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A modified Lax–Wendroff correction for wave propagation in media described by Zener elements

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SUMMARY

A modified Lax–Wendroff correction for wave propagation in attenuating and dispersive media described by Zener elements is presented. As opposed to the full correction, this new technique is explicit and offers large computational savings. The technique may be applied to a wide variety of hyperbolic problems. Here, the concept is illustrated for wave propagation in visco-acoustic media.

Key words: attenuation, electromagnetic modelling, finite-difference methods, seismic modelling, viscoelasticity.

INTRODUCTION

Explicit finite-difference schemes are commonly employed to solve wave-propagation problems. The temporal accuracy of such schemes can be increased by employing a Lax–Wendroff correction (e.g. Lax & Wendroff 1964; Dablain 1986; Gustafsson, Sjöberg & Abrahamsson 1988; Sei 1991). In the literature, this technique is also referred to as a modified scheme or, somewhat ambiguously, a compact scheme. The approach yields drastic improvements in numerical characteristics of finite-difference schemes (Dablain 1986).

During the last decade, convolutional formulations of constitutive relations to model wave propagation in attenuating and dispersive media have been employed extensively (e.g. Emmerich & Korn 1987; Carcione, Kosloff & Kosloff 1988; Kunz & Luebbers 1993; Robertsson, Blanch & Symes 1994). By employing an array of Zener elements (standard linear solids in visco-acoustics), an essentially arbitrary quality factor, Q, versus frequency, f, behaviour can be obtained (Blanch, Robertsson & Symes 1995). The quality factor, $Q \equiv \mathcal{R}e(c^2)/\mathcal{I}m(c^2)$, where $c(f)$ is the phase velocity as a function of frequency, roughly corresponds to the number of wavelengths that a wave can propagate before its amplitude decays by a factor of $e^{-\pi}$ (White 1992). Unfortunately, as we will show in this paper, the full Lax–Wendroff correction is generally not a good candidate for this class of scheme.

We present a modified Lax–Wendroff correction with numerical properties as good as or even better than the full correction. Additionally, the computational cost in terms of calculations per gridpoint of the new scheme is only increased by roughly 30 per cent. The technique may be applied to various hyperbolic problems, including acoustic, elastic and electromagnetic wave propagation. Here, we illustrate the method for the 1-D viscoacoustic case, where we increase the accuracy in time from second to fourth order.

LAX–WENDROFF FINITE-DIFFERENCE SCHEME

To increase the temporal accuracy of a finite-difference scheme without expanding the temporal stencil, a Lax–Wendroff correction may be used (Lax & Wendroff 1964). For a function $u(t, x)$ described by a hyperbolic system of partial differential equations, the temporal accuracy of a staggered (Virieux 1986) second-order accurate finite-difference approximation, $u_{\Delta t} = (u^{n+1/2} - u^{n-1/2})/\Delta t$, may be increased by expressing $u_{,ttt}$ through spatial derivatives, since

$$u_{,t} = u_{\Delta t} - \frac{\Delta t^2}{24} u_{,ttt} + O(\Delta t^4), \tag{1}$$

where Δt denotes the discretization in time and n denotes the discrete time level.

In particular, the following equations describe visco-acoustic wave propagation in one dimension:

$$\begin{cases} p_{,t} = -\Pi_{,t} * v_{,x} \\ v_{,t} = -\dfrac{1}{\rho} p_{,x} \end{cases}, \tag{2}$$

where p is the pressure, v the velocity, ρ the density and $\Pi(t)$ is the relaxation function (Robertsson *et al.* 1994). Wave propagation in real-Earth media may be modelled through an array of L Zener elements (Blanch *et al.* 1995):

$$\Pi(t) = K\left(1 + \sum_{l=1}^{L} (\tau_l e^{-t/\tau_{\sigma l}})\right) H(t), \tag{3}$$

where K is the relaxed bulk modulus, τ_l is a measure of

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attenuation strength, $\tau_{\sigma l}$ is a stress-relaxation time defining the frequency for the maximum attenuation for element l, and $H(t)$ is the Heaviside function (Blanch *et al.* 1995). Introducing a memory variable, r, for the special case of $L=1$ yields (Robertsson *et al.* 1994):

$$\begin{cases} p_{,t} = -K(1+\tau)v_{,x} - r \\ v_{,t} = -\dfrac{1}{\rho}p_{,x} \\ r_{,t} = -\dfrac{1}{\tau_\sigma}(r + K\tau v_{,x}) \end{cases} . \tag{4}$$

The full Lax–Wendroff corrections for increasing a second-order accurate approximation of eq. (4) to fourth-order accuracy are

$$\begin{cases} p_{,ttt} = -K^2(1+\tau)^2\dfrac{1}{\rho}v_{,xxx} - \dfrac{K(1+\tau)}{\rho}r_{,xx} - \dfrac{1}{\tau_\sigma^2}r \\ \qquad -K\dfrac{1}{\tau_\sigma^2}\tau v_{,x} - K\dfrac{1}{\tau_\sigma}\dfrac{\tau}{\rho}p_{,xx} \\ v_{,ttt} = -\dfrac{1}{\rho^2}K(1+\tau)p_{,xxx} - \dfrac{1}{\rho}\dfrac{1}{\tau_\sigma}r_{,x} - K\tau\dfrac{1}{\rho}\dfrac{1}{\tau_\sigma}v_{,xx} . \\ r_{,ttt} = -\dfrac{1}{\tau_\sigma^3}r - K\tau\dfrac{1}{\tau_\sigma^3}v_{,x} - K\dfrac{\tau}{\rho}\dfrac{1}{\tau_\sigma^2}p_{,xx} \\ \qquad -K^2\dfrac{1}{\tau_\sigma}\tau(1+\tau)\dfrac{1}{\rho}v_{,xxx} - K\dfrac{\tau}{\rho}\dfrac{1}{\tau_\sigma}r_{,xx} \end{cases} \tag{5}$$

The full $O(4, 4)$ scheme is obtained by adding the correction terms in eq. (5) and approximating the first-order spatial derivatives by fourth-order accurate centred differences, and the spatial derivatives in the correction terms by second-order accurate approximations. Here, $O(4, 4)$ denotes a scheme fourth-order accurate in both time and space.

An alternative approach is to use eq. (2) to calculate the correction terms:

$$\begin{cases} p_{,ttt} = -\dfrac{1}{\rho}\Pi_{,t}*(\Pi_{,t}*v_{,xxx}) \\ v_{,ttt} = -\dfrac{1}{\rho^2}\Pi_{,t}*p_{,xxx} \end{cases} . \tag{6}$$

The factor $\Pi_{,t}$ contains one term with a Heaviside function and one term with a Dirac function. Since the Dirac function dominates the Heaviside function, we neglect the Heaviside function in the following approximation, which will effectively yield the full Lax–Wendroff correction for an acoustic equation, that is a wave equation without dispersion and attenuation. Hence, concentrating on wave propagation,

$$\begin{cases} p_{\Delta t} = -\Pi_{,t}*\left(v_{,x} + \dfrac{K(1+\tau)}{\rho}\dfrac{\Delta t^2}{24}v_{,xxx}\right) \\ v_{\Delta t} = -\dfrac{1}{\rho}\left(p_{,x} + \dfrac{K(1+\tau)}{\rho}\dfrac{\Delta t^2}{24}p_{,xxx}\right) \end{cases} . \tag{7}$$

Again, the convolution is eliminated by introducing a memory variable. A so-called *pseudo-O(4, 4) scheme* can be obtained by adding the correction terms as described in eq. (7).

The Lax–Wendroff corrections derived above are strictly only applicable to homogeneous media. To construct a Lax–Wendroff correction which is strictly applicable to heterogeneous media is a simple task (e.g. Sei & Symes 1994). The main purpose of this paper is not to explain the Lax–Wendroff correction itself, but to show that in the case of a medium described by a set of relaxation mechanisms (Zener elements) a large part of the full Lax–Wendroff correction can be neglected. In other words, the purpose of the paper is to make the use of a Lax–Wendroff correction feasible for media described by Zener elements.

NUMERICAL PROPERTIES

The correction terms for the full $O(4, 4)$ scheme (eq. 5) are considerably more computationally expensive than the 'pseudo' corrections in eq. (7). This would become particularly cumbersome in higher dimensions. Moreover, the terms with spatial second derivatives have to be implemented implicitly for the scheme to obtain the desired accuracy and hence require an even larger amount of computational effort. Additionally, the same terms in eq. (5) would lead to an extra stability condition, since they correspond to a heat equation which grows exponentially in time. Hence certain combinations of Δt, τ and τ_σ would lead to an ill-posed problem.

A thorough stability analysis of schemes of the same type as the pseudo-$O(4, 4)$ scheme is presented by Blanch (1995). The stability criterion for the pseudo-$O(4, 4)$ scheme is

$$\sqrt{\frac{K(1+\tau)}{\rho}}\frac{\Delta t}{\Delta x} < \frac{1}{\sqrt{D}}, \tag{8}$$

where Δx is the spatial discretization and D is the dimension in space. The value of the left-hand side of eq. (8) is commonly referred to as the Courant number. Note that we have increased the stability limit by a factor of 7/6 compared to the analogous scheme that is second-order accurate in time [referred to as the $O(2, 4)$ scheme; Robertsson *et al.* (1994)].

We derived the numerical dispersion relations for the $O(2, 4)$, the $O(4, 4)$ and the pseudo-$O(4, 4)$ schemes. Fig. 1 shows a set of dispersion and attenuation curves for a medium with one Zener element with $\tau = 0.04$ and $\eta = \Delta t/\tau_\sigma = 0.4$ for different fractions of the maximum stable Courant number. The phase velocity of the pseudo-$O(4, 4)$ scheme is nearly identical to that of the $O(4, 4)$ scheme (Figs 1a–c), clearly displaying the characteristic $O(4, 4)$ behaviour. The pseudo-$O(4, 4)$ scheme is superior overall to the $O(2, 4)$ scheme. In particular, close to the stability limit, an almost perfect fit to the analytical curve is obtained for the pseudo-$O(4, 4)$ scheme. In contrast to the $O(2, 4)$ scheme, an important quality of the pseudo-$O(4, 4)$ scheme is that the phase velocity is rarely overestimated (for most values of η and τ). This has practical consequences for the picking of first-breaks, for instance. Although numerical attenuation generally has less severe influence on the simulation results compared to numerical dispersion, the pseudo-$O(4, 4)$ scheme displays the best performance (Figs 1d–f).

In Fig. 2 we illustrate the insensitivity of the dispersion characteristics on η for the pseudo-$O(4, 4)$ scheme. Fig. 2(a) shows the difference between the numerical and the analytical dispersion for a wide range of η. Even though a value of $\eta = 100$ is unrealistically large and therefore of little interest, the curves are very similar (such large values of η correspond to peak attenuation frequencies of the Zener elements far above the simulation-source frequency range). Fig. 2(b) shows the corresponding curves for the attenuation. The error is higher for a value of $\eta = 100$, but overall the curves are similar.

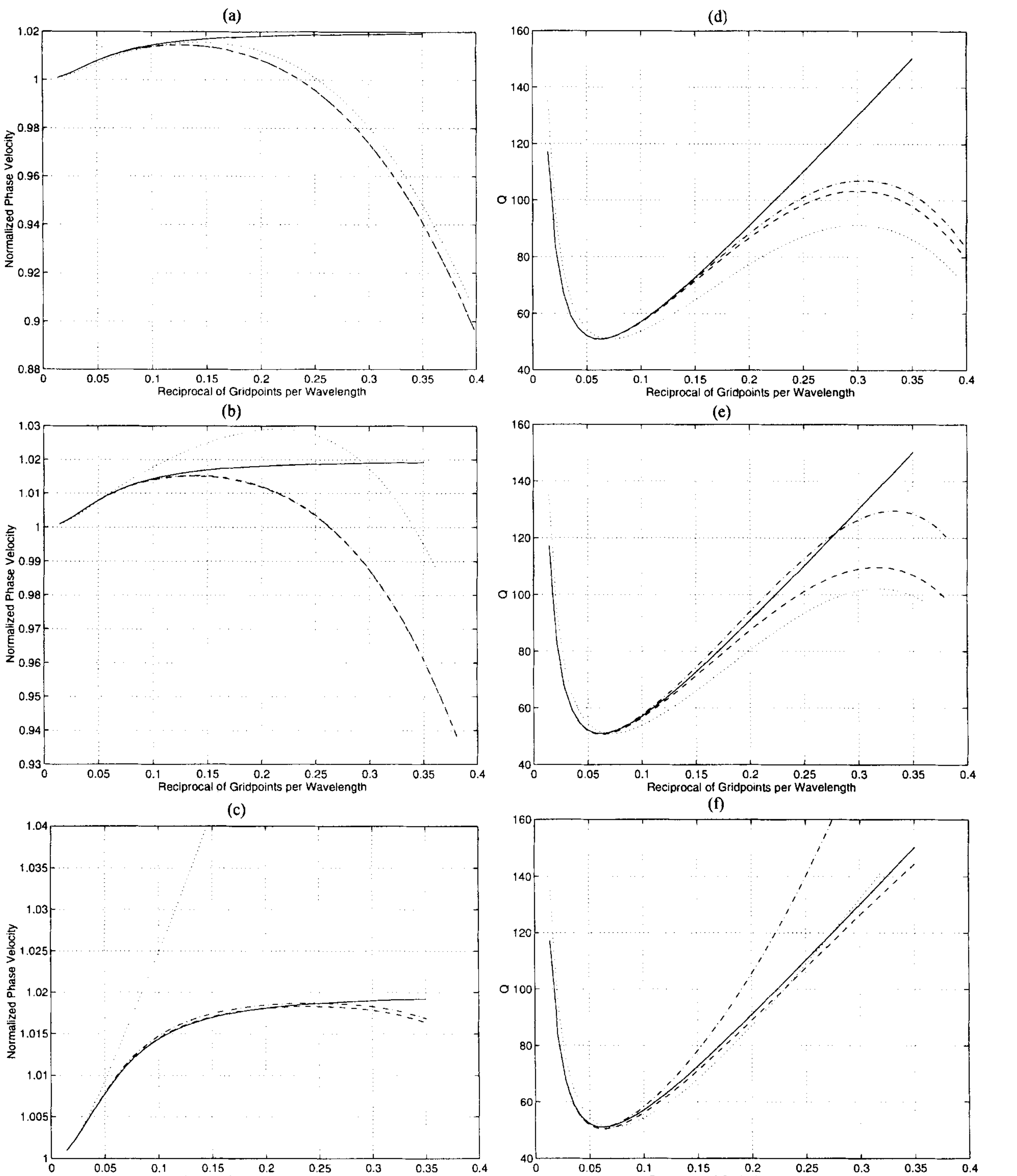

Figure 1. Dispersion (a–c) and attenuation (d–f) curves for an analytical solution (solid), the pseudo-$O(4, 4)$ scheme (dashed), the $O(4, 4)$ scheme (dash-dotted) and the $O(2, 4)$ scheme (dotted). $\tau = 0.04$. $\eta = 0.4$. Courant numbers: 33 per cent of stability limit (a and d), 66 per cent of stability limit (b and e) and 99 per cent of stability limit (c and f).

The value of τ is approximately $2/Q$ (Blanch *et al.* 1995). Fig. 3 shows the dispersion errors, which illustrate the insensitivity of the dispersion characteristics on τ for the pseudo-$O(4, 4)$ scheme. Fig. 3(a) corresponds to the acoustic case of $\tau = 0$, whereas in Fig. 3(b), a τ of 0.3 ($Q \approx 6$) is used. Curves corresponding to the same fraction of maximum stable Courant number display similar characteristics in Figs 3(a) and (b).

Since the scheme is relatively insensitive to both η and τ, it is possible to derive a general wavefield sampling criterion. For the pseudo-$O(4, 4)$ scheme, five gridpoints per wavelength are required to achieve less than 1 per cent numerical

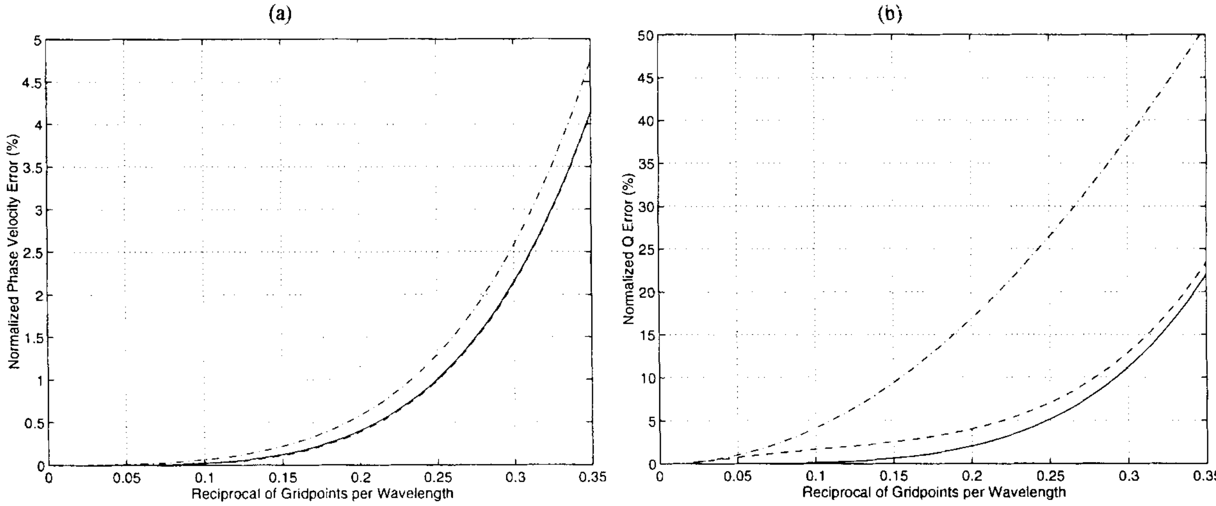

Figure 2. Dispersion (a) and attenuation (b) error curves in per cent for the pseudo-$O(4,4)$ scheme for various values of η at 80 per cent of the stability limit. $\tau = 0.04$. $\eta = 0.001$ (solid), $\eta = 0.5$ (dashed), $\eta = 100$ (dash-dotted).

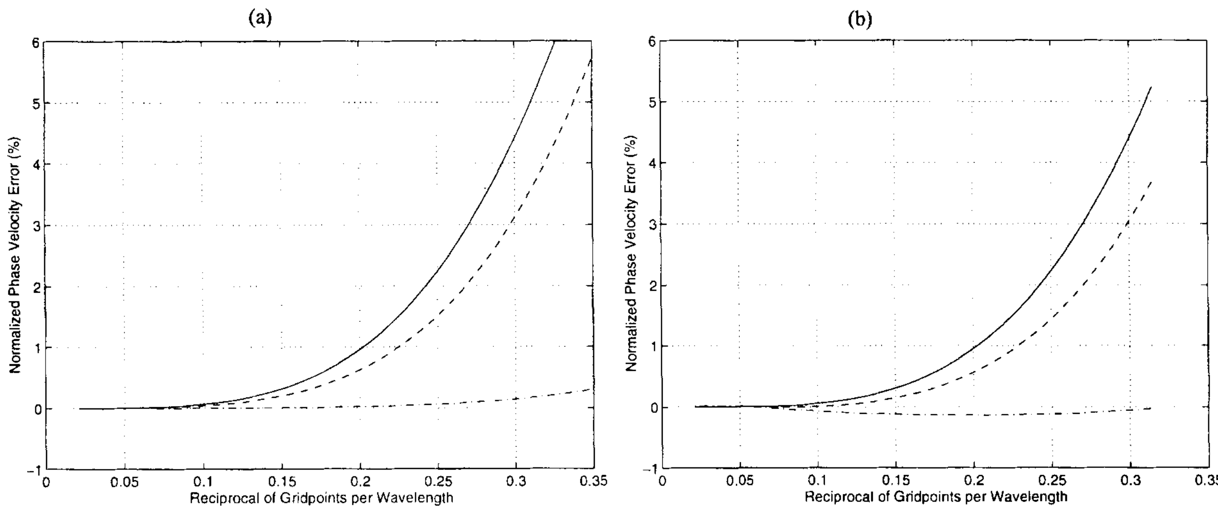

Figure 3. Dispersion error curves in per cent for the pseudo-$O(4,4)$ scheme for $\tau = 0$ (a) and $\tau = 0.3$ (b). $\eta = 0.5$. Courant numbers: 33 per cent of stability limit (solid), 66 per cent of stability limit (dashed) and 99 per cent of stability limit (dash-dotted).

dispersion, four gridpoints per wavelength yield less than 2.5 per cent and three gridpoints per wavelength yield less than 5 per cent. The $O(2,4)$ scheme requires eight gridpoints per wavelength to achieve less than 2 per cent numerical dispersion and five gridpoints per wavelength to achieve less than 5 per cent (Robertsson *et al.* 1994). In terms of computations per gridpoint, the pseudo-$O(4,4)$ scheme is roughly 30 per cent more expensive than the $O(2,4)$ scheme in one dimension.

To demonstrate the accuracy and applicability of the new scheme we computed a seismogram with an analytical solution based on Fourier methods and compared the result to two solutions computed with the pseudo-$O(4,4)$ and $O(2,4)$ finite-difference schemes. The solutions obtained through finite differences were computed with the same computational cost to make comparisons more fair. The $O(2,4)$ scheme has its best properties around a Courant number of 0.4, but using such a low Courant number would result in a higher computational cost for the solution obtained through the $O(2,4)$ scheme. Fig. 4 shows the good agreement between the analytical and the pseudo-$O(4,4)$ solutions. The solution obtained through the $O(2,4)$ scheme contains a prominent precursor, which can be expected from the dispersion curves (Fig. 1). The finite-difference solutions are obtained for a spatially quite sparsely sampled medium (six grid-points per minimum wavelength) and will thus show the shortcomings of the two schemes more prominently.

NUMERICAL ANISOTROPY IN TWO DIMENSIONS

In two or higher dimensions the discretization will cause numerical anisotropy. That is the computed wavefield will propagate with different velocities in different directions. The modified Lax–Wendroff correction for a scheme in two or more dimensions is

$$\begin{cases} p_{\Delta t} = -\Pi_{,t} * \left(\nabla \cdot \bar{v} + \dfrac{K(1+\tau)}{\rho} \dfrac{\Delta t^2}{24} \nabla \cdot (\nabla(\nabla \cdot \bar{v})) \right) \\ \bar{v}_{\Delta t} = -\dfrac{1}{\rho} \left(\nabla p + \dfrac{K(1+\tau)}{\rho} \dfrac{\Delta t^2}{24} \nabla(\nabla \cdot (\nabla p)) \right) \end{cases} . \qquad (9)$$

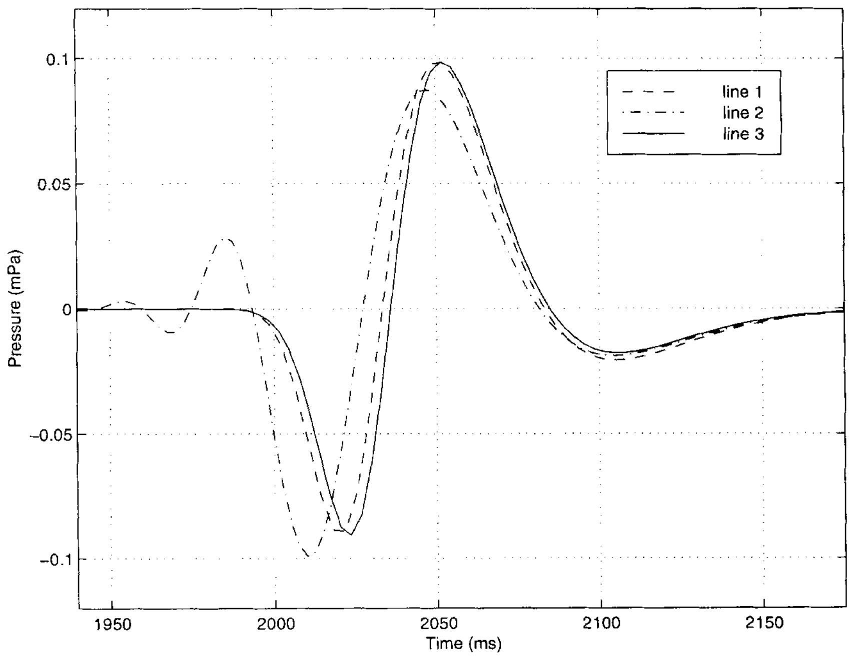

Figure 4. Numerical and analytical seismogram solutions for a Ricker wavelet. The numerical simulations were performed at 97 and 74 per cent of the stability limit. The Ricker wavelet had a centre frequency of 20 Hz with an original peak amplitude of 1 mPa. The wavelet propagated 2.0 km, which is 40 wavelengths for the centre frequency and approximately 88 wavelengths for the highest frequency, using approximately six gridpoints per wavelength for the highest frequency. The temporal steps were 3.8 ms for the modified $O(4,4)$ scheme and 2.5 ms for the $O(2,4)$ scheme. Spatial step 4.0 m, medium velocity 1.0 km s^{-1}, $\eta \approx 0.48$, $\tau = 0.04$. Numerical simulation results: pseudo-$O(4,4)$—line 1, $O(2,4)$—line 2, analytical result—line 3.

To investigate the numerical anisotropy we computed the dispersion relation for the scheme with the modified Lax-Wendroff correction in two dimensions. The numerical anisotropy is shown in Figs 5 and 6 for a range of propagation angles. The result is similar to those of other investigations (e.g. Sei 1991). Propagation along one of the grid directions is effectively the same as using a lower Courant number. Hence, similar behaviour in terms of accuracy and computational cost for one dimension can be expected in higher dimensions.

CONCLUSIONS

A modified Lax–Wendroff correction to increase the numerical accuracy of common finite-difference schemes for the simu-

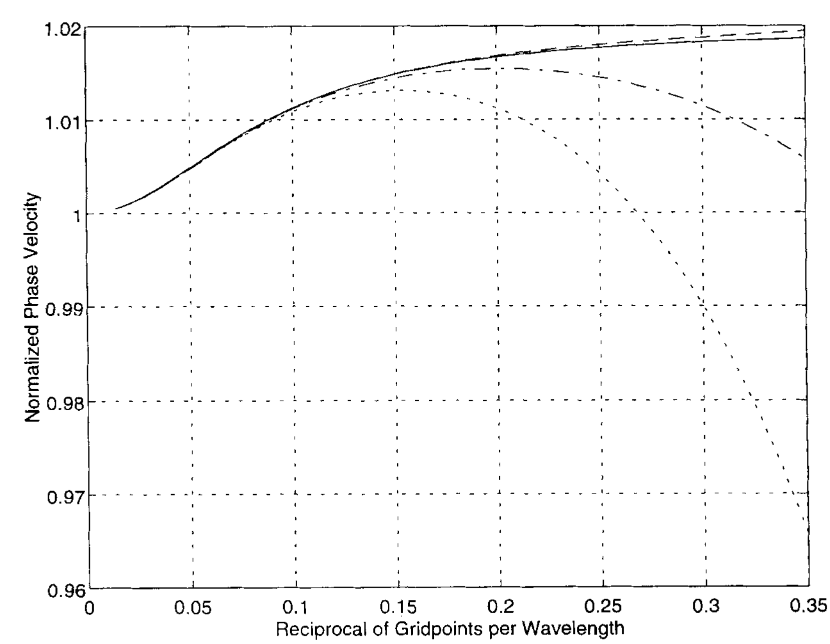

Figure 5. Numerical velocity anisotropy curves for the 2-D pseudo-$O(4,4)$ scheme for different propagation angles with a Courant number at 99 per cent of the stability limit. $\eta = 0.5$. Angles to one of the grid directions: 0° (dotted), 15° (dash-dotted), 45° (dashed), analytical (solid).

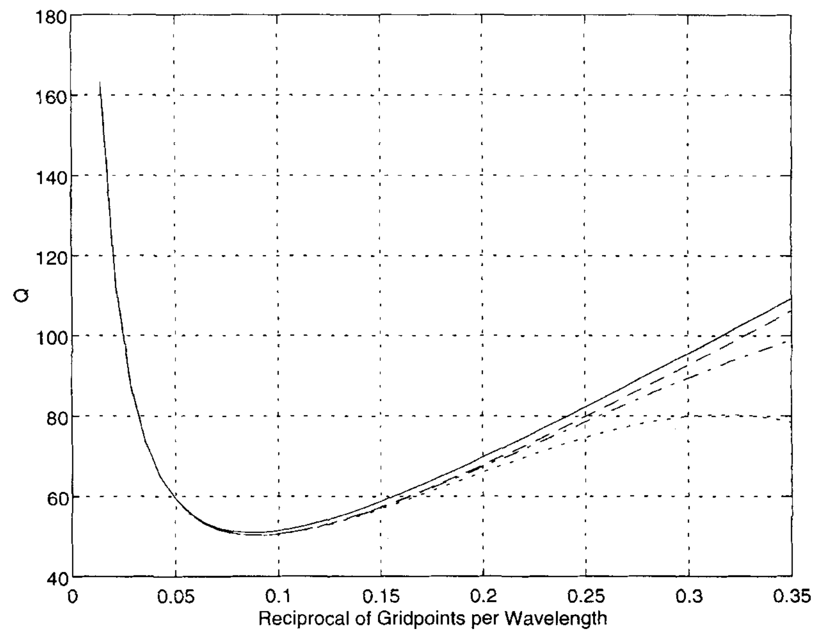

Figure 6. Numerical attenuation anisotropy curves for the 2-D pseudo-$O(4, 4)$ scheme for different propagation angles with a Courant number at 99 per cent of the stability limit. $\eta = 0.4$, $\tau = 0.04$. Angles to one of the grid directions: 0° (dotted), 15° (dash-dotted), 45° (dashed), analytical (solid).

lation of wave propagation in dispersive and attenuating media has been presented. A full Lax–Wendroff correction would result in an implicit scheme. Moreover, the full Lax–Wendroff scheme requires a much larger number of spatial derivatives to be calculated compared to the new scheme.

We have investigated the numerical properties of this new modified Lax–Wendroff correction applied to an $O(2, 4)$ scheme. The scheme, which is referred to as the pseudo-$O(4, 4)$ scheme, displays typical $O(4, 4)$ characteristics. A larger time step may be used in the pseudo-$O(4, 4)$ scheme compared to the $O(2, 4)$ scheme, since the stability limit is 7/6 times higher. To achieve the same level of numerical accuracy, it is sufficient to use approximately 40 per cent fewer gridpoints per wavelength in the pseudo $O(4, 4)$ scheme than in the $O(2, 4)$ scheme. Moreover, the pseudo-$O(4, 4)$ scheme is roughly only 30 per cent more computationally expensive than the $O(2, 4)$ scheme. In terms of computational cost, this amounts to large savings, particularly in two and three dimensions.

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