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EIGENVALUES OF THE RADIALLY SYMMETRIC p-LAPLACIAN IN $\mathbb{R}^n$

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Abstract

For the p-Laplacian $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$, $p>1$, the eigenvalue problem $-\Delta_p v + q(|x|)|v|^{p-2}v = \lambda|v|^{p-2}v$ in $\mathbb{R}^n$ is considered under the assumption of radial symmetry. For a first class of potentials $q(r)\to\infty$ as $r\to\infty$ at a sufficiently fast rate, the existence of a sequence of eigenvalues $\lambda_k\to\infty$ if $k\to\infty$ is shown with eigenfunctions belonging to $L^p(\mathbb{R}^n)$. In the case $p=2$, this corresponds to Weyl's limit point theory. For a second class of power-like potentials $q(r)\to-\infty$ as $r\to\infty$ at a sufficiently fast rate, it is shown that, under an additional boundary condition at $r=\infty$, which generalizes the Lagrange bracket, there exists a doubly infinite sequence of eigenvalues λ_k with $\lambda_k\to\pm\infty$ if $k\to\pm\infty$. In this case, every solution of the initial value problem belongs to $L^p(\mathbb{R}^n)$. For $p=2$, this situation corresponds to Weyl's limit circle theory.

1. *Introduction and main results*

The spectral theory of $(-\Delta + q(x))$ on bounded and unbounded domains $\Omega\subset\mathbb{R}^n$ with homogeneous boundary conditions is well established (cf. Edmunds and Evans [**13**]). The linear theory is the starting point for the investigation of nonlinear perturbations

$$(-\Delta + q(x))v = f(x,v) \text{ in } \Omega, \quad Bv = 0 \text{ on } \partial\Omega,$$

where Bv stands for homogeneous boundary conditions, for example zero Neumann or Dirichlet conditions. For nonlinearities of the form $f(x,s)=\lambda s + o(s)$ as $s\to 0$, we mention bifurcation theory (cf. Chow and Hale [**7**]) as a representative theory which has its origin in linear spectral theory. Bifurcation theory exhibits the eigenvalues λ_i of $-\Delta + q(x)$ as those parameter values where multiplicity changes in the set of solutions can be expected.

The p-Laplacian operator $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ with $p>1$ has recently attracted similar attention to that formerly paid to the Laplace operator. For those following in the footprints of the development of bifurcation theory, it seems natural to ask first for eigenvalues λ_i of $-\Delta_p v + q(x)|v|^{p-2}v = \lambda|v|^{p-2}v$ with appropriate boundary conditions and then to pass to the perturbed problem $-\Delta_p v + q(x)|v|^{p-2}v = f(x,v)$, where $f(x,s)=\lambda|s|^{p-2}s + o(|s|^{p-2}s)$ as $s\to 0$. Indeed, for a bounded domain Ω, this direction was successfully pursued by Guedda and Veron [**15**] and DelPino and Manásevich [**10**, **11**] . The cases considered in these papers included the bounded one-dimensional interval, the radially symmetric case on balls and annuli, and the general multidimensional case with bifurcation from the first eigenvalue λ_1.

These results depended on a sufficient understanding of the eigenvalue theory for the p-Laplacian. For the case $p\neq 2$, a complete eigenvalue theory is not available. However, in the case of a bounded interval or for radially symmetric eigenfunctions

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on balls and annuli, such an eigenvalue theory exists (see Elbert [**14**], DelPino and Manásevich, [**10**] and Reichel and Walter [**19**]). On general domains, the understanding of the eigenvalues is still sparse; cf. De Thelin [**9**], Barles [**4**], Bhattarcharya [**5**] and Anane [**1**].

The goal of this paper is to extend the eigenvalue theory to the radially symmetric problem

$$-\Delta_p v + q(|x|)|v|^{p-2}v = \lambda |v|^{p-2}v \text{ on } \mathbb{R}^n, \qquad v \in L^p(\mathbb{R}^n), \tag{1}$$

with a radially symmetric potential $q(|x|)$. The value $\lambda \in \mathbb{R}$ is called an eigenvalue if a solution $v \neq 0$ of (1) exists. We seek only radially symmetric eigenfunctions v. For radially symmetric functions $v(x) = u(r)$ with $r = |x|$, we find $(\Delta_p v)(x) = (L_p u)(r)$ with $L_p u = r^{1-n}(r^{n-1}|u'|^{p-2}u')'$ being the radially symmetric p-Laplacian. The notation $s^{(t)} = |s|^{t-1}s$, $s \in \mathbb{R}$, $t > 0$ for the odd tth-power-function is used throughout the paper. The eigenvalue problem (1) is then

$$\begin{aligned} -L_p u + q(r)u^{(p-1)} &= \lambda u^{(p-1)} \text{ in } (0,\infty), \\ u'(0) = 0, \quad & u \in L^p(0,\infty; r^{n-1}), \end{aligned} \tag{2}$$

where $L^p(0,\infty;r^{n-1})$ is the set of all measurable functions u on $(0,\infty)$ such that $\int_0^\infty r^{n-1}|u(r)|^p\,dr < \infty$. Depending on the nature of the potential q, we may have to impose a second boundary condition at ∞. After multiplication with r^{n-1}, (2) takes the form

$$\begin{aligned} -\big(r^{n-1}u'^{(p-1)}\big)' + r^{n-1}q(r)u^{(p-1)} &= \lambda r^{n-1}u^{(p-1)} \text{ in } (0,\infty), \\ u'(0) = 0, \quad & u \in L^p(0,\infty; r^{n-1}). \end{aligned} \tag{3}$$

We will always assume that the potential q is continuous on $[0,\infty)$. Solutions of (2) are supposed to satisfy $u \in C^1[0,\infty)$, $r^{n-1}u'^{(p-1)} \in C^1[0,\infty)$. The condition $u'(0) = 0$ makes the initial value problem $u'(0) = 0$, $u(0) = c$ well defined for (2) (cf. Section 2.2). Thus $r = 0$ is no longer considered a singular endpoint.

1.1. *A limit point type situation*

For the case $p = 2$, the famous theory of Weyl [**22**] classifies the endpoint $r = \infty$ as a *limit point* endpoint if the initial value problem for (2) at $r = 1$ has at most one linearly independent solution $u \in L^2(1,\infty;r^{n-1})$. It follows that at $r = \infty$, a boundary condition on u is neither necessary nor admissible. It suffices to consider the problem

$$-L_2 u + q(r)u = \lambda u \text{ in } (0,\infty), \quad u'(0) = 0, \qquad u \in L^2(0,\infty;r^{n-1}) \tag{4}$$

to obtain full spectral information. If the potential $q(r) \geqslant -\text{const.}\ r^2$ at ∞, then $r = \infty$ is indeed a limit point endpoint [**8**, Chapter 9.3] . If $q(r) \to \infty$ as $r \to \infty$, then (4) has a discrete spectrum; see Titchmarch [**20**]. (Note that $u(r)r^{(n-1)/2} = v(r)$ transforms (4) into $-v'' + (q(r) + ((n^2 + 3 - 4n)/4)v = \lambda v$ on $(0,\infty)$. It follows that $u \in L^2(1,\infty;r^{n-1})$ if and only if $v \in L^2(1,\infty)$. In this form the criteria from [**8**, **20**] apply.)

For the radially symmetric p-Laplacian we do not have a theory of similar generality. However, we can exhibit a *class of potentials* q tending sufficiently fast to $+\infty$ as $r \to \infty$ for which the sole requirement of L^p-integrability is enough to obtain a discrete set of eigenvalues.

THEOREM 1. *Suppose that* $q:[0,\infty)\longrightarrow\mathbb{R}$ *is a given* C^1*-potential with the following two properties.*

(i) *There exist* $\alpha>0$ *and* $\beta>\max\{(p-n)/(p-1),0\}$ *such that* $q(r)\geqslant\alpha r^\beta$ *for large* r.

(ii) $q'(r)/q(r)^{1+1/p}\to 0$ *as* $r\to\infty$.

Then, without an additional boundary condition at ∞, *problem* (2) *has a countable number of simple eigenvalues* $\lambda_1<\lambda_2<\dots$, $\lim_{k\to\infty}\lambda_k=\infty$, *and no other eigenvalues. The corresponding eigenfunction* u_k *has* $k-1$ *simple zeroes in* $(0,\infty)$. *Between* $r=0$ *and the first zero of* u_k, *between any two consecutive zeros of* u_k, *and also to the right of the last zero of* u_k, *there is exactly one zero of* u_{k+1}.

REMARK 2. (a) Notice that (i) is satisfied if $q(r)\geqslant\alpha r^\beta$ with $\beta>1$ and $\alpha>0$. Hence (i) and (ii) are both satisfied for $q(r)=\alpha r^\beta$ with $\beta>1$ and $\alpha>0$.

(b) It is interesting to compare Theorem 1 with the very similar conditions from the classical result of Titchmarch [**20**], where for $p=2$, a discrete spectrum is found provided that $q(r)\to\infty$ as $r\to\infty$, $q'>0$ at ∞, q'' has one sign at ∞, and $q'(r)=O(q(r)^c)$ for $0<c<3/2$.

1.2. *A limit circle, oscillatory type situation*

Again in the case $p=2$, Weyl's theory [**22**] classifies the endpoint $r=\infty$ as a *limit circle* endpoint if every solution u of the initial value problem for (2) at $r=1$ is in $L^2(1,\infty;r^{n-1})$. For a suitable spectral theory, it is necessary to require an additional boundary condition at $r=\infty$. This is done in the following way: choose a non-zero function $u_0:[0,\infty)\longrightarrow\mathbb{R}$ with $u_0'(0)=0$ and $u_0, r^{n-1}u_0'$ absolutely continuous such that

$$u_0\in L^2(0,\infty;r^{n-1}),\, r^{1-n}(-(r^{n-1}u_0')'+r^{n-1}qu_0)\in L^2(0,\infty;r^{n-1}).$$

Then the Lagrange bracket

$$[u,u_0]_\infty:=\lim_{r\to\infty} r^{n-1}(u'(r)u_0(r)-u(r)u_0'(r))$$

is used to define the eigenvalue problem

$$-L_2u+q(r)u=\lambda u \text{ in } (0,\infty),\quad u'(0)=0,\quad [u,u_0]_\infty=0.$$

Each admissible function u_0 creates a different boundary condition at ∞. Equivalently (see Zettl [**23**]), one may choose a value $\lambda_0\in\mathbb{R}$ and define $u_0:[0,\infty)\longrightarrow\mathbb{R}$ as a solution of the initial value problem $-L_2u_0+qu_0=\lambda_0u_0$ with $u_0'(0)=0$ on $(0,\infty)$. Each different value of λ_0 creates a different boundary condition at ∞.

All problems with limit circle endpoints have discrete spectrum (cf. Coddington and Levinson [**8**, Chapter 9.4]). Potentials which generate a limit-circle endpoint at $r=\infty$ are given by the class $q\in C^1[1,\infty)$ such that $q(r)\leqslant 0$, $W(r)=q'(r)/|q(r)|^{3/2}\in BV(0,\infty)$, $W(r)\to 0$ as $r\to\infty$ and $1/|q(r)|^{1/2}\in L^1(1,\infty)$ (cf. Hille [**16**, Section 10]). Important examples are $q(r)=-r^\alpha$ with $\alpha>2$. Then the discrete spectrum of (4) extends to $\pm\infty$.

For the radially symmetric p-Laplacian we do not have such a general theory. However, we can exhibit again a class of power-like potentials q tending sufficiently fast to $-\infty$ as $r\to\infty$, for which we can formulate a substitute for the Lagrange bracket as a boundary condition at $+\infty$.

THEOREM 3. *Suppose that $q(r) = -r^\alpha$ for large r with $\alpha > p/p-1$. Then every solution of the initial value problem (2) lies in $L^p(0,\infty; r^{n-1})$. For a fixed λ_0, let u_0 be a reference solution of $-L_p u + q(r)u^{(p-1)} = \lambda_0 u^{(p-1)}$ in $[0,\infty)$ with $u_0'(0) = 0$. Then (2) together with the boundary condition*

$$\lim_{r\to\infty} \left(r^{(n-1)p/(p-1)}(-q(r))\right)^{(p-2)/p^2} r^{(n-1)/(p-1)}(u'(r)u_0(r) - u(r)u_0'(r)) = 0 \quad (5)$$

has a countable number of simple eigenvalues $\{\lambda_i\}_{i\in\mathbb{Z}}$, $\lim_{k\to\infty}\lambda_k = \infty$, $\lim_{k\to-\infty}\lambda_k = -\infty$ and no other eigenvalues. Each eigenfunction has an infinite number of zeros. Different choices of λ_0 lead to different boundary conditions at ∞, and hence to different sets of eigenvalues.

REMARK 4. (a) Note that in the case $p = 2$, the new boundary condition (5) is precisely $[u, u_0]_\infty = 0$. Only for the cases $p \neq 2$ does a new factor involving the potential q explicitly appear.

(b) For $p = 2$, the above theorem reproduces the familiar restriction $\alpha > 2$.

We mention that we could have included a weight $w(|x|)$ in the right-hand side of (1) or (2). In order to keep the conditions in Theorem 1 and Theorem 3 simple, we decided to consider only the case $w \equiv 1$.

2. *The generalized Prüfer transformation*

Instead of working directly with equation (3), we transform it into a more convenient representation via the generalized Prüfer-transformation, which we discuss in what follows.

2.1. *The generalized sine function*

Generalized sine functions have been well studied in the literature (see Lindqvist [**17**]). The generalized sine function $\sin_p$ is first defined locally as the solution of the differential equation

$$u'^p + \frac{u^p}{p-1} = 1, \qquad u(0) = 0, u'(0) = 1. \quad (6)$$

Equation (6) arises as a first integral of $(u'^{(p-1)})' + u^{(p-1)} = 0$. The solution defines the function $S_p(\phi) = \sin_p(\phi)$ as long as it is increasing, that is, for $\phi \in [0, \pi_p/2]$, where

$$\frac{\pi_p}{2} = \int_0^{(p-1)^{1/p}} \frac{dt}{1 - t^p/(p-1)^{1/p}} = \frac{(p-1)^{1/p}}{p\sin(\pi/p)}\pi. \quad (7)$$

Since $S_p'(\pi_p/2) = 0$, we define S_p on the interval $[\pi_p/2, \pi_p]$ by $S_p(\phi) = S_p(\pi_p - \phi)$, and for $\phi \in (\pi_p, 2\pi_p]$ we put $S_p(\phi) = -S_p(2\pi_p - \phi)$ and extend S_p as a $2\pi_p$-periodic function on $\mathbb{R}$. In the special case $p = 2$, $S_2(x) = \sin x$ and $\pi_2 = \pi$. The following properties of S_p will be used frequently.

LEMMA 5. *For $p > 1$, the generalized sine functions S_p have the following properties.*

(i) *$S_p, S_p'^{(p-1)}$ are C^1-functions on $\mathbb{R}$ with L^∞-norms $\|S_p\|_\infty = (p-1)^{1/p}$, $\|S_p'\|_\infty = 1$ and $\|(S_p'^{p-1})'\|_\infty = (p-1)^{(p-1)/p}$.*

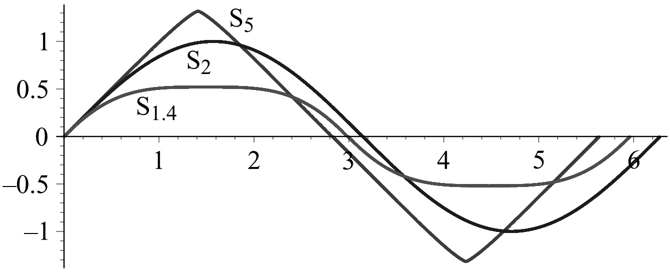

FIGURE 1. S_p, $p=1.4, 2, 5$.

(ii) *S_p solves $|S_p'|^p + |S_p|^p/(p-1) = 1$ on $\mathbb{R}$.*
(iii) *$|S_p'(t+(\pi_p/2))|^p \leqslant (p-1)|t|^{p/(p-1)}$ for $t \in \mathbb{R}$.*
(iv) *For $1<p\leqslant 2$, the function S_p' is C^1, whereas for $p \geqslant 2$, the function S_p' is $1/(p-1)$-Hölder continuous.*

Proofs can be obtained from the results of Lindqvist [**17**]. We show in Figure 1 the graphs of the function S_p for $p=1.4, 2, 5$. As $p \to \infty$, the function S_p converges to $1-|x-1|$ and as $p\to 1$, it approaches 0.

2.2. *Generalized phase-plane coordinates*

The main sources for the results in this section are Reichel and Walter [**19**] and Brown and Reichel [**6**]. With the help of the generalized sine function, we transform any solution of (2) into phase space via generalized polar coordinates ρ and ϕ.

$$\begin{cases} r^{n-1}u'(r)^{(p-1)} = \rho(r) S_p'(\phi(r))^{(p-1)}, \\ Q(r)^{(p-1)/p} u(r)^{(p-1)} = \rho(r) S_p(\phi(r))^{(p-1)}. \end{cases} \tag{8}$$

Here $Q:[0,\infty) \longrightarrow (0,\infty)$ is an arbitrary positive C^1-function, which will be chosen later. A calculation using the defining properties of S_p and S_p' as in Lemma 5(ii) leads to the pair of equations

$$\begin{aligned} \phi' &= \frac{r^{n-1}}{p-1}(-q(r)+\lambda)Q(r)^{(1-p)/p}|S_p(\phi)|^p + \frac{Q'(r)}{pQ(r)} S_p(\phi)S_p'(\phi)^{(p-1)} \\ &\quad + r^{(1-n)/(p-1)}Q(r)^{1/p}|S_p'(\phi)|^p, \end{aligned} \tag{9}$$

$$\begin{aligned} \rho' &= \rho\Big\{ \big(r^{n-1}(q(r)-\lambda)Q(r)^{(1-p)/p} + r^{(1-n)/(p-1)}Q(r)^{1/p}\big) S_p(\phi)^{(p-1)} S_p'(\phi) \\ &\quad + \frac{Q'(r)}{pQ(r)}|S_p(\phi)|^p \Big\}. \end{aligned} \tag{10}$$

The main feature of the system is the fact that the ϕ-equation (9) is independent of ρ. Moreover, the ρ-equation (10) is linear in ρ. Solutions of (9) and (10) are denoted by $\phi(r;\lambda)$ and $\rho(r;\lambda)$, respectively.

Radially symmetric solutions of (2) satisfy $u'(0)=0$. This amounts (in a suitable normalization) to $\phi(0)=\pi_p/2$. The value $\rho(0)$ may be chosen arbitrarily. This reflects the invariance of (2) under scaling.

A more detailed analysis shows that the angular function $\phi(r;\lambda)$ of a radially symmetric solution u of (2) satisfies $\phi(r;\lambda)-\pi_p/2 = O(r^n)$ as $r\to 0$ (cf. Reichel and

Walter [**19**]). Therefore, a solution of the ϕ-equation (9) with $\phi(r;\lambda)-\pi_p/2=O(r^n)$ will be called a *tamed solution*. In the next lemma we recall from Brown and Reichel [**6**, Lemma 9] that (9) is uniquely solvable in the class of tamed solutions. As an aside, we note that (9) also has solutions $\phi(r;\lambda)-\pi_p/2=O(r^{n-p})$, which do *not* correspond to radially symmetric solutions on balls.

LEMMA 6. (i) *For a tamed solution $\phi(r;\lambda)$ of* (9), *we find the relation*

$$\lim_{r\to 0}(\phi(r;\lambda)-\pi_p/2)r^{-n}=\frac{1}{n}(-q(0)+\lambda)Q(0)^{(1-p)/p}.$$

(ii) *The initial value problem* (9) *with initial value $\pi_p/2$ at $r=0$ has a unique tamed solution.*

REMARK 7. The restriction to tamed solutions of the ϕ-equation is equivalent to the well-posedness of the initial value problem for u at $r=0$ with $u(0)=c$, $u'(0)=0$. Alternatively, we could have considered $r=0$ as a singular endpoint itself. For $n\geqslant p$, the endpoint $r=0$ is of limit-point type. The L^p-integrability condition at 0 uniquely selects the solution with $u'(0)=0$. For $1<p<n$, the endpoint $r=0$ is of limit-circle type (cf. the weakly regular case in Brown and Reichel [**6**]), that is, the initial value problem at $r=0$ is well defined for arbitrary values of $u(0)$ and $u'(0)$.

As a basic tool for our analysis we use a simple comparison principle between upper and lower solutions for first-order differential equations. Such comparison principles can be found in detail in Walter [**21**].

LEMMA 8 (comparison principle). *Assume that $g(r,s)$ is defined on $(a,b)\times\mathbb{R}$. If the functions ϕ,ψ are C^1-functions on (a,b), continuous in $[a,b]$ with $\phi(a)\leqslant\psi(a)$, and*

$$\phi'\leqslant g(r,\phi)\quad\text{and}\quad\psi'\geqslant g(r,\psi)\quad\text{in }(a,b),$$

then ϕ is called a lower solution and ψ is called an upper solution. If $g(r,s)$ is uniformly Lipschitz-continuous with respect to s on compact subsets of $[a,b]\times\mathbb{R}$, then the conclusion $\phi(r)\leqslant\psi(r)$ in $[a,b]$ holds.

Proof. On a finite interval $[a,b]$, the functions ϕ,ψ attain their values in the interval $[-M,M]$. Let L be the Lipschitz constant of g with respect to the second variable on the compact set $[a,b]\times[-M,M]$. The difference $\xi=\psi-\phi$ satisfies $\xi'\geqslant g(r,\psi)-g(r,\phi)\geqslant -L|\xi|$ on intervals $[a,b]$. This shows that $\xi e^{-L(r-a)}$ is increasing on intervals where ξ is negative. Since $\xi(a)\geqslant 0$, we get $\xi\geqslant 0$ on $[a,b]$. A more refined version of this result can be found in [**19**]. □

3. *The limit point type situation*

For the proof of Theorem 1 we choose $Q\equiv 1$ in the generalized Prüfer transformation. Thus the Prüfer equations (9) and (10) simplify to

$$\phi' = \frac{r^{n-1}}{p-1}(-q(r)+\lambda)|S_p(\phi)|^p+r^{(1-n)/(p-1)}|S_p'(\phi)|^p, \tag{11}$$

$$\rho' = \rho\big(r^{n-1}(q(r)-\lambda)+r^{(1-n)/(p-1)}\big)S_p(\phi)^{(p-1)}S_p'(\phi). \tag{12}$$

LEMMA 9. *Suppose that* $\phi(r;\lambda_1)$ *and* $\phi(r;\lambda_2)$ *are tamed solutions of* (11) *with* $\lambda_1<\lambda_2$. *Then* $\phi(r;\lambda_1)<\phi(r;\lambda_2)$ *for all* $r>0$.

The proof is essentially a consequence of the comparison principle of Lemma 8. Details can be found in Reichel and Walter [**19**].

LEMMA 10. *Suppose that* $\phi(r;\lambda)$ *is a tamed solution of* (11). *Then there exists* $k\in\mathbb{N}$ *depending on* λ *such that*

$$\limsup_{r\to\infty}\phi(r;\lambda),\liminf_{r\to\infty}\phi(r;\lambda)\in[(k-1)\pi_p,k\pi_p].$$

Proof. Every solution $\phi(r;\lambda)$ of (11) passes through multiples of π_p only from below. In particular, every solution of (11) is bounded below by 0. Let R be so large that $-q(r)+\lambda\leqslant 0$ for $r\geqslant R$. At R, we have $\phi(R;\lambda)\in[(2l-1)\pi_p/2,(2l+1)\pi_p/2]$ for some $l\in\mathbb{N}_0$. Since $\psi\equiv(2l+1)\pi_p/2$ serves as an upper solution on $[R,\infty)$, we find that ϕ is bounded. Since ϕ passes through multiples of π_p only from below, the possible accumulation points of $\phi(r;\lambda)$ at infinity must lie between two consecutive multiples of π_p. □

Given any tamed solution $\phi(r;\lambda)$, we will denote its limiting interval at ∞ as in Lemma 10 by $[(k-1)\pi_p,k\pi_p]$ without explicitly mentioning the λ-dependence of the value k.

LEMMA 11. *Let* $J\subset\mathbb{R}$ *be a bounded interval and suppose that* $\lambda\in J$. *For every* $r_0>0$ *sufficiently large, there exists* $\epsilon=\epsilon(r_0)$ *with the following properties.*

(i) *For a tamed solution* $\phi(r;\lambda)$ *of* (11) *with* $\lambda\in J$ *and* $(k-1)\pi_p\leqslant\phi(r_0)\leqslant k\pi_p-\epsilon$, $\phi(r)\to(k-1)\pi_p$ *as* $r\to\infty$.

(ii) $\epsilon(r_0)=O(r_0^{-\gamma})$ *uniformly for* $\lambda\in J$ *with*

$$\gamma=\frac{\beta}{p}+\min\left\{\frac{n}{p},\frac{n-1}{p-1}\right\}.$$

Proof. By the hypothesis on $q(r)$, we can assume that r_0 is so large that $q(r)-\lambda\geqslant(\alpha/2)r^\beta$ for $r\geqslant r_0$. We construct an upper solution ψ which converges to $(k-1)\pi_p$. As an ansatz, we take

$$\psi(r)=(k-1)\pi_p+A(\alpha r^{n-1+\beta})^{-1/p},$$

with $A=(\alpha r_0^{n-1+\beta})^{1/p}(\pi_p-\epsilon)$ and ϵ to be chosen later. Property (i) then follows from the comparison principle of Lemma 8. It remains to verify that ψ is indeed an upper solution, and to determine ϵ as a function of r_0. The condition for an upper solution is

$$\psi'\geqslant\frac{r^{n-1}}{p-1}(-q(r)+\lambda)|S_p(\psi)|^p+r^{(1-n)/(p-1)}|S_p'(\psi)|^p,\qquad r\geqslant r_0.\tag{13}$$

Since ψ only attains values in $[(k-1)\pi_p,k\pi_p-\epsilon]$, there exists a value c_p depending only on p such that $|S_p(\psi)|\geqslant c_p\epsilon|\psi-(k-1)\pi_p|$. Therefore the upper solution condition (13) is fulfilled provided that

$$\psi'\geqslant-\underbrace{c_p^p\frac{\alpha}{2(p-1)}}_{=:c(p,\alpha)}r^{n-1+\beta}\epsilon^p|\psi-(k-1)\pi_p|^p+r^{(1-n)/(p-1)},\qquad r\geqslant r_0.\tag{14}$$

After insertion of the particular form of ψ, this is equivalent to

$$\frac{n-1+\beta}{p}\alpha^{-1/p}Ar^{((1-n-\beta)/p)-1} \leqslant c(p,\alpha)\epsilon^p\alpha^{-1}A^p - r^{(1-n)/(p-1)}, \qquad r \geqslant r_0. \quad (15)$$

Due to the monotonicity of the two sides of (15) with respect to r, it is sufficient that (15) holds with equality at r_0. After insertion of the choice of A, this amounts to

$$\frac{n-1+\beta}{p}(\pi_p-\epsilon)r_0^{-1} + r_0^{(1-n)/(p-1)} = c(p,\alpha)\epsilon^p r_0^{n-1+\beta}(\pi_p-\epsilon)^p. \quad (16)$$

Equation (16) defines the value $\epsilon = \epsilon(r_0)$. The asymptotic behaviour of ϵ as $r_0 \to \infty$ is obtained from (16) by replacing $(\pi_p - \epsilon)$ by a constant. □

LEMMA 12. *Suppose that $\phi(r;\lambda)$ is a tamed solution of* (11) *with $\phi(r;\lambda) \geqslant (k-1)\pi_p$ at ∞ and $\lim_{r\to\infty}\phi(r;\lambda) = (k-1)\pi_p$. Then there exists a constant $\delta > 0$, which may depend on ϕ, such that, for large r,*

$$\phi(r;\lambda) - (k-1)\pi_p \geqslant \delta r^{(1-n)/(p-1)}q(r)^{-1/p}.$$

Proof. We show that $\psi(r) = (k-1)\pi_p + \delta r^{(1-n)/(p-1)}q(r)^{-1/p}$ is a lower solution for $\delta > 0$ small. The condition for a lower solution is

$$\psi' \leqslant \frac{r^{n-1}}{p-1}(-q(r)+\lambda)|S_p(\psi)|^p + r^{(1-n)/(p-1)}|S_p'(\psi)|^p. \quad (17)$$

Let us assume that R is so large that $-q(r)+\lambda \geqslant -2q(r)$ for all $r \geqslant R$. Moreover, assume that δ is so small that $|S_p'(\psi(r))|^p \geqslant 1/2$ for $r \geqslant R$. Since $|S_p(s)| \geqslant C_p(s-(k-1)\pi_p)$ for $s \in [(k-1)\pi_p, (k-1/2)\pi_p]$ it is sufficient for (17) to have

$$\psi' \leqslant \frac{r^{n-1}}{p-1}2C_p^p(-q(r))(\psi-(k-1)\pi_p)^p + \frac{1}{2}r^{(1-n)/(p-1)}. \quad (18)$$

If we insert the special form of ψ, then this amounts to

$$\delta\left(\frac{1-n}{p-1}r^{(1-n)/(p-1)-1}q(r)^{-1/p} - \frac{1}{p}r^{(1-n)/(p-1)}q(r)^{-1-1/p}q'(r)\right) \leqslant -\frac{2C_p^p}{p-1}\delta^p r^{(1-n)/(p-1)} + \frac{1}{2}r^{(1-n)/(p-1)} \quad (19)$$

for $0 < \delta < \delta_0$ and $r \geqslant R$. Notice that by the decay assumption (ii) on $q'/q^{1+1/p}$ from Theorem 1, the decay rate of the left-hand side of (19) is faster than $r^{(1-n)/(p-1)}$. The positive part of the right-hand side decays exactly like $r^{(1-n)/(p-1)}$. If we choose δ_0 to be sufficiently small, inequality (19) holds for $r \geqslant R$ sufficiently large uniformly for all $\delta \in (0,\delta_0)$. This shows that ψ is indeed a lower solution. By choosing δ_0 to be even smaller, we can achieve $\psi(R) < \phi(R;\lambda)$ for all $\delta \in (0,\delta_0)$. The comparison principle of Lemma 8 implies the claimed lower bound for $\phi(r;\lambda)$. □

LEMMA 13. *Suppose that $\phi(r;\lambda)$ is a tamed solution of* (11) *such that $\liminf_{r\to\infty}\phi(r;\lambda)$, $\limsup_{r\to\infty}\phi(r;\lambda) \in [(k-1)\pi_p, k\pi_p]$. Suppose further that $\phi(r;\lambda) \not\to (k-1)\pi_p$ as $r\to\infty$. Then $\phi(r) \to k\pi_p$ as $r \to \infty$ and*

$$0 \leqslant k\pi_p - \phi(r;\lambda) \leqslant O(r^{-\gamma}),$$

with γ as in Lemma 11.

Proof. If $\phi(r;\lambda)\not\to(k-1)\pi_p$, then the convergence condition of Lemma 11(i) must be violated, that is, $k\pi_p-\phi(r_0;\lambda)\leqslant\epsilon(r_0)$ for all sufficiently large r_0. □

We know from Lemma 10 that any tamed solution $\phi(r;\lambda)$ has its accumulation points at ∞ inside an interval $[(k-1)\pi_p,k\pi_p]$. Due to Lemma 11 and Lemma 13, we know more: $\phi(r;\lambda)$ converges either to $(k-1)\pi_p$ from above or to $k\pi_p$ from below.

PROPOSITION 14. *For $k\in\mathbb{N}$, define the set*

$$S_k=\left\{\lambda\in\mathbb{R}:\lim_{r\to\infty}\phi(r;\lambda)<k\pi_p\right\}.$$

Then S_k is an open right-bounded interval, that is, $S_k=(-\infty,\lambda_k)$. Moreover, λ_k is the unique value of λ such that $\phi(r;\lambda)\leqslant k\pi_p$ for all $r\in[0,\infty)$ and $\lim_{r\to\infty}\phi(r;\lambda)=k\pi_p$.

Proof. First we show that S_k is non-empty. On compact intervals $[0,R]$, it was shown by Reichel and Walter [**19**] that $\|\phi(\cdot;\lambda)\|_{\infty,[0,R]}\to0$ as $\lambda\to-\infty$. By Lemma 11, this implies that $\|\phi(\cdot;\lambda)\|_{\infty,[0,\infty)}\to0$ also as $\lambda\to-\infty$. Moreover, the λ-monotonicity of $\phi(\cdot;\lambda)$ from Lemma 9 shows that S_k is an interval extending to $-\infty$. Since for any fixed $R>0$ we also have $\phi(R;\lambda)\to+\infty$ as $\lambda\to+\infty$, and since once $\phi(r;\lambda)$ has crossed a multiple of π_p its stays above that multiple of π_p, we see that S_k is a right-bounded interval, that is, $S_k=(-\infty,\lambda_k\}$. At this stage, it is not clear whether $\lambda_k\in S_k$ or not. We now show that S_k is open. Suppose that $\tilde\lambda\in S_k$, that is, $\lim_{r\to\infty}\phi(r;\tilde\lambda)=l\pi_p$ with $l<k$. In particular, there exists a large value $r_0>0$ such that $\phi(r_0;\tilde\lambda)<k\pi_p-\epsilon(r_0)$. By continuous dependence on λ, we find that $\phi(r_0;\lambda)<k\pi_p-\epsilon(r_0)$ also for $\lambda\in(\tilde\lambda-\tau,\tilde\lambda+\tau)$. By Lemma 11, this implies that $(\tilde\lambda-\tau,\tilde\lambda+\tau)\subset S_k$, that is, S_k is an open interval $(-\infty,\lambda_k)$, and $\phi(r;\lambda_k)\leqslant k\pi_p$ and $\lim_{r\to\infty}\phi(r;\lambda_k)=k\pi_p$. It remains to prove the uniqueness property of λ_k. Suppose that there exists a second value $\tilde\lambda_k>\lambda_k$ such that $\phi(r;\tilde\lambda_k)$ also approaches $k\pi_p$ from below. We will show that for small $\delta>0$, the function $\psi(r):=\phi(r;\lambda_k)+\delta$ is a lower solution to $\phi(r;\tilde\lambda_k)$ on the interval $[R_0,\infty)$ with sufficiently large R_0. The comparison principle of Lemma 8 then implies that $\phi(r;\lambda)+\delta\leqslant\phi(r;\tilde\lambda_k)$, and hence $\phi(r;\tilde\lambda_k)$ must cross the level $k\pi_p$, which is a contradiction. To show that $\psi(r)$ is a lower solution to $\phi(r;\tilde\lambda_k)$, we need to verify that

$$\psi'=\phi' = \frac{r^{n-1}}{p-1}(-q+\lambda_k)|S_p(\phi)|^p+r^{(1-n)/(p-1)}|S_p'(\phi)|^p, \tag{20}$$

$$\leqslant \frac{r^{n-1}}{p-1}(-q+\tilde\lambda_k)|S_p(\phi+\delta)|^p+r^{(1-n)/(p-1)}|S_p'(\phi+\delta)|^p. \tag{21}$$

To verify the inequality between (20) and (21), notice first that $\lambda_k<\tilde\lambda_k$. Next we suppose $r\geqslant R_0$ to be so large that $-q+\tilde\lambda_k\leqslant0$ on $[R_0,\infty)$. Finally, as long as $\phi,\phi+\delta$ attain values in $[(k-1/2)\pi_p,k\pi_p]$, we have the inequalities

$$|S_p(\phi)|\geqslant|S_p(\phi+\delta)|\quad\text{and}\quad|S_p'(\phi)|\leqslant|S_p'(\phi+\delta)|.$$

This establishes (20) and (21) for $r\geqslant R_0$ and as long as $\phi,\phi+\delta$ attain values in $[(k-1/2)\pi_p,k\pi_p]$. The comparison principle of Lemma 8 implies that for small δ the lower solution $\phi(r;\lambda_k)+\delta$ pushes $\phi(r;\tilde\lambda_k)$ up to the level $k\pi_p$ on a finite interval $[R_0,R_1]$. This contradiction proves the uniqueness of λ_k, and completes the proof of the proposition. □

Proposition 14 already exhibits the special property of the eigenvalue λ_k. The following proposition adds the L^p-integrability property and finishes the proof of Theorem 1.

PROPOSITION 15. *Let $\phi(r;\lambda)$ be a tamed solution of* (11) *and let $\rho(r;\lambda)$ be the corresponding solution of* (12). *The function $u(r;\lambda)$ obtained from $\phi(r;\lambda)$ and $\rho(r;\lambda)$ by the Prüfer transformation* (8) *belongs to $L^p(0,\infty;r^{n-1})$ if and only if $\lambda=\lambda_k$.*

Proof. We begin by integrating the ρ-equation (12):

$$\rho(r;\lambda)=\rho_0\exp\int_0^r\big(t^{n-1}(q-\lambda)+t^{(1-n)/(p-1)}\big)S_p(\phi)^{(p-1)}S_p'(\phi)\,dt. \tag{22}$$

The L^p-integrability condition on u is expressed as

$$\int_0^\infty \rho^{p/(p-1)}|S_p(\phi)|^p r^{n-1}\,dr<\infty. \tag{23}$$

Part 1: Suppose that $\lambda=\lambda_k$. Since $\phi(r;\lambda_k)$ approaches $k\pi_p$ from below, we have $S_p(\phi)^{(p-1)}S_p'(\phi)<0$ at $r=\infty$. Thus for large r the integrand in (22) is negative, and hence the exponential in (22) is bounded, that is, $\rho(r;\lambda_k)$ is bounded. By Lemma 13, we have $0\leqslant k\pi_p-\phi(r;\lambda)\leqslant O(r^{-\gamma})$, with γ as in Lemma 11. Hence

$$|S_p(\phi(r;\lambda))|^p\leqslant c_p|\phi(r;\lambda)-k\pi_p|^p\leqslant O(r^{-\gamma p}).$$

Thus the integrability condition (23) amounts to the verification of $n-1-\gamma p<-1$, that is, $\gamma>n/p$, which holds true due to the values of γ from Lemma 11 and the condition on the exponent β in Theorem 1.

Part 2: Suppose that $\lambda_{k-1}<\lambda<\lambda_k$. Since $\phi(r;\lambda)$ approaches $(k-1)\pi_p$ from above, we have $S_p(\phi)^{(p-1)}S_p'(\phi)>0$ at $r=\infty$. The integrand in (22) is positive for large r. By Lemma 12, we have $\phi(r;\lambda)-(k-1)\pi_p\geqslant\delta r^{(1-n)/(p-1)}q(r)^{-1/p}$ for large r, and hence

$$S_p(\phi(r;\lambda))^{(p-1)}S_p'(\phi(r;\lambda))\geqslant c_p r^{1-n}q(r)^{(1-p)/p}.$$

This implies the following lower bound for ρ.

$$\rho(r;\lambda)\geqslant c_p\exp\int_0^r q(t)^{1/p}\,dt.$$

Lemma 12 also implies that $|S_p(\phi)|^p\geqslant C_p r^{(1-n)p/(p-1)}q(r)^{-1}$ for large r. Altogether this results in the lower bound

$$\begin{aligned}\int_0^\infty |u(r;\lambda)|^p r^{n-1}\,dr &= \int_0^\infty r^{n-1}\rho(r;\lambda)^{(p-1)/p}|S_p(\phi(r;\lambda)|^p\,dr\\ &\geqslant C_p\int_0^\infty r^{(1-n)/(p-1)}q(r)^{-1}\exp\left(\int_0^r q(t)^{1/p}\,dt\right)dr.\end{aligned} \tag{24}$$

The decay assumption (ii) of Theorem 1 shows that $q'(r)\leqslant o(1)q(r)^{1+1/p}$, where $o(1)\to 0$ as $r\to\infty$. Hence, given any value $M>0$, there exists $r_0>0$ such that

Values of λ
2.708 092 43
2.708 092 432 5
2.708 092 432 561
2.708 092 432 562

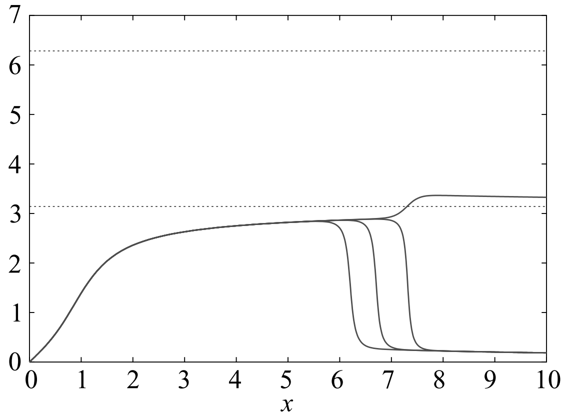

FIGURE 2. $\phi(x;\lambda)$ *for* λ *near* λ_1.

$q(r)^{1/p} \geqslant M(\log q(r))'$ for $r \geqslant r_0$. Inserting this estimate in (24), we get

$$\begin{aligned}\int_0^\infty &|u(r;\lambda)|^p r^{n-1}\,dr\\ &\geqslant C_{p,r_0}\int_0^\infty r^{(1-n)/(p-1)}q(r)^{-1}\exp\left(\int_0^r M(\log q(t))'\,dt\right)dr\\ &\geqslant C_{p,r_0,M}\int_0^\infty r^{(1-n)/(p-1)}q(r)^{-1}\exp(M\log q(r))\,dr.\end{aligned}$$

The remaining integrand is divergent, since $q(t) \geqslant \alpha r^\beta$ and $M > 0$ can be chosen suitably large. Hence u does not belong to $L^p(0,\infty;r^{n-1})$.

This completes the proof of the proposition. □

To illustrate the way in which Proposition 14 characterizes eigenvalues, we give the following example.

EXAMPLE 16. The previous theory was devised to suit the radially symmetric situation. In one space dimension, we can choose the boundary condition at the regular endpoint 0 to be arbitrary, that is, we do not have to require that $u'(0) = 0$. No part of the above theory changes. Consider the one-dimensional example

$$-u'' + x^{3/2}u = \lambda u \text{ in } (0,\infty), \qquad u(0) = 0.$$

As a first step, we compute the eigenvalues λ_1 and λ_2 using the Sleign2 package (cf. Bailey, Everitt and Zettl [**3**]). On the basis of these guesses, we compute solutions $\phi(x;\lambda)$ of the initial value problem (11) for various values of λ by using an enclosure-based initial value solver Awa (cf. Lohner [**18**]). The computed solutions consist of small number intervals (typically of size 10E-14) enclosing the true solutions. In Figure 2 and Figure 3, we have plotted $\phi(x;\lambda)$ on the x-interval $[0,10]$, for values of λ near λ_1 and λ_2. It can be seen that λ_1, λ_2 is precisely the largest value of λ such that $\phi(x;\lambda)$ stays below $\pi, 2\pi$, respectively.

The values computed using Sleign2 are

$$\lambda_1 \approx 2.708\,092\,21, \quad \lambda_2 \approx 5.585\,662\,36.$$

Values of λ
5.585 662 56
5.585 662 560 44
5.585 662 560 462
5.585 662 560 463

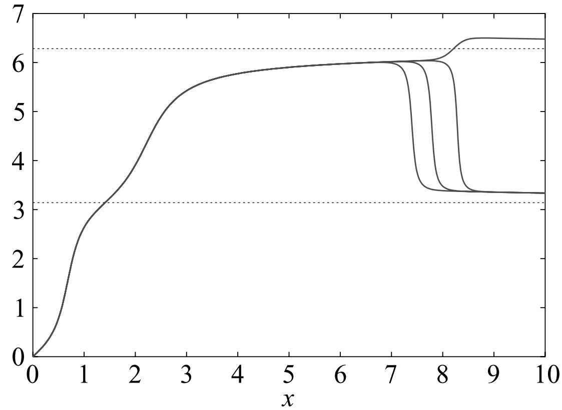

FIGURE 3. $\phi(x;\lambda)$ *for* λ *near* λ_2.

The enclosure method improves the Sleign2 values by 10^{-7}, and gives the precise estimates

$$\lambda_1 \in 2.708\,092\,432\,56_1^2, \quad \lambda_2 \in 5.585\,662\,560\,46_2^3.$$

4. *The limit circle type situation*

For the proof of Theorem 3, we choose in the generalized Prüfer transform $Q(r) = -q(r)r^{(n-1)p/(p-1)}$ for large $r \geqslant R_0$ and Q an arbitrary smooth positive C^1-function on $[0, R]$. Thus the Prüfer equations (9) and (10) simplify for large $r \geqslant R_0$ to

$$\begin{aligned} \phi' &= (-q)^{1/p} + \frac{\lambda}{p-1}(-q)^{(1-p)/p}|S_p(\phi)|^p + \frac{Q'}{pQ}S_p(\phi)S_p'(\phi)^{(p-1)}, \\ \rho' &= \rho\left(-\lambda(-q)^{(1-p)/p}S_p(\phi)^{(p-1)}S_p'(\phi) + \frac{Q'}{pQ}|S_p(\phi)|^p\right). \end{aligned}$$

In view of the form of the potential $q(r) = -r^\alpha$ for large $r \geqslant R_0$, the equations simplify further to

$$\phi' = r^{\alpha/p} + \frac{\lambda}{p-1}r^{\alpha(1-p)/p}|S_p(\phi)|^p + \frac{\beta}{r}S_p(\phi)S_p'(\phi)^{(p-1)}, \tag{25}$$

$$\rho' = \rho\left(-\lambda r^{\alpha(1-p)/p}S_p(\phi)^{(p-1)}S_p'(\phi) + \frac{\beta}{r}|S_p(\phi)|^p\right), \tag{26}$$

for $r \geqslant R_0$, where

$$\beta = \frac{\alpha}{p} + \frac{n-1}{p-1}.$$

LEMMA 17. *Suppose that* $\phi(r;\lambda)$ *is a tamed solution of* (25). *Then the following holds.*

(i) *For* $r \geqslant R_0$, *we have*

$$\phi(r;\lambda) = \phi(R_0;\lambda) + \frac{p}{\alpha+p}\left(r^{(\alpha+p)/p} - R_0^{(\alpha+p)/p}\right) + \lambda K_1(r;\lambda,R_0) + K_2(r;\lambda,R_0),$$

where $K_i(\lambda,R_0) := \lim_{r\to\infty} K_i(r;\lambda,R_0)$, $i = 1, 2$ *exists and is continuous in* λ. *Moreover,* $K_1(\lambda,R_0) \geqslant 0$ *and* $K_i(\lambda,R_0) \to 0$, $i = 1, 2$ *as* $R_0 \to \infty$ *uniformly in* $\lambda \in \mathbb{R}$.

(ii) $\phi'(r;\lambda) = r^{\alpha/p} + O(1/r)$, $|\phi''(r;\lambda)| = O(r^{(\alpha-p)/p})$ *as* $r \to \infty$ *uniformly for* λ *in bounded intervals.*

Proof. (i) Integration of (25) yields

$$
\begin{aligned}
&\phi(r;\lambda) - \phi(R_0;\lambda) \\
&\quad = \left(r^{(\alpha+p)/p} - R_0^{(\alpha+p)/p}\right)\frac{p}{\alpha+p} + \lambda \underbrace{\frac{1}{p-1}\int_{R_0}^{r} s^{\alpha(1-p)/p}|S_p(\phi(s;\lambda))|^p\, ds}_{:=K_1(r;\lambda,R_0)} \\
&\qquad + \underbrace{\beta \int_{R_0}^{r} \frac{1}{s} S_p(\phi(s;\lambda)) S_p'(\phi(s;\lambda))^{(p-1)}\, ds.}_{:=K_2(r;\lambda,R_0)}
\end{aligned}
$$

The integrand of $K_1(r;\lambda,R_0)$ is bounded above by $(p-1)s^{\alpha(1-p)/p}$. The latter is integrable over (R_0,∞) since $\alpha > p/(p-1)$ by hypothesis. Therefore $K_1(\lambda,R_0) = \lim_{r\to\infty} K_1(r;\lambda,R_0)$ exists, and continuity with respect to λ follows from Lebesgue's theorem of dominated convergence. It also follows that $\lim_{R_0\to\infty} K_1(\lambda,R_0) = 0$. Next we show the same for $K_2(r;\lambda,R_0)$. By using the ϕ-equation (25) one more time, we find that

$$
\begin{aligned}
\frac{1}{\beta}K_2(r;\lambda,R_0) \;=\; & \int_{R_0}^{r} \frac{1}{s^{(\alpha+p)/p}}\biggl(\phi' - \frac{\lambda}{p-1}s^{\alpha(1-p)/p}|S_p(\phi)|^p \\
& - \frac{\beta}{s}S_p(\phi)S_p'(\phi)^{(p-1)}\biggr) S_p(\phi)S_p'(\phi)^{(p-1)}\, ds.
\end{aligned}
$$

If we denote by $\Sigma_p(x)$ a primitive of $S_p(x)S_p'(x)^{(p-1)}$, then integration by parts results in

$$
\begin{aligned}
\frac{1}{\beta}K_2(r;\lambda,R_0) \;=\; & \frac{\Sigma_p(\phi(s;\lambda))}{s^{(\alpha+p)/p}}\bigg|_{R_0}^{r} + \frac{\alpha+p}{p}\int_{R_0}^{r} \frac{\Sigma_p(\phi(s;\lambda))}{s^{(\alpha+2p)/p}}\, ds \\
& - \frac{\lambda}{p-1}\int_{R_0}^{r} \frac{S_p(\phi)^{(p+1)}S_p'(\phi)^{(p-1)}}{s^{1+\alpha}}\, ds \\
& - \beta\int_{R_0}^{r} \frac{S_p(\phi)^2|S_p'(\phi)|^{2(p-1)}}{s^{(\alpha+2p)/p}}\, ds.
\end{aligned}
$$

The first term vanishes at $r=\infty$ since $\Sigma_p(x)$ grows at most linearly in x. The remaining integrals have integrands which are bounded in modulus by integrable functions, uniformly with respect to λ. As before, Lebesgue's theorem of dominated convergence shows that $K_2(\lambda,R_0) = \lim_{r\to\infty} K_2(r;\lambda,R_0)$ is continuous in λ, and tends to 0 as $R_0 \to \infty$.

(ii) The behaviour of $\phi'(r;\lambda)$ can be read off directly from (25) and the fact that $\alpha > p/(p-1)$. The behaviour of $\phi''(r;\lambda)$ can be read off from the differentiated version of (25) and the previous information on ϕ'. $\square$

The next two results are needed to show that every solution of (2) belongs to $L^p(0,\infty;r^{n-1})$.

LEMMA 18. *Suppose that $\phi(r;\lambda)$ is a tamed solution of* (25). *Then*

$$\int_0^r \frac{|S_p(\phi(s;\lambda)|^p}{s}\,ds = \frac{p-1}{p}\log r + O(1) \quad \text{as } r\to\infty$$

uniformly for λ in bounded intervals. Here $O(1)$ denotes a quantity which is bounded for $r\in(0,\infty)$.

Proof. We calculate first the following integral, which is clearly related to the one in question.

$$\begin{aligned}\int_0^r \frac{|S_p'(\phi)|^p}{s}\,ds &= \int_0^r \phi' S_p'(\phi)\frac{S_p'(\phi)^{(p-1)}}{s\phi'}\,ds\\ &= S_p(\phi)\frac{S_p'(\phi)^{(p-1)}}{s\phi'}\Big|_0^r - \int_0^r S_p(\phi)\frac{\big(S_p'^{\,(p-1)}\big)'(\phi)}{s}\,ds\\ &\quad+\int_0^r \frac{S_p'(\phi)^{(p-1)}}{s^2\phi'}+\frac{S_p'(\phi)^{(p-1)}\phi''}{s(\phi')^2}\,ds.\end{aligned}$$

As $r\to 0$, we have by Lemma 6(i) that $\phi(r)=\pi_p/2+Cr^n+\mathrm{HOT}$, $\phi'(r)=nCr^{n-1}+\mathrm{HOT}$ and $S_p'(\phi)^{(p-1)}=C'r^n+\mathrm{HOT}$ (where HOT means higher order terms). Hence the first term on the right-hand side above is continuous at $r=0$. Moreover, it tends to 0 at $r=\infty$, that is, it is bounded in r. Similarly, by using the growth rate for ϕ', ϕ'' from Lemma 17, we can see that the last two integrals are bounded in r. If we use the differential equation satisfied by S_p (cf. Section 2.1), we have $(S_p'^{\,(p-1)})'=-S_p^{(p-1)}$. This yields

$$\int_0^r \frac{|S_p'(\phi)|^p}{s}\,ds = O(1) + \int_0^r \frac{|S_p(\phi)|^p}{s}\,ds. \tag{27}$$

If one observes that $|S_p(\phi)|^p/(p-1)=1-|S_p'(\phi)|^p$, then (27) gives the claimed result. □

LEMMA 19. *Suppose that $\phi(r;\lambda)$ is a tamed solution of* (25). *Then the corresponding solution $\rho(r;\lambda)$ of* (26) *satisfies*

$$c(\lambda)r^{\beta(p-1)/p} \leqslant \rho(r;\lambda) \leqslant C(\lambda)r^{\beta(p-1)/p} \quad \text{for large } r\geqslant R_0,$$

where $c(\lambda), C(\lambda)$ are positive constants.

Proof. Integration of (26) leads to

$$\rho(r;\lambda)=\rho(R_0;\lambda)\exp\left(\int_{R_0}^r -\lambda s^{\alpha(1-p)/p}S_p(\phi)^{(p-1)}S_p'(\phi)+\frac{\beta}{s}|S_p(\phi)|^p\,ds\right).$$

The first integral is of order $r^{1+(\alpha(1-p)/p)}$, which is a negative power by hypotheses. Hence the dominating term is the second integral, which by Lemma 18 yields

$$\rho(r;\lambda)\approx \exp\int_{R_0}^r \frac{\beta}{s}|S_p(\phi)|^p\,ds \approx r^{\beta(p-1)/p} \quad \text{for large } r,$$

which is the claimed relation. □

PROPOSITION 20. *Let $\phi(r;\lambda)$ be a tamed solution of* (25) *and let $\rho(r;\lambda)$ be the corresponding solution of* (26). *The function $u(r;\lambda)$ obtained from $\phi(r;\lambda)$ and $\rho(r;\lambda)$ by the Prüfer transformation* (8) *belongs to $L^p(0,\infty;r^{n-1})$ for every $\lambda\in\mathbb{R}$.*

Proof. Consider $\int_0^\infty r^{n-1}|u|^p\,dr$. The integrand is $r^{n-1}\rho^{p/(p-1)}Q^{-1}|S_p|^p$. The choice of Q and Lemma 19 show that, for large r,

$$r^{n-1}|u|^p \leqslant C(\lambda)r^{\alpha(1-p)/p}.$$

By the assumption that $\alpha > p/(p-1)$, the last quantity is integrable at ∞. □

PROPOSITION 21. *Let $\lambda_0, \lambda \in \mathbb{R}$ be two given values, and let $\phi = \phi(r;\lambda)$, $\phi_0 = \phi(r;\lambda_0)$. Moreover let $u = u(r;\lambda)$ and $u_0 = u(r;\lambda_0)$ be obtained by the Prüfer transformation as in Proposition* 20. *The following are equivalent.*

(i) $\lim_{r\to\infty} \phi(r;\lambda) - \phi(r;\lambda_0) = 0 \mod \pi_p$.

(ii) $\lim_{r\to\infty}(r^{(n-1)p/(p-1)}(-q(r)))^{(p-2)/p^2} r^{(n-1)/(p-1)}(u'(r)u_0(r) - u(r)u_0'(r)) = 0$.

Proof. By the Prüfer transform, we have, for large r,

$$\begin{aligned} &u'(r)u_0(r) - u(r)u_0'(r) \\ &\quad = \underbrace{Q^{-1/p}r^{(1-n)/(p-1)}\rho^{1/(p-1)}\rho_0^{1/(p-1)}}_{=:h(r)}(S_p'(\phi)S_p(\phi_0) - S_p(\phi)S_p'(\phi_0)). \end{aligned}$$

The choice of Q and the asymptotics of ρ, ρ_0 from Lemma 19 show that for large r, the function $h(r)$ is bounded above and below by positive constants times $(r^{(n-1)p/(p-1)+\alpha})^{(2-p)/p^2} r^{(1-n)/(p-1)}$. Hence (ii) is equivalent to $S_p'(\phi)S_p(\phi_0) - S_p(\phi)S_p'(\phi_0) \to 0$ as $r \to \infty$. Since S_p'/S_p is a π_p-periodic functions, this is equivalent to $\phi(r) - \phi_0(r) = 0 \mod \pi_p$ at $r = \infty$. □

Proof of Theorem 3. By Proposition 21, eigenvalues are characterized by

$$D(\lambda;\lambda_0) := \lim_{r\to\infty} \phi(r;\lambda) - \phi(r;\lambda_0) = l\pi_p, \qquad l \in \mathbb{Z}. \tag{28}$$

By Lemma 17, we know that $D(\lambda;\lambda_0)$ is a continuous function of λ. The comparison principle from Lemma 8 shows, moreover, that $D(\lambda;\lambda_0)$ is strictly increasing in λ. More precisely, we have

$$\begin{aligned} D(\lambda;\lambda_0) = \phi(R_0;\lambda) - \phi(R_0;\lambda_0) + \lambda K_1(\lambda, R_0) - \lambda_0 K_1(\lambda_0, R_0) \\ + K_2(\lambda, R_0) - K_2(\lambda_0, R_0). \end{aligned}$$

Recall that $K_1 \geqslant 0$. We choose R_0 to be so large that $|K_2(\lambda, R_0)| \leqslant 1$ uniformly in $\lambda \in \mathbb{R}$. For such a fixed R_0, one knows that $\phi(R_0;\lambda) \to \infty$ as $\lambda \to \infty$ (cf. Reichel and Walter [**19**]). Thus $D(\lambda;\lambda_0) \to \infty$ as $\lambda \to \infty$.

It remains to investigate the limit as $\lambda \to -\infty$. We need to show that $D(\lambda;\lambda_0)$ can be made smaller than any constant $-M < 0$. Therefore we choose R_0 to be so large that $\phi(R_0;\lambda_0) \geqslant 2M$ and $K_1(\lambda;R_0)$, $|K_2(\lambda;R_0)|$ are smaller than $\epsilon > 0$, uniformly in λ. Finally, since we know from [**19**] that $\phi(R_0;\lambda) \to 0$ as $\lambda \to -\infty$, we can choose λ to be so negative that $\phi(R_0;\lambda) < \epsilon$. Altogether, we obtain $D(\lambda;\lambda_0) \leqslant \epsilon - 2M + \lambda\epsilon + 2\epsilon$, which is smaller than $-M$ for small $\epsilon > 0$. Thus we have shown that $D(\lambda,\lambda_0)$ is a continuous, strictly increasing function of λ with $\lim_{\lambda\to\pm\infty} D(\lambda;\lambda_0) = \pm\infty$. Hence (28) is fulfilled for a doubly infinite sequence of eigenvalues λ_k tending to $\pm\infty$ as $k \to \pm\infty$. This completes the proof of Theorem 3. □

TABLE 1. *Limit circle eigenvalues of* (29).

Index	Eigenvalue	Error estimate
-1	$-0.992\,475\,700$E+01	0.128 11E-03
0	$-0.537\,591\,815$E+00	0.378 03E-05
1	0.847 746 563E+01	0.215 20E-04
2	0.200 756 512E+02	0.713 20E-04

4.1. *A numerical example for* $p=2$

Theorem 3 and Proposition 21 characterize eigenvalues by the behaviour of the Prüfer angle $\phi(r;\lambda)$ with respect to the angle $\phi(r;\lambda_0)$ of a reference solution. To see the viability of this approach, we finish our discussion with a numerical example.

Finding limit circle oscillatory eigenvalues for potentials $q(r)=-r^\alpha$, $\alpha>2$ is numerically very hard, since solutions oscillate more and more rapidly as r increases. Indeed, Sleign2 cannot compute limit circle eigenvalues for $q(r)=-r^3$ and $p=2$. To find a good numerical example, we choose the problem

$$-\Delta u - u = \lambda w(r) u \text{ in } \mathbb{R}^2 \setminus B_1(0), \quad u=0 \text{ on } \partial B_1(0),$$

with $w(r)=1/r^2$. Here $B_1(0)$ is the unit ball in $\mathbb{R}^2$. Radially symmetric solutions satisfy

$$-(ru')' - ru = \lambda r w(r) u \text{ in } (1,\infty), \quad u(1)=0. \tag{29}$$

Problem (29) is known as the Sears–Titchmarch equation. Due to the factor $w(r)$ on the right-hand side, it is not covered by Theorem 3. Nevertheless, we will show that the same arguments as those in the proof of Theorem 3 can be used to characterize the eigenvalues in terms of phase-plane coordinates.

4.2. *Eigenvalues via standard limit circle theory*

Eigenvalues are characterized (cf. Zettl [**23**]) by the two boundary conditions

$$u(1)=0 \quad \text{and} \quad [u,u_0]_\infty = \lim_{r\to\infty} r(u'(r)u_0(r) - u(r)u_0'(r)) = 0,$$

where u_0 is a maximal domain function, that is,

$$u_0 \in L^2(0,\infty, rw(r)), \quad (w(r)r)^{-1}(-(ru_0')' - ru_0) \in L^2(0,\infty; w(r)r).$$

For (29), such functions are given by linear combinations of $\cos(t)/\sqrt{t}$ and $\sin(t)/\sqrt{t}$, which arise as solutions of the differential equation (29) for $\lambda=-1/4$. We choose the particular function

$$u_0 = \sin(1)\cos(t)/\sqrt{t} - \cos(1)\sin(t)/\sqrt{t}$$

since it also satisfies the Dirichlet condition at $t=1$. Given this information, Sleign2 computes the eigenvalues shown in Table 1 (we have shifted the numbering in order to match our notation).

4.3. *Eigenvalues via phase-plane coordinates*

With $Q(r)=r^2$, the Prüfer transformation is given by

$$ru' = \rho\cos(\phi), \quad ru = \rho\sin(\phi) \tag{30}$$

and the pair of equations

$$\phi' = 1 + \frac{\lambda}{r^2}\sin(\phi)^2 + \frac{1}{r}\sin(\phi)\cos(\phi), \tag{31}$$

$$\rho' = \rho\Big(-\frac{\lambda}{r^2}\sin(\phi)\cos(\phi) + \frac{1}{r}\sin(\phi)^2\Big). \tag{32}$$

We restate the results from Lemma 17–19, Proposition 20 and 21. The proof is the same as before.

LEMMA 17′. *Suppose that $\phi(r;\lambda)$ is a tamed solution of* (31).

(i) *For $r \geqslant R_0$, we have*

$$\phi(r;\lambda) = \phi(R_0;\lambda) + (r - R_0) + \lambda K_1(r;\lambda,R_0) + K_2(r;\lambda,R_0),$$

where K_1, K_2 have the same properties as in Lemma 17.

(ii) $\phi'(r;\lambda) = r + O(1/r)$, $|\phi''(r;\lambda)| = O(1/r)$ *as $r \to \infty$ uniformly for λ in bounded intervals.*

LEMMA 18′. *Suppose that $\phi(r;\lambda)$ is a tamed solution of* (31). *Then*

$$\int_0^r \frac{|\sin(\phi(s;\lambda))|^2}{s}\,ds = \frac{1}{2}\log r + O(1) \quad \text{as } r \to \infty$$

uniformly for λ in bounded intervals.

LEMMA 19′. *Suppose that $\phi(r;\lambda)$ is a tamed solution of* (31) *and $\rho(r;\lambda)$ is the corresponding solution of* (32). *Then $c(\lambda)\sqrt{r} \leqslant \rho(r;\lambda) \leqslant C(\lambda)\sqrt{r}$ for large $r \geqslant R_0$.*

PROPOSITION 20′. *For every $\lambda \in \mathbb{R}$, the function $u(r;\lambda)$ obtained from $\phi(r;\lambda)$ and $\rho(r;\lambda)$ by the Prüfer transformation* (30) *belongs to the space $L^2(0,\infty;w(r)r)$.*

PROPOSITION 21′. *Let $\lambda_0, \lambda \in \mathbb{R}$ be two given values, and let $\phi = \phi(r;\lambda)$, $\phi_0 = \phi(r;\lambda_0)$. Moreover let $u = u(r;\lambda)$ and $u_0 = u(r;\lambda_0)$ be obtained by the Prüfer transformation as in Proposition* 20′. *The following are equivalent.*

(i) $\lim_{r\to\infty} \phi(r;\lambda) - \phi(r;\lambda_0) = 0 \bmod \pi$.

(ii) $[u,u_0]_\infty = \lim_{r\to\infty} r(u'(r)u_0(r) - u(r)u_0'(r)) = 0$.

On the basis of part (ii) of Proposition 21′, we started the following calculations: guaranteed enclosures of solutions $\phi(x;\lambda)$ of the initial value problem (31) are computed for various values of λ by the Awa code (cf. Lohner [**18**]), as described in Example 16. In Figure 4, we have plotted $\phi(r;\lambda)$ on the interval $[1,100]$ for the given values of λ. The labelled line represents $\phi(r;\lambda_0)$ for $\lambda_0 = -1/4$. Since solutions depend in a very weak way on the parameter λ, the figure does not contain much information. It is more illustrative to plot the difference $\phi(r;\lambda) - \phi(r;\lambda_0)$, as shown in Figure 5. However, since the λ-dependence is so subtle, we can only give bounds on λ_i. The given values for $\lambda_1, \lambda_2, \lambda_{-1}$ are such that

$$\begin{aligned}
\phi(R_0;\lambda_1) - \phi(R_0;\lambda_0) &> \pi && \text{at } R_0 = 4.35E+003,\\
\phi(R_0;\lambda_2) - \phi(R_0;\lambda_0) &> 2\pi && \text{at } R_0 = 2.90E+003,\\
\phi(R_0;\lambda_{-1}) - \phi(R_0;\lambda_0) &< -\pi && \text{at } R_0 = 1.93E+003.
\end{aligned}$$

Together with the monotonicity of $\phi(r;\lambda)$ with respect to λ, this provides the guaranteed bounds as shown in Figure 5. By comparison with the Sleign2 results,

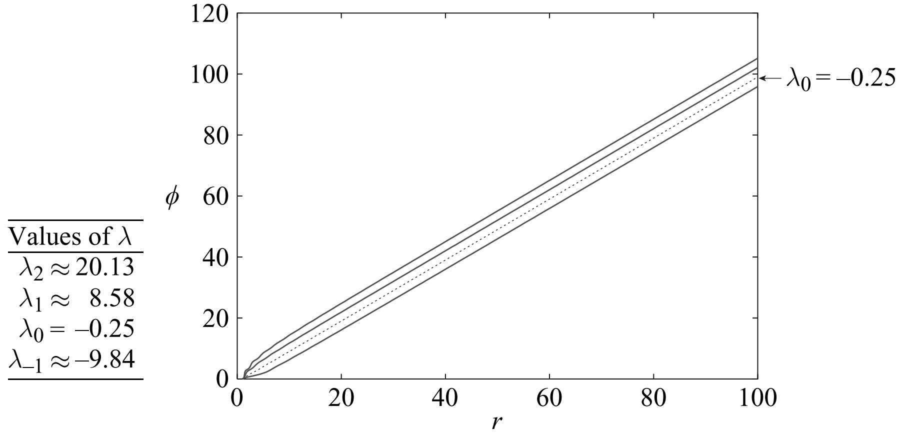

FIGURE 4. $\phi(r;\lambda)$ *for* λ_{-1} *up to* λ_2.

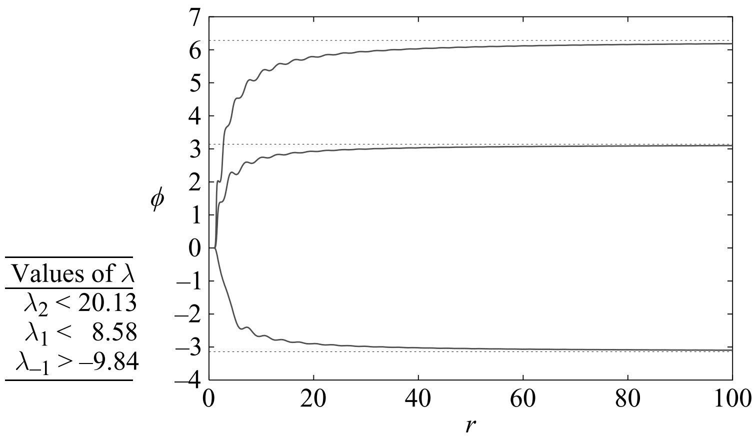

FIGURE 5. *Difference of* $\phi(r;\lambda)$ *from the reference solution* $\phi(r;\lambda_0)$.

we see that the phase-plane approach reasonably confirms $\lambda_{-1}, \lambda_1, \lambda_2$. However, λ_0 is clearly computed with considerable error by Sleign2, since its known exact value is -0.25.

5. *Open questions*

We finish with a selection of questions which remain open.

(1) Can one generalize Theorem 1 to potentials $q(r)$ satisfying only $q(r) \to \infty$ as $r \to \infty$?

(2) Can one find in Theorem 3 a class of potential that is more general than pure powers?

(3) What is the asymptotic distribution of eigenvalues in Theorems 1 and 3?

(4) Can one approximate the eigenvalues obtained in Theorem 1 by sequences of eigenvalues of regular problems over $[0, R]$ with $u'(0) = u(R) = 0$ by letting $R \to \infty$? This is the case for $p = 2$ (cf. Bailey *et al.* [**2**]).

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