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The Carathéodory–Fejér Extension of a Finite Geometric Series

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It is shown that the Carathéodory–Fejér extension of a finite geometric series can be given explicitly up to a simple polynomial equation in an auxiliary variable. This result allows us to analyse the Carathéodory–Fejér approximation method in the case where the quotients of successive Maclaurin coefficients of the given function tend to a limit.

0. Introduction

THE THEOREM of Carathéodory & Fejér (1911) (or, briefly, CF theorem) states: *Given complex numbers $c_0, \ldots, c_K$, there is among all functions g that are analytic in the unit disc D and satisfy*

$$g(w) = c_0 + c_1 w + c_2 w^2 + \ldots + c_K w^K + O(w^{K+1}) \tag{0.1}$$

as $w \to 0$ a uniquely determined one for which the supremum norm

$$\|g\| := \sup\{|g(w)|;\ w \in D\}$$

attains a minimum. The optimal function $g = g^*$, which is called the CF extension of $c_0 + \ldots + c_K w^K$, is a scalar multiple of a finite Blaschke product with at most degree K. It is the only function of this type which satisfies (0.1).

(Note that one may assume $c_0 \neq 0$; otherwise the problem can be reduced to one with smaller K.) As is well known (Takagi, 1924/25)

$$g^*(w) = \sigma \frac{\bar{u}_0 w^K + \ldots + \bar{u}_K}{u_K w^K + \ldots + u_0} \tag{0.2}$$

is obtained by solving the singular-value problem $Cu = \sigma\bar{u}$, where

$$C := \begin{pmatrix} c_K & c_{K-1} & \cdots & c_0 \\ c_{K-1} & & ⋰ & \\ \vdots & ⋰ & & \\ c_0 & & & 0 \end{pmatrix}, \quad u := \begin{pmatrix} u_0 \\ \vdots \\ u_K \end{pmatrix}, \tag{0.3}$$

and σ must be chosen to be the *greatest* singular value of C.

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This singular-value problem is of course equivalent to a characteristic equation plus a linear system. However, in this paper we show that in the case $c_j = c^j$ ($c \neq 0$ fixed), it can be reduced to a particularly simple polynomial equation in an auxiliary variable; the singular value σ and the corresponding singular vector u are simple functions of the solution of this equation.

This result allows us to analyse a particular case of the Carathéodory–Fejér approximation method (briefly called CF method), which was proposed by Trefethen (1981*a*) and has since been generalized in various directions. (For more details and references see Ellacott & Gutknecht, 1983; Gutknecht, 1983; Trefethen, 1981*b*.)

The smaller singular values of C yield examples for Takagi's generalization of the CF theorem (Takagi, 1924/25; Gutknecht, 1983). There one tries to determine a function g satisfying (0.1) that is meromorphic in $\bar{D}$, has at most a fixed number ν ($\leqslant K$) of poles in D, is bounded near the unit circle, and has minimal norm on D. In general, there exist irregular cases where the singular-value problem leads to a function g that does not match all the given derivatives at zero, but this can be excluded in our example.

1. The CF Extension of a Finite Geometric Series

The basic result of this paper is:

THEOREM 1.1 (i) *The finite geometric series* $1+cw+c^2w^2+\ldots c^Kw^K$ $(c>0, K\geqslant 0)$ *has the CF extension*

$$g^*(w) := \sigma_0 \frac{U_K(x_0)w^K+\ldots+U_0(x_0)}{U_0(x_0)w^K+\ldots+U_K(x_0)}, \tag{1.1}$$

where U_l *denotes the lth Chebyshev polynomial of the second kind (which is of degree* l*),* x_0 *is the (algebraically) largest zero of* $U_{K+1}(x)-cU_K(x)$*, and*

$$\sigma_0 := U_K(x_0) \geqslant \max\{1, c^K\}.$$

(ii) *If* $c>1$*, then as* $K\to\infty$

$$\frac{g^*(w)}{c^Kw^{K+1}} \to \frac{c^2}{c^2-1}\cdot\frac{c-w}{cw-1} \tag{1.2}$$

uniformly on $\bar{D}$.

Note that the general case of a finite geometric series with complex ratio $c\neq 0$ can be reduced to the case $c>0$ treated here by replacing w by $w\,e^{i\arg c}$.

The proof of Theorem 1 is deferred to Section 3. It will be seen there that the algebraically smaller roots $x_j < x_0$ of $U_{K+1}(x) = cU_K(x)$ belong to singular values $\sigma_j = U_K(x_j) < \sigma_0$ of C (in the same order), that all these singular values are simple (for every $K\geqslant 0$), and that the singular vectors too are obtained by formally replacing x_0 by x_j. Hence, it is clear that Takagi's irregular case, which requires multiple singular values of the matrix obtained by deleting the first row and the last column of C, cannot occur here (cf. Takagi, 1924/25; Gutknecht, 1983). Therefore, Takagi's "regular" generalization of the CF theorem holds here and yields an extension of part (i) of Theorem 1.1:

THEOREM 1.2 *The unique function g^* that*

(i) *is meromorphic in $\bar{D}$,*
(ii) *satisfies* (0.1),
(iii) *has at most v poles in D $(0 \leqslant v \leqslant K)$,*
(iv) *is bounded in some annulus $\{w \in \mathbb{C};\ 1 \leqslant |w| < \rho(g)\}$,*

and minimizes $\|g\|$ among all functions g satisfying (i)–(iv) *is*

$$g^*(w) := (-1)^v \sigma_v \frac{U_K(x_v)w^K + \ldots + U_0(x_v)}{U_0(x_v)w^K + \ldots + U_K(x_v)}, \tag{1.3}$$

where $x_0 > x_1 > \ldots > x_K$ *denote the* $K+1$ *distinct zeros of* $U_{K+1}(x) - cU_K(x)$, *and* $\sigma_v := (-1)^v U_K(x_v)\ (>0)$.

Likewise the function g^* is the solution of the maximum problem related to a theorem of Landau cited by Takagi (1924/25, p. 90) and Gutknecht (1983, Th. 1.1 (v)).

Notable formulae following from our proof in Section 3 are

$$x_v = \frac{1}{2}\left(c + \frac{1}{c} - \frac{1}{\sigma_v^2 c^{2K+1}}\right), \quad v = 0, \ldots, K, \tag{1.4}$$

cf. Equation (3.2b), and the inequalities

$$\xi_1^{(K+1)} < x_0, \tag{1.5a}$$

$$\xi_{v+1}^{(K+1)} < x_v < \xi_v^{(K)}, \quad v = 1, \ldots, K, \tag{1.5b}$$

where $\xi_1^{(l)} > \xi_2^{(l)} > \ldots > \xi_l^{(l)}$ denote the zeros of U_l. By solving (1.4) for σ_v^2 and using (1.5) one obtains simple lower and (except for $v = 0$) upper bounds for σ_v.

A useful observation for testing programs is that if c is chosen to be $c = (K+2)/(K+1)$, then $x_0 = 1$ and the coefficients $U_l(1)$ in (1.1) equal $l+1$.

2. Application to the CF Method

Let $\mathscr{P}_m$ denote the space of complex polynomials p of degree m. $\mathscr{P}_m$ is a subspace of the space $\tilde{\mathscr{P}}_m$ of functions $\tilde{p}$ that are analytic in $\Omega := \{w \in \mathbb{C};\ 1 < |w| < \infty\}$, are bounded in every bounded subset of Ω, and have a pole of order at most m at ∞. Let f be a given function that is analytic in the unit disc D, and let

$$f_M(w) := \sum_{k=0}^{M} a_k w^k \quad (w \in D) \tag{2.1}$$

denote the Mth partial sum of its Maclaurin series. The polynomial CF method of Trefethen (1981*a*) for approximating f by a polynomial of degree $m < M$ is based on extending

$$f_M(w) - f_m(w) = a_{m+1} w^{m+1} + \ldots + a_{M+1} w^M$$

backwards to a function $q \in \tilde{\mathscr{P}}_M$ according to the CF theorem, i.e. $w \mapsto w^M q(1/w)$ is the CF extension of $a_M + \ldots + a_{m+1} w^{M-m-1}$. It is easy to see that $\tilde{p} := f_M - q \in \tilde{\mathscr{P}}_m$ is the best approximation of f_m out of $\tilde{\mathscr{P}}_m$. By deleting the terms with negative index in the series for $\tilde{p}$, one ends up with the *CF approximation* $p^{cf} \in \mathscr{P}_m$ that is normally very close to the best approximation of f_M, and hence of f if M was big enough.

Our results of Section 1 allow us to analyse the model problem with $a_j = a^j$, i.e.

$$f(w) := \frac{1}{1-aw}, \qquad f_M(w) := \frac{1-(aw)^{M+1}}{1-aw}. \tag{2.2}$$

The best approximation $p^* \in \mathscr{P}_m$ to f is known explicitly (Al'per, 1959; Rivlin, 1967),

$$p^*(w) = 1 + aw + \ldots + (aw)^{m-1} + \frac{(aw)^m}{1-|a|}. \tag{2.3}$$

The error curve

$$f(w) - p^*(w) = \frac{a^{m+1}}{1-|a|^2} w^m \frac{w-\bar{a}}{1-aw} \tag{2.4}$$

is exactly circular. (This is just an example where the generalization of the CF theorem due to Adamjan, Arov & Krein (1971) yields $\tilde{p} \in \mathscr{P}_m$, i.e. $\tilde{p} = p^*$.) For finite M, $\tilde{p} \notin \mathscr{P}_m$ and the error function $q = f_M - \tilde{p}$ looks far less simple. But since q is obtained by extending $a^{m+1}w^{m+1} + \ldots + a^M w^M$ backwards according to the CF theorem, we can just apply Theorem 1.1 after having divided by $(aw)^M$ and made two simple substitutions ($w \to 1/w, c := 1/a$). Assuming $0 < a < 1$ for simplicity again, we obtain:

THEOREM 2.1 *Let f_M be given by* (2.2) *with* $0 < a < 1$. *Let* $K := M - m - 1 \geq 0$, *and let x_0 be the* (*algebraically*) *largest root of* $aU_{K+1}(x) - U_K(x) = 0$. *Then the best approximation $\tilde{p}$ to f_M out of $\mathscr{P}_m$ is given by*

$$\tilde{p}(w) := f_M(w) - \sigma_0 a^M w^M \frac{U_K(x_0) + \ldots + U_0(x_0)w^K}{U_0(x_0) + \ldots + U_K(x_0)w^K}, \tag{2.5}$$

where $\sigma_0 := U_K(x_0) > 0$. *As* $m, M \to \infty$ *with* $M - m$ *fixed*, $\tilde{p}$ *converges uniformly on the unit circle* ∂D *to* p^* *given by* (2.3).

As mentioned, σ_0 is a simple singular value of a real symmetric matrix and $u := (U_0(x_0), \ldots, U_K(x_0))^T$ is a corresponding singular vector. Hence σ_0 and u are stable under perturbations of the matrix, and even for non-hermitian perturbations, cf. Stewart (1973). In particular, σ_0 and u are the limits of the corresponding quantities in the singular value problem belonging to any more general f whose coefficients a_j satisfy

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = a \tag{2.6}$$

with $0 < a < 1$, an assumption under which the CF method is known to work well in practice. In view of $a_{M-j} \sim a_M a^{-j}$ we obtain:

THEOREM 2.2 *Let* $\{a_j\}_{j=0}^{\infty}$ *satisfy* (2.6) *with* $0 < a < 1$, *let f_M be defined by* (2.1), *and let $\tilde{p}$ denote the best approximation to f_M out of $\mathscr{P}_m$. If* $m, M \to \infty$ *in such a way that* $M - m$ *is fixed, then*

$$\frac{f_M(w) - \tilde{p}(w)}{a_M w^M} \to \sigma_0 \frac{U_K(x_0) + \ldots + U_0(x_0)w^K}{U_0(x_0) + \ldots + U_K(x_0)w^K} \tag{2.7}$$

uniformly on ∂D, *where x_0 and σ_0 are the same as in Theorem* 2.1.

In particular, this theorem implies that as $m, M \to \infty$, $M-m = K+1$ fixed, the poles of all Blaschke products $q/\|q\| = (f_M - \tilde{p})/\|f_M - \tilde{p}\|$ lie in a disc $|w| \leqslant \xi$ with fixed $\xi < 1$. (In the error analyses of Hollenhorst (1976) and Trefethen (1981*a*), who require weaker assumptions than (2.6), this is only guaranteed if $|a| < 0{\cdot}43426\ldots$ or $\frac{1}{36}$, respectively.) By applying Lemma 1.2 of Ellacott & Gutknecht (1983), which is based on Cauchy's coefficient estimate, it follows that *the relative truncation error* $\|p^{cf} - \tilde{p}\|/\|f_M - \tilde{p}\|$ *tends to* 0 *geometrically as* $m, M \to \infty$ $(M-m$ *fixed*): for any $R \in (\xi, 1)$

$$\frac{\|p^{cf} - \tilde{p}\|}{\|f_M - \tilde{p}\|} = O(R^m),$$

where $\|f_M - \tilde{p}\| \sim \sigma_0 a_M$ according to (2.7). (Note that $K = M-m-1$, $q^- = p^{cf} - \tilde{p}$, and $\sigma = \|f_M - \tilde{p}\|$ in the lemma we refer to.) For further comments on this and related error estimates see Section 1 of Ellacott & Gutknecht (1983).

3. Proof of Theorem 1.1

The matrix C defined in (0.3) is real and symmetric since $c_j = c^j$ with $c > 0$. Hence, the singular-value problem $Cu = \sigma\bar{u}$ can be reduced to an eigenvalue problem $Cv = \tilde{\lambda}v$, where $\sigma = |\tilde{\lambda}|$ and $u = v$ if $\tilde{\lambda} > 0$, $u = iv$ if $\tilde{\lambda} < 0$. The formulae simplify if we substitute $a := 1/c$, $\lambda := \tilde{\lambda}/c^K$. Then $Cv = \tilde{\lambda}v$ is equivalent to

$$a^l \sum_{j=0}^{K-l} a^j v_j = \lambda v_l, \quad l = 0, \ldots, K.$$

Defining $v_{K+1} := 0$, $v_{-1} := v_0/a$ we obtain

$$\lambda(av_l - v_{l+1}) = a^{K+1} v_{K-l}, \quad l = 0, \ldots, K, \tag{3.1}$$

whence

$$\lambda(av_l - v_{l+1}) - \lambda(av_{l-1} - v_l)/a = a^K(av_{K-l} - v_{K-l+1}) = a^{2K+1} v_l/\lambda\cdot$$

or, after reordering,

$$v_{l+1} - 2x(a, \lambda)v_l + v_{l-1} = 0, \quad l = 0, \ldots, K, \tag{3.2a}$$

where

$$x := x(a, \lambda) := \frac{1}{2}\left(a + \frac{1}{a} - \frac{a^{2K+1}}{\lambda^2}\right). \tag{3.2b}$$

It is well known that the Chebyshev polynomials

$$T_l(x) = \cos(l \arccos x) \quad \text{and} \quad U_l(x) = \frac{\sin((l+1)\arccos x)}{\sin(\arccos x)},$$

$(l = 0, 1, \ldots)$ form together a fundamental set of solutions of (3.2a). Since (3.2a) is symmetric in v_{l+1} and v_{l-1}, this is also true for the reverse recursion starting at v_K, i.e. (3.2a) has the general solution $v_{K-j} = \alpha T_j(x) + \beta U_j(x)$. Moreover, since $U_0(x) \equiv 1$, $U_1(x) = 2x$, the recursion for U_l holds for $l = 0$ if we set $U_{-1}(x) :\equiv 0$. Hence, in view of $v_{K+1} = 0$,

$$v_j = U_{K-j}(x) \quad (j = -1, \ldots, K+1) \tag{3.3}$$

is the solution sought. [Note that a constant factor would cancel in (0.2).] Since

$v_{-1} = v_0/a$, x and a are directly related by

$$\frac{U_K(x)}{U_{K+1}(x)} = a = \frac{1}{c}. \tag{3.4}$$

The rational function U_K/U_{K+1} has exactly $K+1$ simple poles and K interlacing zeros in $(-1, 1)$ and is strictly decreasing on the whole real line except at these $K+1$ poles. For $x > 1$ ($x < -1$) it is positive (negative). Hence it is easy to see from the graph of this function that (3.4) (with $a > 0$) has exactly $K+1$ solutions $x_0 > x_1 > \ldots > x_K > -1$, which satisfy (1.5). Obviously, at most $x_0 > 1$; in fact $x_0 > 1$ iff $a < (K+1)/K+2)$ and $x_0 = 1$ iff $a = (K+1)/(K+2)$. If we denote the eigenvalue corresponding to x_ν by λ_ν, then by (3.2b), $|\lambda_0| > |\lambda_1| > \ldots > |\lambda_K|$. Moreover, from (3.1) with $l = K$ we conclude that $\lambda v_K = a^K v_0$, hence by (3.3)

$$\lambda_\nu = a^K U_K(x_\nu), \quad \tilde{\lambda}_\nu = U_K(x_\nu), \quad \nu = 0, \ldots, K. \tag{3.5}$$

Since the zeros of U_K interlace with the points x_ν, the eigenvalues λ_ν alternate in sign, λ_0 being positive. Hence $\sigma_\nu = (-1)^\nu \tilde{\lambda}_\nu$ and $u_l = v_l$ if ν is even, $u_l = iv_l$ if ν is odd. Finally, applying Cauchy's coefficient estimate to $g^*(w)$ yields

$$\sigma_0 = \|g^*\| \geqslant \max |c_l| \geqslant \max \{1, c^K\}.$$

This establishes the first part of Theorem 1.1 and the ingredients mentioned in Section 1 for the proof of Theorem 1.2.

Of course, v_j can also be written in terms of the powers of the reciprocal roots η and $1/\eta$ of the auxiliary equation $z^2 - 2x(a, \lambda)z + 1 = 0$ of recursion (3.2a) since these powers form another fundamental set of solutions. If $x > 1$, η and $1/\eta$ are easily seen to be real and positive: say, $\eta \in (0, 1)$. Moreover, it is well known [and in view of the relation $x = \frac{1}{2}(\eta + 1/\eta)$ easy to check] that $U_l(x) = (\eta^{l+1} - \eta^{-l-1})/(\eta - \eta^{-1})$ if $|x| \neq 1$. Thus,

$$v_j = \frac{\eta^{K-j+1} - \eta^{-K+j-1}}{\eta - \eta^{-1}} \quad (j = -1, \ldots, K+1). \tag{3.6}$$

In the case $c > 1$,

$$(1 + \ldots + c^K w^K)/c^K w^K \to 1/(w - c^{-1}) = 1/(w-a)$$

on ∂D as $K \to \infty$. Since $\tilde{\lambda}_0^2 = c^{-K}\sigma_0 \geqslant 1$, it follows from (3.2b) that $x_0 \to \frac{1}{2}[a + (1/a)] > 1$. Consequently,

$$\begin{aligned} \eta &= x_0 - (x_0^2 - 1)^{\frac{1}{2}} \to a, \\ \frac{\tilde{\lambda}_0}{c^K} &= \lambda_0 = a^K U_K(x_0) \to \frac{1}{1-a^2}, \end{aligned} \tag{3.7}$$

and $v_j/v_0 \to a^j$ (for j fixed). In fact,

$$|x_0 - \tfrac{1}{2}[a + (1/a)]| \leqslant a^{2K+1};$$

hence, there exist $\gamma > 0$ and $K_0 > 0$ such that for all $K > K_0$

$$|\eta - a| \leqslant \gamma a^{2K+1} \quad \text{and} \quad 0 < \eta < b := \tfrac{1}{2}(1+a).$$

Thus,

$$|\eta^j - a^j| = |\eta - a| \sum_{k=0}^{j} \eta^k a^{j-k} \leqslant (j+1)\gamma a^{2K+1},$$
$$\sum_{j=0}^{K} |\eta^j - a^j| \leqslant \tfrac{1}{2}\gamma a^{2K+1}(K+2)^2, \tag{3.8}$$

and this tends to 0 as $K \to \infty$. In view of (3.6),

$$|v_j/v_0 - a^j| = \left| \frac{\eta^j - \eta^{-j+2K+2}}{1-\eta^{2K+2}} - a^j \right| \leqslant \frac{|\eta^j - a^j| + b^{2K+2}(a^j + b^{-j})}{1-b^{2K+2}}, \quad j = 0, \ldots, K.$$

So, by using (3.8) we get for all $w \in \bar{D}$

$$\left| \sum_{j=0}^{K} (v_j/v_0) w^j - \sum_{j=0}^{\infty} (aw)^j \right| \leqslant \sum_{j=0}^{K} |v_j/v_0 - a^j| + \sum_{j=K+1}^{\infty} a^j$$
$$\leqslant \frac{1}{1-b^{2K+2}} \left[\tfrac{1}{2}\gamma a^{2K+1}(K+2)^2 + \frac{b^{2K+2} + a^{K+1}}{1-a} + \frac{b^{K+2}}{1-b} \right].$$

It follows that $\sum (v_j/v_0) w^j$ tends uniformly to $1/(1-aw)$ as $K \to \infty$. Likewise, $\sum (v_j/v_0) w^{-j} \to w/(w-a)$. Hence by (0.2) and (3.7)

$$\frac{g^*(w)}{c^K w^{K+1}} \to \frac{1}{1-a^2} \cdot \frac{1-aw}{w-a} = \frac{c^2}{c^2-1} \cdot \frac{c-w}{cw-1}$$

uniformly on $\bar{D}$. This concludes the proof of Theorem 1.1. ∎

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