c}$, and

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$\alpha \in F$, or α is an axiom of T (and therefore there is no $\vec{c}$ in it). Both cases can be handled simultaneously. We proceed by induction with respect to the length of the quantifier string of $\alpha(\vec{m})$ (in prenex normal form). If α has no quantifiers the assertion holds because M_c is a substructure of M.

(a) $\alpha(\vec{c})$ is of the form $\exists x\beta(\vec{c}, x)$. It follows that $\beta(\vec{z}, x)$ is in F. By Lemma III.9 there is a c_k such that $T_\infty \vdash \beta(\vec{c}, c_k)$. $\beta(\vec{z}, x)$ having a shorter quantifier string than $\alpha(\vec{z})$, we conclude $M_c \vdash \beta(\vec{m}, m_k)$, i.e. $M_c \vdash \exists x\beta(\vec{m}, x)$.

(b) $\alpha(\vec{c})$ is of the form $\forall\vec{x}\beta(\vec{c}, \vec{x})$, where $\beta(\vec{z}, \vec{x})$ is in F. $T_\infty \vdash \forall\vec{x}\beta(\vec{c}, \vec{x})$ implies $T_\infty \vdash \beta(\vec{c}, \vec{d})$ for each $\vec{d}$. $\beta(\vec{z}, \vec{x})$ having a shorter quantifier string than $\alpha(\vec{z})$ we conclude $M_c \vdash \beta(\vec{m}, \vec{n})$ for each $\vec{n}$ in M_c, i.e. $M_c \vdash \forall\vec{x}\beta(\vec{m}, \vec{x})$.

For the remaining axioms (cf. III.7) one can proceed in exactly the same way. Q.E.D.

IV.8. Definition. Let T be an F-theory (cf. IV.1) and I an m-ary F_G-prime character. By $T_{I\langle\vec{c}\rangle}$ we denote the theory T enriched by an m-tuple $\vec{c}$ of new individual constants and the axioms $\gamma(\vec{c})$, $\gamma \in I$. T will be called an F_G-theory if for any I and any m-ary formula $\forall\vec{x}\beta(\vec{y}, \vec{x})$ in F (β in prenex normal form) the formula $\forall\vec{x}\beta(\vec{c}, \vec{x})$ is provable in $T_{I\langle\vec{c}\rangle}$ in case every model M of $T_{I\langle\vec{c}\rangle}$ contains a substructure M' of M (in the sense of the language of $T_{I\langle\vec{c}\rangle}$) which is also a model of $T_{I\langle\vec{c}\rangle}$ and which fulfills $M' \vdash \forall\vec{x}\beta(\vec{c}, \vec{x})$.

IV.9. *Remark.* In order to check if some F-theory T is an F_G-theory it suffices to check all the formulas of the form $\forall\vec{x}\beta(\vec{y}, \vec{x})$ which lie in some inducing set E (cf. I.2). The analogous simplification works for showing that some theory is an F-theory.

IV.10. *Complete theories.* Any complete theory T is an $L(T)_{L(T)}$-theory. What is left to be shown (cf. IV.3) is $T_{I\langle\vec{c}\rangle} \vdash \forall\vec{x}(\vec{c}, \vec{x})$, under the assumptions of Definition IV.8, $I\langle\vec{c}\rangle$ can be replaced by $\alpha(\vec{c})$ where α is an atom (cf. I.15). Thus it can be shown that $T \vdash \forall\vec{y}(\alpha(\vec{y}) \Rightarrow \forall\vec{x}\beta(\vec{y}, \vec{x}))$, i.e. $T_{I\langle\vec{c}\rangle} \vdash \forall\vec{x}\beta(\vec{c}, \vec{x})$.

IV.11. *Universal theories.* Ω_n is an F_G-theory because it is an F-theory (cf. IV.4) and because F is induced by a set E of formulas which does not contain any universal quantifier (cf. IV.9).

IV.12. *Complemented lattices.* It is obvious that the theory $\Sigma_n^{(m)}$ is an F_G-theory.

IV.13. Lemma. *Let T be an F_G-atomistic and F_G-complete F_G-theory and O a model of $\bar{T}_\infty$. For any $\alpha \in F$ and any m-tuple $\vec{c}$ of elements of O_c we have $T_\infty \vdash \alpha(\vec{c})$ if and only if $O_c \vdash \alpha(\vec{c})$.*

Proof. What is left to be shown is that $O_c \vdash \alpha(\vec{c})$ implies $T_\infty \vdash \alpha(\vec{c})$ (cf. Theorem IV.7). Applying induction with respect to the length of the quantifier string of α (in prenex normal form) we have two different cases:

(a) $\alpha(\vec{y})$ is of the form $\exists x\beta(\vec{y}, x)$. $O_c \vdash \alpha(\vec{m})$ implies that there is an $n \in O_c$ such that $O_c \vdash \beta(\vec{m}, n)$. $\vec{m}$ corresponds to $\vec{c} \in L(T_\infty)$ and n to some d. By induction we have $T_\infty \vdash \beta(\vec{c}, d)$ i.e. $T_\infty \vdash \exists x\beta(\vec{c}, x)$. Thus we have $T_\infty \vdash \alpha(\vec{c})$.

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(b) $\alpha(\vec{y})$ is of the form $\forall\vec{x}\beta(\vec{y}, \vec{x})$. $O_c \vdash \alpha(\vec{m})$ implies that for all $\vec{n}$ in O_c we have $O_c \vdash \beta(\vec{m}, \vec{n})$. Again $\vec{m}$ corresponds to $\vec{c}$ and each $\vec{n}$ to some $\vec{d}$ in $L(T_\infty)$. Thus for all $\vec{d}$ in $L(T_\infty)$ we have, by induction $T_\infty \vdash \beta(\vec{c}, \vec{d})$. It follows that any model M of T_∞ contains a submodel M_c (cf. IV.6 and IV.7) such that $M_c \vdash \forall\vec{x}\beta(\vec{c}, \vec{x})$. Let now I be the F_G-atom consisting of all formulas $\gamma(\vec{x})$ such that $T_\infty \vdash \gamma(\vec{c})$ (cf. III.8). The theory T_∞ is a strictly conservative extension of the theory $T_{I\langle\hat{c}\rangle}$ (cf. III.11 and IV.8). Thus every model M of $T_{I\langle\hat{c}\rangle}$ has a submodel M_c such that $M_c \vdash \forall\vec{x}\beta(\vec{c}, \vec{x})$. Since T is an F_G-theory it follows that $T_{I\langle\hat{c}\rangle} \vdash \forall\vec{x}\beta(\vec{c}, \vec{x})$, hence $T_\infty \vdash \forall\vec{x}\beta(\vec{c}, \vec{x})$. Q.E.D.

IV.14. COROLLARY. *O_c is a model of $\bar{T}_\infty$.*

PROOF. As O_c is a model of T_∞ (cf. IV.7) it only has to be shown that the axioms of the form $\neg\alpha(\vec{c})$ of $\bar{T}_\infty$ ($\alpha \in F$) are fulfilled in O_c. Assume such an axiom $\neg\alpha(\vec{c})$ were not fulfilled in O_c, thus $O_c \vdash \alpha(\vec{c})$. This would imply (by IV.13) that $T_\infty \vdash \alpha(\vec{c})$, thus $\bar{T}_\infty \vdash \alpha(\vec{c})$, in contradiction to Lemma III.13. Q.E.D.

IV.15. THEOREM. *Let T be an F_G-complete, F_G-atomistic F_G-theory and O a model of $\bar{T}_\infty$ (cf.* III.12) (*which exists because of* III.13). *Thus it follows that O_c is an F_G-atomic (denumerable) model of T.*

PROOF. It is to be shown that the F-type I of any m-tuple $\vec{c}$ of O_c is an F_G-atom. This follows from Lemma III.8 and IV.13. Q.E.D.

IV.16. *Remark.* If O' is another model of $\bar{T}_\infty$, Theorem II.3 implies that O'_c is isomorphic to O_c.

It is also possible to describe an F_G-atomic model O of T in the following way: the underlying set of O is the set of constants c_i of $L(T_\infty)$ modulo the relation: $c_i \sim c_j$ if $T_\infty \vdash c_i = c_j$. The m-ary predicate constant α is determined by: $\alpha(c_{i_1}, \cdots, c_{i_m})$ holds if $T_\infty \vdash \alpha(c_{i_1}, \cdots, c_{i_m})$. For this construction we have to assume that all predicate constants of T and all the formulas $x_0 = f(x_1, \cdots, x_n)$ are contained in the set F, determining the maps between the models.

IV.17. PROPOSITION. *For an F_G-complete and F_G-elementary F_G-theory the following statements are equivalent*

(a) *T has an F_G-prime model,*

(b) *T has an F_G-atomic model,*

(c) *T is F_G-atomistic.*

PROOF. (a) implies (c) because of Theorem II.16. (c) implies (b) because of Theorem IV.15. (b) implies (a) because of Proposition I.25.

IV.18. *Remark.* An F_G-atomic model of T is an F_G-model, i.e. an F_G-prime model. But an F_G-(prime) model of T need not be an F_G-atomic model, in general. It only guarantees the existence of an F_G-atomic (prime) model.

The following is an example of a theory T which fulfills the assumption of Proposition IV.17 and which has an F_G-(prime) model which is not F_G-atomic. The individual constants of T are $d_1, d_2, \cdots, d_n, \cdots, n < N$, where N is finite or ω. There is one unary predicate constant $p(x)$. F is induced by $p(x)$; and G is F itself. The only axiom is $\exists x p(x)$. The following structure O is an F_G-prime model which is not F_G-atomic. The underlying set of O consists of the elements $m_1, m_2, \cdots, m_n, \cdots$, $n < N$ (corresponding to the d_n's) and two elements a and b. p is defined by

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$O \vdash p(a)$, $O \vdash \neg p(b)$ and $O \vdash \neg p(m_n)$, $n < N$. By omitting the element b one gets an F_G-atomic model of T.

IV.19. *Complete theories.* Proposition IV.17 is a generalization of Vaught's theorem [7, 3.5].

IV.20. *Σ-structures.* If the theory Σ_n is an F_G-theory which has an F_G-prime model O (i.e. the free Σ-structure generated by the n elements $a_1, \cdots, a_n$ exists) then O can be constructed according to Remark IV.16.

IV.21. *Complemented lattices* are an example for IV.20 (cf. also IV.12). That $\Sigma_n^{(m)}$ is F_G-complete and F_G-elementary follows from I.17 and II.12 respectively.

The existence of the free m-complemented lattices generated by n elements (cf. Chen and Grätzer [1]) implies that the theory $\Sigma_n^{(m)}$ is F_G-atomistic. Using the uniqueness theorem [1, §2, Theorem 3], it is easy to see that the F_G-atomic model is the free m-complemented lattice generated by the n elements $a_1, \cdots, a_n$.

IV.22. DEFINITION. An F_G-initial model O is an F_G-prime model of T which has precisely one F-map $h: O \to M$ into any model M of T.

IV.23. THEOREM. *An F_G-initial model O of some F_G-elementary and F_G-complete F_G-theory is an F_G-atomic model where each m-ary F_G-atom is realized by precisely one m-tuple of elements in O.*

PROOF. By Proposition IV.17 there is an F_G-atomic model O' of T. There are also F-maps $h: O \to O'$ and $k: O' \to O$. kh is the identity of O because O is F_G-initial. This implies that for any m-tuple $\vec{c}$ of elements in O, the F-type $I\langle\vec{c}\rangle$ of $\vec{c}$ in O coincides with the F-type J of $f(\vec{c})$ in O'. As O' is F_G-atomic, so is O. Now assume that some m-ary F_G-atom were realized by two different m-tuples $\vec{c}$ and $\vec{d}$. According to Theorem II.7 there would be an F-map carrying $\vec{c}$ into $\vec{d}$. Thus $\mathrm{id}_0: O \to O$ would not be the only F-map $O \to O$. Q.E.D.

IV.24. THEOREM. *An F_G-atomic model O is an F_G-initial model if and only if any m-ary F_G-atom is realized in any model M of T by precisely one m-tuple of elements ($m \geq 1$), provided that T is F_G-complete.*

PROOF. If each m-ary F_G-atom is realized by precisely one m-tuple of elements in each model M then there can be only one F-map $h: O \to M$.

If for some model M there would be two different m-tuples $\vec{c}$ and $\vec{d}$ which realize some m-ary F_G-atom I then one could construct two different F-maps $h, k: O \to M$. Namely let $\vec{a}$ be an m-tuple of elements in O realizing the F_G-atom I. Consider the maps which send $\vec{a}$ into $\vec{c}$ and $\vec{a}$ into $\vec{d}$ respectively. They both can be extended to F-maps by the procedure applied in the proof of I.25. Q.E.D.

IV.25. PROPOSITION. *An F_G-complete and F_G-elementary F_G-theory T has an F-initial model if and only if it is F_G-atomistic and each m-ary F_G-atom is realized by at most one m-tuple of elements in each model ($m \geq 1$).*

This follows from Theorems IV.23 and 24. Applying Remark I.13 we can prove

IV.26. COROLLARY. *An F_G-complete and F_G-elementary F_G-theory T has an F-initial model if and only if T is F_G-atomistic and each m-ary F_G-atom is equivalent to an m-ary formula $\mu(\vec{x})$ which is preserved by F-maps and which is fulfilled by only one m-tuple of elements in each model of T ($m > 0$). It even can be shown that μ is a formula of F itself (cf. Fittler* [2]).

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